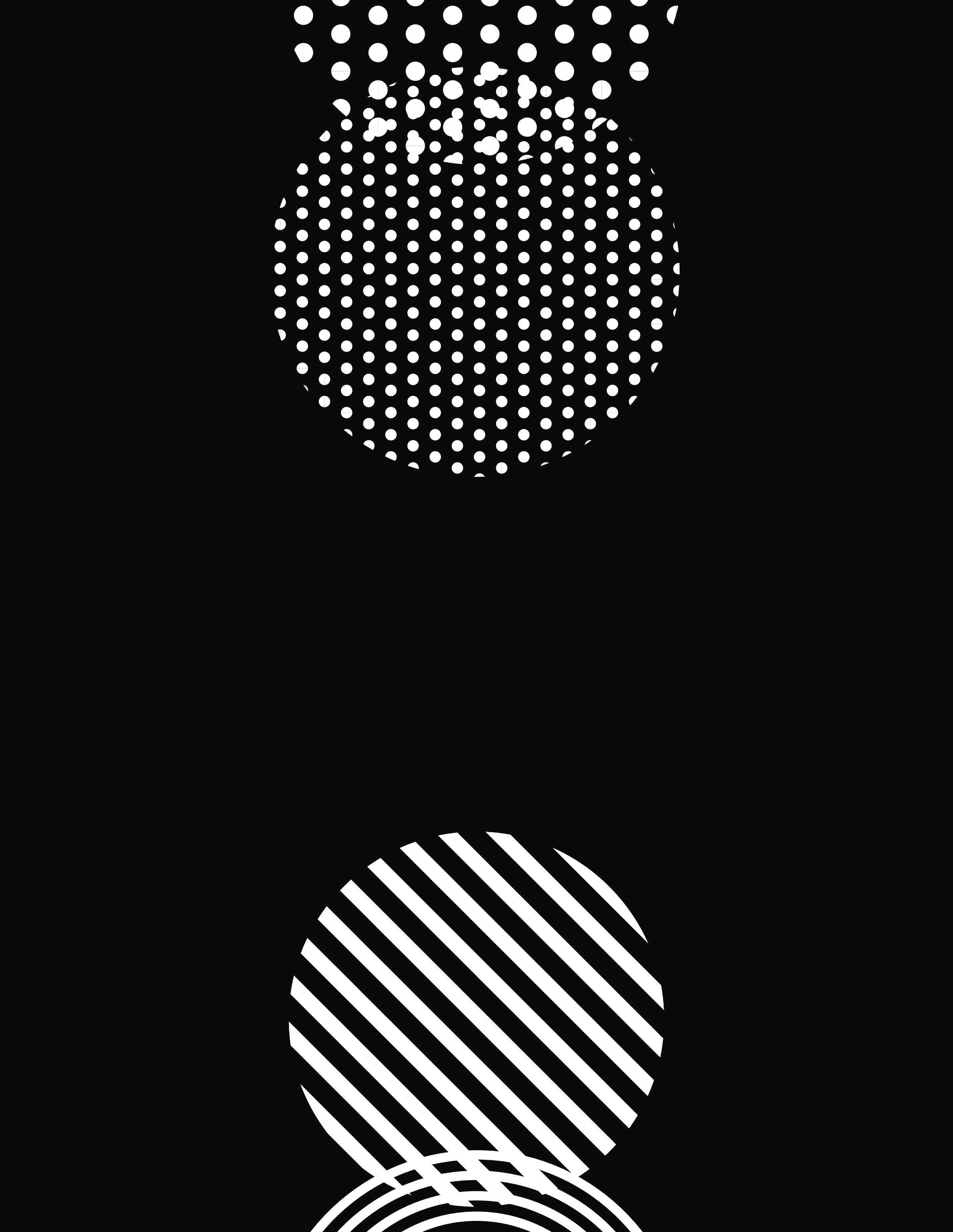


Problem 1. 1d hard-rod gas.

① The partition function is

$$Q_N(L,T) = \frac{1}{N!\,h^N} \int d^N x\, d^N p\; e^{-\beta H}$$

where

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<j} u(x_i - x_j)$$

It factorizes into "momentum" and "position" parts:

$$Q_N(L,T) = Z_{mom} \times Z_{pos}$$

where

$$Z_{mom} = \frac{1}{h^N} \prod_{i=1}^{N} \left[\int_{-\infty}^{+\infty} dp_i\, e^{-\frac{\beta}{2m} p_i^2} \right] = \left(\frac{2\pi m}{h^2 \beta} \right)^{N/2}$$

$$Z_{pos} = \frac{1}{N!} \int d^N x\; e^{-\beta \sum_{i<j} u(x_i - x_j)}$$

$$= \frac{1}{N} \int_{\text{ordered}} d^N x\; e^{-\beta \sum_{i<j} u(x_i - x_j)}$$

$$\boxed{Q_N(L,T) = \lambda^N \cdot Z_N(L)}$$

② We evaluate now the configurational integral.

For $N=1$ we have:

$$Z_1 = \int_0^L dx_1 = L$$

$$Z_2 = \frac{1}{2}\int_0^L dx_1 \cdot \int_{\substack{0 \\ |x_2 - x_1| > D}}^{L} dx_2 = \frac{1}{2} L \cdot \int_D^{L-D} dx_2 = \frac{1}{2} L \cdot (L - 2D)$$

$$Z_3 = \frac{1}{3} L \cdot \int_D^{L-2D} dx_2 \int_{x_2+D}^{L-D} dx_3 = \frac{1}{3} L \cdot \int_D^{L-2D} dx_2 \left(L - 2D - x_2\right)$$

$$= \frac{1}{3} L \left[(L-2D)^2 - \frac{1}{2}(L-2D)^2 - (L-2D) D + \frac{1}{2} D^2\right]$$

$$= \frac{1}{2 \cdot 3} L \left[(L-2D)(L-4D) + D^2\right] = \frac{L}{3!}(L-3D)^2$$

From which we infer the general expression:

$$\boxed{Z_N = \frac{L}{N!}(L - ND)^{N-1}}$$

③ The Helmholtz free energy is now:

$$\lambda = \left(\frac{2\pi m kT}{h^2}\right)^{1/2}$$

$$Q = \left(\frac{2\pi m kT}{h^2}\right)^{N/2} \cdot \frac{L(L-ND)^{N-1}}{N!} \quad ; \quad \ln N! = N\ln N - N$$

$$A = -kT\ln Q = -NkT\ln\lambda - kT\left(\ln\frac{L^N}{N!} - (N-1)\ln\left(1-\frac{ND}{L}\right)\right)$$

$$= -NkT\left(\ln\lambda + \ln\frac{L}{N} + \ln\left(1-\frac{ND}{L}\right)\right)$$

$$= -NkT\ln\left(\lambda\rho^{-1}\left(1-\frac{ND}{L}\right)\right)$$

$$P = -\frac{\partial A}{\partial L} = NkT \cdot \frac{\frac{1}{N}\left(1-\frac{ND}{L}\right) + \frac{D}{N}\cdot\frac{ND}{L^2}}{\rho^{-1}\left(1-\frac{ND}{L}\right)}$$

$$= \frac{NkT}{L}\left(1-\frac{ND}{L}\right)^{-1}$$

$$\boxed{P\left(L-(N-1)D\right) = NkT}$$

④ The internal energy:

$$A = U - TS$$

$$S = -\frac{\partial A}{\partial T} = Nk\ln\left[\lambda\rho^{-1}\left(1-\rho D\right)\right] + \frac{NkT}{\lambda}\cdot\underbrace{\frac{\partial\lambda}{\partial T}}_{\frac{\lambda}{2T}}$$

$$U = A + TS = \frac{NkT}{2} \leftarrow \text{equipartition theorem}$$

compressibility:

$$\Rightarrow \frac{\partial P}{\partial L} = \frac{\partial}{\partial L}\left(\frac{NkT}{L-(N-1)D}\right) = -\frac{NkT}{\left(L-(N-1)D\right)^2} = -P\Big/\left(L-(N-1)D\right)$$

the compressibility is then:

$$k_T = -\frac{1}{L}\frac{\partial L}{\partial P} = \frac{1-\frac{N-1}{L}D}{P}$$

the compressibility vanishes for $\frac{N}{L} \sim D^{-1}$.

Problem 2 Bose gas in a harmonic trap

What's the density of states? The energy levels are:

$$\epsilon_{l_1,l_2,l_3} = \hbar[l_1\omega_1 + l_2\omega_2 + l_3\omega_3] + \text{const}, \quad l_\alpha = 0,1,\ldots$$

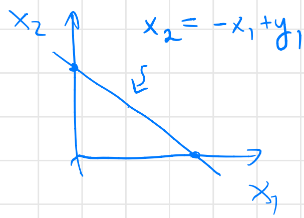
$$g(\epsilon) = \int_0^\infty dl_1 dl_2 dl_3 \, \delta(\epsilon - \hbar\omega_1 l_1 - \hbar\omega_2 l_2 - \hbar\omega_3 l_3)$$

$$= \frac{1}{\hbar^3\omega_1\omega_2\omega_3}\int_0^\infty dx_1 dx_2 dx_3 \, \delta(\epsilon - x_1 - x_2 - x_3) = \frac{\epsilon^2}{2(\hbar\omega_0)^3}$$

In 1d $\int_0^\infty dx \, \delta(\epsilon - x) = 1$ for $\epsilon > 0$

In 2d $\int_0^\infty dx_1 dx_2 \, \delta(\epsilon - x_1 - x_2) = \left\{ \begin{array}{l} y_1 = x_1 + x_2 \\ y_2 = x_2 \end{array} \right\}$

$$= \int_0^\infty dy_1 \, \delta(\epsilon - y_1) \cdot \int_0^{y_1} dy_2 = \epsilon$$



In 3D $\int_0^\infty dx_1 dx_2 dx_3 \, \delta(\epsilon - x_1 - x_2 - x_3)$

$$= \int_0^\infty dy_1 \, \delta(\epsilon - y_1) \int_0^{y_1} dy_2 \int_0^{y_2} dy_1 = \int_0^\epsilon dy_2 \cdot y_2 = \frac{1}{2}\epsilon^2.$$

Total particle number

$$N = \sum_{\epsilon} \frac{1}{z^{-1} e^{\beta \epsilon} - 1} = \overbrace{\frac{1}{z^{-1} - 1}}^{N_0} + \int g(\epsilon) \frac{d\epsilon}{z^{-1} e^{\beta \epsilon} - 1}$$

$$= N_0 + \int_0^\infty d\epsilon \frac{\epsilon^2}{z^{-1} e^{\beta\epsilon} - 1} \cdot \frac{1}{2(\hbar \omega_0)^3} = \left\{ \begin{array}{l} x = \beta \epsilon \\ d\epsilon = kT\, dx \\ \epsilon = kT x \end{array} \right\}$$

$$= N_0 + \left(\frac{kT}{\hbar\omega_0}\right)^3 \cdot \underbrace{\frac{1}{2} \int_0^\infty dx \frac{x^2}{z^{-1} e^{x} - 1}}_{g_3(z)} = N_0 + \left(\frac{kT}{\hbar\omega_0}\right)^3 g_3(z).$$

Number of particles in the excited states:

$$N_{exc} = N - N_0 = \left(\frac{kT}{\hbar\omega_0}\right)^3 g_3(z).$$

As we decrease the temperature, the fugacity $z \to 1$ which is its maximal value. Therefore maximal number of particles in the excited state is

$$N_{exc}^{max} = \left(\frac{kT}{\hbar\omega_0}\right)^3 g_3(1)$$ and is achieved

at the critical temperature:

$$N = N_{exc}^{max} = \left(\frac{kT_c}{\hbar\omega_0}\right)^3 \overbrace{g_3(1)}^{\zeta(3)}$$

$$\Rightarrow \frac{kT_c}{\hbar\omega_0} = \left(\frac{N}{\zeta(3)}\right)^{1/3}.$$

For $T < T_c$ the number of atoms in N_{exc} is

$$\frac{N_{exc}}{N} = \left(\frac{T}{T_c}\right)^{1/3} \quad \text{hence} \quad \frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{1/3}.$$

The energy is

$$U = \sum_{\epsilon} \frac{\epsilon}{z^{-1}e^{\beta\epsilon} - 1} = \frac{1}{2(\hbar\omega_0)^3} \int_0^\infty d\epsilon \, \frac{\epsilon^3}{z^{-1}e^{\beta\epsilon} - 1} = \left\{ \begin{array}{l} x = \beta\epsilon \\ dx = \beta\, d\epsilon \\ \epsilon = kTx \end{array} \right.$$

$$= \frac{(kT)^4}{2(\hbar\omega_0)^3} \int_0^\infty dx \, \frac{x^3}{z^{-1}e^{x} - 1} = \frac{(kT)^4}{2(\hbar\omega_0)^3} \Gamma(4)\, g_4(z)$$

$$\Gamma(4) = 3 \cdot 2$$

$$= 3\frac{(kT)^4}{(\hbar\omega_0)^3} g_4(z)$$

The heat capacity is discontinuous at critical temperature

Problem 3 2d bosons and fermions

The density of states:

Number of states with momentum equal or less P:

$$\Sigma(P) = \frac{1}{\hbar^2}\int d^2q\, d^2p = \frac{A}{\hbar^2}\pi p^2$$

the density of states: $a(p)\,dp = \frac{2A\pi}{\hbar^2}\, p\, dp$

expressed in terms of the energy $p = \sqrt{2m\epsilon}$

$dp = \frac{p\, d\epsilon}{2\epsilon}$ we get

$$a(\epsilon)\,d\epsilon = \frac{2A\pi}{\hbar^2}\cdot\frac{p^2}{2\epsilon}\,d\epsilon = \frac{2\pi m}{\hbar^2}\cdot A\, d\epsilon$$

For fermions:

$$N = \sum_\epsilon \frac{1}{z^{-1}e^{\beta\epsilon}+1} = \int_0^\infty d\epsilon\,\frac{a(\epsilon)}{z^{-1}e^{\beta\epsilon}+1} = \frac{2\pi m A}{\hbar^2}\cdot\int_0^\infty \frac{d\epsilon}{z^{-1}e^{\beta\epsilon}+1}$$

$$= \left\{\begin{array}{l} x = \beta\epsilon \\ d\epsilon = kT\,dx \end{array}\right\} = \frac{2\pi m A}{\hbar^2}\,kT\cdot f_1(z)$$

$$= \frac{A}{\lambda^2} f_1(z) \qquad \lambda = \sqrt{\frac{\hbar^2}{2\pi m kT}}$$

For the energy:

$$U_F = \int d\epsilon \, \frac{a(\epsilon)\,\epsilon}{z^{-1} e^{\beta\epsilon} + 1} = \frac{A\,kT}{\lambda^2} f_2(z)$$

We have:

$$f_1(z_F) = g_1(z_B)$$

$$\ln(1+z_F) = -\ln(1-z_B) \Rightarrow (1+z_F)(1-z_B) = 1$$

from which follows

$$z_F = \frac{z_B}{1-z_B} \Rightarrow 1+z_F = \frac{1-z_B+z_B}{1-z_B} = \frac{1}{1-z_B} \quad \checkmark$$

We use now that $z \frac{\partial}{\partial z} f_2(z) = f_1(z)$

therefore

$$f_2(z_F) = \int_0^{z_F} \frac{\ln(1+z)}{z}\, dz = \left\{ \begin{array}{l} z = \frac{z'}{1-z'} \; ; \; 1+z = \frac{1}{1-z'} \\ dz = \frac{1-z'+z'}{(1-z')^2}\, dz' \\ \quad = \frac{1}{(1-z')^2}\, dz' \end{array} \right.$$

$$= -\int_0^{z_B} \frac{\ln(1-z')}{z'(1-z')}\, dz' =$$

$$= \underbrace{-\int_0^{z_B} \frac{\ln(1-z')}{z'}\, dz'}_{g_2(z_B)} - \int_0^{z_B} \frac{\ln(1-z')}{1-z'}\, dz' = g_2(z_B) + \frac{1}{2}\log^2(1-z_B).$$

Therefore:

$$U_F = \frac{AkT}{\lambda^2} f_2(z_F) = \frac{AkT}{\lambda^2}\left(g_2(z_B) + \frac{1}{2}\log^2(1-z_B)\right)$$

$$= U_B + \frac{AkT}{2\lambda^2}\log^2(1-z_B)$$

but $\log^2(1-z_B) = g_1^2(z_B) = \frac{N^2}{A^2}\lambda^4$

$$U_F = U_B + \frac{AkT}{2\lambda^2}\cdot\frac{N^2}{A^2}\lambda^4 = U_B + \frac{kTN^2}{2A^2}\cdot\lambda^2$$

$$= U_B + \frac{\cancel{kT}N^2}{2A^2}\cdot\frac{h^2}{2\pi m\cancel{kT}} = U_B + \left(\frac{N}{A}\right)^2\cdot\frac{h^2}{4\pi m}.$$