HOLOGRAPHIC HARMONY: A TOPOLOGICAL ATLAS OF THE DECAPHONIC MUSICAL UNIVERSE

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Dedicated to Leszek Możdżer

ABSTRACT. Motivated by the recent construction of the acoustic Decaphonic Piano, this paper presents a comprehensive mathematical theory of harmony within the 10-tone equal temperament (10-TET). We generalize the classical notion of a triad by treating musical systems as solutions to the diophantine constraint $t+s \equiv q \pmod{10}$, where a generator interval q is partitioned into major (t) and minor (s) thirds. By systematically scanning the landscape of possible harmonies—particularly those generated by the 10-TET analog of the perfect fifth ($q = 7$) and the more acoustic choice ($q = 6$)—we uncover a rich topological diversity that defies the intuition of the standard 12-tone system.

Our analysis reveals that these harmonic worlds fall into four distinct topological classes, governed by the incidence geometry of Levi graphs. We encounter systems that mirror the famous Desargues configuration, "wormhole" topologies that fold the musical space, and degenerate systems that fracture the harmonic universe into disconnected, chiral realities. We culminate with the *Holographic Theorem*, which proves that the global topology of any such musical universe is encoded entirely within the local structure of a single chord. Finally, we provide a complete classification of all musical systems in 10-TET and demonstrate how affine isomorphisms can be used by composers to "translate" musical works between mathematically equivalent, yet perceptually distinct, harmonic realms.

1. Introduction

The number ten has long held a mystical status in the history of thought—from the Pythagorean *tetractys* to the metric completeness of the decimal system—yet in the evolution of Western music, it has remained a path not taken. For centuries, the dominance of the 12-tone equal temperament (12-TET) has conditioned our ears and our mathematics to a specific set of harmonic symmetries. But what happens when we step outside this familiar cage?

This paper is motivated by a tangible reality: the recent physical realization of the **Decaphonic Piano** [2], (see Appendix A for more details), a fully acoustic grand piano tuned to 10-tone equal temperament (10-TET). While the system's theoretical foundations [16] were established by Sethares [14] — who demonstrated its unique capacity for novel timbral consonance—and practical interest has grown through compositions by Hunt [7], Senpai [13], Sevish [15], and Hideya [6], the existence of this *acoustic* instrument forces a transition from theoretical speculation to practical necessity. How does one compose for such a world? If the familiar structural pillars of 12-TET—the circle of fifths, the mirror symmetry of major and minor triads—are removed, what new structures emerge to take their place?

1.1. The Search for Harmony.

In this work, we propose a generalized framework for harmony in any n-tone system, with a specific focus on the rich landscape of $n = 10$. We define a harmonic system not by acoustic approximation, but by combinatorial constraints: a musical system is determined by a partition of a "Fifth" (generator q) into a "Major Third" (t) and a "Minor Third" (s), satisfying the modular equation:

$$t + s \equiv q \pmod{n}.$$

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By fixing the generator q and scanning through all possible differences $\Delta = t - s$ (the "harmonic color"), we act as cartographers of this new universe. We pay particular attention to the "natural" generator ($q = 7$, the 10-TET perfect fifth) and the more "acoustic" generator ($q = 6$, corresponding to the interval of 720 cents), discovering that the mathematical soil of 10-TET yields a variety of behaviors far more exotic than the standard 12-tone model.

Crucially, while we focus here on $n = 10$, **this methodology provides a general blueprint for exploring harmonic topologies in any equal temperament.** The same analytical lens can be applied to the standard 12-TET—potentially revealing structurally distinct subsystems hidden within the familiar Western chromatic scale—or to higher-density microtonal systems like 31-TET. In this sense, the decaphonic system serves as a "laboratory": complex enough to exhibit non-trivial topological phenomena, yet compact enough to admit the complete classification we present in Section 10.

1.2. **A Landscape of Strange Behaviors.** Our exploration uncovers systems that behave like standard tonal music, and others that defy conventional logic. We find:

- **Geometric Crystallinity:** Three of the identified systems (including one based on the tritone) reveal a rigid, highly symmetric structure. We show that these are "cousins" of the famous Desargues configuration (10_3) from projective geometry, offering a harmonic space where navigation is governed by the laws of incidence geometry.
- **Universe Splitting and Modal Degeneracy:** We encounter systems where the "Circle of Fifths" breaks. In the most extreme cases, the harmonic world fractures into disconnected "universes"—a composer starting in one cannot reach the other without breaking the system's rules. We show how this degeneracy can be repaired by introducing a rigorous distinction between Major and Minor polarities, restoring connectivity at the cost of complexity.
- **Wormhole Topologies:** We identify systems where the graph of chordal connections folds onto itself, creating "wormholes" that allow for rapid modulation between distant tonal centers.

1.3. **The Holographic Principle.** The central mathematical contribution of this "minimonograph" is the application of graph theory to classify these disparate worlds. We utilize **Levi graphs**—structures traditionally used to study point-line incidence—to model the *Tonnetz* (the network of harmonic relations).

This topological approach leads to a surprising unification: the **Holographic Theorem**. We prove that the "shape" of the global musical universe (the graph of all possible modulations) is mathematically isomorphic to the "shape" of a single chord. The macrocosm is encoded in the microcosm.

1.4. **From Math to Music.** While the tools we use—affine geometry, combinatorics, and graph theory—are mathematical, our target is the practicing artist. In the final sections, we provide a complete classification of all possible 10-TET systems, reducing the chaos of possibilities to exactly four canonical topological classes.

This classification offers a powerful new tool for composition: **Isomorphic Translation**. We demonstrate that a piece of music composed in a "standard" system can be rigorously transformed via affine mapping into a completely different harmonic system. This preserves the logical syntax of the composition (the tension and release) while radically altering its semantic content (the sound and mood). Thus, the mathematician provides the map, but it remains the privilege of the musician to explore the territory.

2. The Choice of the Generator: Defining the "Fifth"

Our primary object of study is the 10-tone equal temperament (10-TET), where the octave is divided into ten logarithmically equal steps of ratio $2^{1/10}$. However, to construct a theory of harmony—to define triads, cycles, and modulations—we require more than just a scale; we need a **generator interval** q, which plays the structural role that the Perfect Fifth ($q = 7$) plays in standard Western music.

Why should one choice of q be privileged over another? While 10-TET is an equal temperament, its musical viability rests on how well its intervals approximate rational harmonic ratios.

2.1. The Acoustic Justification for $q = 7$.

Previous mathematical analysis of Pythagorean systems [10, 11] has established that among all small-step temperaments, the case $n = 10$ is distinguished by its remarkable approximation of the 13th harmonic. Specifically, the rational interval $g = 13/8 = 1.625$, or 840.528 cents, allows for the construction of a Pythagorean tuning that minimizes comma error more effectively than even the 12-tone approximation of 3/2.

When we map this rational generator onto the equal tempered grid, we find that:

$$2^{7/10} \approx 1.6245, \text{ or 840 cents.}$$

The approximation is exceptionally close. Thus, the interval of **7 steps** in 10-TET naturally assumes the role of the "Perfect Fifth"—the fundamental building block of harmonic stability.

2.2. A Structural Parallel.

There is a striking numerical coincidence that makes this choice even more compelling for the theorist.

In the classical 12-TET system, the generator ($g \approx 3/2$) corresponds to the interval index **7** (seven semitones). In the 10-TET system, the generator ($g \approx 13/8$) corresponds to the interval index **7** (seven deca-steps).

System	Steps per Octave	Generator Ratio	Generator Index (q)
Standard	$n = 12$	$3/2 = 1.5$	**7**
Decaphonic	$n = 10$	$13/8 = 1.625$	**7**

Table 1. The structural analogy between the generator in 12-TET and 10-TET. In both systems, the fundamental harmonic interval is found at index 7.

This alignment is explicitly demonstrated by comparing the full sets of Pythagorean intervals for both systems. In the $n = 10$ system (Table 2), the generator $g = 13/8$ falls precisely at Label 7:

Label	0	1	2	3	4	5	6	**7**	8	9
Interval	1	$\frac{2197}{2048}$	$\frac{32768}{28561}$	$\frac{16}{13}$	$\frac{169}{128}$	$\frac{371293}{262114}$	$\frac{256}{169}$	$\frac{13}{8}$	$\frac{28561}{16384}$	$\frac{4096}{2197}$

Table 2. Intervals of the 10-TET Pythagorean system. The generator (Label 7) is highlighted.

We observe the exact same structural placement in the classical $n = 12$ system with the generator $g = 3/2$:

Label	0	1	2	3	4	5	6	**7**	8	9	10	11
Interval	1	$\frac{256}{243}$	$\frac{9}{8}$	$\frac{32}{27}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{729}{512}$	$\frac{3}{2}$	$\frac{128}{81}$	$\frac{27}{16}$	$\frac{16}{9}$	$\frac{243}{128}$

Table 3. Intervals of the 12-TET Pythagorean system. Note that the generator is also Label 7.

This observation suggests that despite the acoustic differences (a "Fifth" based on the 13th harmonic rather than the 3rd), the structural logic of the 10-TET universe shares a deep functional affinity with standard Western harmony. This parallel forms the basis for our chord definitions in the following sections.

Accordingly, we adopt **$q = 7$** as the standard generator for defining our basic chords. However, unlike classical theory which rigidly adheres to the fifth, the combinatorial nature of our study will also lead us to explore the "acoustic" generator **$q = 6$** (720 cents interval), which, as we shall see, unlocks the geometric secrets of the system.

3. Generalization of Major and Minor Chords

To construct a harmonic theory for the 10-TET system, we must first establish a formal definition of "Major" and "Minor" chords in the language of modular arithmetic. We begin by rigorously defining these concepts for the classical 12-TET system, which will serve as our reference model.

3.1. Mathematical Definition of Chords.
In an arbitrary n-TET system, steps are labeled $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. Western harmony relies on three structural intervals:

(1) The perfect fifth, denoted by q.
(2) The major third, denoted by t.
(3) The minor third, denoted by s.

A **Major chord** (D_r) with root $r \in \mathbb{Z}_n$ is defined as the ordered triple:

$$D_r = (r,\ r+t,\ r+q) \pmod{n}. \tag{1}$$

A **Minor chord** (M_r) is structurally defined via the "inverse interval" logic. The minor third s must be the complement of the major third t within the fifth q:

$$s + t \equiv q \pmod{n}. \tag{2}$$

The minor chord with root r is then:

$$M_r = (r,\ r+s,\ r+q) \pmod{n}. \tag{3}$$

4. The Reference Model: The Classical 12-TET System

Before applying this framework to the 10-TET system, we must exhaustively describe the properties of the classical 12-TET system. This will provide the necessary baseline for comparison.

4.1. Solutions for Thirds in 12-TET.
In 12-TET, we are given $n = 12$ and the Pythagorean generator implies $q = 7$. A defining characteristic of the diatonic scale is that the major third is exactly one step larger than the minor third. We impose this constraint:

$$t - s \equiv 1 \pmod{12}.$$

Combined with $t + s \equiv 7$, we solve for t:

$$2t \equiv 8 \pmod{12}.$$

This congruence admits exactly two mathematical solutions:

(1) $2t = 8 \implies t = 4$. Consequently $s = 3$.
(2) $2t = 20 \implies t = 10$. Consequently $s = 9$.

From the perspective of Western harmony, the second solution ($t = 10$) corresponds to a minor seventh interval, which is too wide to function as a third. Thus, classical theory exclusively utilizes the first solution, defining the Major Third as 4 semitones and the Minor Third as 3 semitones.

4.2. **12-TET Chords and the Tonnetz.** Using the solution ($t = 4, s = 3$) and $q = 7$, we generate the 12-TET chords. In the standard chromatic scale, we map the note names to integers modulo 12, such that $C = 0, C\sharp = 1, D = 2, \ldots, B = 11$.

Label	0	1	2	3	4	5	6	7	8	9	10	11
Name	C	$C\sharp$	D	$D\sharp$	E	F	$F\sharp$	G	$G\sharp$	A	$A\sharp$	B

Table 4. Mapping of 12-TET note names to integer labels.

In this notation the 12-TET chords are:

All 12 Major Chords (12-TET)

Name	Root	Chord Tones
D_0	C, 0	{0, 4, 7}
D_1	$C\sharp$, 1	{1, 5, 8}
D_2	D, 2	{2, 6, 9}
D_3	$D\sharp$, 3	{3, 7, 10}
D_4	E, 4	{4, 8, 11}
D_5	F, 5	{5, 9, 0}
D_6	$F\sharp$, 6	{6, 10, 1}
D_7	G, 7	{7, 11, 2}
D_8	$G\sharp$, 8	{8, 0, 3}
D_9	A, 9	{9, 1, 4}
D_{10}	$A\sharp$, 10	{10, 2, 5}
D_{11}	B, 11	{11, 3, 6}

All 12 Minor Chords (12-TET)

Name	Root	Chord Tones
M_0	C, 0	{0, 3, 7}
M_1	$C\sharp$, 1	{1, 4, 8}
M_2	D, 2	{2, 5, 9}
M_3	$D\sharp$, 3	{3, 6, 10}
M_4	E, 4	{4, 7, 11}
M_5	F, 5	{5, 8, 0}
M_6	$F\sharp$, 6	{6, 9, 1}
M_7	G, 7	{7, 10, 2}
M_8	$G\sharp$, 8	{8, 11, 3}
M_9	A, 9	{9, 0, 4}
M_{10}	$A\sharp$, 10	{10, 1, 5}
M_{11}	B, 11	{11, 2, 6}

Figure 1. Complete enumeration of 12-TET chords.

4.3. **12-TET Chords and Their Tonnetz Assignments. Explanation of the assignment method:**

The **Euler-Riemann Tonnetz** for 12-TET is constructed by connecting chords that share exactly **two common tones**. In this system, every major chord is adjacent to exactly three minor chords:

- **Parallel (P):** $D_r \leftrightarrow M_r$ (share Root, Fifth).
- **Relative (R):** $D_r \leftrightarrow M_{r-3}$ (share Root, Third).
- **Leading-Tone (L):** $D_r \leftrightarrow M_{r+4}$ (share Third, Fifth).

In short, the method states that:

- Replace exactly one note of the chord with another note to form a minor chord.
- The resulting chord must share exactly two common tones (modulo 12) with the original chord.

Example 1: The chord D_0 (C Major) = {0, 4, 7}

- P transformation: $D_0 = \{0, 4, 7\} \to \{0, 3, 7\} = M_0$ (C minor). Common tones: {0, 7}.
- R transformation: $D_0 = \{0, 4, 7\} \to \{9, 0, 4\} = M_9$ (A minor). Common tones: {0, 4}.
- L transformation: $D_0 = \{0, 4, 7\} \to \{4, 7, 11\} = M_4$ (E minor). Common tones: {4, 7}.

Example 2: The chord D_7 (G Major) = {7, 11, 2}

- P transformation: $D_7 = \{7, 11, 2\} \to \{7, 10, 2\} = M_7$ (G minor). Common tones: {7, 2}.
- R transformation: $D_7 = \{7, 11, 2\} \to \{4, 7, 11\} = M_4$ (E minor). Common tones: {7, 11}.
- L transformation: $D_7 = \{7, 11, 2\} \to \{11, 2, 6\} = M_{11}$ (B minor). Common tones: {11, 2}.

Similarly for minor chords: we replace one note to form a major chord, preserving two common tones. **Example 3: The chord M_2 (D minor) = {2,5,9}**

- P transformation: $M_2 = \{2, 5, 9\} \to \{2, 6, 9\} = D_2$ (D Major). Common tones: {2, 9}.
- R transformation: $M_2 = \{2, 5, 9\} \to \{0, 2, 5\} = D_{10}$ ($A\sharp$ Major). Common tones: {2, 5}.
- L transformation: $M_2 = \{2, 5, 9\} \to \{5, 9, 0\} = D_5$ (F Major). Common tones: {5, 9}.

Example 4: The chord M_9 (A minor) = {9, 0, 4}

- P transformation: $M_9 = \{9, 0, 4\} \to \{9, 1, 4\} = D_9$ (A Major). Common tones: {9, 4}.
- R transformation: $M_9 = \{9, 0, 4\} \to \{5, 9, 0\} = D_5$ (F Major). Common tones: {9, 0}.
- L transformation: $M_9 = \{9, 0, 4\} \to \{0, 4, 7\} = D_0$ (C Major). Common tones: {0, 4}.

This leads to the following Tonnetz neighbor tables in integer notation:

Major chords → three minor chords

Major Chord	Three Minor Neighbors
D_0 (C)	M_0, M_4, M_9
D_1 (C♯)	M_1, M_5, M_{10}
D_2 (D)	M_2, M_6, M_{11}
D_3 (D♯)	M_3, M_7, M_0
D_4 (E)	M_4, M_8, M_1
D_5 (F)	M_5, M_9, M_2
D_6 (F♯)	M_6, M_{10}, M_3
D_7 (G)	M_7, M_{11}, M_4
D_8 (G♯)	M_8, M_0, M_5
D_9 (A)	M_9, M_1, M_6
D_{10} (A♯)	M_{10}, M_2, M_7
D_{11} (B)	M_{11}, M_3, M_8

Minor chords → three major chords

Minor Chord	Three Major Neighbors
M_0 (Cm)	D_0, D_8, D_3
M_1 (C♯m)	D_1, D_9, D_4
M_2 (Dm)	D_2, D_{10}, D_5
M_3 (D♯m)	$D_3, D_1 1, D_6$
M_4 (Em)	D_4, D_0, D_7
M_5 (Fm)	D_5, D_1, D_8
M_6 (F♯m)	D_6, D_2, D_9
M_7 (Gm)	D_7, D_3, D_{10}
M_8 (G♯m)	D_8, D_4, D_{11}
M_9 (Am)	D_9, D_5, D_0
M_{10} (A♯m)	D_{10}, D_6, D_1
M_{11} (Bm)	D_{11}, D_7, D_2

FIGURE 2. The connectivity of the 12-TET Tonnetz in integer notation.

These relationships produce the hexagonal honeycomb structure on the plane known as *the Euler – Riemann Tonnetz*, composed of triangles where each vertex is a chord, and adjacent chords share two tones. This is illustrated below in Figure 3 quoted from [3].

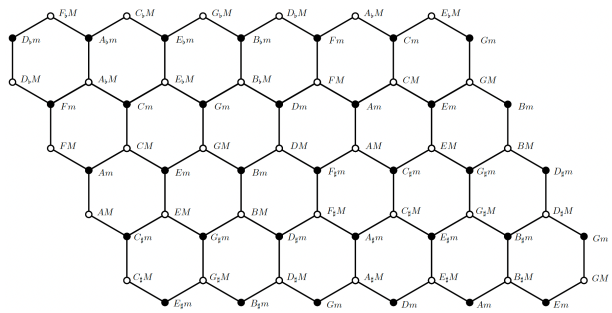


FIGURE 3. The 12-TET Tonnetz.

This Tonnetz can be also visualized as a Levi bipartite graph of a 12_3 configuration [3]:

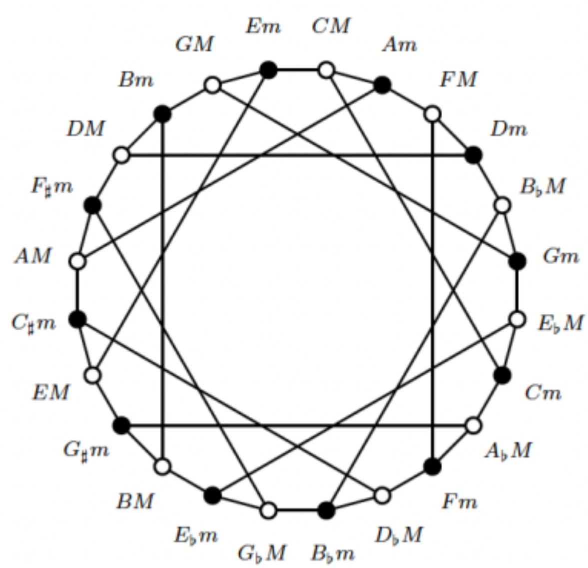


FIGURE 4. The 12-TET Tonnetz as a Levi bipartite graph of 12 points and 12 lines, sharing the incidence relation where each line contains 3 points, and every three lines meet at precisely one point [3].

5. The 10-TET System and the Choice of Thirds

We now return to Section 3.1 to define chords in 10-TET system. We have established that for the 10-TET the generator $g = 13/8$ dictates the fifth $q = 7$. However, the value of the major third t is a free parameter. To define a harmonic system analogous to Western diatonic harmony, we must determine the relationship between the major third (t) and the minor third (s).

5.1. **Possible gaps between the "thirds".** We recall equation (2) defining the main characteristic of the fifth in any n-TET system as the sum of thirds. When applied to 10-TET, this gives:

$$t + s \equiv 7 \pmod{10}. \tag{4}$$

We now systematically explore possible constraints on the difference $\Delta = t - s$.

5.1.1. *Hypothesis 1: The $\Delta = 1$ Difference.* The classical 12-TET system satisfies $t - s = 1$. Imposing this on 10-TET yields:

$$2t \equiv 8 \pmod{10}.$$

This admits two solutions:

- $t = 4$ (implies $s = 3$). This solution is analyzed in Section 6.1. We call the corresponding musical systems as **"The Narrow Thirds System"**.
- $t = 9$ (implies $s = 8$). This solution, which we call **"The Far Narrow Thirds System"**, is analyzed in Section 6.4.

5.1.2. *Hypothesis 2: The Even Difference.* One might ask if the major and minor thirds could differ by a larger step, for instance, $\Delta = 2$. This leads to the system:

$$\begin{cases} t + s \equiv 7 \\ t - s \equiv 2 \end{cases} \tag{5}$$

Summing the equations gives $2t \equiv 9 \pmod{10}$. In the ring of integers modulo 10, the quantity $2t$ must always be even. Since 9 is odd, **no integer solution exists**. Thus, it is mathematically impossible to construct a Major/Minor system in 10-TET where the thirds differ by 2 steps.

Of course the same argument shows that any even number $\Delta \in \mathbb{Z}_{10}$ makes the system

$$t + s \equiv 7 \quad \& \quad t - s \equiv \Delta \quad \& \quad \Delta \in \mathbb{Z}_{10} \text{ with } \Delta \text{ even},$$

incompatible. Thus, we have **no solutions for all Δ even**.

5.1.3. *Hypothesis 3: The $\Delta = 3$ Difference.* Next, we test the difference $\Delta = 3$:

$$\begin{cases} t + s \equiv 7 \\ t - s \equiv 3 \end{cases} \tag{6}$$

Summing gives $2t \equiv 10 \equiv 0 \pmod{10}$. The solutions are:

- $t = 0$ (implies $s = 7$). Trivial solution (unison and fifth).
- $t = 5$ (implies $s = 2$). This system is analyzed in Section 6.3. For reasons that will be clear in Section 6.3, we call this system **"The Tritone Systsem"**.

The solution $(t = 5, s = 2)$ is mathematically valid. However, the "Major Third" $t = 5$ corresponds to exactly half the octave ($\sqrt{2}$, or the tritone). A chord constructed this way would be $C = \{0, 5, 7\}$.

5.1.4. *Hypothesis 4: The $\Delta = 5$ Difference.* Here solutions are

- $t = 6$ (implies $s = 1$). This system, which we call **"The Wide Thirds System"** is analyzed in Section 6.2.
- $t = 1$ (implies $s = 6$). This is the same as $(t, s) = (6, 1)$.

5.2. **Structure of the Analysis: Musical Intuition and Mathematical Rigor.** The following analysis in Sections 6 and 7 conducts a systematic scan of 10-TET harmonic systems by varying the interval choices defined by the thirds t and s, and their difference Δ. Our primary objective in these sections is to explore the *phenomenology* of these harmonic worlds—focusing on how the resulting chordal networks (Tonnetze) are perceived and navigated by a musician.

To maintain the flow of musical intuition and ensure accessibility for a music-theoretical audience, we deliberately postpone the rigorous definitions of the underlying combinatorial structures to **Section 8**. While Sections 6 and Section 7 will frequently employ graph-theoretic terminology—such as cycles, paths, and symmetries—we use these concepts here in their heuristic, geometric sense as maps of harmonic space.

The mathematically inclined reader is advised that the formal underpinning of these "harmonic maps" is established in Section 8, where we rigorously define them as Levi graphs of specific point-line configurations and analyze their incidence geometry, girth, and automorphism groups. This separation allows the musical narrative to motivate the mathematics, ensuring that the formal definitions in Section 8 are seen as necessary tools to solve specific musical problems, rather than abstract prerequisites.

6. 10-TET systems with the fifth $q = 7$

The enumeration given in Section 5.1 exhausts all possibilities for systems with the 'fifth' $q = 7$. This provides the way to proceed, and to analyze in detail all these cases one after the other.

6.1. The "Narrow Thirds" Solution (4,3).

We first fully analyse the **Narrow Thirds** case $(t, s) = (4, 3)$. This choice is the **unique** solution that:

(1) Is mathematically solvable (unlike Δ is even).
(2) It is **narrow** in the distance between the thirds that are not far from the 12TET thirds, providing the closest mathematical analogy to the standard 12-TET-system.

Therefore first, in the next section, we proceed with the analysis of the solution $(t = 4, s = 3)$.

6.1.1. *Explicit Enumeration and the Discovery of Identity.* We formally define the chords for 10-TET using $t = 4$ and $s = 3$:

$$D_r = \{r, r+4, r+7\} \pmod{10}$$
$$M_r = \{r, r+3, r+7\} \pmod{10}$$

Let us list the chords explicitly as sets of integers to inspect their structure.

Major Chords (D_r)

Name	Root	**Set**
D_0	0	**{0, 4, 7}**
D_1	1	{1, 5, 8}
D_2	2	{2, 6, 9}
D_3	3	{3, 7, 0}
D_4	4	{4, 8, 1}
D_5	5	{5, 9, 2}
D_6	6	{6, 0, 3}
D_7	7	{7, 1, 4}
D_8	8	{8, 2, 5}
D_9	9	{9, 3, 6}

Minor Chords (M_r)

Name	Root	**Set**
M_0	0	{0, 3, 7}
M_1	1	{1, 4, 8}
M_2	2	{2, 5, 9}
M_3	3	{3, 6, 0}
M_4	4	{4, 7, 1}
M_5	5	{5, 8, 2}
M_6	6	{6, 9, 3}
M_7	7	**{7, 0, 4}**
M_8	8	{8, 1, 5}
M_9	9	{9, 2, 6}

FIGURE 5. Explicit pitch sets of 10-TET chords. Note the highlighted entries.

6.1.2. *Theorem of Modal Degeneracy.* Inspection of the tables above reveals a startling fact. Compare, for example, the major chord rooted at 0 with the minor chord rooted at 7:

- $D_0 = \{0, 4, 7\}$
- $M_7 = \{7, 0, 4\}$

These are the same set. This is not an isolated anomaly but a general property of the system.

Theorem 1 (Modal Identity)**.** *In the 10-TET system with* $(t = 4, s = 3)$*, the set of pitch classes forming a major chord* D_r *is identical to the set forming the minor chord* M_{r+7}*.*

$$D_r \equiv M_{r+7} \pmod{10}.$$

Proof. $D_r = \{r, r+4, r+7\}$. $M_{r+7} = \{(r+7), (r+7)+3, (r+7)+7\} = \{r+7, r+10, r+14\} \equiv \{r+7, r, r+4\}$. The sets are identical. □

6.1.3. *Musical Interpretation of Modal Degeneracy.* This mathematical identity has profound musical consequences for the 10-TET system.

- **Structural Uniqueness:** Unlike in 12-TET, where a Major triad (e.g., C-E-G) and its relative Minor (A-C-E) are distinct objects sharing two tones, in 10-TET they are the *same physical object* sharing all three tones. In fact, there are only 10 unique triads in the entire $(4, 3)$ solution 10-TET Universe.
- **Context-Dependent Perception:** Since the interval set $\{4, 3, 3\}$, corresponding to a passage $0 \to 4 \to 7 \to 0 \equiv 10$, is invariant under cyclic permutation, the perceived quality of the chord depends entirely on inversion and context (specifically, which note is heard as the root or bass):
  - If the pitch r is established as the root, the intervals above it are $4 + 3$, and the chord is heard as **Major** (D_r).
  - If the pitch $r + 7$ is established as the root, the intervals above it are $3 + 4$, and the chord is heard as **Minor** (M_{r+7}).
- **The Janus-Faced Triad:** Physically, the 10-TET triad is an ambiguous, "Janus-faced" harmonic object. It functions as a bridge between the major and minor modes not by changing notes (voice leading), but by changing the listener's perspective (re-interpreting the root).

6.1.4. *The Degenerate Tonnetz and its Levi Graph.* To understand the topology of this system, we introduce the standard Neo-Riemannian transformations adapted to 10-TET. For every Major chord D_r and Minor chord M_r, we define the operations P (Parallel), L (Leading-Tone), and R (Relative) as follows:

- **Parallel (P):** Preserves the root and fifth.

$$P(D_r) = M_r, \quad P(M_r) = D_r$$

- **Leading-Tone (L):** Preserves the minor third (Major chord) or major third (Minor chord).

$$L(D_r) = M_{r+4}, \quad L(M_r) = D_{r-4}$$

- **Relative (R):** Preserves the major third (Major chord) or minor third (Minor chord).

$$R(D_r) = M_{r-3}, \quad R(M_r) = D_{r+3}$$

All indices are taken modulo 10. These transformations map the chords as shown in the tables below. Note that the R transformation is crossed out because, in this specific system, it acts as the identity operation on the set of chords.

Input	P	L	R
D_0	M_0	M_4	~~M_7~~
D_1	M_1	M_5	~~M_8~~
D_2	M_2	M_6	~~M_9~~
D_3	M_3	M_7	~~M_0~~
D_4	M_4	M_8	~~M_1~~
D_5	M_5	M_9	~~M_2~~
D_6	M_6	M_0	~~M_3~~
D_7	M_7	M_1	~~M_4~~
D_8	M_8	M_2	~~M_5~~
D_9	M_9	M_3	~~M_6~~

Figure 6. Transformations on Major Chords. The R column is crossed out because $M_{r-3} \equiv D_r$ is a trivial identity.

Input	P	L	R
M_0	D_0	D_6	~~D_3~~
M_1	D_1	D_7	~~D_4~~
M_2	D_2	D_8	~~D_5~~
M_3	D_3	D_9	~~D_6~~
M_4	D_4	D_0	~~D_7~~
M_5	D_5	D_1	~~D_8~~
M_6	D_6	D_2	~~D_9~~
M_7	D_7	D_3	~~D_0~~
M_8	D_8	D_4	~~D_1~~
M_9	D_9	D_5	~~D_2~~

Figure 7. Transformations on Minor Chords. The R column is crossed out because $D_{r+3} \equiv M_r$ is a trivial identity.

This identity fundamentally alters the topological construction of the Tonnetz. In 12-TET, the "Relative" transformation connects a chord to a distinct neighbor ($R : D_r \to M_{r-3}$), creating a graph where every vertex has degree 3. In 10-TET, the "Relative" transformation ($D_r \to M_{r+7}$) maps the chord to itself. Consequently, the edge corresponding to R becomes a loop, and the functional degree of each vertex drops to 2.

To visualize this structure globally, we construct the **Levi graph** of the configuration. We treat the set of 10 major chords as "points" and the set of 10 minor chords as "lines". An incidence (edge) exists if a major chord transforms into a minor chord via a Parallel (P) or Leading-Tone (L) motion.

Since each major chord connects exactly to two minor chords, and each minor chord connects to two major chords, the global structure forms a **Levi graph of a (10_2) configuration**.

We present this graph below by arranging all 20 chords on a single circle. To highlight the structural relationships, we arrange them in the sequence: $D_0, M_4, D_1, M_5, D_4, M_8, \ldots$. The connections reveal the hidden topology.

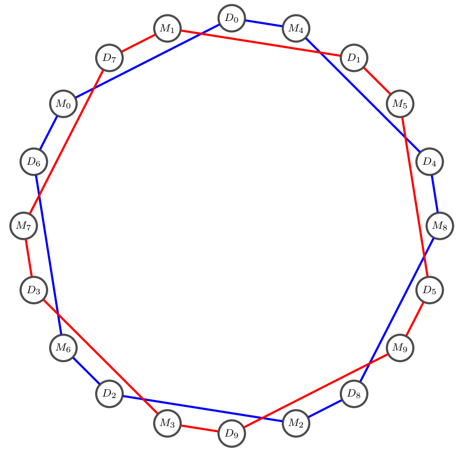


FIGURE 8. The full Levi graph of the 10-TET Tonnetz arranged on a circle. The blue edges correspond to the connections between "even" chords, while the red edges connect "odd" chords. The interleaving of the nodes visually demonstrates the symmetry of the system.

Topological Decomposition. The coloring in Figure 8 makes the fundamental property of the system immediately visible. Although constructed as a single harmonic system, the graph naturally separates into two disjoint, non-intersecting subgraphs.

The blue path traces a closed cycle involving only the chords with even roots (for D) and their associated minors. The red path traces a similar cycle for the odd roots. No blue edge ever meets a red edge.

Theorem 2 (Topological Decomposition)**.** *The 10-TET Tonnetz for the solution* $(t = 4, s = 3)$ *is isomorphic to the Levi graph of a* (10_2) *configuration, which decomposes into two disjoint Levi graphs of* (5_2) *configurations (cycles of length 10).*

This means the harmonic Universe of $(4, 3)$ solution 10-TET consists of two parallel "harmonic realities" that never intersect. Geometrically, the second graph is a perfect copy of the first, rotated by an angle of $\pi/10$ (one step in the circular arrangement). We display the separated components below, preserving their original geometric positions from Figure 8 to highlight the rotation.

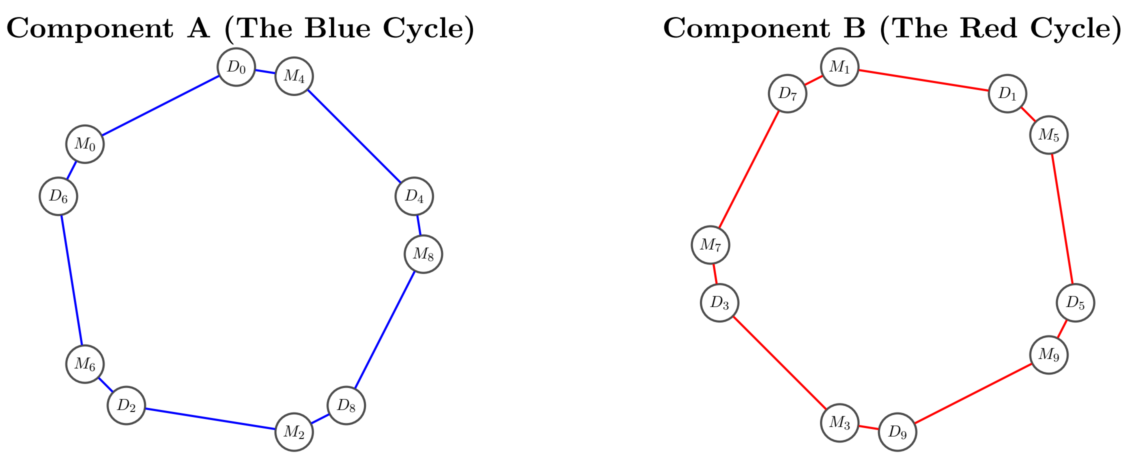


FIGURE 9. The decomposition of the 10-TET Tonnetz into two independent (5_2) Levi graphs. Note that the geometric shape of the red cycle is identical to the blue cycle but rotated by 18°, occupying the gaps left by the blue component.

6.1.5. *The Functional Perspective for the* $(4,3)$ *System: Restoring Connectivity in 3D.* Our analysis of "modal degeneracy" in Theorem 1 relied on a strict set-theoretical perspective: since the pitch-class sets of D_r and M_{r+7} are identical, we treated them as the same vertex in the Tonnetz. This reduced the graph valence to 2 and caused the topological splitting described in Section 6.1.4.

However, from a functional (Riemannian) perspective, a chord is defined not just by its pitch content, but by its root. In this view, the major triad $D_0 = \{0, 4, 7\}$ rooted at 0 is distinct from the minor triad $M_7 = \{7, 0, 4\}$ rooted at 7, despite comprising the same notes. The "Relative" transformation (R) is no longer an identity map, but a bijection – *a wormhole* – mapping a chord from the "even universe" to the "odd universe" (since r and $r + 7$ have opposite parity).

If we adopt this functional differentiation, the vertex set doubles from 10 unique sets to 20 functional chords. The topology undergoes a dimensional lift:

(1) The "Parallel" (P) and "Leading-Tone" (L) transformations continue to link chords within the same parity cycle (forming the Blue and Red rings derived in Section 7).
(2) The "Relative" (R) transformation now acts as a transversal edge, connecting every node D_r in the Blue cycle to a node M_{r+7} in the Red cycle (and vice versa).

We explicitly list these restored functional connections in the tables below. Unlike the degenerate case, the R-column now provides valid, distinct outputs that link the previously disjoint cycles.

Input	P	L	R
D_0	M_0	M_4	M_7
D_1	M_1	M_5	M_8
D_2	M_2	M_6	M_9
D_3	M_3	M_7	M_0
D_4	M_4	M_8	M_1
D_5	M_5	M_9	M_2
D_6	M_6	M_0	M_3
D_7	M_7	M_1	M_4
D_8	M_8	M_2	M_5
D_9	M_9	M_3	M_6

FIGURE 10. Functional Transformations on Major Chords. The R transformation is now active, mapping D_r to the functionally distinct M_{r-3}.

Input	P	L	R
M_0	D_0	D_6	D_3
M_1	D_1	D_7	D_4
M_2	D_2	D_8	D_5
M_3	D_3	D_9	D_6
M_4	D_4	D_0	D_7
M_5	D_5	D_1	D_8
M_6	D_6	D_2	D_9
M_7	D_7	D_3	D_0
M_8	D_8	D_4	D_1
M_9	D_9	D_5	D_2

FIGURE 11. Functional Transformations on Minor Chords. The R transformation is now active, mapping M_r to the functionally distinct D_{r+3}.

Consequently, the graph valence is restored to 3 (as in 12-TET), and the disconnected components merge into a single, connected 3-dimensional structure. The two disjoint cycles of the 2D projection become the bases of a **prism graph** (or cylindrical lattice) in 3D space.

We visualize this "Functional 10-TET Cylinder" below. The Blue and Red cycles form the top and bottom bases. To enhance readability and avoid visual overlapping, we apply a rotational offset to the Upper (Red) ring, creating a "twisted" cylinder where the functional R-connections form inclined bridges.

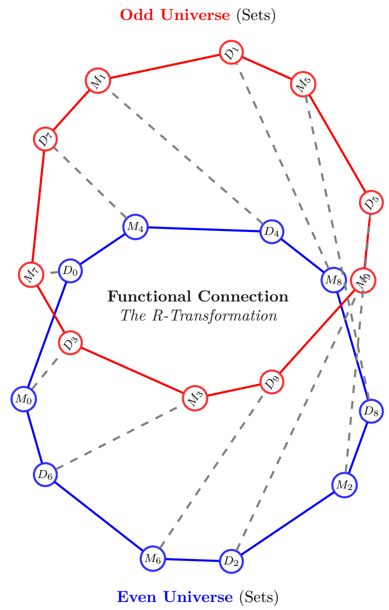


FIGURE 12. The "Functional Cylinder" of 10-TET. The Blue and Red cycles form the geometric bases, visualized here with alternating edge lengths to reflect the distinction between Parallel (P) and Leading-Tone (L) transformations. The gray dashed lines represent the Relative (R) transformation, acting as a wormhole between the disjoint pitch-class set universes.

6.1.6. *Musical Implications: The Phantom Modulation.* This functional restoration of connectivity in 3D has profound musical implications, extending the concept of "Janus-faced" triads introduced in Section 6.3.

In classical 12-TET harmony, modulation typically involves changing pitch classes (e.g., introducing a sharp or flat). However, in the $(4,3)$ solution 10-TET, the R-transformation represented by the vertical edges of the cylinder corresponds to a **phantom modulation**. The physical sound remains identical (the pitch-class set is invariant), but the listener is instantaneously transported from the "Even Universe" to the "Odd Universe" purely by re-interpreting the root.

This suggests that the $(4,3)$ solution 10-TET separates physical motion (changing notes) from functional motion (changing keys) more radically than 12-TET. Traversing the blue or red rings corresponds to physical changes in pitch, while traversing the vertical "wormholes" corresponds to purely cognitive shifts in harmonic function. The cylinder acts as a topological map of this duality, unifying the disjoint set-theoretical cycles into a single functional object.

6.1.7. *The Combinatorial Structure: The Cubic Graph.* By explicitly distinguishing between D_r and M_{r+7} and treating the identity relation as a valid edge (R), the system reveals a highly regular internal geometry. The interactions of P, L, and R generate a unified graph that can be analyzed using the language of graph theory.

6.1.8. *The Geometric Structure.* In this restored view, every Major chord D_r connects to exactly three Minor chords:

- M_r via the Parallel operation P (intra-universe),
- M_{r+4} via the Leading-tone operation L (intra-universe),
- M_{r+7} via the Wormhole operation R (inter-universe).

Symmetrically, every Minor chord connects to exactly three Major chords. This transforms our graph into a connected, cubic (3-valent) bipartite graph with $V = 20$ vertices (10 Majors and 10 Minors).

This structure constitutes a **connected cubic bipartite graph**. We can interpret the 10 Minor chords as "Points" and the 10 Major chords as "Lines". The incidence relation is defined by our harmonic connections: each Line (Major chord) passes through exactly 3 Points (the Minors it connects to), and each Point (Minor chord) lies on exactly 3 Lines.

However, it should be mentioned that these are not geometric points and lines in the strict sense. As we will see in the Section 6.1.10, in this interpretation the graph admits that two points lie on two different lines and two lines can meet in more than one point.

6.1.9. *Visualization: The Levi Graphs.* The incidence graph of a configuration is formally known as a **Levi graph**. In Figure 13, we present this graph in a layout corresponding to the spectral analysis from the previous sections. The Blue and Red universes form two concentric rings, with the P and L operations creating the perimeter of each ring. The R operations (gold dashed lines) act as radial wormholes fusing the two worlds together.

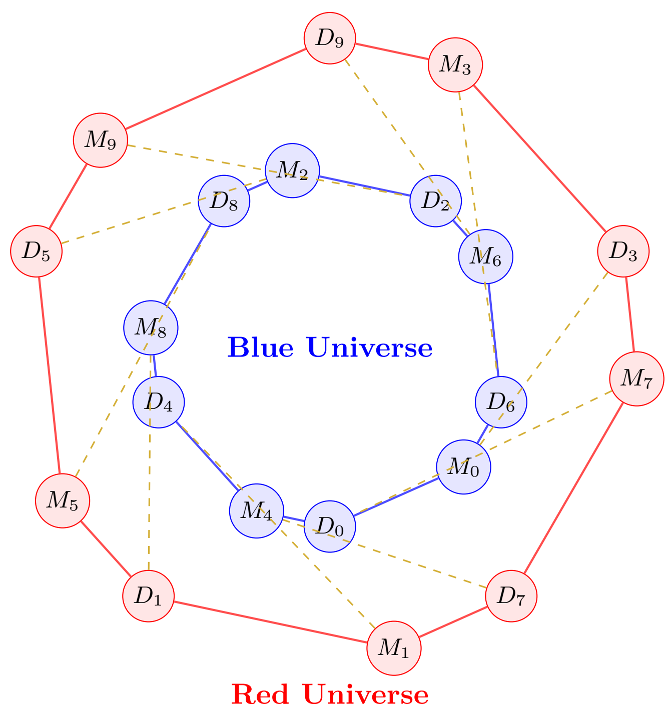


FIGURE 13. The Levi graph representing the (10_3) configuration. The Blue and Red universes are visualized as concentric rings, connected by radial R-wormholes (gold dashed lines), uniting the system.

In Figure 14, we present two visualizations of the graph. On the left, chords are arranged in the standard sequential notation $(D_0, M_0, D_1, M_1 \dots)$, highlighting the local alternating nature of the Blue and Red subnetworks. On the right, the chords are arranged according to the Hamiltonian cycle (see Section 6.1.12) found in the system $(D_0, M_0, D_3, M_3 \dots)$, revealing the global symmetry of the connections.

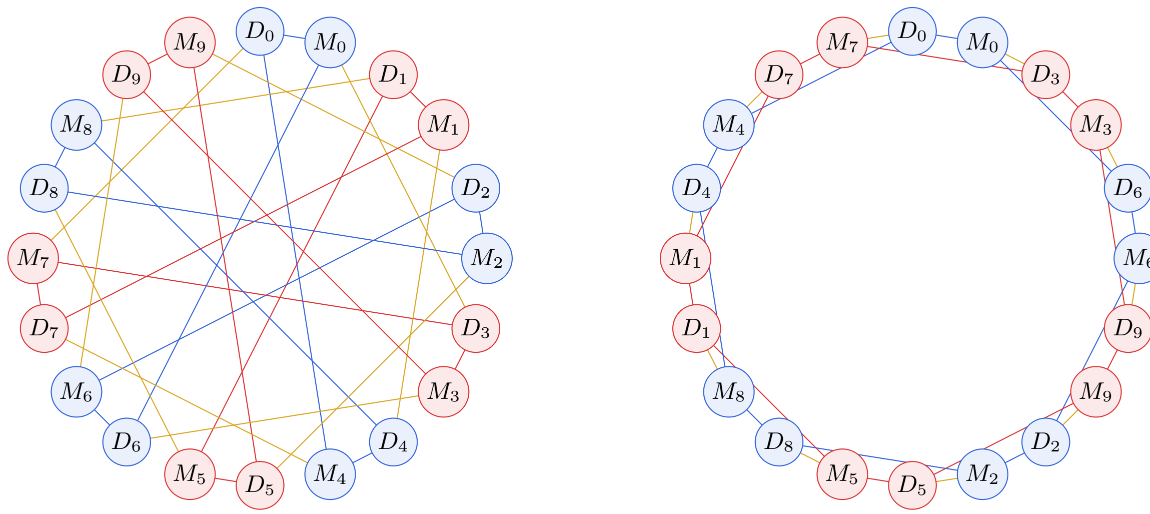


FIGURE 14. The Connected Levi Graph visualized in two arrangements. (a) Standard sequential notation. (b) Hamiltonian perimeter arrangement. Blue and Red nodes denote the parity of the root; dashed gold lines denote the Identity Wormholes connecting them.

6.1.10. *Minimal Chiral Cycles.* The introduction of wormholes significantly reduces the girth (shortest cycle length) of the graph. While the intra-universe graphs have a girth of 10, the restored cubic graph allows for tight loops of length 4.

Theorem 3. *In the restored cubic system, there are exactly two isomorphism classes of minimal cycles (of length 4) under the rotational symmetry of $\mathbb{Z}_{10}$. These classes are distinguished by their chirality (direction of winding through the wormhole).*

Proof. A cycle must alternate between Major and Minor chords. A length-2 cycle implies a double edge, which is impossible as P, L, R are distinct operations. Thus, the minimal length is 4. A 4-cycle must involve at least two wormhole crossings to leave and return to the starting universe. Algebraically, we look for a sequence of operations that results in the identity element. The two minimal solutions correspond to the two directions of traversing the wormhole:

(1) **The Violet Cycle (Right-handed):**

$$S = P \cdot R_{+7} \cdot L_{-4} \cdot R_{+7}$$

Tracing the index change: $P(0) \to R(+7) \to L(-4) \to R(+7)$. The total displacement is $+7 - 4 + 7 = +10 \equiv 0 \pmod{10}$.

(2) **The Orange Cycle (Left-handed):**

$$S' = P \cdot R_{+3} \cdot L_{+4} \cdot R_{+3}$$

Note that the inverse of the wormhole $D \to M_{+7}$ is $M \to D_{-7} = D_{+3}$. Tracing the index change: $P(0) \to R(+3) \to L(+4) \to R(+3)$. The total displacement is $+3+4+3 = +10 \equiv 0 \pmod{10}$.

These two sequences are non-isomorphic because they traverse the graph with opposite chirality, effectively winding around the toroidal structure of the graph in opposite directions. □

Figure 15 illustrates these two fundamental cycles embedded within the Levi graph, depicted with straight edges and directional arrows to emphasize the geometric path.

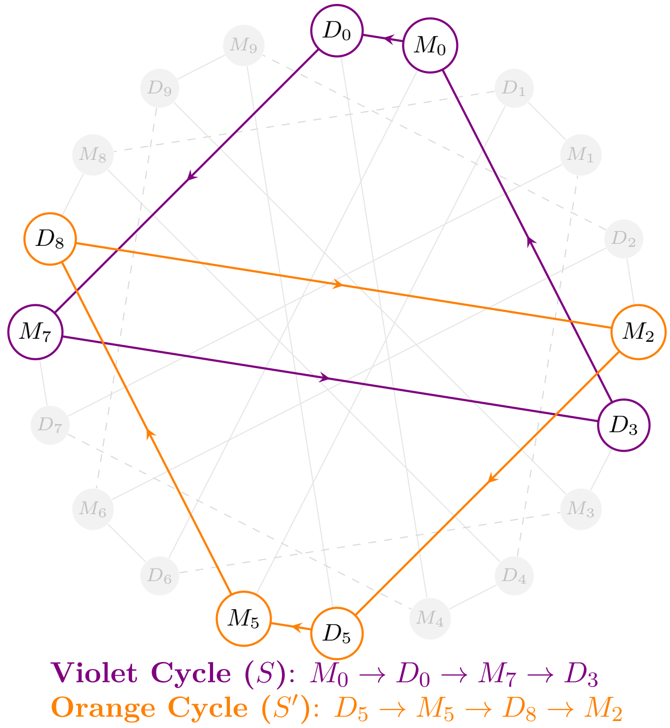


FIGURE 15. Visualisation of the two chiral minimal cycles. Straight edges connect the nodes, with arrows placed at the midpoint of each segment to indicate direction. These cycles represent the fundamental loops of the cubic graph.

6.1.11. *No geometric realization as points and lines on the plane.* The **girth 4 of this system forbids its geometric interpretation as a** 10_3 **configuration** of straight lines and points in $\mathbb{R}^2$. Historically (see Section 6.2.1 for more details about that), graphs as the one considered here, appeared as a visualisation of configurations of straight 10 lines $\mathcal{L}$ and 10 points $\mathcal{P}$ on the plane, with an **incidence** property that each line $L \in \mathcal{L}$ contained three points $P \in \mathcal{P}$, and that at each point $P \in \mathcal{P}$ three lines $L \in \mathcal{L}$ met. In Euclidean geometry two points may lie on at most one straight line. So our Levi graph with girth equal 4, which has 10 points (say Major chords) and 10 lines (Minor chords), with its connections interpreted as incidence, can not be a geometric configuration as e.g. girth four implies an existence of a cycle $D_{r_1} \to M_{r_2} \to D_{r_3} \to M_{r_4} \to D_{r_1}$ where two 'points' D_{r_1} and D_{r_3} lie both on two distinct 'lines' M_{r_2} and M_{r_4}.

6.1.12. *Algebraic Characterization and Graph Invariants.* To properly classify the functional 10-TET graph within the family of cubic graphs, we must first establish its fundamental algebraic invariants. By inspection of the graph constructed in Section 7.2, we identify the following properties:

(1) **Order and Valence:** The graph has $V = 20$ vertices and is 3-regular (cubic).
(2) **Bipartiteness:** The graph is bipartite (Levi graph), with the two partitions corresponding to Major chords (D) and Minor chords (M).
(3) **Girth:** As proven in Theorem 2, the graph has a girth of $g = 4$ (containing 4-cycles such as $M_0 \to D_0 \to M_7 \to D_3$).

Remark: The existence of these 4-cycles implies that in the point-line interpretation, two distinct "lines" (e.g., D_0 and D_3) intersect at two distinct "points" (e.g., M_0 and M_7). Consequently, two distinct points can determine more than one line. This violates the axioms of a linear configuration, distinguishing this graph from the rigid geometric structures discussed later.

(4) **Hamiltonicity:** The graph is Hamiltonian. A complete tour of all 20 chords can be constructed by alternating the Wormhole (R) and Leading-Tone (L) operations.

6.1.13. *Hamiltonian Connectivity: Two Paths.* The graph is Hamiltonian, meaning there exists a path that visits every chord exactly once before returning to the start. We identify two distinct Hamiltonian cycles generated by alternating operations, each revealing a different aspect of the system's symmetry.

The Perimeter Cycle ($R \cdot P$). The first cycle corresponds to the perimeter arrangement shown previously in Figure 14(b). It is generated by alternating the Parallel (P) and Wormhole (R) transformations:

$$C_{PR} = (R \cdot P)^{10}$$

Starting from D_0, the sequence proceeds: $D_0 \xrightarrow{P} M_0 \xrightarrow{R} D_3 \xrightarrow{P} M_3 \xrightarrow{R} D_6 \dots$. This path traces the local inversion (P) followed by a global shift (R), naturally unfolding the graph's maximal symmetry into a ring.

The Grand Tour ($L \cdot R$). We can construct a second, distinct Hamiltonian cycle by alternating the Wormhole (R) and Leading-Tone (L) operations:

$$C_{LR} = (L \cdot R)^{10}$$

This path effectively transposes the chord by a minor third (interval class 3) at each step:

$$D_0 \xrightarrow{R} M_7 \xrightarrow{L} D_3 \xrightarrow{R} M_0 \xrightarrow{L} D_6 \dots$$

While the algebraic formula is as simple as the perimeter cycle, its geometric realization in the standard sequential layout $(D_0, M_0, D_1 \dots)$ is strikingly different. Instead of stepping between neighbors, the Grand Tour weaves a complex, star-like pattern, jumping across the circle to stitch the entire manifold together.

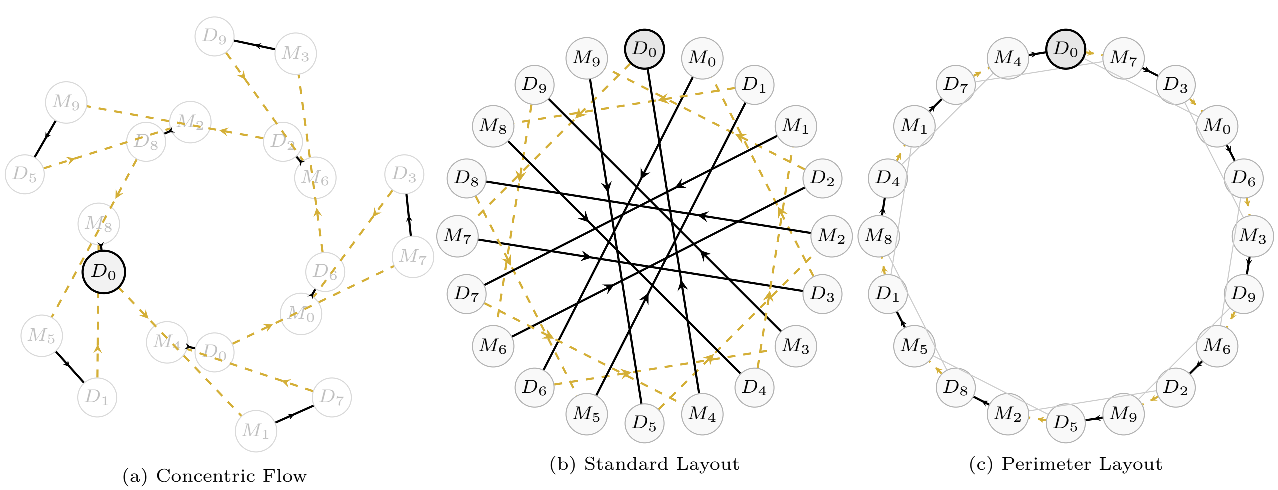


FIGURE 16. The "Grand Tour" Hamiltonian Cycle $(L \cdot R)^{10}$. (a) Concentric flow. (b) Star-polygon structure in standard layout. (c) Perimeter layout: the nodes are re-indexed such that the $(L \cdot R)^{10}$ Hamiltonian cycle forms the outer boundary. The remaining P operations are shown as internal grey chords to represent the full Levi graph structure for completeness.

6.2. **The "Wide" Choice of Thirds: A close Desargues cousin.** We now analyze the "Wide Thirds" 10-TET system defined by the parameters $(t, s) = (6, 1)$. While earlier sections focused on the topological continuity of the "narrow" systems (where $\Delta = 1$), the $\Delta = 5$ case offers a different kind of beauty. It satisfies the rigorous conditions of a linear incidence geometry, turning out to be one of the 10_3 **configurations**—structures first encountered in the 17th century in the **Desargues Theorem** of projective geometry. At this point, a historical interlude is in order.

6.2.1. *The Evolution of* 10_3 *Configurations.* The concept of a 10_3 configuration first appeared in the 17th century in the work of **Girard Desargues** [4]. The incidence structure of the 10 points and 10 lines involved in his theorem was recognized to possess a remarkable self-dual property, allowing the roles of points and lines to be interchanged.

For more than two centuries following its discovery, the Desargues configuration remained the unique prototype of such a system. The abstract combinatorial definition of a 10_3 configuration was formulated much later. In 1881, **Seligmann Kantor** [8] broke this solitude by discovering a second, non-isomorphic 10_3 configuration. However, Kantor's configuration (often denoted $(10_3)_8$ in older literature) posed a geometric anomaly: unlike the Desargues configuration, it cannot be realized by points and straight lines in the real projective plane.

Kantor's discovery spurred a century-long effort to classify all possible 10_3 structures. Over time, other non-equivalent and realizable configurations were identified. The classification was finally completed and proven in the late 20th century. In 2000, Betten, Brinkmann, and Pisanski published the definitive catalogue [1], proving there are precisely 10 distinct isomorphism classes.

In this classification, the configurations are distinguished, in particular, by their automorphism groups. Remarkably, the **Desargues configuration** is identified as type $(10_3)_8$, possessing the largest group of symmetries (flag-transitive, but not cyclic). The only 10_3 configuration in the entire classification that admits a **cyclic symmetry group** acting transitively on the points is designated as:

Configuration $(10_3)_3$

This implies that any "Harmonic" realization generated purely by a cyclic group C_{10} is combinatorially isomorphic to configuration $(10_3)_3$, a **cyclic cousin** of the classical Desargues configuration $(10_3)_8$.

Closing this historical aside, it is worthwhile to note that although this configuration is only a relative of the Desargues configuration, it can also be realized as a system of 10 straight lines and 10 points on the real plane. It obeys the rules that (a) each point meets three lines, (b) each line contains three points, and (c) the axioms of linear incidence geometry are satisfied (specifically, that any two distinct points determine at most one line).

In the following analysis, we demonstrate that the "Wide Thirds" musical system constitutes a valid combinatorial configuration of type $(10_3)_3$. This result bridges the gap between harmonic theory and combinatorial geometry, revealing that this specific choice of intervals generates a graph with a girth of $g = 6$, satisfying the Axiom of Incidence required for a true point-line geometry.

6.2.2. *Definition of Chords.* We first solve the system of congruences:

$$\begin{cases} t + s \equiv 7 \\ t - s \equiv 5 \end{cases} \pmod{10} \tag{7}$$

Summing the equations yields $2t \equiv 12 \equiv 2 \pmod{10}$, which implies $t = 1$ or $t = 6$. Since a major third of 1 step is counter-intuitive, we select:

$$\mathbf{t = 6}, \quad \mathbf{s = 1}.$$

Using the solution $(6, 1)$, we define the chords for every root $r \in \{0, \ldots, 9\}$.

Major Chords (D_r): Defined as $(r, r + t, r + q) = \{r, r + 6, r + 7\}$.
Minor Chords (M_r): Defined as $(r, r + s, r + q) = \{r, r + 1, r + 7\}$.

Table 5. Table of Major Chords (D_r) for $t = 6$

Name	Root	Set
D_0	0	{0, 6, 7}
D_1	1	{1, 7, 8}
D_2	2	{2, 8, 9}
D_3	3	{3, 9, 0}
D_4	4	{4, 0, 1}
D_5	5	{5, 1, 2}
D_6	6	{6, 2, 3}
D_7	7	{7, 3, 4}
D_8	8	{8, 4, 5}
D_9	9	{9, 5, 6}

Table 6. Table of Minor Chords (M_r) for $s = 1$

Name	Root	Set
M_0	0	{0, 1, 7}
M_1	1	{1, 2, 8}
M_2	2	{2, 3, 9}
M_3	3	{3, 4, 0}
M_4	4	{4, 5, 1}
M_5	5	{5, 6, 2}
M_6	6	{6, 7, 3}
M_7	7	{7, 8, 4}
M_8	8	{8, 9, 5}
M_9	9	{9, 0, 6}

The interval vectors for Major $[6, 1, 3]$ and Minor $[1, 6, 3]$ are distinct. Thus, **no modal degeneracy exists**.

6.2.3. *Connectivity and the Tonnetz.* We construct the Tonnetz by connecting chords that share exactly **two common tones**. The system is perfectly 3-valent.

P (Parallel): $D_r \leftrightarrow M_r$ (Share root and fifth).
L (Leading): $D_r \leftrightarrow M_{r+6}$ (Share third and fifth).
R (Relative): $D_r \leftrightarrow M_{r+9}$ (Share root and third).

D_r **Connections**

Chord	Neighbors $\{M_i, M_j, M_k\}$
D_0	M_0, M_6, M_9
D_1	M_1, M_7, M_0
D_2	M_2, M_8, M_1
D_3	M_3, M_9, M_2
D_4	M_4, M_0, M_3
D_5	M_5, M_1, M_4
D_6	M_6, M_2, M_5
D_7	M_7, M_3, M_6
D_8	M_8, M_4, M_7
D_9	M_9, M_5, M_8

M_r **Connections**

Chord	Neighbors $\{D_i, D_j, D_k\}$
M_0	D_0, D_4, D_1
M_1	D_1, D_5, D_2
M_2	D_2, D_6, D_3
M_3	D_3, D_7, D_4
M_4	D_4, D_8, D_5
M_5	D_5, D_9, D_6
M_6	D_6, D_0, D_7
M_7	D_7, D_1, D_8
M_8	D_8, D_2, D_9
M_9	D_9, D_3, D_0

Figure 17. Adjacency tables for the Wide System ($\Delta = 5$). Note that the graph mixes indices differently than the compact system, ensuring global connectivity.

6.2.4. *The Levi Graph.* The resulting graph connects all 20 chords into a single structure. We visualize it by placing chords on a circle: $D_0, M_0, D_1, M_1 \ldots$. The P and R transformations form the perimeter, while L forms the internal chords.

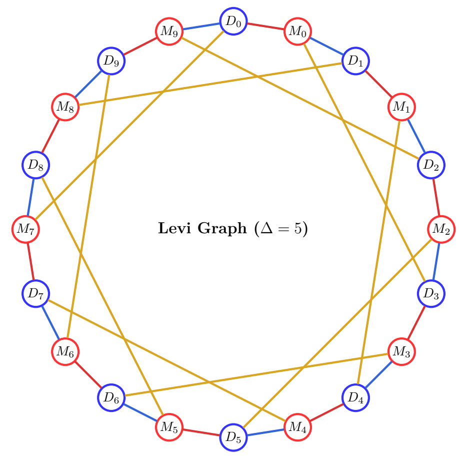


FIGURE 18. The Levi Graph with colored perimeter segments (D_iM_i red, M_iD_{i+1} blue) and chords (M_iD_{i+3} gold).

6.2.5. *Topologgy of Cycles: Girth 6.* A fundamental question in the topology of this harmonic network is the length of the shortest path to return to the starting chord (the **girth**). For the **"Wide Thirds System"** we have the following Theorem.

Theorem 4. *The girth of the* ***"Wide Thirds"*** $(6, 1)$ ***System*** *is* ***6***.

Proof. Our proof uses the "Shift Method".
Parity Constraint. Our harmonic graph is **bipartite**: edges only connect Major (D) chords to Minor (M) chords. Consequently, any walk starting at a Major chord will be at a Minor chord after an odd number of steps (1, 3, 5...). To return to the starting type (Major), the path length must necessarily be **even**. Therefore, cycles of length 3 or 5 cannot exist. We need only examine lengths $2k$ (i.e., 2, 4, 6...).
The "Shift Method" for Even Cycles. We analyze the path as a sequence of "double-steps": $D_{start} \to M \to D_{next}$. Each double-step shifts the root index by a specific amount modulo 10. To close a cycle, the sum of these shifts must be $0 \equiv 10 \pmod{10}$.

There are exactly 6 possible distinct double-steps starting from D_r:

(1) **PR**: $D_r \to M_{r+6} \to D_{r+6}$. Shift: **6**.
(2) **PL**: $D_r \to M_{r-1} \to D_{r-1}$. Shift: **9** (since $-1 \equiv 9$).
(3) **LP**: $D_r \to M_r \to D_{r+1}$. Shift: **1**.
(4) **LR**: $D_r \to M_{r+6} \to D_{r+7}$. Shift: **7**.
(5) **RP**: $D_r \to M_r \to D_{r+4}$. Shift: **4**.
(6) **RL**: $D_r \to M_{r-1} \to D_{r+3}$. Shift: **3**.

Proof that Length 4 is Impossible. A cycle of length 4 consists of exactly **two** double-steps. For a non-trivial cycle (one that does not simply backtrack like PR then RP), we must select two *different* non-inverse shifts that sum to 10. Let us examine the possible sums of distinct shifts from our set $\{1, 3, 4, 6, 7, 9\}$:

- $1 + 3 = 4$, $1 + 4 = 5$, $1 + 6 = 7$, $1 + 7 = 8$.
- $3 + 4 = 7$, $3 + 6 = 9$.
- $4 + 6 = 10$. This looks promising! However, shift 4 is RP and shift 6 is PR. The sequence would be PR followed by RP, i.e., $RP \cdot PR = RP^2R$. Since $P^2 = Identity$, this collapses to a $P \to P$ path. This is a backtracking path, not a cycle. The same story is when we do RP followed by PR, since $R^2 = Identity$.
- The same logic applies to $1 + 9 = 10$ (PL and LP) and $3 + 7 = 10$ (RL and LR).

Since no two distinct, non-inverse shifts sum to 10, **cycles of length 4 do not exist.**

Proof that Length 6 Exists. It is enough to show at least one such cycle. Examples of two paths of length 6 of oposite chirality through the harmonic space is visualized below.

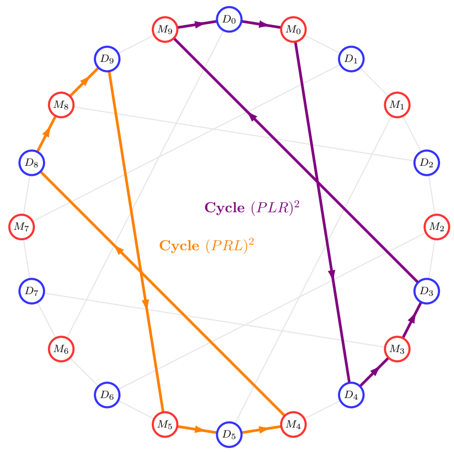


FIGURE 19. Visualization of the two chiral shortest cycles (girth 6). To avoid visual overlap, we present the Left-Handed cycle (Violet) starting at D_0, and the Right-Handed cycle (Orange) starting at D_8. Arrows indicate the direction of the flow, revealing the distinct chiral nature of each path.

□

6.2.6. *"Wide Thirds" System is a* 10_3 *Configuration.* The girth 6 of the considered system, proofs that the **"Wide Thirds" System is a geometric** 10_3 **configuration** of line and points on the plane. Contrary to the $(4,3)$ system, because of girth 6, no incidence of two points on two different lines is possible, so in the (Major Chords, Minor chord) – (Points, Lines) interpretation, the fundamental Euclidean geometry requirement that two points define a line and that two lines meet at single point is satisfied.

6.2.7. *Identification as a Cyclic Configuration.* We now list structural properties of the derived Tonnetz which are easy to detect. We first determine the "offsets" – the specific intervals that connect Major chords to Minor chords in the graph.

By examining the common tones defined in Section 6.2.3, we observe that every Major chord D_r shares two notes with exactly three Minor chords:

(1) $M_{r+\mathbf{0}}$ (sharing $\{r, r+7\}$): An offset of **0**.
(2) $M_{r+\mathbf{6}}$ (sharing $\{r+6, r+7\}$): An offset of **6**.
(3) $M_{r+\mathbf{9}}$ (sharing $\{r, r+6\}$): An offset of **9**.

Thus, the connection set for the system is $S = \{0, 6, 9\}$. This set of jumps defines the graph's invariants:

(1) **Cubic (3-valent):** Every chord has exactly 3 neighbors determined by the offsets.
(2) **Bipartite:** Connections only exist between Major and Minor chords.
(3) **Order 20:** There are 10 Major and 10 Minor chords.
(4) **Girth 6:** The shortest cycle has length 6. A concrete example of such a hexagon, is e.g. generated by the alternating sum of offsets $(9-6+0-9+6-0=0)$:

$$D_0 \xrightarrow{9} M_9 \xrightarrow{6} D_3 \xrightarrow{0} M_3 \xrightarrow{9} D_4 \xrightarrow{6} M_0 \xrightarrow{0} D_0$$

Arrows denote the offset magnitude connecting the nodes; this is the opposite 6-cycle to the violet cycle in Figure 19.

(5) **Hamiltonicity** System is visibly Hamiltonian having all points on the perimeter of its Levi graph connected (seeFigure 18).

(6) **Cyclic Symmetry:** The graph admits a cyclic group action C_{10}, that maps $D_r \to D_{r+k}$ and $M_r \to M_{r+k}$ and which is generated by a clockwise rotation by 36 degrees: $D_r \to D_{r+1}$ and $M_r \to M_{r+1}$. This **action is transitive** and preserves the offset structure, prob=ving that the **cyclic group C_{10} is a subgroup of the full symmetry group** of the “Wide Thirds” system.

With the geometric invariants and a cyclic subgroup of the automorphism group of the Wide Thirds system, we return to the classification context established in Section 6.2.1.

Recall that the census by Betten, Brinkmann, and Pisanski [1] identifies the **Cyclic** 10_3 Configuration as a singular entity: it is the only one, denoted by $(10_3)_3$, among the ten distinct classes to admit a cyclic symmetry group acting transitively on the points. Since our analysis has demonstrated that the Wide 10-TET system is both a valid (10_3) configuration and inherently generated by the cyclic group C_{10}, the identification follows immediately from this uniqueness.

Theorem 5. *The Levi graph of the Wide 10-TET system ($\Delta = 5$) is isomorphic to the **Unique Cyclic Configuration (10_3)**, labeled as $(10_3)_3$ in the Betten census.*

This identification reveals the precise nature of the “Wide” system. Unlike the degenerate $(4, 3)$ system (which contains squares and thus fails the axioms of a linear geometry), the $(6, 1)$ system satisfies the condition of Girth 6, making it a valid configuration of points and lines.

It is a **Self-Dual** geometry: the graph structure is invariant under the exchange of Major and Minor chords. However, its most significant feature is its distinctness from the **Desargues configuration** known from the Desargues theorem in projective geometry. While Desargues is highly symmetric (Group symmetry order 240, flag-transitive), the Wide system is the **unique** configuration among the ten combinatorial types of (10_3) that admits a cyclic automorphism group. Thus, it represents the singular intersection between the axioms of combinatorial geometry and the cyclic symmetry inherent in musical theory.

6.3. The Tritone Solution: System (5,2).

We now pass to the system from Section 5.1.3 characterized by the difference $\Delta = 3$.

$$\begin{cases} t + s \equiv 7 \\ t - s \equiv 3 \end{cases} \implies 2t \equiv 10 \pmod{10} \tag{8}$$

This yields the solution $t = 5$ and $s = 2$.

Mathematically, this system is characterized by its huge symmetry. Analysis of the underlying graph reveals a **Girth of 4**, indicating the immediate presence of squares (4-cycles) in the chordal network.

6.3.1. *Chord Definitions and Chirality.*

We define the chords with root r using the intervals $t = 5$ and $s = 2$:

$$D_r = \{r, r + 5, r + 7\} \pmod{10} \tag{9}$$

$$M_r = \{r, r + 2, r + 7\} \pmod{10} \tag{10}$$

A unique feature of this system is the **intervallic symmetry** of the chords. For a Major chord $D_0 = \{0, 5, 7\}$, the interval vector is $(5, 2, 3)$. For a Minor chord $M_0 = \{0, 2, 7\}$, the interval vector is $(2, 5, 3)$. Geometrically, these triads are mirror images (chiral reflections) of each other within the chromatic circle, sharing the perfect fifth $\{0, 7\}$ as a rigid spine.

The complete list of chords generated by this system is provided in Tables 7 and 8.

Table 7. Major Chords $D_r = \{r, r+5, r+7\}$

Chord	**Pitch Class Set**
D_0	$\{0,5,7\}$
D_1	$\{1,6,8\}$
D_2	$\{2,7,9\}$
D_3	$\{3,8,0\}$
D_4	$\{4,9,1\}$
D_5	$\{5,0,2\}$
D_6	$\{6,1,3\}$
D_7	$\{7,2,4\}$
D_8	$\{8,3,5\}$
D_9	$\{9,4,6\}$

Table 8. Minor Chords $M_r = \{r, r+2, r+7\}$

Chord	**Pitch Class Set**
M_0	$\{0,2,7\}$
M_1	$\{1,3,8\}$
M_2	$\{2,4,9\}$
M_3	$\{3,5,0\}$
M_4	$\{4,6,1\}$
M_5	$\{5,7,2\}$
M_6	$\{6,8,3\}$
M_7	$\{7,9,4\}$
M_8	$\{8,0,5\}$
M_9	$\{9,1,6\}$

6.3.2. *Adjacency and Graph Connectivity.* We determine the Tonnetz connectivity by identifying neighbors that share exactly two common tones.

(1) **The P-Transformation (Parallel)**: D_r shares $\{r, r+7\}$ (Root, Fifth) with a minor chord. Looking at $M_r = \{r, r+2, r+7\}$, we see an immediate match.
$$D_r \leftrightarrow M_r$$
This operation preserves the root and fifth, moving the third by $\Delta = 3$ steps.

(2) **The R-Transformation (Relative)**: D_r shares $\{r, r+5\}$ (Root, Third) with a minor chord. We seek $M_k = \{k, k+2, k+7\}$ containing $\{r, r+5\}$. Setting $k = r+8$ (which is $r-2$), we get $M_{r+8} = \{r+8, r, r+5\}$.
$$D_r \leftrightarrow M_{r+8}$$
This operation connects the Major chord at index r to the Minor chord shifted by -2 steps.

(3) **The L-Transformation (Leading Tone)**: D_r shares $\{r+5, r+7\}$ (Third, Fifth) with a minor chord. We seek M_k containing these tones. Setting $k = r+5$, we get $M_{r+5} = \{r+5, r+7, r+2\}$.
$$D_r \leftrightarrow M_{r+5}$$
This is a **tritone jump**. The L transformation connects a Major chord to the Minor chord rooted at its tritone antipode.

The full set of connections for the system is given in Tables 9 and 10.

Table 9. Major Chord Connections ($D_r \to M_k$)

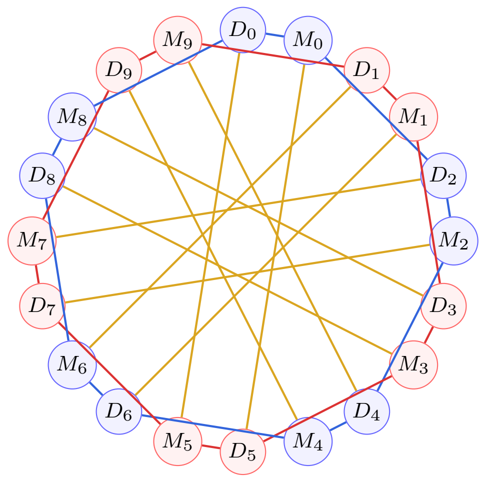
Chord	**P** (M_r)	**L** (M_{r+5})	**R** (M_{r-2})
D_0	M_0	M_5	M_8
D_1	M_1	M_6	M_9
D_2	M_2	M_7	M_0
D_3	M_3	M_8	M_1
D_4	M_4	M_9	M_2
D_5	M_5	M_0	M_3
D_6	M_6	M_1	M_4
D_7	M_7	M_2	M_5
D_8	M_8	M_3	M_6
D_9	M_9	M_4	M_7

Table 10. Minor Chord Connections ($M_r \to D_k$)

Chord	**P** (D_r)	**L** (D_{r+5})	**R** (D_{r+2})
M_0	D_0	D_5	D_2
M_1	D_1	D_6	D_3
M_2	D_2	D_7	D_4
M_3	D_3	D_8	D_5
M_4	D_4	D_9	D_6
M_5	D_5	D_0	D_7
M_6	D_6	D_1	D_8
M_7	D_7	D_2	D_9
M_8	D_8	D_3	D_0
M_9	D_9	D_4	D_1

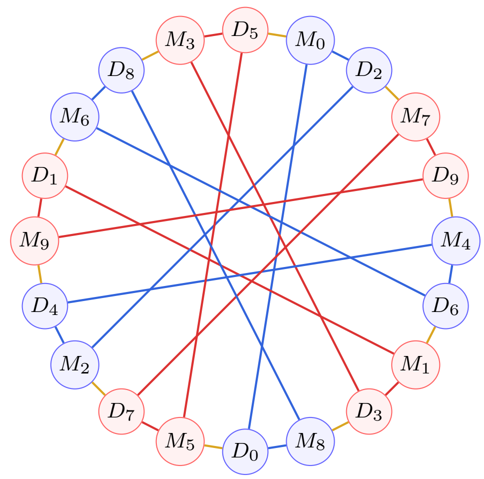
6.3.3. *The Levi Graph.* Since the chord sets D and M are disjoint, the resulting structure is a cubic bipartite graph of order 20. The P and R transformations preserve the parity of the root index ($D_{2k} \leftrightarrow M_{2k}$ and $D_{2k} \leftrightarrow M_{2k-2}$), creating two disjoint cycles of 10 chords each. These are the "Even Universe" (Blue) and the "Odd Universe" (Red). The system becomes fully connected via the L transformation, which maps $D_r \leftrightarrow M_{r+5}$, effectively linking an even root to an odd root.

The visualization of this graph is presented in Figure 20. Figure 21 rearanges this graph to a form visibly Hamiltonian, with a Hamiltonian cycle along the perimeter.



Standard Levi Graph

FIGURE 20. The standard Levi graph.



Rearranged Levi Graph

FIGURE 21. The rearranged Levi graph showing the Red/Blue cycles and Gold chords.

6.3.4. *Minimal Cycles.* To characterize the local topology of the graph, we analyze the minimal cycles (girth). Unlike the $(4, 3)$ system which possessed a hexagonal structure ($g = 6$), the $(5, 2)$ system exhibits a tighter connectivity.

Theorem 6. *The Tritone graph has a girth of $g = 4$. The minimal cycles are of the form $(PL)^2$, forming squares that connect antipodal chords.*

Proof. Consider the sequence of operations starting from D_r:

(1) $D_r \xrightarrow{P} M_r$.
(2) $M_r \xrightarrow{L} D_{r+5}$.
(3) $D_{r+5} \xrightarrow{P} M_{r+5}$.
(4) $M_{r+5} \xrightarrow{L} D_{r+5+5} = D_{r+10} = D_r$.

This sequence $D_r \to M_r \to D_{r+5} \to M_{r+5} \to D_r$ forms a closed cycle of length 4. Since the graph is bipartite, it contains no odd cycles (length 3). Therefore, $g = 4$. □

Geometrically, these cycles appear as "bow-tie" structures crossing the center of the circular Levi graph, linking an Even pair (D_r, M_r) with its antipodal Odd pair (D_{r+5}, M_{r+5}).

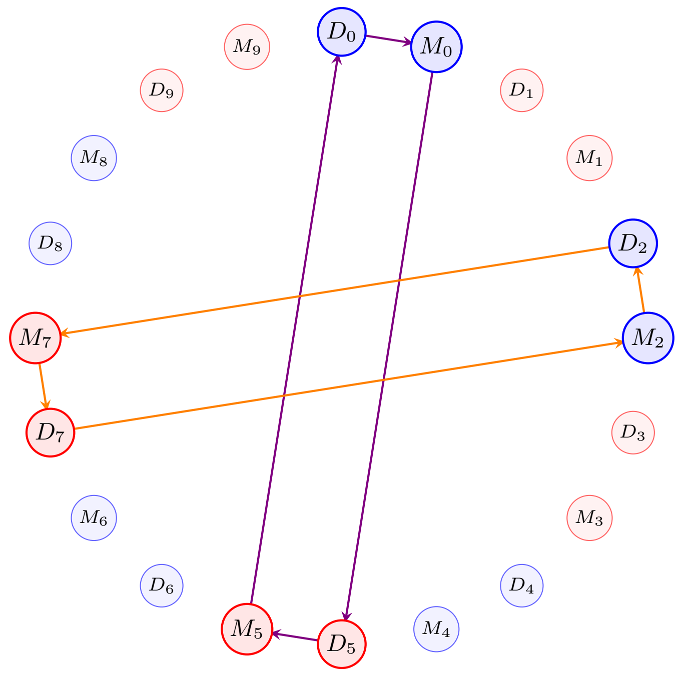


FIGURE 22. Two minimal cycles (girth $g = 4$) of opposite chirality. The Violet path ($D_0 \to M_0 \to D_5 \to M_5$) traverses locally clockwise, while the Orange path ($M_2 \to D_2 \to M_7 \to D_7$) traverses locally counter-clockwise.

6.3.5. *Algebraic Characterization.* We summarize the graph invariants for the (5,2) solution:

(1) **Order and Valence:** The graph has $V = 20$ vertices and is 3-regular (cubic).
(2) **Bipartiteness:** The graph is bipartite (Levi graph).
(3) **Girth:** The girth is $g = 4$, characterized by the $(PL)^2$ squares.
(4) **Hamiltonicity:** The graph is Hamiltonian, as haviung an obvious Hamiltonian perimeter cycle in Figure 21, or alternatively, by the construction of the "Tritone Tour" in the following section.

6.3.6. *The Tritone Tour.* Because the system is connected, finding a Hamiltonian cycle is straightforward. We can traverse the "Even Universe" almost entirely, cross via a tritone bridge (L) to the "Odd Universe", traverse it, and bridge back.

A valid Hamiltonian sequence is:

$$L \cdot R \cdot (P \cdot R)^4 \cdot L \cdot R \cdot (P \cdot R)^4$$

The Path Trace:

(1) **Even Sector:** $M_8 \to D_0 \to M_0 \to D_2 \to M_2 \to D_4 \to M_4 \to D_6 \to M_6 \to D_8$.
(2) **Bridge:** $D_8 \xrightarrow{L} M_3$ (Switch to Odd).
(3) **Odd Sector:** $M_3 \to D_5 \to M_5 \to D_7 \to M_7 \to D_9 \to M_9 \to D_1 \to M_1 \to D_3$.
(4) **Return Bridge:** $D_3 \xrightarrow{L} M_8$ (Switch to even).

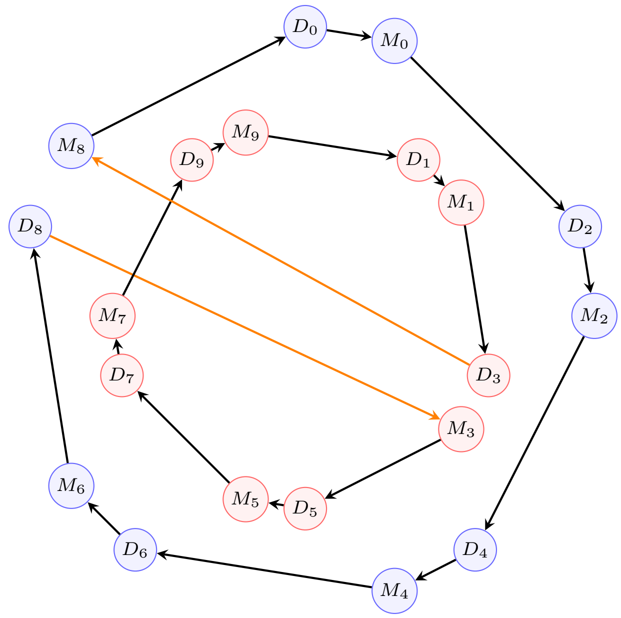


FIGURE 23. The Hamiltonian "Tritone Tour" in the (5,2) system. The path traverses the blue even ring, jumps via an orange tritone bridge (L) to the red odd ring, and returns to close the cycle.

6.4. **Analysis of the "Far Narrow Thirds" System: Solution $(9, 8)$.** We now examine the $\Delta = 1$ harmonic system determined by the partition of $q = 7$ into $(t, s) = (9, 8)$. Here, the "Major Third" is the interval $t = 9$ (a descending semitone) and the "Minor Third" is $s = 8$ (a descending whole tone).

6.4.1. *Chords.* We define the chords according to the standard generation rules for this partition:

- **Major Chords (D_r):** $\{r, r + 9, r + 7\}$
- **Minor Chords (M_r):** $\{r, r + 8, r + 7\}$

where $r \in \{0, 1, \ldots, 9\}$.

Chord	**Pitch Class Set**
D_0	$\{0, 9, 7\}$
D_1	$\{1, 0, 8\}$
D_2	$\{2, 1, 9\}$
D_3	$\{3, 2, 0\}$
D_4	$\{4, 3, 1\}$
D_5	$\{5, 4, 2\}$
D_6	$\{6, 5, 3\}$
D_7	$\{7, 6, 4\}$
D_8	$\{8, 7, 5\}$
D_9	$\{9, 8, 6\}$

Chord	**Pitch Class Set**
M_0	$\{0, 8, 7\}$
M_1	$\{1, 9, 8\}$
M_2	$\{2, 0, 9\}$
M_3	$\{3, 1, 0\}$
M_4	$\{4, 2, 1\}$
M_5	$\{5, 3, 2\}$
M_6	$\{6, 4, 3\}$
M_7	$\{7, 5, 4\}$
M_8	$\{8, 6, 5\}$
M_9	$\{9, 7, 6\}$

TABLE 11. The parallel lists of Major and Minor chords in the $(9, 8)$ system.

Interestingly, contrary to another $Delta = 1$ system $(4, 3)$, a direct comparison of the two tables reveals that **no Major chord set is identical to any Minor chord set**. The modal degeneracy encountered in the $(4, 3)$ system is absent here; the universe of chords remains strictly bipartite.

6.4.2. *The Tonnetz and PLR Transformations.* We establish the connectivity of the harmonic space (the Tonnetz) by identifying chords that share two pitch classes (dyads). For every Major chord D_r, there are exactly three Minor chords that share a dyad with it. These connections define the three primary transformations P, L, and R:

(1) **Parallel (P):** D_r shares the dyad $\{r, r+7\}$ (Root and Fifth) with M_r.
(2) **Leading (L):** D_r shares the dyad $\{r+9, r+7\}$ (Third and Fifth) with M_{r+9}.
(3) **Relative (R):** D_r shares the dyad $\{r, r+9\}$ (Root and Third) with M_{r+2}.

Thus, the neighborhood of a Major chord D_r is $\{M_r, M_{r+9}, M_{r+2}\}$. Conversely, the neighborhood of a Minor chord M_r is $\{D_r, D_{r+1}, D_{r+8}\}$.

Table 12. Major Chord Connections ($D_r \to M_k$)

Chord	**P** (M_r)	**L** (M_{r+9})	**R** (M_{r+2})
D_0	M_0	M_9	M_2
D_1	M_1	M_0	M_3
D_2	M_2	M_1	M_4
D_3	M_3	M_2	M_5
D_4	M_4	M_3	M_6
D_5	M_5	M_4	M_7
D_6	M_6	M_5	M_8
D_7	M_7	M_6	M_9
D_8	M_8	M_7	M_0
D_9	M_9	M_9	M_1

Table 13. Minor Chord Connections ($M_r \to D_k$)

Chord	**P** (D_r)	**L** (D_{r+1})	**R** (D_{r+8})
M_0	D_0	D_1	D_8
M_1	D_1	D_2	D_9
M_2	D_2	D_3	D_0
M_3	D_3	D_4	D_1
M_4	D_4	D_5	D_2
M_5	D_5	D_6	D_3
M_6	D_6	D_7	D_4
M_7	D_7	D_8	D_5
M_8	D_8	D_9	D_6
M_9	D_9	D_0	D_7

6.4.3. *Graph Structure and Hamiltonian Analysis.* The resulting structure is a **cubic bipartite Levi graph**. The graph forms a single connected component and is presented on the figure below.

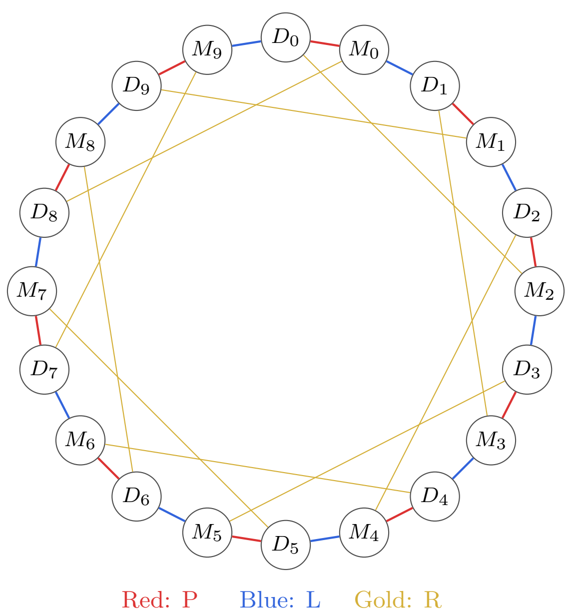


Figure 24. The Levi graph of the $(9, 8)$ system, arranged to highlight the Hamiltonian cycle on the perimeter. The Red (P) and Blue (L) edges form the continuous outer ring (the "Necklace"), while the Gold (R) edges provide the internal cross-connectivity.

6.4.4. *Girth and Minimal Cycles.* To determine the girth, we first check for the existence of 4-cycles. By using the shift method as in Sections 6.2.5 and 6.2.5, we find that **no cycles of length 4 exist**.

We can, however, find a cycle of length 6. By alternating P and R combined with L, we find closed loops such as:

$$D_0 \xrightarrow{R} M_2 \xrightarrow{L^{-1}} D_3 \xrightarrow{P} M_3 \xrightarrow{R^{-1}} D_1 \xrightarrow{L} M_0 \xrightarrow{P^{-1}} D_0$$

Sum of offsets: $2+1+0-2-1-0=0$. Thus, the girth of the graph is **6**, giving it a local hexagonal geometry similar to the 12-TET Tonnetz.

6.4.5. *Hamiltonian Cycle.* A Hamiltonian cycle visits every chord exactly once. Due to the connectivity provided by the L transformation (offset 9, which steps by -1), a simple alternating P-L chain traverses the entire group:

$$D_0 \xrightarrow{P} M_0 \xrightarrow{L} D_1 \xrightarrow{P} M_1 \xrightarrow{L} D_2 \dots \xrightarrow{P} M_9 \xrightarrow{L} D_0$$

This sequence generates the cycle of roots: $0 \to 0 \to 1 \to 1 \to 2 \dots$, covering all 20 chords in the system.

7. The Alternative Harmonic Universe: The Case of the Acoustic Fifth $q = 6$

In the preceding sections, our analysis adhered to the paradigm where the system's "Fifth" (q) was determined by the generator of the Pythagorean tuning ($q = 7$). However, a purely acoustic perspective suggests an alternative foundation. In 10-TET, the interval labeled **6** corresponds to 720 cents. This interval is the best approximation of the Just Perfect Fifth ($3/2 \approx 702$ cents) available in the system, significantly closer than the interval 7 (which in 10TET corresponds to 840 cents).

If we abandon the generator-based logic in favor of this acoustic proximity, we must solve the fundamental harmonic equation for a new fifth $q = 6$:

$$t + s \equiv 6 \pmod{10} \tag{11}$$

subject to the difference parameter Δ:

$$t - s \equiv \Delta \pmod{10} \tag{12}$$

Adding these equations yields $2t \equiv 6 + \Delta \pmod{10}$. For integer solutions to exist, **Δ must be even**.

To maintain a distinct harmonic hierarchy similar to classical theory, we impose the strict ordering condition: $s < t$, $t = 1, 2, \dots, 9$. This condition ensures the Major Third is wider than the Minor Third. This constraint naturally excludes the neutral case $\Delta = 0$ (where $t = s = 3$, $t = s = 8$) and the inverted cases where $t > s$.

We are thus left with two feasible solutions for Δ, namely: $\Delta = 2$ and $\Delta = 4$, and **three solutions for** (t, s), namely:

(1) **The "Degenerate" System** $(t, s) = (\mathbf{4}, \mathbf{2})$,
(2) **The "Tritone Like" System** $(t, s) = (\mathbf{5}, \mathbf{1})$, and
(3) **The "Geometric" System**$(t, s) = (\mathbf{9}, \mathbf{7})$.

7.1. **The Degenerate System: $\Delta = 2$, $(t, s) = (4, 2)$.** We now turn our attention to the solution $(t, s) = (4, 2)$. In this scenario, the Major Third is 480 cents and the Minor Third is 240 cents. The interval of the Fifth is fixed at $q = 6$ (720 cents). As we will see this system exhibits a totally new feature that was not present in systems with $q = 7$: it genuinly splits into two connected components, or in the 'cosmological language' of previous sections, it splits into two disconected Universes that can not be unified by any (wormhole like) connection defined by the system itself.

7.1.1. *Chords and Modal Degeneracy.* Let us list the chords explicitly as sets of integers to inspect their structure. We have:

- Major chords: $D_r = \{r, r + 4, r + 6\}$,
- Minor chords: $M_r = \{r, r + 2, r + 6\}$,

or explicitly:

Major Chords (D_r)		
Name	Root	**Set**
D_0	0	**{0, 4, 6}**
D_1	1	{1, 5, 7}
D_2	2	{2, 6, 8}
D_3	3	{3, 7, 9}
D_4	4	{4, 8, 0}
D_5	5	{5, 9, 1}
D_6	6	{6, 0, 2}
D_7	7	{7, 1, 3}
D_8	8	{8, 2, 4}
D_9	9	{9, 3, 5}

Minor Chords (M_r)		
Name	Root	**Set**
M_0	0	{0, 2, 6}
M_1	1	{1, 3, 7}
M_2	2	{2, 4, 8}
M_3	3	{3, 5, 9}
M_4	4	**{4, 6, 0}**
M_5	5	{5, 7, 1}
M_6	6	{6, 8, 2}
M_7	7	{7, 9, 3}
M_8	8	{8, 0, 4}
M_9	9	{9, 1, 5}

FIGURE 25. Explicit pitch sets of 10-TET chords in the $(4, 2)$ system. Note the highlighted entries.

Looking at these tables, we find a behavior in the currently considered system $(4, 2)$ similar to that of the system $(4, 3)$: In both systems, we encounter a **Modal Degeneracy** related to the fact that some major chords coincide as **sets of notes** with some minor chords; e.g., the major chord rooted at 0 with the minor chord rooted at 4:

- $D_0 = \{0, 4, 6\}$
- $M_4 = \{4, 6, 0\}$

This can be formalized in the following theorem.

Theorem 7 (Modal Identity)**.** *In the 10-TET system with* $(t = 4, s = 2)$*, the set of pitch classes forming a major chord* D_r *is identical to the set forming the minor chord* M_{r+4}*.*

$$D_r \equiv M_{r+4} \pmod{10}.$$

Proof. $D_r = \{r, r+4, r+6\}$. $M_{r+4} = \{(r+4), (r+4)+2, (r+4)+6\} = \{r+4, r+6, r+10\} \equiv \{r+4, r+6, r\}$. The sets are identical. □

7.1.2. *Tonnetz.* The similarity between systems $(4, 2)$ and $(4, 3)$ can be further explored – with quite a **surprise** at the end – by considering the Tonnetz connections in system $(4, 2)$. As before, they are generated by three transformations: P – **Parallel**, L – **Leading Tone**, and R – **Relative**, defined in the following table:

Input	**Parallel** (P)	**Leading** (L)	**Relative** (R)
D_r	M_r	M_{r+4}	M_{r+8}
M_r	D_r	D_{r+6}	M_{r+2}

or in the explicit tables below:

Input	P	L	R
D_0	M_0	~~M_4~~	M_8
D_1	M_1	~~M_5~~	M_9
D_2	M_2	~~M_6~~	M_0
D_3	M_3	~~M_7~~	M_1
D_4	M_4	~~M_8~~	M_2
D_5	M_5	~~M_9~~	M_3
D_6	M_6	~~M_0~~	M_4
D_7	M_7	~~M_1~~	M_5
D_8	M_8	~~M_2~~	M_6
D_9	M_9	~~M_3~~	M_7

FIGURE 26. Transformations on Major Chords. The L column is crossed out because $M_{r+4} \equiv D_r$ is a trivial identity.

Input	P	L	R
M_0	D_0	~~D_6~~	D_2
M_1	D_1	~~D_7~~	D_3
M_2	D_2	~~D_8~~	D_4
M_3	D_3	~~D_9~~	D_5
M_4	D_4	~~D_0~~	D_6
M_5	D_5	~~D_1~~	D_7
M_6	D_6	~~D_2~~	D_8
M_7	D_7	~~D_3~~	D_9
M_8	D_8	~~D_4~~	D_0
M_9	D_9	~~D_5~~	D_1

FIGURE 27. Transformations on Minor Chords. The L column is crossed out because $D_{r-4} \equiv M_r$ is a trivial identity.

Looking at these connections, we confirm that, parallel to our observations regarding system $(4,3)$, the system $(4,2)$ also **identifies (some) Major and Minor chords**; this time it happens via the **wormhole** L **connection**. Also in these tables, we find that, to some extent, system $(4,2)$ is **topologically similar** to system $(4,3)$: If in the current $(4,2)$ case we forget about the L connections, the **system** $(4,2)$ **splits into two separate subsystems** consisting of chords D_r and M_r with even indices r (the blue Universe), and chords (Major and Minor) with odd indices r (the red Universe). See figures below.

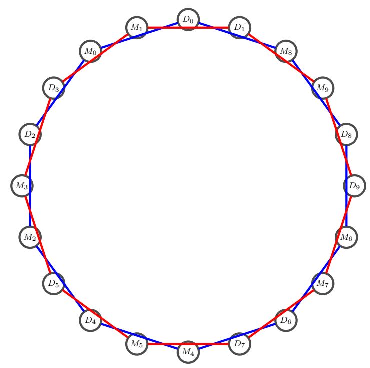


FIGURE 28. The full Levi graph of the 10-TET Tonnetz arranged on a circle. The blue edges correspond to the connections between “even” chords, while the red edges connect “odd” chords. The interleaving of the nodes visually demonstrates the symmetry of the system.

FIGURE 29. The decomposition of the 10-TET Tonnetz into two independent (5_2) Levi graphs for the $(4,2)$ system. Note that the geometric shape of the red cycle is identical to the blue cycle but rotated by 18°, occupying the gaps left by the blue component.

7.1.3. *Generic split into two separate Universes.* So far we only mentioned similarities between systems $(4,2)$ and $(4,3)$. These systems, although similar in features mentioned so far, have one important difference. This is evident if we compare three chords connected to a given Major/Minor chord in the $(4,3)$ system:

$$\{P(D_r) = M_r,\ L(D_r) = M_{r+4},\ R(D_r) = M_{r+7}\}$$
$$\{P(M_r) = D_r,\ L(M_r) = D_{r+6},\ R(M_r) = D_{r+3}\}$$

and three chords connected to a given Major/Minor chord in the $(4,2)$ system:

$$\{P(D_r) = M_r,\ L(D_r) = M_{r+4},\ R(D_r) = M_{r+8}\}$$
$$\{P(M_r) = D_r,\ L(M_r) = D_{r+6},\ R(M_r) = D_{r+2}\}.$$

It follows from these transformations that, without the wormhole (golden) connections R (in the $(4,3)$ case) and L (in the $(4,2)$ case), in both systems $(4,3)$ and $(4,2)$, chords with indices of a given parity are connected with chords with indices of the same parity. Thus, without the golden wormhole connections, in both systems we have two separate Tonnetze which constitute two independent Blue and Red Universes.

However, if we admit **the golden wormhole R connections** in the $(4,3)$ system, they change parity: if D_r and M_r have indices of parity 0, then $R(D_r)$ and $R(M_r)$ have an index of parity 1. As a result in system $(4,3)$, the Blue and Red Universes are glued together via the wormhole connections R, and in this way the $(4,3)$ system gets equipped with a connected Tonnetz, in which one can travel from the Blue subsystem to the Red one and the other way around.

The situation is very different in the $(4,2)$ system: Allowing the wormhole transitions via the L transformations will facilitate travel through the Blue Universe—enabling jumps from any node in the blue figure to its antipodal node—and it will do the same for travel within the Red Universe; however, it will not connect one Universe to the other, as in this system the node $L(D_r)$ has the same parity of its index as nodes $P(D_r)$ and $R(D_r)$. Thus, contrary to the $(4,3)$ system, the system $(4,2)$ consists of **two totally independent** highly connected subsystems that never interact with each other. See Figures 30 and 31 below.

FIGURE 30. The Blue and Red Universes with all wormhole L connections included (shown in gold). Note that both graphs are cubic (every node is connected to exactly three others: two cycle neighbors and one antipodal neighbor) and both have girth 4, which is evident from the highlighted minimal cycles (orange in the Blue Universe, violet in the Red Universe).

FIGURE 31. The Blue and Red Universes with rearranged perimeters. The coloring logic remains consistent with the previous figure: Blue/Red lines indicate the cycle connections, while Gold lines indicate wormhole connections.

7.1.4. *Musical Interpretation: Palindromic Ambiguity and Isolated Realities.* The mathematical properties derived above—specifically the identity $D_r \equiv M_{r+4}$ and the topological split—translate into unique musical characteristics that distinguish the $(4, 2)$ system from the previously analyzed $(4, 3)$ case.

- **The Palindromic Triad:** In the $(4, 3)$ system, the interval vector was $\{4, 3, 3\}$. In the current $(4, 2)$ system, the intervals between consecutive pitch classes of a chord (e.g., $0, 4, 6, 0$) form the vector $\{4, 2, 4\}$. This structure is a *palindrome*.
  - This symmetry implies that the chord is invariant under retrograde inversion.
  - Unlike standard triads which have a clear "bottom-heavy" or "top-heavy" structure, the $(4, 2)$ triad is perfectly balanced around its central interval (the Minor Third, $s = 2$).
- **The Janus-Faced Ambiguity:** Because of the identity $D_r \equiv M_{r+4}$, the distinction between Major and Minor is purely a function of the acoustic bass or context, not the pitch content.
  - If r is the bass, the intervals are $4 + 2$ (Major Third + Minor Third), and the chord functions as **Major** (D_r).
  - If $r + 4$ is the bass, the intervals are $2 + 4$ (Minor Third + Major Third), and the chord functions as **Minor** (M_{r+4}).

  The "Wormhole" transformation L (Leading-tone) exploits this ambiguity. In this system, L connects a Major chord to its antipodal Minor chord (e.g., $D_0 \leftrightarrow M_4$). Musically, this represents a radical re-interpretation of the *same* set of notes, instantly shifting the harmonic function to the opposite pole of the circle of fifths.
- **Absolute Harmonic Segregation:** The most profound difference from the $(4, 3)$ system lies in the global topology. In the $(4, 3)$ case, the "wormhole" connections allowed modulation between the Even (Blue) and Odd (Red) sets of chords.

  In the $(4, 2)$ system, however, all functional transformations (P, R, L) preserve the parity of the root index.
  - **Two Independent Universes:** A composition beginning in the **Blue Universe** (even roots) serves as a "closed system." It is harmonically rich—fully connected internally via cubic graph connections—but it is impossible to modulate to the **Red Universe** (odd roots) using standard harmonic moves.
  - **Musical Consequences:** The 10-TET composer working in the $(4, 2)$ system must choose a "Universe" at the outset. To switch between them would require a non-functional event (e.g., a "chromatic slip" or unmotivated transposition), effectively breaking the laws of the local harmonic physics.

7.2. **The "Tritone Like" System: $\Delta = 4$, $(t, s) = (5, 1)$.** We now examine the harmonic system determined by the partition $(t, s) = (5, 1)$ with the alternative "acoustic" generator $q = 6$. Here, the "Major Third" is the interval $t = 5$ (the tritone) and the "Minor Third" is $s = 1$ (the ascending semitone). The difference is $\Delta = |5 - 1| = 4$.

7.2.1. *Chords.* We define the chords according to the standard generation rules for this partition with $q = 6$:

- **Major Chords (D_r):** $\{r, r + 5, r + 6\}$
- **Minor Chords (M_r):** $\{r, r + 1, r + 6\}$

where $r \in \{0, 1, \dots, 9\}$.

Chord	Pitch Class Set
D_0	$\{0,5,6\}$
D_1	$\{1,6,7\}$
D_2	$\{2,7,8\}$
D_3	$\{3,8,9\}$
D_4	$\{4,9,0\}$
D_5	$\{5,0,1\}$
D_6	$\{6,1,2\}$
D_7	$\{7,2,3\}$
D_8	$\{8,3,4\}$
D_9	$\{9,4,5\}$

Chord	Pitch Class Set
M_0	$\{0,1,6\}$
M_1	$\{1,2,7\}$
M_2	$\{2,3,8\}$
M_3	$\{3,4,9\}$
M_4	$\{4,5,0\}$
M_5	$\{5,6,1\}$
M_6	$\{6,7,2\}$
M_7	$\{7,8,3\}$
M_8	$\{8,9,4\}$
M_9	$\{9,0,5\}$

TABLE 14. The parallel lists of Major and Minor chords in the $(5,1)$ system.

A direct comparison of the two tables reveals that **no Major chord set is identical to any Minor chord set**. The modal degeneracy encountered in the $(4,2)$ system is absent here; the universe of chords remains strictly bipartite.

7.2.2. *The Tonnetz and PLR Transformations.* We establish the connectivity of the harmonic space (the Tonnetz) by identifying chords that share two pitch classes (dyads). For every Major chord D_r, there are exactly three Minor chords that share a dyad with it. These connections define the three primary transformations P, L, and R:

(1) **Parallel (P):** D_r shares the dyad $\{r, r+6\}$ (Root and Fifth) with M_r. (Offset $A=0$).
(2) **Leading (L):** D_r shares the dyad $\{r+5, r+6\}$ (Third and Fifth) with M_{r+5}. (Offset $B=5$).
(3) **Relative (R):** D_r shares the dyad $\{r, r+5\}$ (Root and Third) with M_{r+9}. (Offset $C=9$).

Thus, the neighborhood of a Major chord D_r is $\{M_r, M_{r+5}, M_{r+9}\}$.

TABLE 15. Major Chord Connections ($D_r \to M_k$)

Chord	**P** (M_r)	**L** (M_{r+5})	**R** (M_{r+9})
D_0	M_0	M_5	M_9
D_1	M_1	M_6	M_0
D_2	M_2	M_7	M_1
D_3	M_3	M_8	M_2
D_4	M_4	M_9	M_3
D_5	M_5	M_0	M_4
D_6	M_6	M_1	M_5
D_7	M_7	M_2	M_6
D_8	M_8	M_3	M_7
D_9	M_9	M_4	M_8

TABLE 16. Minor Chord Connections ($M_r \to D_k$)

Chord	**P** (D_r)	**L** (D_{r+5})	**R** (D_{r+1})
M_0	D_0	D_5	D_1
M_1	D_1	D_6	D_2
M_2	D_2	D_7	D_3
M_3	D_3	D_8	D_4
M_4	D_4	D_9	D_5
M_5	D_5	D_0	D_6
M_6	D_6	D_1	D_7
M_7	D_7	D_2	D_8
M_8	D_8	D_3	D_9
M_9	D_9	D_4	D_0

7.2.3. *Graph Structure and Hamiltonian Analysis.* The resulting structure is a **cubic bipartite Levi graph**. The graph forms a single **connected component**.

The graph is visualized below. The P and R transformations link adjacent chords on the circle (offsets 0 and -1), creating the perimeter, while the L transformation (offset 5) creates "wormhole" connections between antipodal points.

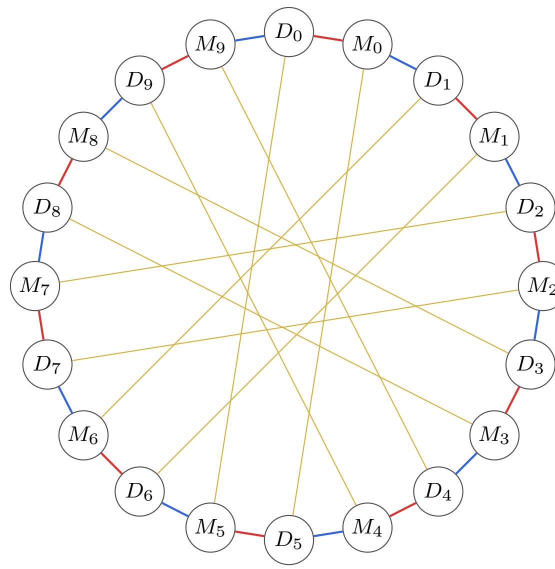


FIGURE 32. The Levi graph of the $(5, 1)$ system. The Red (P) and Gold (R) edges form the Hamiltonian perimeter cycle. The Blue (L) edges connect antipodal chords across the center.

7.2.4. *Girth and Minimal Cycles.* The graph has an obvious girth **4**. For example, the sequence $P \cdot L \cdot P \cdot L$:

$$D_r \xrightarrow{P} M_r \xrightarrow{L} D_{r+5} \xrightarrow{P} M_{r+5} \xrightarrow{L} D_{r+5+5} = D_r$$

forms 4-cycle. Thus, the graph contains 4-cycles (rectangles). This indicates a "boxy" local geometry.

7.2.5. *Hamiltonian Cycle.* Since the graph is connected and cubic, we can look for a Hamiltonian cycle. The sequence of alternating P and R transformations traverses the perimeter of the graph perfectly:

$$D_0 \xrightarrow{P} M_0 \xrightarrow{R^{-1}} D_1 \xrightarrow{P} M_1 \xrightarrow{R^{-1}} D_2 \ldots \xrightarrow{P} M_9 \xrightarrow{R^{-1}} D_0$$

(Note: R maps $D_r \to M_{r-1}$, so R^{-1} maps $M_k \to D_{k+1}$). This cycle visits all 20 chords exactly once, confirming the system is Hamiltonian.

7.3. **The "Geometric" System: $\Delta = 2$, $(t, s) = (9, 7)$.** We solve the system of congruences with the acoustic generator $q = 6$:

$$\begin{cases} t + s \equiv 6 \\ t - s \equiv 2 \end{cases} \pmod{10} \tag{13}$$

Summing the equations yields $2t \equiv 8 \pmod{10}$, which implies $t = 4$ or $t = 9$. If $t = 4$, then $s = 2$, which corresponds to the degenerate $(4, 2)$ system. Thus, we select the solution:

$$\mathbf{t = 9}, \quad \mathbf{s = 7}.$$

7.3.1. *Definition of Chords.* Using the solution $(9, 7)$ with $q = 6$, we define the chords for every root $r \in \{0, \ldots, 9\}$.

Major Chords (D_r): Defined as $(r, r + t, r + q) = \{r, r + 9, r + 6\}$.
Minor Chords (M_r): Defined as $(r, r + s, r + q) = \{r, r + 7, r + 6\}$.

Table 17. Table of Major Chords (D_r) for $t = 9$

Name	**Root**	**Set**
D_0	0	{0, 9, 6}
D_1	1	{1, 0, 7}
D_2	2	{2, 1, 8}
D_3	3	{3, 2, 9}
D_4	4	{4, 3, 0}
D_5	5	{5, 4, 1}
D_6	6	{6, 5, 2}
D_7	7	{7, 6, 3}
D_8	8	{8, 7, 4}
D_9	9	{9, 8, 5}

Table 18. Table of Minor Chords (M_r) for $s = 7$

Name	**Root**	**Set**
M_0	0	{0, 7, 6}
M_1	1	{1, 8, 7}
M_2	2	{2, 9, 8}
M_3	3	{3, 0, 9}
M_4	4	{4, 1, 0}
M_5	5	{5, 2, 1}
M_6	6	{6, 3, 2}
M_7	7	{7, 4, 3}
M_8	8	{8, 5, 4}
M_9	9	{9, 6, 5}

The interval vectors for Major $[9, 3, 7]$ and Minor $[7, 1, 9]$ are distinct (noting that interval $9 \equiv 1$, $7 \equiv 3$). A check of the specific sets confirms that **no modal degeneracy exists**.

7.3.2. *Connectivity and the Tonnetz.* We construct the Tonnetz by connecting chords that share exactly **two common tones**. The system is perfectly 3-valent.

P (Parallel): $D_r \leftrightarrow M_r$ (Share root and fifth, $\{r, r + 6\}$). Offset **0**.
L (Leading): $D_r \leftrightarrow M_{r+9}$ (Share third and fifth, $\{r + 9, r + 6\}$). Offset **9**.
R (Relative): $D_r \leftrightarrow M_{r+3}$ (Share root and third, $\{r, r + 9\}$). Offset **3**.

D_r **Connections**

Chord	Neighbors $\{M_i, M_j, M_k\}$
D_0	M_0, M_9, M_3
D_1	M_1, M_0, M_4
D_2	M_2, M_1, M_5
D_3	M_3, M_2, M_6
D_4	M_4, M_3, M_7
D_5	M_5, M_4, M_8
D_6	M_6, M_5, M_9
D_7	M_7, M_6, M_0
D_8	M_8, M_7, M_1
D_9	M_9, M_8, M_2

M_r **Connections**

Chord	Neighbors $\{D_i, D_j, D_k\}$
M_0	D_0, D_1, D_7
M_1	D_1, D_2, D_8
M_2	D_2, D_3, D_9
M_3	D_3, D_4, D_0
M_4	D_4, D_5, D_1
M_5	D_5, D_6, D_2
M_6	D_6, D_7, D_3
M_7	D_7, D_8, D_4
M_8	D_8, D_9, D_5
M_9	D_9, D_0, D_6

Figure 33. Adjacency tables for the $(9, 7)$ System. The offsets $\{0, 9, 3\}$ ensure a fully connected bipartite graph.

7.3.3. *The Levi Graph.* The resulting graph connects all 20 chords into a single structure. We visualize it by placing chords on a circle: $D_0, M_0, D_1, M_1 \dots$. The P and L transformations form the perimeter (Offsets 0 and 9), while R forms the internal chords (Offset 3).

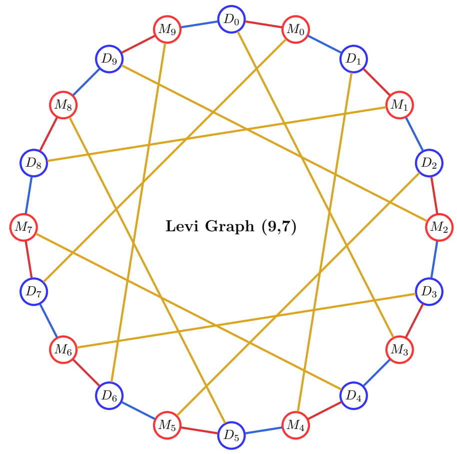


FIGURE 34. The Levi Graph of the $(9, 7)$ system. The perimeter is formed by P (red) and L (blue), while R (gold) forms the internal connections.

7.4. **Topology of Cycles: Girth 6.** We analyze the length of the shortest path to return to the starting chord (the **girth**) using the "Shift Method".

Parity Constraint. Our harmonic graph is **bipartite**. Consequently, any cycle must have an even length. We examine lengths $2k$ (i.e., 2, 4, 6...).

The "Shift Method" for Even Cycles. We analyze the path as a sequence of "double-steps": $D_{start} \to M \to D_{next}$. There are exactly 6 possible distinct double-steps starting from D_r, defined by the offsets $\{0, 9, 3\}$:

(1) **PL**: $D_r \to M_r \to D_{r+1}$ (Inverse L). Shift: **1**.
(2) **PR**: $D_r \to M_r \to D_{r-3}$ (Inverse R). Shift: **7** (-3).
(3) **LP**: $D_r \to M_{r+9} \to D_{r+9}$. Shift: **9**.
(4) **LR**: $D_r \to M_{r+9} \to D_{r+9-3}$. Shift: **6**.
(5) **RP**: $D_r \to M_{r+3} \to D_{r+3}$. Shift: **3**.
(6) **RL**: $D_r \to M_{r+3} \to D_{r+3+1}$. Shift: **4**.

Proof that Length 4 is Impossible. A cycle of length 4 consists of two distinct non-inverse shifts summing to $10 \equiv 0 \pmod{10}$. The set of shifts is $\{1, 3, 4, 6, 7, 9\}$. Let us check the sums:

- $1 + 9 = 10$. This corresponds to shifts 1 (PL) and 9 (LP). However, $LP \cdot PL = L(P^2)L^{-1}$. Since $P^2 = id$, this is $L \cdot L^{-1} = id$. This is a backtracking path, not a true cycle.
- $3 + 7 = 10$. Corresponds to RP and PR. Backtracking.
- $4 + 6 = 10$. Corresponds to RL and LR. Backtracking.

Since no non-trivial pair sums to 10, **cycles of length 4 do not exist.**

Proof that Length 6 Exists. We can find a cycle of length 6 using offsets $0, 3, 9$. Consider the path P, R, L:

$$D_0 \xrightarrow{P(0)} M_0 \xrightarrow{R^{-1}(-3)} D_7 \xrightarrow{L(9)} M_6 \xrightarrow{P^{-1}(0)} D_6 \xrightarrow{R(3)} M_9 \xrightarrow{L^{-1}(-9)} D_0$$

Net shift: $0 - 3 + 9 + 0 + 3 - 9 = 0$. The system has Girth 6.

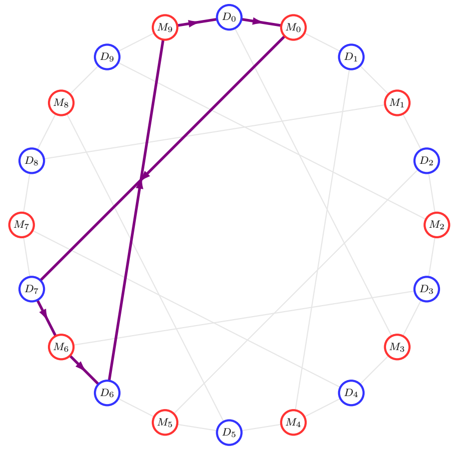


FIGURE 35. Visualization of a 6-cycle in the $(9, 7)$ system. This cycle confirms the hexagonal local geometry.

7.4.1. *Identification as a Cyclic Configuration.* We list structural properties of the derived Tonnetz:

(1) **Connection Set:** $S = \{0, 3, 9\}$.
(2) **Cubic and Bipartite:** 3-valent, Major-Minor only.
(3) **Order 20:** 10 Major and 10 Minor chords.
(4) **Girth 6:** No squares, but hexagons exist.
(5) **Hamiltonicity:** The perimeter cycle (P and L alternating) visits all nodes.
(6) **Isomorphism:** The connection set $\{0, 3, 9\}$ is equivalent to $\{0, 1, 3\}$ under the multiplication by 7 (mod 10) (since $0 \cdot 7 = 0, 3 \cdot 7 = 21 \equiv 1, 9 \cdot 7 = 63 \equiv 3$).

Thus, this system is structurally isomorphic to the $(6, 1)$ system analyzed previously.

Theorem 8. *The Levi graph of the $(9, 7)$ system is isomorphic to the **Unique Cyclic Configuration $(10_3)_3$**.*

This confirms the hypothesis that the solution $t - s = 2$ (with $q = 6$) yields the same high-symmetry geometric configuration as the "Wide" system $t - s = 5$ (with $q = 7$), providing a second musical realization of this unique combinatorial geometry.

8. LEVI GRAPHS AND CONFIGURATION THEORY

In this section, we provide precise definitions for the geometric and graph-theoretic notions used intuitively in Sections 6 and 7. As previously noted, the earlier exposition was designed to be accessible to musicians; here, we rigorously formalize these structures to place our 10-TET harmonic systems on a firm mathematical footing.

To ensure clarity, we analyze detailed examples of Levi graphs $L_n(a, b, c)$, specifically focusing on the toy model $L_5(a, b, c)$ and our primary object of study, $L_{10}(a, b, c)$. The subsequent subsections provide a full classification of the cyclic Levi graphs of $L_{10}(a, b, c)$, demonstrating that there exist exactly four non-isomorphic classes. Crucially, every specific harmonic system encountered in Sections 6 and 7 corresponds to one of these four fundamental classes.

This classification serves as the theoretical foundation for the final part of this work. In **Section 9**, we will utilize these results to characterize all musical systems resulting from our generation scheme $t + s = q$ subject to the difference constraint $t - s = \Delta$.

8.1. Connection Sets and Levi Graphs.

Definition 1 (Offset). *Let n be a positive integer. An **offset** (or connection set) S is defined as a three-element subset of the ring of integers modulo n, denoted by $\mathbb{Z}_n$:*

$$S = \{a, b, c\} \subset \mathbb{Z}_n, \quad |S| = 3.$$

Definition 2 (Cyclic Cubic Bipartite Levi Graph). *Given an offset $S = \{a, b, c\}$, the **Levi graph** $L_n(S)$ is a bipartite graph with vertex set $V = \mathcal{P} \cup \mathcal{L}$, where:*

$$\mathcal{P} = \{P_0, P_1, \dots, P_{n-1}\} \quad \textit{and} \quad \mathcal{L} = \{L_0, L_1, \dots, L_{n-1}\}.$$

The edge set E is defined constructively using all elements of S:

$$E = \{(P_i, L_{i+s}) \mid i \in \mathbb{Z}_n, \, \forall s \in S\}.$$

In other words, every point P_i is connected to the three lines $L_{i+a}, L_{i+b}, L_{i+c}$.

Historically, the set $\mathcal{P}$ in a graph $L_n(S)$ is called the set of its **points**, and the set $\mathcal{L}$ is called the set of its **lines**.

8.2. **Cycles and the Girth.** A $k \geq 4$ passage $P_{i_0} \to L_{i_1} \to P_{i_2} \to L_{i_3} \to \cdots \to L_{i_{k-1}} \to P_{i_k} = P_{i_0}$ in a graph $L_n(S)$ along a loop through its edges, is called a **cycle of length** k. The minimal length of all cycles in the graph is called its **girth**.

If a graph $L_n(S)$ has a 4–cycle (girth 4), then two points from $\mathcal{P}$ are connected with two distinct lines from $\mathcal{L}$. This forbids a geometric interpretation for such a graph as a configuration of straight lines and points in $\mathbb{R}^2$, for in Euclidean geometry two points may lie on at most one straight line. So Levi graphs $L_n(S)$ with **girth equal to four** can **not** form **a geometric configuration**.

8.3. **Isomorphism and Symmetry.**

Definition 3 (Bipartite Graph Isomorphism and Symmetry). *Two Levi graphs $L_n(S)$ and $L_n(S')$ are **isomorphic**, denoted $L_n(S) \cong L_n(S')$, if there exists a bijection $\phi : V(L_n(S)) \to V(L_n(S'))$ which preserves adjacency and the bipartite structure. Specifically, ϕ must satisfy:*

$$u \sim v \textit{ in } L_n(S) \iff \phi(u) \sim \phi(v) \textit{ in } L_n(S'),$$

and one of the following two conditions holds:

*(1) **Structure Preserving:** $\phi(\mathcal{P}) = \mathcal{P}'$ and $\phi(\mathcal{L}) = \mathcal{L}'$.*
*(2) **Structure Swapping:** $\phi(\mathcal{P}) = \mathcal{L}'$ and $\phi(\mathcal{L}) = \mathcal{P}'$.*

*An isomorphism ϕ of a graph $G = L_n(S)$ to itself is called a **symmetry** (or automorphism). The set $\mathrm{Aut}(G)$ of all symmetries forms the **symmetry group** of $L_n(S)$.*

8.4. **Affine Isomorphisms and Canonical Offsets.**

8.4.1. *The Action of the Affine Group.* We consider the group of affine transformations on $\mathbb{Z}_n$, denoted by $\mathrm{Aff}(\mathbb{Z}_n)$. An element $g \in \mathrm{Aff}(\mathbb{Z}_n)$ is defined by a pair (α, k) where $\alpha \in \mathbb{Z}_n^*$ and $k \in \mathbb{Z}_n$, acting on the ring elements as:

$$g(x) = \alpha x + k \pmod{n}.$$

This group acts naturally on the set of all offsets (subsets of size 3) by applying the transformation element-wise:

$$g \cdot \{s_1, s_2, s_3\} = \{\alpha s_1 + k, \alpha s_2 + k, \alpha s_3 + k\}.$$

Proposition 1 (Affine Isomorphism). *If two offsets S and S' belong to the same orbit under the action of $\mathrm{Aff}(\mathbb{Z}_n)$ (i.e., $S' = \alpha S + k$), then the graphs $L_n(S)$ and $L_n(S')$ are isomorphic. Such an isomorphism is called an **affine isomorphism**.*

Proof. We construct the vertex mapping $\phi : V(L_n(S)) \to V(L_n(S'))$ as follows:

$$\phi(P_i) = P'_{\alpha i},$$
$$\phi(L_j) = L'_{\alpha j+k}.$$

Consider an edge (P_i, L_{i+s}) in $L_n(S)$ where $s \in S$. The image of P_i is $P'_{\alpha i}$. The image of L_{i+s} is $L'_{\alpha(i+s)+k} = L'_{\alpha i+(\alpha s+k)}$. Let $s' = \alpha s + k$. By definition, $s' \in S'$. Thus, the edge maps to $(P'_{\alpha i}, L'_{\alpha i+s'})$, which is a valid edge in $L_n(S')$. Since α is invertible, this mapping is a bijection and explicitly preserves the bipartition ($\mathcal{P} \to \mathcal{P}'$, $\mathcal{L} \to \mathcal{L}'$). □

8.4.2. *Algorithm: Orbit Enumeration.* To classify all Levi graphs up to affine isomorphism, we must partition the set of all normalized offsets into disjoint orbits. We employ a procedure that generates the full set of normalized members for each orbit and identifies the canonical representative:

(1) Let $\mathcal{N}$ be the set of all normalized offsets (offsets containing 0) in $\mathbb{Z}_n$, sorted in lexicographical order.
(2) Initialize a set of visited offsets $\mathcal{V} = \emptyset$ and a list of orbit classes $\mathcal{C}$.
(3) Iterate through each offset $S \in \mathcal{N}$:
   - If $S \in \mathcal{V}$, skip it (it belongs to an orbit already generated).
   - If $S \notin \mathcal{V}$, then S is the **canonical representative** of a new orbit.
   - **Generate the full affine orbit** $\mathcal{O}_S$ containing all normalized equivalents of S:
     (a) Initialize the orbit set $\mathcal{O}_S = \emptyset$.
     (b) For every unit $\alpha \in \mathbb{Z}_n^*$:
       (i) Compute the scaled set $S_\alpha = \{0, \alpha x, \alpha y\}$.
       (ii) Generate the three normalized forms of S_α by shifting each of its elements to 0:
         – The set S_α itself (already contains 0).
         – The set shifted by $-\alpha x$: $\{0, -\alpha x, \alpha y - \alpha x\}$.
         – The set shifted by $-\alpha y$: $\{0, -\alpha y, \alpha x - \alpha y\}$.
       (iii) Sort the elements of each form to ensure standard representation (increasing order) and add them to $\mathcal{O}_S$.
   - Record the class as $(S, \mathcal{O}_S)$ in $\mathcal{C}$ and mark all elements in $\mathcal{O}_S$ as visited (add to $\mathcal{V}$).

The final list $\mathcal{C}$ contains the disjoint affine equivalence classes, each fully listed and represented by its lexicographically smallest member.

8.4.3. *Non-Affine Isomorphisms.* It is important to note that affine equivalence does not capture all possible isomorphisms. There exist pairs of Levi graphs which are **isomorphic** but **affinely nonequivalent** (their canonical offsets distinct). Such "exceptional" isomorphisms correspond to combinatorial symmetries that do not arise from the cyclic structure of $\mathbb{Z}_n$. However, it is worthwhile to mention that for square-free orders n of $\mathbb{Z}_n$, like $n = 5$ and $n = 10$, it is known [9] that the affine classification coincides with the full isomorphism classification.

8.5. **Example: The case $\mathbb{Z}_5$.**

8.5.1. *Detailed Derivation of Affine Orbits.* Let $n = 5$. The group of units is $\mathbb{Z}_5^* = \{1, 2, 3, 4\}$. The total number of normalized offsets in $\mathbb{Z}_5$ is $\binom{4}{2} = 6$. The set of all normalized offsets is:

$$\mathcal{N} = \{\{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \{0,2,3\}, \{0,2,4\}, \{0,3,4\}\}.$$

We apply our algorithm step-by-step.
Step 1: We pick the first offset $S = \{0, 1, 2\}$. This is our first canonical offset.

Step 2: We generate the orbit of $S = \{0, 1, 2\}$ by applying all $\alpha \in \mathbb{Z}_5^*$ and normalizing the results.

- **Case** $\alpha = 1$**:** $S_1 = \{0, 1, 2\}$.
  - Shift by 0: $\{0, 1, 2\}$.
  - Shift by -1: $\{-1, 0, 1\} \equiv \{4, 0, 1\}$. Sorted: **{0, 1, 4}**.
  - Shift by -2: $\{-2, -1, 0\} \equiv \{3, 4, 0\}$. Sorted: **{0, 3, 4}**.
- **Case** $\alpha = 2$**:** $S_2 = \{0, 2, 4\}$.
  - Shift by 0: $\{0, 2, 4\}$. Sorted: **{0, 2, 4}**.
  - Shift by -2: $\{-2, 0, 2\} \equiv \{3, 0, 2\}$. Sorted: **{0, 2, 3}**.
  - Shift by -4: $\{-4, -2, 0\} \equiv \{1, 3, 0\}$. Sorted: **{0, 1, 3}**.
- **Case** $\alpha = 3$**:** $S_3 = \{0, 3, 6\} \equiv \{0, 3, 1\}$. (Equivalent to Case $\alpha = 2$ due to symmetry).
- **Case** $\alpha = 4$**:** $S_4 = \{0, 4, 8\} \equiv \{0, 4, 3\}$. (Equivalent to Case $\alpha = 1$ due to symmetry).

Conclusion: The orbit of $\{0, 1, 2\}$ contains the set:

$$\text{Orbit} = \{\{0, 1, 2\}, \{0, 1, 4\}, \{0, 3, 4\}, \{0, 2, 4\}, \{0, 2, 3\}, \{0, 1, 3\}\}.$$

This set contains exactly **all 6** possible normalized offsets in $\mathbb{Z}_5$. Therefore, there is only **one** affine equivalence class.

Result: There is a unique cyclic cubic bipartite Levi graph for $n = 5$: $\mathbf{L_5(0, 1, 2)}$.

8.6. Example: The case $\mathbb{Z}_{10}$.

8.6.1. *Affine equivalence of offsets.* For $n = 10$, we have $\mathbb{Z}_{10}^* = \{1, 3, 7, 9\}$. Applying the same orbit enumeration algorithm, we find that the sets merge into exactly **four** affine equivalence classes.

Class	Normalized Offsets in the class	Levi G	Order of Aut(G)
1	$\{\mathbf{0,1,2}\}, \{0,1,9\}, \{0,3,6\}, \{0,3,7\}, \{0,4,7\}, \{0,8,9\}$	$\mathbf{L_{10}(0,1,2)}$	40
2	$\{\mathbf{0,1,3}\}, \{0,1,4\}, \{0,1,7\}, \{0,1,8\}, \{0,2,3\}, \{0,2,9\}, \{0,3,4\}, \{0,3,9\}, \{0,6,7\}, \{0,6,9\}, \{0,7,8\}, \{0,7,9\}$	$\mathbf{L_{10}(0,1,3)}$	20
3	$\{\mathbf{0,1,5}\}, \{0,1,6\}, \{0,2,5\}, \{0,2,7\}, \{0,3,5\}, \{0,3,8\}, \{0,4,5\}, \{0,4,9\}, \{0,5,6\}, \{0,5,7\}, \{0,5,8\}, \{0,5,9\}$	$\mathbf{L_{10}(0,1,5)}$	320
4	$\{\mathbf{0,2,4}\}, \{0,2,6\}, \{0,2,8\}, \{0,4,6\}, \{0,4,8\}, \{0,6,8\}$	$\mathbf{L_{10}(0,2,4)}$	800

8.6.2. *Symmetry Groups of $\mathbb{Z}_{10}$ Levi Graphs.* The Levi graph of configuration $\mathbb{Z}_{10}$ with offsets S is a bipartite graph with 20 vertices (10 Points P_i, 10 Lines L_j) and edges defined by $L_{i+s} \sim P_i$ for each $s \in S$. The structure of the automorphism group depends heavily on the arithmetic properties of the offset set S.

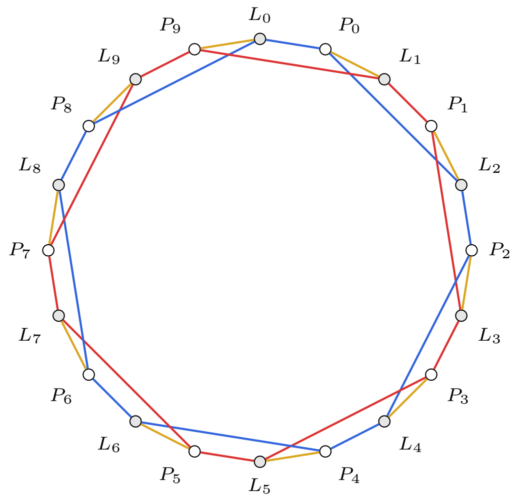
Below, we detail the derivation of the symmetry order for the four distinct isomorphism classes.

Class 1: The Rigid Palindrome (Order 40). **Offset:** $S = \{0, 1, 2\}$

This graph represents the "standard" geometry. To calculate its symmetry order, we look at the operations that preserve the adjacency structure $L_{i+s} \sim P_i$.

1. **Rotational Symmetry (×10):** Since the graph is built on $\mathbb{Z}_{10}$, we can rotate the indices by any integer $k \in \{0, \dots, 9\}$. The map $\sigma(x) = x + k$ preserves the offsets because the difference between neighbors remains constant. This gives us a base factor of **10**.
2. **Reflectional Symmetry (×2):** The set of offsets $\{0, 1, 2\}$ consists of intervals $\{1, 1\}$. This structure is a "palindrome." We can apply a reflection map $x \to -x$ (modulo 10). Because the difference $1 - 0 = 1$ mirrors the difference $2 - 1 = 1$, the graph looks identical if read backwards. This adds a factor of **2**.
3. **Duality (×2):** Levi graphs are bipartite (P vs L). However, this specific graph is *self-dual.* We can swap the roles of Points and Lines (mapping $P_i \to L_{-i}$ and $L_i \to P_{-i}$) without breaking the graph structure. This contributes the final factor of **2**.

Total Order: $10 \times 2 \times 2 = \mathbf{40}$.

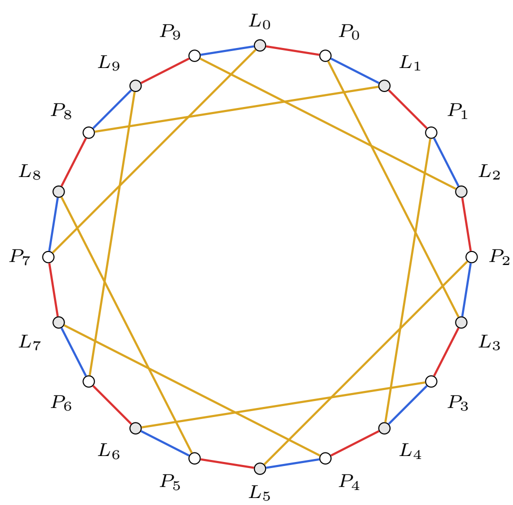


Class 2: The Chiral Drum (Order 20). **Offset:** $S = \{0, 1, 3\}$

By changing the last offset to 3, we alter the intervals between connections. The gaps are now $1 - 0 = 1$ and $3 - 1 = 2$. The set of intervals is $\{1, 2\}$.

(1) **Rotational Symmetry ($\times 10$):** As with all $\mathbb{Z}_{10}$ graphs, the cyclic group acts freely. We still have our base factor of **10**.
(2) **Broken Reflection ($\times 1$):** The sequence of intervals $\{1, 2\}$ is **not** a palindrome (flipping it gives $\{2, 1\}$). Consequently, we cannot reflect the graph onto itself; the structure is "chiral" (handed). We lose the reflection factor found in Class 1.
(3) **Duality ($\times 2$):** Despite the chirality, the graph remains self-dual. We can still swap Points and Lines provided we also invert the rotation direction appropriately to match the offset sums. This preserves the factor of **2**.

Total Order: $10 \times 1 \times 2 = \mathbf{20}$.



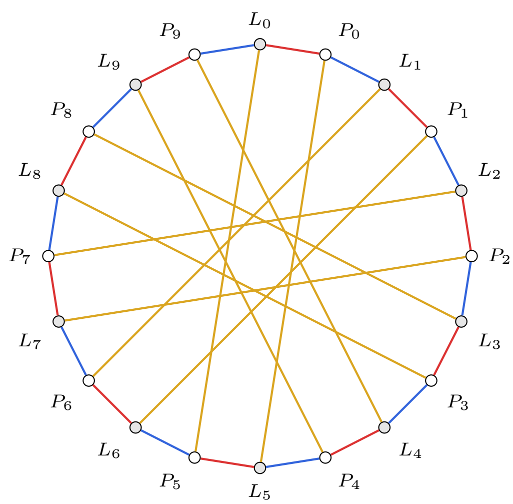
Class 3: The Wormhole Ring (Order 320). **Offset:** $S = \{0, 1, 5\}$

This case represents a massive jump in symmetry due to the special properties of the offset 5 (which is half of 10).

(1) **Structure of Squares:** The edges from offset 0 ($P_i \sim L_i$) and offset 5 ($P_i \sim L_{i+5}$) form a series of disjoint 4-cycles (squares). For example, $P_0 - L_0 - P_5 - L_5 - P_0$ is a closed loop. There are exactly 5 such squares.

(2) **The Flexible Ring:** The offset 1 (Blue) edges connect these squares into a ring. However, because the Gold connections (offset 5) are "wormholes" crossing strictly antipodal points, the structure has a high degree of freedom. We can "twist" or flip individual squares.
(3) **Combinatorial Explosion (×16):** We can flip the squares independently, subject only to a global parity constraint (we cannot flip just one square in isolation without breaking the ring, but we can flip any even combination). The number of valid configurations is $2^{5-1} = 2^4 = 16$. This multiplier of **16** is responsible for the large order.
(4) **Base Symmetries:** We retain the standard rotation (×10) and duality (×2).

Total Order: $10 \times 16 \times 2 = \mathbf{320}$.



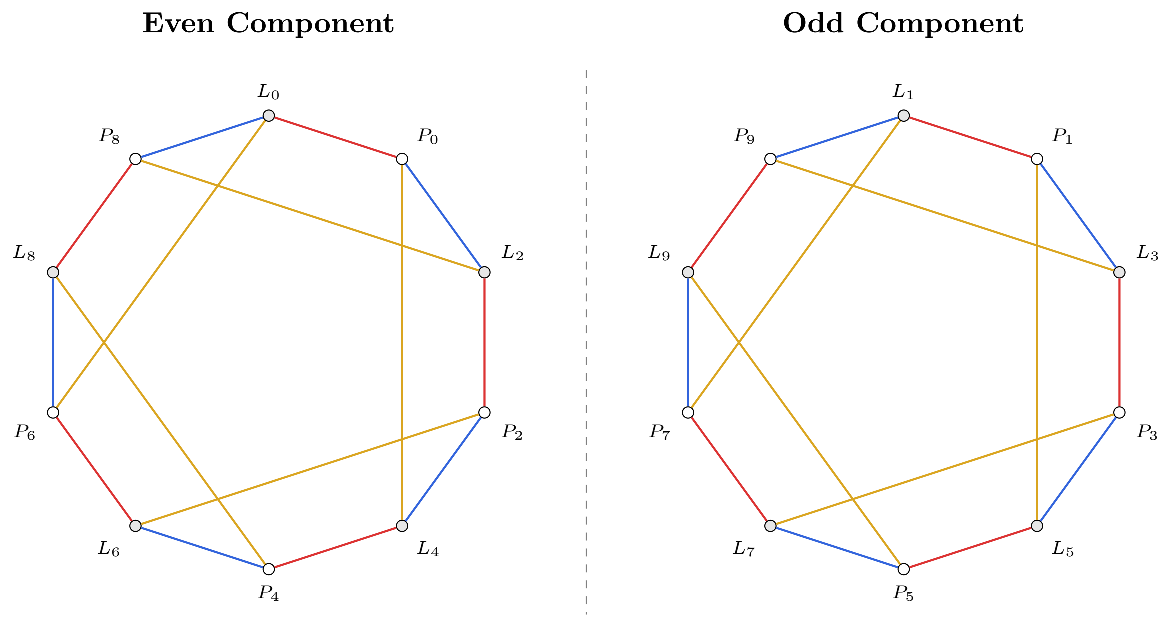
Class 4: Parallel Universes (Order 800). **Offset:** $S = \{0, 2, 4\}$

The maximum order occurs here because the graph breaks apart. Every offset in S is an **even number**.

(1) **Disconnection:**
- Even Points P_{2k} connect only to Even Lines ($2k + \text{even} = \text{even}$).
- Odd Points P_{2k+1} connect only to Odd Lines ($2k + 1 + \text{even} = \text{odd}$).

There is strictly no path between an even index and an odd index. The graph splits into two identical, disconnected components (Component A and Component B), each isomorphic to the Levi graph of $\mathbb{Z}_5$ with offsets $\{0, 1, 2\}$.
(2) **Symmetry of One Component (×20):** A single component is a Class 1 graph but in the world of modulo 5. Its order is 5 (Rot) × 2 (Refl) × 2 (Dual) = 20.
(3) **Independent Action:** Since the components are disconnected, we can rotate or reflect Component A without touching Component B, and vice-versa. This gives us 20×20 combinations.
(4) **The Swap (×2):** Since Component A and Component B are identical copies, we can physically swap the entire even set with the entire odd set. This contributes a final factor of 2.

Total Order: 20 (Even) × 20 (Odd) × 2 (Swap) = **800**.



9. Identification of Musical Systems as Levi Graphs $L_{10}(S)$

In Sections 6 and 7, we explored seven distinct musical systems arising from the constraint $t+s=q$ and the choice of the difference $\Delta = t - s$. While these systems were analyzed primarily through their harmonic properties—defined by the pitch classes $\{0,t,q\}$ constituting a Major chord—we now turn to their topological structure.

It is crucial to distinguish between the *chord definition* and the *graph definition*. The Levi graph represents the connections between Major chords D_r and Minor chords M_k. As detailed in the preamble to Section 8, a Major chord $D_r = \{r, r+t, r+q\}$ shares two common tones with the Minor chords at indices $k \in \{r, r+t, r-s\}$. Therefore, the geometry of the Tonnetz is determined not by the chord shape $\{0,t,q\}$, but by the **offset set** $S = \{0, t, n-s\}$ (taken modulo n). It is this set S that defines the "jumps" in the graph and allows us to classify the systems using the results of Section 8.6.

9.1. **Characterization Theorems.** Table 19 summarizes the seven systems, their defining intervals, and their corresponding offset sets S.

System Name	**Thirds** (t,s)	**Fifth** q	**Offset Set** $S=\{0,t,n-s\}$	**Canonical Class**
Wide Thirds	$(6,1)$	7	$\{0,6,9\}$	$L_{10}(0,1,3)$
Far Narrow Thirds	$(9,8)$	7	$\{0,9,2\}$	$L_{10}(0,1,3)$
The Geometric System	$(9,7)$	6	$\{0,9,3\}$	$L_{10}(0,1,3)$
Tritone System	$(5,2)$	7	$\{0,5,8\}$	$L_{10}(0,1,5)$
Tritone Like System	$(5,1)$	6	$\{0,5,9\}$	$L_{10}(0,1,5)$
Narrow Thirds	$(4,3)$	7	$\{0,4,7\}$	$L_{10}(0,4,7)$
Degenerate System	$(4,2)$	6	$\{0,4,8\}$	$L_{10}(0,1,2)$

Table 19. Classification of 10-TET Harmonic Systems by Levi Graph Offsets

9.1.1. *The Geometric Class: Systems (6,1), (9,8), and (9,7).* We first examine the class containing the "Wide Thirds" system. Despite their apparent differences—one generated by a Fifth ($q=7$) and another by a Tritone ($q=6$)—these systems share an identical underlying geometry.

Theorem 9. *The musical systems defined by the interval pairs* $(6,1)$*,* $(9,8)$*, and* $(9,7)$ *are pairwise isomorphic. They are all representations of the Levi graph of the geometric configuration* 10_3*, belonging to the equivalence class* $L_{10}(0,1,3)$*.*

Proof. We demonstrate the equivalence by constructing explicit affine maps $\phi(x) = ax + b \pmod{10}$ that transform the offset sets of each system into the canonical set $\{0,1,3\}$ or between the systems themselves.

(1) **(6,1) to Canonical:** The offset set is $S_{(6,1)} = \{0,6,9\}$. The map $\phi(x) = 3x + 3$ transforms:

$$\{0,6,9\} \xrightarrow{\times 3} \{0,18,27\} \equiv \{0,8,7\} \xrightarrow{+3} \{3,1,0\} = \{0,1,3\}.$$

(2) **(9,8) to Canonical:** The offset set is $S_{(9,8)} = \{0,9,2\}$. The map $\phi(x) = x + 1$ transforms:

$$\{0,9,2\} \xrightarrow{+1} \{1,0,3\} = \{0,1,3\}.$$

(3) **(9,7) to Canonical:** The offset set is $S_{(9,7)} = \{0,9,3\}$. The map $\phi(x) = 7x$ transforms:

$$\{0,9,3\} \xrightarrow{\times 7} \{0,63,21\} \equiv \{0,3,1\}.$$

Isomorphisms between the musical systems are visibly implied. For instance, the "Wide Thirds" system (6,1) is isomorphic to the "Geometric" system (9,7) via the map $\phi(x) = 9x + 9$. This map transforms $\{0,6,9\}$ into $\{9,3,0\}$, which is exactly the offset set of the (9,7) system. □

This result illuminates the structural equivalences visible in Figures 18, 24, and 34. Musically, this is a surprising result: it implies that the "Geometric" system $(9,7)$, which is built upon a generic tritone generator ($q = 6$), shares the exact same navigational logic as the (6,1) and (9,8) systems, which are built upon the perfect fifth ($q = 7$).

9.1.2. *The Tritone Class: Systems (5,2) and (5,1).*

Theorem 10. *The "Tritone" system* $(5,2)$ *and the "Tritone Like" system* $(5,1)$ *are isomorphic. They belong to the equivalence class of the Levi graph* $L_{10}(0,1,5)$*.*

The canonical offset $\{0,1,5\}$ is characterized by the presence of the element 5 (the tritone), which creates the specific "tritone bridges" observed in the analysis.

Proof. The offset set for (5,1) is $\{0,5,9\}$. The map $\phi(x) = 3x + 5$ transforms it into the offset set of (5,2), which is $\{0,5,8\}$:

$$\{0,5,9\} \xrightarrow{\times 7} \{0,35,63\} \equiv \{0,5,3\} \xrightarrow{+5} \{5,0,8\} \equiv \{0,5,8\}.$$

□

9.1.3. *The Narrow Thirds Class: System (4,3).*

Theorem 11. *The "Narrow Thirds" system* $(4,3)$ *forms a unique equivalence class, isomorphic to the Levi graph* $L_{10}(0,4,7)$*.*

This system is not equivalent to any other studied here. Its offset set $\{0,4,7\}$ (where $n - s = 7$) coincides with its chord definition set, a unique feature arising because $q = n - s = 7$. This graph is characterized by the "wormhole" connectivity discussed in Section 8, distinguishing it from the geometric class.

9.1.4. *The Degenerate Class: System (4,2).*

Theorem 12. *The "Degenerate" system* $(4,2)$ *forms a unique equivalence class, isomorphic to the Levi graph* $L_{10}(0,1,2)$*.*

The offset set $\{0,4,8\}$ transforms to the canonical set $\{0,1,2\}$ via the multiplier 3 (since $\{0,12,24\} \equiv \{0,2,4\}$, which is $\{0,1,2\}$ scaled by 2). The disconnected nature of this graph—splitting into two isomorphic "universes"—is a defining topological invariant that separates it from all other cases.

9.2. **The Law of Holographic Harmony.** We conclude our identification of musical systems with a fundamental observation regarding the relationship between the *internal* structure of a chord and the *external* structure of the Tonnetz it generates.

Throughout our analysis, we distinguished between the chord definition sets (Major $D = \{0, t, q\}$, Minor $M = \{0, s, q\}$) and the graph offset sets defining their connections ($S_D = \{0, t, -s\}$, $S_M = \{0, s, -t\}$). A careful inspection reveals that these distinctions are merely a matter of perspective: all four sets belong to the **same equivalence class** under affine transformations.

Theorem 13 (The Holographic Theorem). *For any 10-TET system defined by thirds (t, s) such that $t + s = q$, the Major chord, the Minor chord, and the topological offsets of their respective graph connections are all affinely isomorphic.*

Proof. We establish the equivalence through explicit affine mappings:

(1) **Major Chord → Minor Offsets (by Translation):** Consider the Major chord pitch classes $D = \{0, t, q\}$. By translating this set by $-t$, we obtain:
$$D - t = \{0 - t, t - t, q - t\} = \{-t, 0, s\}.$$
This is identical to the offset set $S_M = \{0, s, -t\}$ describing the connections of a Minor chord.

(2) **Minor Chord → Major Offsets (by Translation):** Consider the Minor chord pitch classes $M = \{0, s, q\}$. By translating this set by $-s$, we obtain:
$$M - s = \{0 - s, s - s, q - s\} = \{-s, 0, t\}.$$
This is identical to the offset set $S_D = \{0, t, -s\}$ describing the connections of a Major chord.

(3) **Major Chord → Minor Chord (by Inversion):** Unlike the previous cases, the Major and Minor chords are generally not translations of each other (unless $t = s$). However, they are reflections. Applying the affine map $\phi(x) = -x + q$, which is the same as $\phi(x) = 9x + q$, to the Major chord $D = \{0, t, q\}$:
$$\{0, t, q\} \xrightarrow{\times 9} \{0, 9t, 9q\} = \{0, -t, -q\} \xrightarrow{+q} \{q, q - t, 0\}.$$
Substituting $s = q - t$, we get $\{q, s, 0\}$, which is the Minor chord set M.

Since all four sets—the two chord shapes and the two graph offset sets—are linked by translations or reflections, they share the same interval vector and define the same class of Levi graph. Musically, this implies a **"holographic" principle**: the global topology of the harmonic universe (the Tonnetz) is encoded entirely within the local structure of a single chord. □

10. General Classification of 10-TET Pythagorean Systems

We are now in a position to generalize our findings into a complete classification of all possible harmonic systems within the 10-tone equal temperament.

Our analysis has shown that every musical system is uniquely determined by the choice of the generator interval (the "Fifth") q and the partitioning of this interval into a Major Third t and a Minor Third s. This partitioning satisfies the condition $t + s = q$ (modulo 10). Each such partition defines a specific harmonic flavor characterized by the difference $\Delta = t - s$.

From a topological perspective, each system (t, s, q) generates a Tonnetz defined by the offset set $\{0, t, n - s\}$ (which, as per the Holographic Theorem, is affinely isomorphic to the chord set $\{0, t, q\}$). Since Section 8 established that there are exactly four non-isomorphic classes of cyclic Levi graphs $L_{10}(a, b, c)$, every possible musical system must fall into one of these four categories.

We can therefore systematically scan all possible generators $q \in \{1, \dots, 9\}$ and all possible integer partitions $t + s = q$ (with $t \geq s$ for uniqueness in labeling) to assign every conceivable 10-TET triad to its fundamental geometric class.

Theorem 14 (The Classification of 10-TET Harmonies)**.** *Every Pythagorean tuning system in 10-TET defined by a generator q and thirds* (t, s) *is isomorphic to one of the four canonical Levi graph structures:*

(1) ***The Narrow Thirds Class*** $L_{10}(0, 1, 2)$*,*
(2) ***The Geometric Class*** $L_{10}(0, 1, 3)$*,*
(3) ***The Tritone Class*** $L_{10}(0, 1, 5)$*,*
(4) ***The Degenerate Class*** $L_{10}(0, 2, 4)$*.*

Table 20 presents the complete atlas of these systems. Each cell contains the tuple $(t, s; \Delta)$ representing the musical system's thirds and their difference. This table unifies the disparate examples analyzed in Sections 6 and 7 (such as the "Wide Thirds" $(6, 1)$ and "Narrow Thirds" $(4, 3)$) into a single coherent framework.

Fifth q	**Narrow Thirds** $L_{10}\{0, 1, 2\}$	**Geometric** $L_{10}\{0, 1, 3\}$	**Tritone** $L_{10}\{0, 1, 5\}$	**Degenerate** $L_{10}\{0, 2, 4\}$
1	(9,2; 7)	(7,4; 3), (8,3; 5)	(6,5; 1)	–
2	–	(9,3; 6)	(7,5; 2)	(8,4; 4)
3	(7,6; 1)	(2,1; 1), (9,4; 5)	(8,5; 3)	–
4	–	(3,1; 2)	(9,5; 4)	(8,6; 2)
5	–	–	(3,2; 1), (4,1; 3), (8,7; 1), (9,6; 3)	–
6	–	**(9,7; 2)**	**(5,1; 4)**	**(4,2; 2)**
7	**(4,3; 1)**	**(6,1; 5)**, **(9,8; 1)**	**(5,2; 3)**	–
8	–	(7,1; 6)	(5,3; 2)	(6,2; 4)
9	(8,1; 7)	(6,3; 3), (7,2; 5)	(5,4; 1)	–

Table 20. Complete Classification of 10-TET Pythagorean Systems. The table lists all 36 possible systems $(t, s; \Delta)$, in the format (Major third t, Minor Third $s; \Delta = t - s$), grouped by the affine class of their Levi graph. Bold entries highlight the systems analyzed in Sections 6 and 7.

This classification reveals the scarcity of "structurally unique" harmonic worlds. Despite the combinatorial possibilities, any composer working in 10-TET is ultimately navigating one of these four topological spaces.

10.1. **Mathematics versus Music.** We conclude this study by drawing a necessary line between the objective reality of the graph and the subjective reality of the ear. From a strictly mathematical point of view, our classification theorem implies a collapse of complexity: there are not thirty-six musical systems in 10-TET, nor even the seven we analyzed in detail. There are, topologically speaking, only four.

To a mathematician, the "Narrow Thirds" system $(4, 3; 1)$ and the "Exotically Wide" system $(8, 1; 7)$ are identical objects. They both reside in the first column of Table 20 (the class $L_{10}\{0, 1, 2\}$), meaning there exists an affine isomorphism that maps the chords and Tonnetz of one perfectly onto the other. The structural invariants—the cycles, the connectivity, the "wormholes"—are the same.

However, isomorphism is not identity in the realm of perception. While the mathematician sees equivalence, the musician hears a chasm of difference. The system $(4, 3)$, with its 4-step Major third and 3-step Minor third, approximates the familiar consonances of 12-TET. In contrast, its isomorphic twin $(8, 1)$ utilizes a Major third of 8 steps (perceptually a minor sixth) and a Minor third of 1 step (a semitone). A chord progression that sounds "natural" and smooth in the former would sound jarring, alien, or perhaps avant-garde in the latter, despite preserving the exact same voice-leading logic.

This creates a new frontier for composition. The explicit isomorphisms $\phi(x) = ax + b$ derived in Section 10 are not merely abstract proofs; they are translation tools. It is now possible to compose a piece in a "standard" system like $(4, 3)$, and then strictly transform it via isomorphism into a mathematically equivalent but sonically distinct system like $(8, 1)$ or $(9, 2)$.

What would such a transformation yield? The harmony would remain syntactically correct—every resolution and common-tone connection would be preserved—but the semantic "color" would shift radically. Whether the result is beautiful, horrible, or simply unrecognizable is not a question for the mathematician. The math provides the map; it is the choice of the musician to decide which landscape is worth inhabiting.

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Appendix A. Historical Motivation and the Acoustic Implementation

As written in the original paper on Pythagorean tunings for n-TETs [10], the author's initial motivation for considering the 10-TET system comes from a live concert experience, summarized below.

In January 2023 I contacted the world renowned jazz pianist Leszek Możdżer, to ask if he would be willing to give a piano recital for the participants of a mathematics conference 'GRIEG meets Chopin' that I was helping to organize. To my surprise Leszek Możdżer's answer to my request was positive, but with one caveat. Specifically, would I help him with the mathematics needed to redesign his Östlind and Almquist concert piano from the usual 12-step equally tempered (TET) scale to the 10-step TET scale? Możdżer further proposed that, at the concert during the mathematical conference, two pianos would be played: his redesigned 10-TET acoustic Östlind and Almquist piano and the usual 12-TET Steinway piano.

Without thinking much about 'why the ten-step scale?' I gladly accepted Możdżer's proposal and in short order prepared a table with data needed to retune the piano from 12-TET to 10-TET. But while the mathematics I used was straightforward the actual retuning of the 12-TET piano to the 10-TET scale encountered many technical issues (the full technical procedure is described in [2]). These were eventually resolved by the combined efforts of two teams of Leszek Możdżer (with Roman Galiński, Jan Grzyśka, Ryszard Mariański, Mirosław Mastalerz, Sławomir Rosa) and mine (Aleksander Bogucki, Andrzej Włodarczyk). My team even filed a patent application with the major ideas of this retuning.

As a result the World Premiere of the Acoustic Decaphonic Piano by Leszek Możdżer took place on July 13, 2023 in the Nowa Miodowa Concert Hall in Warsaw, Poland. In a brilliant program he premiered a number of his compositions written for two pianos, both traditional 12-TET and the decaphonic 10-TET, some of which were played on both instruments simultaneously. He also performed a number of world's piano masterpieces paraphrased for the 10-TET acoustic piano. In the opinion of many of those in attendance, he proved that with his virtuosity and for his musical compositions/paraphrases, the 10-TET piano is a wonderful instrument.

Leszek Możdżer's answer to my question 'why you want a 10-scale piano?' is beyond the scope of this note; shortly: it was quite unsatisfactory for me. So, since January 2023 I have been looking for a mathematical argument that would characterize the 10-step musical scale among all other scales. The present paper is an attempt for such a characterization.

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