1. General description of the SCREAM proposal

1.1. **Introduction.** This project is concerned with Differential Geometry, which during the last 150 years had been mainly identified with the areas of Riemannian geometry, or its specializations such as Kähler geometry, sometimes considered in other than Riemannian signatures. While research focussed on these areas has led to spectacular successes, there are many other geometries, even starting from two dimensions, which although noticed and well motivated (for example in alternative theories of gravity) were not sufficiently studied.

Recently, there has been a growing interest in reviving these other than Riemanian geometries, due to their emergence in various applications, ranging from areas of pure mathematics, such as geometric analysis (Ricci solitons) and complex analysis (uniformly Levi-degenerate surfaces), through physics: classical field theories, general relativity, string theory (near horizon geometry), to geometric control theory and robotics (rolling balls [1], simple nonholonomic systems).

In particular, in promoting the revival of these more general geometries, the Principal Investigator (PI) of this proposal, together with Michael Eastwood from Australian National University, Wojciech Kryński from Institute of Mathematics of the Polish Academy of Sciences (IMPAN), and Benjamin Warhurst from the University of Warsaw, organized a 'Simon's Semester', a new mathematical initiative of the Simons Foundation, `https://www.simonsfoundation.org`, entitled 'Symmetry and Geometric Structures'. This event, held at IMPAN in Warsaw between 1.09-30.11.2017, gathered together about 100 mathematicians from 15 countries, and posed many questions which are the main topics in this proposal.

Earlier this year, the Norwegian Principal Investigator (Prof. Boris Kruglikov) organized together with Dennis The (Tromsø), José Figueroa-O'Farrill (Edinburgh), Sigbjørn Hervik (Stavanger), Irina Markina (Bergen), Jan Slovák (Brno), and Bent Ørsted (Aarhus), the Abel Symposium 2019, `https://abelsymposium.no/2019`, entitled 'Geometry, Lie Theory and Applications'. The Abel Symposium is a prestigious annual event established by the Niels Henrik Abel Memorial Fund and the Norwegian Mathematical Society. This event again gathered about 50 mathematicians from the entire world with broad expertise in the realm of general geometries, not limited to the (pseudo)Riemannian paradigm.

Before passing to the detailed description of the proposal and its goals, we show an excerpt from the PI's application for the 2017 Simons Semester 'Symmetry and Geometric Structures', which expresses our belief that now is the time for the revival of this kind of geometries. It also gives a brief idea of what these 'other-than-Riemannian geometries' are, and how they appear in the lowest dimensions:

> When speaking of differential geometry, it is Riemannian geometry that first springs to mind. But there are many other notions of geometry even in two dimensions. They are equally well motivated but are often overlooked in differential geometry, despite their historical pedigree and intrinsic interest. In higher dimensions, alternative geometric structures abound. There is an international resurgence of interest in this area and we believe a Simons Semester would be most timely, both in stimulating new developments and in disseminating our current knowledge to the next generation.
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> A key principle and unifying theme is one of symmetry. To justify its study from this viewpoint, a geometry should have a most symmetrical incarnation with an abundance of symmetries. So it will be a homogeneous space, often referred to as the flat model, many characteristics of which will persist in the curved setting. Even in two dimensions this principle leads to interesting geometries. The round two-sphere, amongst two-dimensional Riemannian manifolds, is one of the most symmetrical: its symmetry group $SO(3)$ is of maximal dimension (although not usually considered as flat, the round sphere is just as good as Euclidean space from the standpoint of symmetry). But $SL(3,\mathbb{R})$ also acts transitively on the two-sphere, an action that preserves its geodesics, the great circles. This observation is crucial in mapping the Earth in a manner suitable for navigation (as is Mercator's projection, which is conformal, i.e. angle-preserving). The Lorentz transformations $SO(3,1)$ act conformally on the two-sphere (as the celestial sphere). As a symplectic manifold, the two-sphere enjoys an infinite-dimensional family of symmetries. Again this is tied to cartography with area-preserving mappings of the Earth often used in geography.

On the three-sphere, the principle of symmetry leads to a host of fascinating geometries, e.g. the following groups and their associated structures.

$SO(4)$	$SL(4,\mathbb{R})$	$SO(4,1)$	$Sp(4,\mathbb{R})$	$SU(2,1)$
Riemannian	projective	conformal	contact-projective	CR

In each case, the listed group acts naturally on the 3-sphere preserving the listed structure. Not only that, but the group in question is characterised as the transformations that preserve the given structure (e.g. $SO(4)$ comprises the orientation-preserving isometries of the round 3-sphere). Finally, there are curved versions of these flat models (e.g. CR geometry in 3 dimensions is an intrinsic structure inherited on real smooth hypersurfaces in 2-dimensional complex manifolds). Apart from Riemannian geometry, these are examples of parabolic geometries, recently developed by various researchers in Central Europe, including Andreas Čap (Vienna) and Jan Slovák (Brno). It has also been noticed that many key elements of Riemannian geometry (e.g. Killing fields) are, in fact, projectively invariant. Thus, many familiar (and in higher dimensions not-so-familiar) differential geometries are captured by the new parabolic theory. In five-dimensions, life becomes very interesting indeed! In particular, Élie Cartan's seminal five variables geometry, often regarded as a real tour-de-force of ingenuity, is now subsumed by the general theory. That is not to say everything is done: far from it! Working out and appreciating the implications of the general theory in any particular case can be a very involved task. Moreover, there is evidence for a theory beyond the parabolic realm, currently supported only by some fascinating examples. All-in-all, this is a thriving field with ill-defined boundaries that should be pushed to their limits.

1.2. **Why SCREAM?.**

1.2.1. *Symmetry.* One of the leading themes of the proposal is that of a *symmetry*. In the context of differential geometry it is realized as follows.

Consider a pair (M,S) of an n-dimensional manifold M equipped with a geometric structure S on it. Typically the structure S may be given by a tensor field on M (e.g. a pseudo-Riemannian metric g), a class of such tensor fields on M (e.g. a conformal structure $[g]$), a field of k-planes D on M (a rank k distribution), or a distribution with additional algebraic structure in D (e.g. a class of tensor fields on D, a split of D in a direct sum/tensor product of l-planes of various ranks, etc.).

A finite *symmetry* of the geometry associated with a pair (M,S) is a diffeomorphisms $\phi : M \to M$ preserving the structure S. The set of all such ϕ is the group G of symmetries. In the important cases, which we are going to study, this group G is in fact a Lie group.

An infinitesimal symmetry is a vector field X on M that preserves S, $\mathscr{L}_X S = 0$. The totality of all infinitesimal symmetries form the *Lie algebra of symmetries* of (M,S). Again, under reasonable assumptions that we adopt, this Lie algebra corresponds to the above Lie group: $\mathfrak{g} = \mathrm{Lie}(G)$. As a rule, it is easier to compute $\mathfrak{g}$, because the defining relation is an overdetermined system of *linear* PDEs. A typical example is the system of Killing equations $\mathscr{L}_X g = 0$ for the metric g.

1.2.2. *Curvature Reduction.* Another approach to geometric structures (M,S) on manifolds and their symmetries is by *defining them* in terms of a *flat model*, i.e. in terms of a homogeneous space $M_0 = G/H$, where G is a Lie group and H is its closed Lie subgroup. Suppose the Lie group G is the full group of symmetries of an invariant geometric structure S on M_0. We then refer to the geometric structure S as the geometric structure typical for the pair of Lie groups (G,H). Given $M_0 = G/H$, together with the geometric structure S typical for the pair (G,H), one then considers pairs (M,S) of manifolds M of the same dimension as M_0 and S sharing all the properties of the geometric structure in the model $M_0 = G/H$. In such a case one says that the geometric structure (M,S) *is modeled* on the homogeneous space $M_0 = G/H$, or is of the type (G,H).

For a vast number of geometric structures (M,S) of type (G,H) there exists a unique principal bundle $H \to \mathscr{G} \to M$, with a canonical $\mathfrak{g}$-valued 1-form ω on $\mathscr{G}$, called a *Cartan connection*, such that

(i) at every point $x \in \mathscr{G}$ the map $\omega : \mathrm{T}_x\mathscr{G} \to \mathfrak{g} = Lie(G)$ is an isomorphism,

(ii) for every fundamental vector field ξ_X, corresponding to an element X in the Lie algebra $\mathfrak{h}$ of H, $\omega(\xi_X) = X$,

(iii) ω is H-equivariant, i.e. the right translation $R_h^*(\omega)$ of ω by the group H coincides with the adjoint action of H in $\mathfrak{g}$, namely $R_h^*(\omega) = h\omega h^{-1}$ for all $h \in H$.

The *curvature* of a Cartan connection ω is defined to be the $\mathfrak{g}$-valued 2-form Ω on $\mathscr{G}$ such that

$$\Omega(X,Y) = \mathrm{d}\omega(X,Y) + \tfrac{1}{2}[\omega(X),\omega(Y)];$$

here X,Y are vector fields on $\mathscr{G}$.

A geometric structure (M,S) of type (G,H) admitting a canonical Cartan connection ω is called a *Cartan geometry* of type (G,H). The *model* $M_0 = G/H$ of such a geometry is characterized by the condition that the *curvature* of its Cartan connection *vanishes*. Thus the curvature of the Cartan connection 'measures' how much a particular Cartan geometry of type (G,H) differs from the homogeneous model $M_0 = G/H$. In this sense the model $M_0 = G/H$ is *flat*.

If the Cartan geometry is non-flat, i.e. if it has *non*vanishing curvature Ω, its curvature may be used to *reduce* the Cartan bundle $H \to \mathscr{G} \to M$ to a bundle over M modeled on a homogeneous space $M_1 = G_1/H_1$, with Lie groups $H_1 \subset G_1$ quite different than G and H. This *curvature reduction* procedure, called the *Cartan reduction*, leads *algorithmically* to the explicit construction of *all*, not only flat(!), homogeneous models of geometries of type (G,H).

1.2.3. *EquivAlence Methods.* The central problem in Differential Geometry is to determine if two geometric structures (M_1,S_1) and (M_2,S_2) are locally or globally equivalent. A solution to the *local equivalence problem* is needed for purely utilitarian reasons: it happened many times in the history that a geometric object was discovered in some coordinate system and then it was rediscovered as a 'new' object in a different coordinate system. Here the prime examples are discoveries of a'priori different solutions to the Einstein field equations of General Relativity, which turned out to be the same after some clever transformation of coordinates.

If one wants to compare two geometric structures of the same kind, and if it happens that their Lie algebras of all local symmetries are nonisomorphic, then they are not locally equivalent. So the knowledge of the full algebra of symmetries helps in solving local equivalence problems. Another case that is tractable for solving equivalence problems occurs when the two geometric structures in question are Cartan geometries. In this case, the *curvature* of the Cartan connection provides the system of all local diffeomorphism invariants, which should match for the equivalent structures.

Typically however, a geometric structure (M,S) modeled on a homogeneous space $M_0 = G/H$ does not admit a Cartan connection; it is not a Cartan geometry. This happens for example during the Cartan reduction when passing from the Cartan geometry of type (G,H) to the *reduced geometric structure* modeled on the homogeneous space $M_1 = G_1/H_1$. It turns out that during this procedure, sometimes one obtains geometries of type (G_1,H_1) that admit a Cartan connection, and sometimes geometries that do not.

Élie Cartan developed an algorithm for finding all local diffeomorphism invariants for a vast class of geometric structures (M,S) of type (G,H) beyond the class of Cartan geometries of type (G,H). Suppose the structure S can be described on M in terms of a system of a coframes (ω^i) on M, perhaps subject to a differential relation, so that a (local) equivalence of structures S_1 and S_2 corresponds to a linear change of their coframes (ω_1^i) and (ω_2^i). Then, under some regularity assumptions on the class of the structures in question, the *Cartan equivalence method* produces all local diffeomorphisms invariants on a derived bundle $\mathscr{G}$ over M, enabling to effectively determine if the two structures S_1 and S_2 are equivalent or not.

It has to be noted that Cartan equivalence method, although conceptually known and successfully applied to interesting examples of geometric structures from the beginning of the XX century, is quite tricky and computationally involved, even in low dimensions. Due to this computational complexity, the method was not widely used during more than 100 years of its life. Recently, however, because of growing computing power of electronic machines, it enjoys a renaissance of its usefulness in geometry. The main researchers in this proposal have contributed to this development (see e.g. [46, 47, 51, 30, 25, 24, 28]).

We point out that the method of Cartan is restricted by the requirement that the structure S has to be of the same pointwise type (which allows to seek for connections preserving this type), and is not easily applicable to such classes of geometric objects as cubic fields, differential operators or non-degenerate almost complex structures. A more general method consists in computing differential invariants of pseudogroup actions associated to the geometry. This has origin in the works of Sophus Lie, Arthur Tresse and was further developed by Lev Ovsiannikov, Donald Spencer, Antonio Kumpera, Peter Olver and others. Recently the fundamental theorem of the theory was proved by Boris Kruglikov and Valentin Lychagin. It states that the algebra of scalar differential invariants is rational and is finitely generated by invariant derivations [35]. This gives another approach to the equivalence problem, via signature of the structures (used in image recognition and shape theory).

Interplay between the two methods should be tested in practice, in particular in the case of parabolic geometries and more general geometries considered in this project.

1.3. **Nonholonomic mechanics and vector distributions.** Our proposal is motivated by the fact that there exists a rich reservoir of interesting geometric structures of a *simple mechanical origin,* and that the framework one builds in order to investigate them has far-reaching applications both in differential geometry and beyond.

These structures arise in a most elementary way from the *kinematics* of systems with *non-holonomic constraints:* that is, mechanical systems whose admissible velocities (i.e. tangent directions to trajectories) are restricted, at each point of the configuration space, to a linear subspace, but do not force the configuration to stay on a lower-dimensional submanifold. Systems of that kind are abundant in the field of *robotics,* where a non-holonomic constraint may arise for example from the presence of a wheel, assumed to roll on a surface without slipping or skidding. Their study – in both kinematical and dynamical aspects – is a central task of *control theory*, thus immediately indicating an important applied point of reference.

The geometric object encoding the constraints is a *vector distribution* $\mathscr{D}$ on the configuration space M: the choice of a linear subspace $\mathscr{D}_x \subset T_xM$ in each tangent space, varying smoothly from point to point (in practice, one may take M to be a dense open subset of the actual configuration space, removing certain 'singular' configurations). Trajectories of the system correspond to curves in M tangent to $\mathscr{D}$ at all points: the *integral curves* of $\mathscr{D}$. We are interested in the case where any two points of M can be connected by an integral curve of $\mathscr{D}$: one then says that $\mathscr{D}$ is *maximally non-holonomic,* or *bracket-generating,* or that the corresponding mechanical system is *controllable.*

We note that this is in general very different from the two kinds of distributions most familiar to the typical geometer, namely integrable and contact ones. The latter two can be, by a suitable choice of local coordinates, transformed to a normal form, and thus carry no interesting local information.

General bracket-generating distributions do, on the contrary, possess non-trivial *local invariants* [57, 16]!

In fact, the fundamental mathematical problem associated with such distributions may be stated as follows: given two bracket-generating distributions $(U,\mathscr{D})$ and $(U',\mathscr{D}')$ on some open subsets of $\mathbb{R}^n$, describe the set of local diffeomorphisms $\varphi : U \to U'$ such that $\mathscr{D}' = \varphi_*\mathscr{D}$. The question whether this set is inhabited is classically that of *equivalence*. On the other hand, in the particular case of $U = U'$, $\mathscr{D} = \mathscr{D}'$, describing the above set is, classically, finding the *symmetries* of the distribution $\mathscr{D}$. It turns out that approaching this problem one discovers a wealth of geometric structure that springs naturally from the seemingly austere datum of a vector distribution.

This *naturally* places the theme of 'nonholonomic mechanics and vector distributions' in the framework of the SCREAM proposal. Nonholonomic mechanical systems with linear velocity constraints define bracket-generating distributions on the configuration space manifold, and generically bracket-generating distributions have *local invariants*, *curvature*, and may have *symmetries*. Thus one is tempted to look for those among these systems whose kinematics, namely the configuration space M and the velocity distribution $\mathscr{D}$, is very symmetric, or is a flat model for some Cartan geometry, or is an example of a homogeneous model for a geometric structure $(M,\mathscr{D})$ of type (G,H), etc. In particular, finding nonholonomic systems whose kinematics $(M,\mathscr{D})$ is a flat model for a *parabolic geometry* is very interesting (see below). If any such system, parabolic, or Cartan geometric in nature, is found (the Polish and the Norwegian PI's encountered examples of such systems in their research [17, 18, 4]), it would associate 'hidden' *algebraic objects*, such as the symmetry, or more generally the pair of Lie groups (G,H), to the mechanical system. Their significance for the physics of the system should be further explored.

1.4. **Tanaka prolongation.** It is now useful to recall an important concept:

A *nilpotent* Lie algebra $\mathfrak{n}$ is said to admit a k-step stratification if $\mathfrak{n} = \mathfrak{n}_{-k} \oplus \mathfrak{n}_{-k+1} \oplus \cdots \oplus \mathfrak{n}_{-1}$ such that $\mathfrak{n}_{-s-1} = [\mathfrak{n}_{-1}, \mathfrak{n}_{-s}]$ for all $s = 1, 2, \ldots, k-1$ and $\mathfrak{n}_{-k}$ is contained in the centre of $\mathfrak{n}$. We will refer to nilpotent Lie algebras with a chosen stratification as nilpotent Lie algebras with a *stratification*.

The crucial fact is that the basic information of any vector distribution $\mathscr{D}$ is encoded, at any point x of the manifold M, in a certain point dependent *nilpotent Lie algebra* $\mathfrak{n}_x$ *with a stratification*, called the *symbol* of $\mathscr{D}$ at x. From now on, we will only consider *strongly regular bracket-generating distributions*, i.e. distributions $\mathscr{D}$ whose symbols are the same (in the stratified Lie algebraic sense) at all points of M.

The symbol of a strongly regular bracket-generating distribution $\mathscr{D}$ is a *diffeomorphism invariant* of the structure $(M,\mathscr{D})$. This divides the set of all such structures into equivalence classes $(M,\mathscr{D},\mathfrak{n})$ of strongly regular bracket-generating distributions $\mathscr{D}$ with a given symbol $\mathfrak{n}$. It follows that within each class $(M,\mathscr{D},\mathfrak{n})$, there are typically still plenty of locally non-equivalent distributions. The question of what is the simplest among them, or equivalently the question of what is the *model* for each class, is answered by performing a purely algebraic procedure on the symbol $\mathfrak{n}$. This procedure is called the *Tanaka prolongation*.

The Tanaka prolongation is defined for any nilpotent Lie algebra with a k-step stratification; thus, in particular, it is well defined for symbols of strongly regular distributions. Given a nilpotent Lie algebra with a stratification $\mathfrak{n}=\mathfrak{n}_{-k}\oplus\mathfrak{n}_{-k+1}\oplus\cdots\oplus\mathfrak{n}_{-1}$, one introduces a notation $\mathfrak{g}_-=\mathfrak{n}$, $\mathfrak{g}_{-s}=\mathfrak{n}_{-s}$, and defines Tanaka prolongation of $\mathfrak{g}_-$ in terms of a sequence $\mathfrak{g}_-$, $\mathfrak{g}^0=\mathfrak{g}_-\oplus\mathfrak{g}_0$, $\mathfrak{g}^1=\mathfrak{g}_-\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, $\mathfrak{g}^2=\mathfrak{g}_-\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2$,..., $\mathfrak{g}^s=\mathfrak{g}_-\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2\cdots\oplus\mathfrak{g}_s$, etc, where $\mathfrak{g}_0$ is the Lie algebra of *derivations* of $\mathfrak{g}_-$ preserving the stratification in it; the vector spaces $\mathfrak{g}_s$ with $s>0$, each of *finite* dimension, are similarly defined in an *algorithmic and purely algebraic* fashion, see e.g. [56]. The obtained sequence $\mathfrak{g}^s$ is *graded* in the sense that $[\mathfrak{g}_\ell,\mathfrak{g}_{\ell'}]\subset\mathfrak{g}_{\ell+\ell'}$ for all $-k\leq\ell,\ell'\leq s$, whenever $-k\leq\ell+\ell'\leq s$, and $[\mathfrak{g}_\ell,\mathfrak{g}_{\ell'}]=\{0\}$ otherwise.

The Tanaka prolongation procedure can have two outcomes: either the sequence $\mathfrak{g}^0$, $\mathfrak{g}^1$, $\mathfrak{g}^2$, ... is infinite, or it terminates at step K, i.e. the procedure results in $\mathfrak{g}_{K+1}=\{0\}$. In the following we will concentrate only on the later case when $K<\infty$, i.e. when the Tanaka prolongation procedure uniquely associates a *finite dimensional* graded Lie algebra $\mathfrak{g}^K$ to the nilpotent Lie algebra with a stratification $\mathfrak{n}$. This ultimate Lie algebra $\mathfrak{g}^K$ is called the *Tanaka prolongation of* $\mathfrak{n}$. We will denote it as $\mathfrak{g}(\mathfrak{n})$. We have the following, crucial, theorem.

Theorem. Among all structures $(M,\mathscr{D},\mathfrak{n})$ of strongly regular bracket generating distributions $\mathscr{D}$ on a manifold M, with symbol $\mathfrak{n}$, whose Tanaka prolongation is *finite*, the model, or what is the same, the structure with maximal symmetry, is given by $M_0=G(N)/P$, where $G(N)$ is a Lie group with the Lie algebra isomorphic to the Tanaka prolongation $\mathfrak{g}(\mathfrak{n})$ of $\mathfrak{n}$, and P is its Lie subgroup having Lie algebra $\mathfrak{p}=\mathfrak{g}_0\oplus\mathfrak{g}_+$. Since the tangent space to M_0 at every point is isomorphic to the symbol $\mathfrak{n}$ of $\mathscr{D}$, the maximally symmetric distribution $\mathscr{D}_0$ on M_0 is defined as a span of vector fields which, at each point $x\in M_0$, are tangent to the vector subspace $\mathfrak{n}_{-1}\subset \mathrm{T}_xM_0$. The symmetry group of the structure $(M_0,\mathscr{D}_0)$ is $G(N)$.

Note that this theorem provides a vast reservoir of geometric structures associated with bracket-generating distributions, which are modeled on homogeneous spaces G/H: to fix a class of geometries of type $(M,\mathscr{D},\mathfrak{n})$, it is enough to choose a nilpotent Lie algebra $\mathfrak{n}$ with a stratification, whose Tanaka prolongation is finite. Then, by choosing $G=G(N)$ and $H=P$, we have a distribution $\mathscr{D}_0$ on $M_0=G(N)/P$, which is the most symmetric among all distributions having constant symbol $\mathfrak{n}$. Studying deformations of these kind of distributions, and their relations to the nonholonomic mechanics, will be yet another objective of this proposal.

1.5. **Parabolic geometries.** As we said before, not all geometric structures (M,S) of type (G,H) admit a Cartan connection, and even if they do, it is not easy, in general, to determine it. *Parabolic geometries* form a large class of structures for which this problem does not exist: the algebraic properties of parabolic geometries guarantee existence and uniqueness of a Cartan connection, making all of them *Cartan geometries*.

One of the simplest ways of defining parabolic geometries, is to say that these are, roughly, geometries $(M,\mathscr{D},\mathfrak{n})$ from the previous sections for which the Tanaka prolongation $\mathfrak{g}(\mathfrak{n})$ of $\mathfrak{n}$ is finite and *semisimple*.

More precisely, general parabolic geometries, in addition to the distribution $\mathscr{D}$ in $\mathrm{T}M$, have more structure in $\mathscr{D}$. Algebraically, this corresponds to having an additional structure s in $\mathfrak{n}$. The structure s can be a tensor in $\mathfrak{n}$, or a property that $\mathfrak{n}$ is composed from specific algebraic objects, such as subalgebras, vector subspaces, etc. There also exists a procedure of Tanaka prolongation of k-step nilpotent Lie algebras with such additional structures. It is almost the same as for the *bare* nilpotent Lie algebras, with the only difference that now $\mathfrak{g}_0$ must preserve not only the strata of $\mathfrak{n}$, but also the structure s in $\mathfrak{n}$. This being said, we *define parabolic geometries* more precisely as structures $(M,\mathscr{D},\mathfrak{n},s)$ for which the Tanaka prolongation preserving the structure s in $\mathfrak{n}$ is *semisimple*[1]. Even if we restrict to the case when the Tanaka prolongation preserving s in $\mathfrak{n}$ is *simple*, there is a vast ZOO of such structures, known before a concept of parabolic geometry was abstracted. These *classical* parabolic geometries include *conformal and projective geometries*, *Cauchy-Riemann geometries*, the *geometry of* $(2,3,5)$ *distributions*, the *geometry of 3rd order ODEs considered modulo contact transformations of variables* and the *geometry of 2nd order ODEs considered modulo point transformations of variables*, etc. But there are *many other* parabolic geometries:

It follows that, any choice of a parabolic subalgebra $\mathfrak{p}$ in a noncompact real form of a simple Lie algebra $\mathfrak{g}$ defines a parabolic geometry, which is a geometric structure (M,S) of type (G,P), where G is a real (semi)simple Lie group with Lie algebra $\mathfrak{g}$ and P is a parabolic subgroup in G with Lie algebra $\mathfrak{p}$. The crucial fact about these geometries is that the algebraic structure of $\mathfrak{g}$ and $\mathfrak{p}$ (semisimplicity of $\mathfrak{g}$ and parabolicity of $\mathfrak{p}$) naturally *equips all such geometries* with a *unique Cartan connection*. Thus *parabolic geometries are always Cartan*

[1]Our definition of parabolic geometries is a rough one, devised for simplicity of exposition of this proposal. It excludes some of the known parabolic geometries, e.g. the projective and the contact projective ones, but this is not crucial here.

geometries. The semisimple Lie group G and a choice of parabolic subgroup P, which models any such geometry as G/P, is usually incorporated to the name: to specify which parabolic geometry from their vast ZOO is considered, we will use the term *parabolic geometry of type* (G,P).

Disclaimer. In the subsequent sections describing *preliminary work* we will often refer to *two particular parabolic geometries* in dimension five, namely to the G_2 *geometry of* $(2,3,5)$ *distributions* and the, quite different, G_2 *contact geometry in dimension five*. Although they on their own create a number of objectives that we will be studying in this proposal, it should be understood that the objectives associated with them have their analogs in other parabolic geometries, which are equally important and will be studied on the same footing.

1.6. **Preliminary work.** The PI and the Norwegian partner have been among the leading researchers in a community that contributed to the revival of the study of geometric structures such as Cartan geometries, and particularly parabolic geometries, in recent years.

The PI was among the first who at the turn of the XXth and XXIst centuries were revitalizing the subject of Cartan geometries. This included work on CR geometry [53, 42, 52, 27], conformal and projective geometries [23, 31], geometry of differential equations [45, 50], and geometry of $(2,3,5)$ distributions [48]. Apart from these predominantly parabolic geometries, he also contributed to studies of more exotic Cartan geometries such as [2, 21, 30, 51]. The main research partner is the author of influential work on differential invariants of geometric structures, geometries of differential equations, integrability and symmetry analysis, and Tanaka theory. A potential Norwegian co-investigator, Dennis The, is a colleague and frequent collaborator of Kruglikov, and he has made significant contributions to the general theory of parabolic geometries and the geometric theory of differential equations. Of particular importance in the context of the project is recent work [37, 36, 38, 14, 15] of the Norwegian research partners on symmetry gaps and classification techniques for homogeneous geometries.

In the following we will outline preliminary work that is directly related to the objectives of the project.

1.6.1. G_2 *geometries, differential equations and mechanics.* Particular work has been done in the field of relations between Cartan geometries and nonholonomic mechanics. Here the best known example is the association between the *kinematics of surfaces* (such as spheres) *rolling on each other without slipping or twisting* and the *parabolic geometry of* $(2,3,5)$ *distributions* [6, 3, 1].

All the three named team members of the proposal have worked on the development of this subject. In particular in [1] new surfaces were found, such that the distribution encoding the nonslipping and nontwisting constraints had the simple Lie group G_2 as the group of all local symmetries. This work was based on the original observation that there is always a $(2,3,5)$ distribution, i.e. a parabolic G_2 geometry in dimension 5, associated with a conformally non flat split signature structure $[g]$ on an oriented 4-dimensional manifold M. The $(2,3,5)$ distribution is defined on a circle bundle $\mathbb{S}^1 \to \mathbb{T}(M) \to M$, called the *circle twistor bundle* $\mathbb{T}(M)$, of all real totally null planes of a given selfduality over M [1]. This *embeds* the nonflat parabolic geometries of conformal 4-dimensional split signature structures into the space of G_2 parabolic geometries in dimension five describing $(2,3,5)$ distributions. The paper [1] created a number of questions, which during the last few years were investigated by Michael Eastwood, Dennis The (a potential Norwegian co-investigator), Katja Sagerschnig (a potential Polish co-investigator) and Paweł Nurowski (PI). These questions are centered around the problem of *characterization of* $(2,3,5)$ *distributions*, with specific subproblems 1), 2) and 3) in our list from Section ??. The preliminary results of Eastwood/The/Sagerschnig/Nurowski indicate that there are curvature obstructions for $(2,3,5)$ distributions to be twistor distributions of the conformal split signature structures in dimension 4.

Another set of problems related to the paper [1] is to find possibly all 4-dimensional conformal split signature manifolds, whose twistor distribution has precisely G_2 symmetry. Preliminary results here are included in the papers of Paweł Nurowski and Gil Bor [4], as well as in the paper of Nurowski and Daniel An (arXives:1302.1910).

A vital source of such geometries are nondegenerate second order PDEs $F(x,u,\partial u,\partial^2 u)=0$ on one function $u(x^1,\dots,x^d)$ of $d=4$ variables, as investigated by Boris Kruglikov (Norwegian PI). Indeed, it was explained in [19] that for both $d=3,4$ any solution u of the PDE carries a canonical conformal structure c_F read off the symbol of the equation. Assuming neutral signature, by [7] there is an associated rank 2 distribution $\widehat{\Pi}$ on the twistor bundle $\mathbb{T}(M_u)$ over the base-manifold lifted to the space of jets, which is projected to the canonical α-null congruence of 2-planes Π. If the PDE possesses a nontrivial dispersionless Lax pair, then $\widehat{\Pi}$ is integrable, but otherwise it has a generic growth and gives rise to a $(2,3,5)$ distribution. For instance, the dispersionless Kadomtsev-Petviashvilli (dKP) equation ($d=3$)

$$F:\ u_{tx}-u_x u_{xx}-u_{yy}=0,\quad c_F=[4u_x dt^2-dy^2+4\,dt\,dx],$$

is integrable (by the Lax pair $\partial_y - \lambda\partial_x + u_{xx}\partial_\lambda$, $\partial_t - (\lambda^2 + u_x)\partial_x + (\lambda u_{xx} + u_{xy})\partial_\lambda$), while its higher-dimensional analog ($d = 4$), the Khokhlov–Zabolotskaya equation

$$F:\ u_{tx} - u_x u_{xx} - u_{yy} + u_{zz} = 0, \quad c_F = [4u_x\,dt^2 - dy^2 + dz^2 + 4\,dt\,dx],$$

is not. Thus symmetry reductions and curvature constraints will lead to special solutions of these PDEs.

Another important G_2 structure is based on contact geometry in dimension 5. This geometry was investigated by Paweł Nurowski (PI), Dennis The and Michael Eastwood. In particular, Eastwood and Nurowski in [17, 18] have shown that the configuration space M of a mechanical system in three dimensions consisting of a disk whose velocity is constrained to be tangent to the plane of a disk, a *flying saucer* as it is called in [17, 18], may be equipped with a *flat* G_2 contact parabolic geometry. It was shown in [17, 18] that such a 'flying saucer' can be also defined in a *curved* 3-dimensional space, in terms of a rather esoteric geometric structure in there. It then raised a problem similar to the one discussed for the twistor distribution and the corresponding 4-manifold: what one has to assume about curvatures of this 'esoteric' 3D geometry for the configuration space of the flying saucer to be equipped with a *flat* G_2 contact geometry, i.e. one with G_2 symmetry.

1.6.2. *Normalizations of geometries with infinite type symbol.* There is a large class of bracket-generating distributions for which the Tanaka prolongation never terminates. Examples are *contact* distributions. Also an *Engel* distribution in dimension 4, i.e. a rank 2 distribution $\mathscr{D}$ such that $\operatorname{rank}([\mathscr{D},\mathscr{D}]) = 3$ and $\operatorname{rank}([\mathscr{D},[\mathscr{D},\mathscr{D}]]) = 4$, has this property.

It is well known that a contact distribution in dimension three is naturally defined on the configuration space M of a skate on the plane. It is a little less known, that viewing the physical system of a skate on the plane properly, one can associate to it a *Cartan geometry*, which turns out to be even a *parabolic geometry*. It is clear that the dimension of M is three, as one needs to have three numbers to determine a position of the skate in the plane and its orientation. The contact distribution $\mathscr{D}$ on M is defined by means of one constraint on velocities which prevents the skate from *skidding*. Although the symbol $\mathfrak{n}$ of this distribution $\mathscr{D}$ has infinite Tanaka prolongation, this leads to a nontrivial parabolic geometry, because the skate enables for two, physically quite different, movements which are compatible with the nonskidding condition. These movements are: the *sliding along the skate blade* and the *evolution named as a pirouette*. These two movements define two lines in $\mathscr{D}$ at every point of M, equipping the configuration space M of the skate with the *split* of $\mathscr{D}$ into rank one subdistributions, $\mathscr{D} = \mathscr{D}_1 \oplus \mathscr{D}_2$. This is reflected in the symbol $\mathfrak{n}$ of the skate distribution, which has a split $\mathfrak{n}_{-1} = \mathfrak{n}_{-1,1} \oplus \mathfrak{n}_{-1,2}$. This 'chops' the Tanaka prolongation of the symbol of $\mathscr{D}$. For the fully understood configuration space of the skate, the Tanaka prolongation procedure has to preserve not only the stratified $\mathfrak{n}$, but also $\mathfrak{n}_{-1,1}$ and $\mathfrak{n}_{-1,2}$. It results in a finite Tanaka prolongation algebra $\mathfrak{g}(\mathfrak{n})$ isomorphic to the simple Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. This leads to the conclusion that the configuration space of a skate is a flat model for a parabolic geometry of type $(SL(3,\mathbb{R}),P)$, with the parabolic subgroup P being the Borel subgroup in $SL(3,\mathbb{R})$.

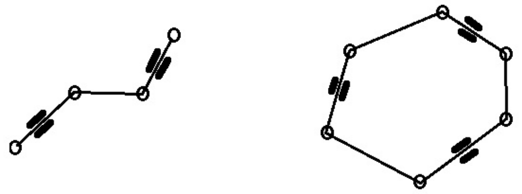
There are more contact parabolic geometries with similar properties. For example, as it is shown in a recent paper of Nurowski and Hill [29], the configuration space of a *car* on the plane defines a flat model for a parabolic geometry of type $(SO(2,3),P)$, with P being the Borel subgroup in $SO(2,3)$. Here the nonholonomic distribution, defined by the wheels preventing the car from skidding, is an *Engel* distribution. It has infinite Tanaka prolongation. But similarly to the skate, this distribution has a *split*, which results in the configuration space of a car to be equipped with a Cartan, actually parabolic, geometry. Also the G_2 contact geometry in dimension five associated with a flying saucer mentioned above is parabolic, due to an additional structure (twisted cubic, enabling a complicated flying saucer's manoeuver) in the contact distribution.

We would like to study more of such examples in the proposal, possibly obtaining an understanding of them at some unifying level.

The above reductions are obtained by additional *tensorial* constraints on the geometry/kinematics. Another source of constraints is of the *curvature type*. This is central in Cartan's equivalence method, under the title of normalization, and it was also used in the study of the symmetry gap problem [37]. However, as mentioned above, this does not necessary lead to a Cartan geometry. For instance, almost complex structures have infinite type as G-structures, yet imposing a non-degeneracy for the Nijenhuis tensor yields a finite type, in particular the symmetry group is always a Lie group. In fact, the largest possible symmetry is G_2 [33], yet in this case it cannot serve as a model space. We will study more reductions of geometries in the project and will aim at understanding the mechanism at a unifying level.

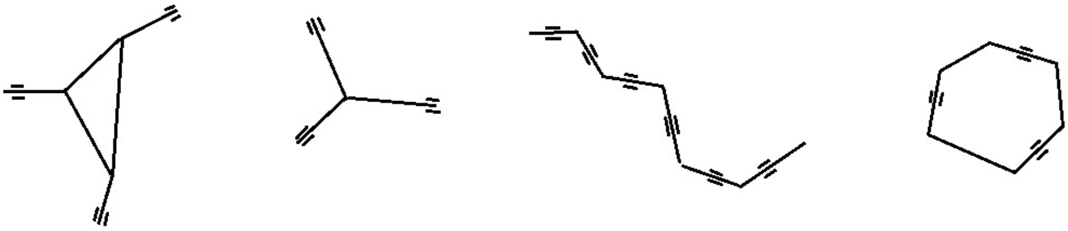
1.6.3. *Snakes and planar robots.* Some years ago, the PI of this project started to generalize the concept of *trailers*, which appeared in control theory based on Goursat flags in multi-dimensions (an example of which is the Engel distribution in dimension four), to the concept of *snakes* or more general *planar robots* – a term coined by Gil Bor and Paweł Nurowski. These are much simpler, and in low dimensions correspond to interesting parabolic geometries.

A planar robot is made out of E edges, joined by V vertices, forming a planar graph with F faces. There are also W wheels, attached to some of the edges; each wheel is aligned parallel to its edge (like the back wheel of a bicycle). During the motion of such a robot, the length of each edge is fixed, but the angles between adjacent edges are allowed to vary. Here are some examples:



V=4, E=3, F=0, W=2 V=E=6, F=1, W=3

Inevitably, some of the robots are nicknamed



"The trident" "The bodyless trident" "The 6-edge snake" "The Benzen"

Each wheel has the effect of preventing the edge to which it is attached from skidding sideways. In other words, the wheel imposes a restriction on the movement of the edge to which it is attached. It is a *non-holonomic constraint*, so that the point on the edge where the wheel is attached can move only in the direction of the edge.

In the physical realization of such a robot, one controls only the direction of the edges by placing controls (motors or muscles) at the vertices joining adjacent edges. (There are no motors at the wheels themselves.) Remarkably, such a robot is typically able to move along the plane quite efficiently by virtue of such controls.

The mathematics of such robots is quite intriguing. For example, the PI and Gil Bor have established a simple result about them, which helps in the following problem.

The wheels of the robot introduce *linear* constraints on its velocity. Hence, similarly to the case of a skate or a car, the configuration space M of the robot is equipped with a vector distribution $\mathscr{D}$ of possible velocities. It follows that for example the *trident snake* from the above figures has M of dimension $n = 6$, and the rank r of its velocity distribution $\mathscr{D}$ is equal to $r = 3$. We will say that the type of the velocity distribution $\mathscr{D}$ of the trident snake is $(r,n) = (3,6)$. It turns out however, that there is another quite different snake, namely the *benzen* snake (see the figure above), which also has the velocity distribution of type $(r,n) = (3,6)$. A similar example is obtained by taking the *bodyless trident snake*, and the 3-*edge snake* with the same number of edges and wheels as the bodyless trident snake, but with a different topology, in which its three edges e_1, e_2, e_3, are arranged such that e_2 is attached to e_1 on its one end, and it is attached to e_3 on its second end, whereas e_1 and e_3 are connected with e_2 only. Both of these snakes have velocity distribution $\mathscr{D}$ of the same type $(r,n) = (2,5)$.

In this project we would like to find *planar robots whose velocity distribution has a simple Lie algebra as its algebra of symmetries*. Hopefully we would like to find all planar robots which are flat models for parabolic geometries associated with distributions of a given rank r on manifolds of a given dimension n. We would first need to select all topologies of snakes whose velocity distribution is of type (r,n). Here is the place for the preliminary, very simple result, which we quote from the unpublished paper of Bor and Nurowski:

Proposition. A velocity distribution of type (r,n) of a planar robot with F faces, V vertices, E edges and W wheels, satisfies (under the independence assumption on the E holonomic and W anholonomic constraints)

$$E = n + 2F - 2, \qquad V = n + F - 1, \qquad W = n - r.$$

This proposition tells us how to relate the fundamental dimensions, n of the configuration space manifold M and r of the velocity distribution $\mathscr{D}$ of the robot, to the number of its faces F, edges E, vertices V and wheels W. We plan to inspect the list of the distributions of rank r in dimension n whose symbols have Tanaka prolongations as simple Lie algebras, and then plan to use this proposition, as a sieve, to pick up only those topologies of planar robots that can have these values of r and n.

We emphasize that this program is not empty. For example, the velocity distribution of a benzen snake is almost everywhere a *generic* $(3,6)$ distribution. This means that it enjoys a parabolic geometry whose model has symmetry $Spin(3,4)$. Similarly, both snakes with $(r,n)=(2,5)$ discussed above, have the distribution $\mathscr{D}$ which almost everywhere is $(2,3,5)$. Thus they both belong to a parabolic geometry with flat model having symmetry G_2. Whether or not there is a way of adjusting the parameters of these snakes, such as the *length* of their edges, to make them $Spin(3,4)$ or G_2 symmetric is an open question worth further studies.

1.6.4. *Contactifications and their generalizations.* As an example of what we mean by a *contactification* we take a $GL(2,\mathbb{R})$ *geometry in dimension four*. It is a structure (M,S) on a 4-dimensional manifold M with S being determined by an assignment of a *twisted cubic curve* $\gamma(x)=(1,x,x^2,x^3)$ at the tangent space T_aM of each point $a\in M$. Alternatively, since the twisted cubic reduces the structure group of each tangent space T_aM from $GL(4,\mathbb{R})$ to the $GL(2,\mathbb{R})$ sitting *maximally* in $GL(4,\mathbb{R})$ (or, what is the same in this case, acting *irreducibly* in $\mathbb{R}^4\cong\mathrm{T}_aM$), the $GL(2,\mathbb{R})$ geometry in dimension 4 can be defined as a geometry on a 4-manifold M with a reduction S of the structure group of the tangent bundle from $GL(4,\mathbb{R})$ to the irreducible $GL(2,\mathbb{R})$.

This geometry was studied by Robert Bryant in [5], but also, with more details devoted to the analysis of the *curvature*, by the PI of this proposal in [49, 24]. In particular, Bryant in [5] has shown that this geometry can be also defined in terms of a certain *symmetric 4th rank tensor* on M, characterized by the requirement that its stabilizer in $GL(4,\mathbb{R})$ is the irreducible $GL(2,\mathbb{R})$. It was also Bryant, who has shown that this geometry is in a one-to-one correspondence with classes of *4th order ODEs considered modulo contact transformations of variables*, restricted by a certain contact invariant conditions, called Wünschmann conditions.

To be more specific, consider a 4th order ODE $y''''=0$. This ODE, considered modulo contact transformation of variables, defines the following coframe

$$\omega^0=-3(\mathrm{d}y-y'\mathrm{d}x),\quad \omega^1=\mathrm{d}y'-y''\mathrm{d}x,\quad \omega^2=-\tfrac{1}{2}(\mathrm{d}y''-y'''\mathrm{d}x),\quad \omega^3=\mathrm{d}y''',\quad \omega^4=\mathrm{d}x \tag{1}$$

on the 5-dimensional space $\mathscr{J}^3$ of 3rd jets of functions $y=f(x)$ of one variable x. This space is foliated by integral curves of the total differential vector field $X=\frac{\mathrm{d}}{\mathrm{d}x}$. The leaf space, whose points are these curves, is the 4-dimensional space M of solutions of the ODE $y''''=0$. A convenient parametrization of M is obtained by passing from coordinates (x,y,y',y'',y''') of $\mathscr{J}^3$ to coordinates (x,a_0,a_1,a_2,a_3) related to the general solution $y=a_0+3a_1x+3a_2x^2+a_3x^3$ of $y''''=0$. Indeed, the transformation $(x,y,y',y'',y''')\to(x,a_0,a_1,a_2,a_3)$ given by

$$x\to x,\quad y\to a_0+3a_1x+3a_2x^2+a_3x^3,\quad y'\to 3a_1+6a_2x+3a_3x^2,\quad y''\to 6(a_2+a_3x),\quad y'''\to 6a_2$$

is a diffeomorphism of $\mathscr{J}^3$. It brings the coframe (1) into the form

$$\omega^0=-3(\mathrm{d}a_0+3x\mathrm{d}a_1+3x^2\mathrm{d}a_2+x^3\mathrm{d}a_3),\quad \omega^1=3(\mathrm{d}a_1+2x\mathrm{d}a_2+x^2\mathrm{d}a_3),\quad \omega^2=-\tfrac{3}{2}(\mathrm{d}a_2+x\mathrm{d}a_3),$$
$$\omega^3=\tfrac{3}{2}\mathrm{d}a_3,\quad \omega^4=\mathrm{d}x.$$

Now the Bryant observation that the 2-form

$$\omega=\omega^0\wedge\omega^3-3\omega^1\wedge\omega^2$$

and the fourth rank symmetric tensor

$$\Upsilon=-3(\omega^1)^2(\omega^2)^2+4\omega^0(\omega^2)^3+4(\omega^1)^3\omega^3+(\omega^0)^2(\omega^3)^2-6\omega^0\omega^1\omega^2\omega^3, \tag{2}$$

although transforming badly by the contact transformations of variables in $\mathscr{J}^3$, projects to the well defined *line* of a 2-form and a *line* of a 4th rank tensor on the solution space M is expressed by the fact that in the new coordinates (x,a_0,a_1,a_2,a_3) on $\mathscr{J}^3$ we have

$$\omega=-9(\mathrm{d}a_0\wedge\mathrm{d}a_3-3\mathrm{d}a_1\wedge\mathrm{d}a_2)\quad\text{and}\quad \Upsilon=81(-3\mathrm{d}a_1^2\mathrm{d}a_2^2+4\mathrm{d}a_0\mathrm{d}a_2^3+4\mathrm{d}a_1^3\mathrm{d}a_3+\mathrm{d}a_0^2\mathrm{d}a_3^2-6\mathrm{d}a_0\mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3).$$

These two objects ω and Υ are clearly defined up to a scale on the space of solutions M parametrized by $a=(a_0,a_1,a_2,a_3)$. Their common stabilizer in $GL(T_aM)$ at every point a of the solution space M of the ODE $y''''=0$, is irreducible $GL(2,\mathbb{R})$, equipping M with a $GL(2,\mathbb{R})$ structure.

What we mean by a *contactification* in this case, is to associate to M a 5-dimensional bundle $\mathbb{T}(M)$, with a (local) typical fiber being an open set $\mathscr{U}\subset\mathbb{R}$, $\mathscr{U}\to\mathbb{T}(M)\xrightarrow{\pi}M$, such that $\mathbb{T}(M)$ is a *contact* manifold equipped with a contact distribution $\mathscr{D}$ having at every point $p\in\mathbb{T}(M)$ the same $GL(2,\mathbb{R})$ structure in $\mathscr{D}_p$ as M has at the tangent space $T_{\pi(p)}M$.

In our case the natural bundle is $\mathbb{T}(M)=\mathscr{J}^3$, with the fibers being tangent to the total differential $X=\frac{\mathrm{d}}{\mathrm{d}x}$. One then defines the desired contactified structure by looking for $\mathscr{D}$ in terms of a suitable contact 1-form λ on $\mathscr{J}^3$. This should satisfy the condition that the contact structure is compatible with the symplectic form ω, i.e.:

$$\big(\mathrm{d}\lambda-b(\omega^0\wedge\omega^3-3\omega^1\wedge\omega^2)\big)\wedge\lambda=0,\quad\text{with some function } b \text{ on } \mathscr{J}^3,$$

and that λ is preserved along the fibers spanned by the total differential $X = \frac{\mathrm{d}}{\mathrm{d}x}$: $(\mathscr{L}_X\lambda)\wedge\lambda = 0$.

The simplest λ realizing this is $\lambda = \mathrm{d}(x + \frac{a_2^3}{a_3}) - 3a_1\mathrm{d}a_2 + a_0\mathrm{d}a_3$, but other choices are also possible. The $GL(2,\mathbb{R})$ contact structure is then obtained on $\mathbb{T}(M) = \mathscr{J}^3$ by distinguishing a contact distribution $\mathscr{D} = \lambda^\perp = \{Y \in \mathrm{T}\mathscr{J}^3 \,|\, \lambda(Y) = 0\}$, and by defining a line of a 4th rank symmetric tensor Υ in $\mathscr{D}$ by the formula (2).

It follows that the so defined structure $(\mathscr{J}^3, \mathscr{D}, \Upsilon)$ on the third jet space $\mathscr{J}^3$ is a *flat model* for the G_2 *contact geometry* in dimension 5. Via the described process of *contactification* it relates the *flat Cartan geometry* of fourth order ODE $y'''' = 0$ considered modulo contact transformations of variables, to the *parabolic contact geometry* with G_2 *symmetry*. It is interesting to perform this construction for the nonflat Wünschmann case, with the general ODE $y'''' = F(x,y,y',y'',y''')$. A natural question immediately arises of *curvature characterization* of those G_2 contact geometries that can be constructed via the contactification of the geometry of ODEs $y'''' = F(x,y,y',y'',y''')$. One can generalize the contactification procedure described here in various directions.

Again, this ODE picture has a counter-part in PDEs. Namely, it turns our that integrable dispersionless hierarchies of PDEs canonically determine a $GL(2,\mathbb{R})$ structure on their solutions [20]. For instance, consider the first three equations of the dKP hierarchy

$$\begin{gathered} u_{xt} - u_x u_{xx} - u_{yy} = 0, \\ u_{xz} - u_{yt} - u_x u_{xy} - u_y u_{xx} = 0, \quad u_{yz} - u_{tt} + u_x^2 u_{xx} - u_y u_{xy} = 0. \end{gathered} \tag{3}$$

Here $u(x,y,t,z)$ is a function on a 4-dimensional manifold M, which we identify with its graph M_u in $\mathscr{J}^\infty$. The characteristic variety of this system is the intersection of three quadrics,

$$p_x p_t - p_y^2 - u_x p_x^2 = 0,\; p_x p_z - p_y p_t - u_x p_x p_y - u_y p_x^2 = 0,\; p_y p_z - p_t^2 + u_x^2 p_x^2 - u_y p_x p_y = 0,$$

specifing a rational normal curve (twisted cubic) in $\mathbb{P}T^*M_u$: $[p_x : p_y : p_t : p_z] = [1 : \lambda : \lambda^2 + u_x : \lambda^3 + 2u_x\lambda + u_y]$. The corresponding α-planes on $\mathbb{T}M_u$ are annihilated by

$$\omega(\lambda) = dx + \lambda dy + (\lambda^2 + u_x)dt + (\lambda^3 + 2u_x\lambda + u_y)dz.$$

This supplies $M_u \simeq M$ with a $GL(2,\mathbb{R})$ geometry, depending on the solution u. Equations (3) are equivalent to the commutativity conditions of the following vector fields, constituting a dispersionless Lax representation

$$\begin{gathered} \partial_y - \lambda\partial_x + u_{xx}\partial_\lambda, \qquad \partial_t - (\lambda^2 + u_x)\partial_x + (\lambda u_{xx} + u_{xy})\partial_\lambda, \\ \partial_z - (\lambda^3 + 2u_x\lambda + u_y)\partial_x + (\lambda^2 u_{xx} + \lambda u_{xy} + u_{xt} + u_x u_{xx})\partial_\lambda. \end{gathered}$$

Projecting integral manifolds of this *holonomic* distribution from $\mathbb{T}M_u$ to M_u we obtain a two-parameter family of α-manifolds of the corresponding $GL(2,\mathbb{R})$ structure, thus establishing its involutivity.

Higher-dimensional generalisations of this construction exists, and it works for all dispersionless hierarchies. Un-expectedly, there is a relation with parabolic geometries. Actually, [39] parametrizes $GL(2,\mathbb{R})$ structures via ODEs satisfying the Wünschmann conditions and these correspond to equations of Cartan's C-class [10] that can be treated by normalizations generalizing the parabolic technique.

2. Objectives

We shall review the objectives of the present proposal, following the above ideas and preliminary results. We note that the problems we consider are frequently controlled by representation theory (in a much deeper way than it happens in e.g. Riemannian geometry), and thus are largely algebraized. This allows for an explicit analysis by means of computer algebra, a significant point of the proposed methodology. Each subsection consists of a minimal introduction, and a number of precise objectives.

2.1. **Geometric robots.** In Sections 1.6.1, 1.6.2, and 1.6.3, we introduced mechanical systems that lead to interesting geometries, and preliminary results about them. Using the collective term 'geometric robots' for all these systems we have the following objectives:

- **Objective 1.** Find geometric robots whose configuration spaces support given geometric structures. Compute local invariants of such structures, in particular their symmetry groups (or infinitesimal symmetry algebras). Identify geometric robots whose configuration spaces admit a transitive (local) action of a semisimple Lie group.
- **Objective 2.** Consider the inverse problem: find mechanical realisations of certain prescribed geometric structures. Derive invariant obstructions for the fixed realization type.
- **Objective 3.** Develop a theory of planar robots as defined in Section 1.6.3, starting with the relation between the number of faces F, edges E, vertices V, and wheels W of such a robot on the one hand, and the dimension of the configuration space of this mechanical system and the rank of the velocity distribution, on the other hand. Compute possible growth vectors of

these distributions depending on configuration parameters of the robot (lengths, position of wheels).

- **Objective 4.** Apply the theory developed in Objective 3 to studies of the 3-edge snake introduced in Section 1.6.3. In this case the dimension of the configuration space M is five, and the velocity distribution $\mathscr{D}$ of a snake, apart from singular points, is a $(2,3,5)$ distribution. Adjust the parameters defining the snake, such as the length of their edges, so that the velocity distribution of the snake has maximal symmetry.
- **Objective 5.** Describe all planar robots whose velocity distribution has a simple Lie algebra as its Lie algebra of symmetries. Give a full list of planar robots corresponding to flat models of parabolic geometries.
- **Objective 6.** Given a cost of motion (e.g. sub-Riemannian metric, a Finsler norm on the symmetry algebra of homogeneous models, distance induced by an embedding of the set of vertices into a Euclidean space, etc) for configurations of robots, determine optimal trajectories. Specify to low-complexity models, and also perform numerical simulations.

2.2. **Homogeneous models, curvature reduction, subgeometries.** The search for homogeneous models is an established theme in the research on geometric structures, in particular maximally non-holonomic distributions. An early example is Cartan's 1910 classification [12] of all $(2,3,5)$-distributions with a multiply transitive local symmetry group. Cartan's equivalence and reduction methods, briefly discussed in Section 1.2.3, have since then been applied successfully to a variety of geometric structures.

There are alternative methods for classifying homogeneous models. A recent approach has been developed by I. Anderson and J. Gutt: they apply the deformation theory of filtered Lie algebras to state a computationally feasible algorithm for a classification of homogeneous models of distributions with a given symbol.

- **Objective 7.** Refine the Cartan's equivalence method (via curvature reduction) and Gutt's recent algorithm (via graded Lie algebra deformations) for classifying homogeneous structures. While Cartan's equivalence method has been known for over 100 years and in principle gives a complete classification, it takes substantial effort to set up. The Anderson–Gutt algorithm is much quicker to set up, but suffers from completeness issues since: (i) curvature is not used beyond harmonic curvature for the initial steps, and (ii) the embedding into the Cartan bundle is not incorporated into the algorithm. Since harmonic curvature is the component of curvature in lowest homogeneity, we would like to aim for an iterative algebraic method proceeding homogeneity-by-homogeneity that classifies candidate homogeneous models, their full curvature, and their corresponding realizations as Cartan geometries.
- **Objective 8.** Classifications of homogeneous models are well-known for some parabolic geometries in low dimensions, e.g. 2nd and 3rd order ODEs, $(2,3,5)$ distributions, etc. However, their realizations as Cartan geometries are completely obscured (or not provided at all). Using Cartan's equivalence method (or its refinement aimed for in Objective 7), calculate these descriptions (via extension functors) and use them to calculate the holonomy of these models.
- **Objective 9.** Find homogeneous models for chosen Cartan geometries considered in this proposal. For example, classify homogenous models for the geometries in Objectives 20 and 21, and the homogeneous models for the G_2 contact geometry in dimension five.

We also propose to study geometries unifying a number of well known low dimensional Cartan geometries. As a specific example consider the contact geometry of a pair of compatible second order PDEs

$$u_{xx} = F(x,y,u,u_x,u_y,u_{xy}), \quad u_{yy} = G(x,y,u,u_x,u_y,u_{xy}).$$

Such systems can be of two kinds: involutive or finite type. The first have infinite-dimensional solution space due to the existence of a Cauchy characteristic, the quotient by which reduces them to a 5-manifold equipped with a $(2,3,5)$-distribution. This transforms contact equivalence to internal equivalence. The second type systems have four-dimensional solution space, with the compatibility given by vanishing of the Mayer bracket $\{F,G\}$, see [34].

These finite type systems, even when incompatible, correspond to a parabolic geometry modelled on $(SL(4),B)$, where B is the Borel subgroup of (upper) triangular determinant one matrices. Via Čap's theory of correspondence / twistor spaces [9], this geometry contains a plethora of sub-geometries: 3-dim projective, 4-dim conformal, 5-dim CR and Legendrian contact, and 5-dimensional path geometry (pairs of 2nd order ODE) [22, 28].

Nevertheless, tools from modern parabolic geometry have not been applied to the study of this geometry in any detail.

- **Objective 10.** (i) Given any one of these sub-geometries, concretely demonstrate how it can be reformulated as a pair of PDE, (ii) What is a Weyl structure for this "master" geometry? (iii) Express the harmonic curvatures of this geometry in terms of a given Weyl structure, (iv) Write some of the simplest BGG equations for this geometry (again in terms of a Weyl structure), (v) Identify the w_{13}-part of the harmonic curvature with the Mayer bracket $\{F,G\}$.

2.3. **Parabolic geometries endowed with additional structure.** The goal here is to investigate classes of geometric structures obtained by enhancing a parabolic geometry by additional geometric data. Examples of geometric structures of this type include:

(1) $(2,3,5)$ distributions $\mathscr{D}$ endowed with a line field. There are two distinct cases of interest:
  (i) $\ell \subset \mathscr{D}$,
  (ii) $\ell \subset [\mathscr{D},\mathscr{D}]$ but transversal to $\mathscr{D}$.
(2) G_2 contact geometries endowed with a line field contained in the twisted cubic cone.
(3) Conformal structures of signature (p,q) endowed with a null k-plane field where $k \leq min(p,q)$.
(4) $(2,3,5)$ distributions $\mathscr{D}$ endowed with a (para-)complex structure.

- **Objective 11.** In examples 1., 2.(i), and 3. the additional geometric data corresponds to a specific reduction of the reductive part G_0 of the structure group P of the given parabolic geometry of type (G,P). Namely, to a Lie subgroup $A_0 \subset G_0$ whose Lie algebra is of the form
$$\mathfrak{a}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{q},$$
where $\mathfrak{q}$ is a parabolic subalgebra of the semisimple part of $\mathfrak{g}_0$. Solve the equivalence problem for these types of structures. Does a canonical Cartan connection exist?

Some progress on these question has already been made. Example 3. was investigated by the PI in collaboration with Gianni Manno and Katja Sagerschnig in [43]. Dennis The, motivated by the study of certain scalar PDE in the plane related to example 2.(i), showed that in the complex case the Lie algebra $\mathfrak{a}$ obtained as the Tanaka prolongation of the data $(\mathfrak{g}_-,\mathfrak{a}_0)$ is *always* a parabolic subalgebra such that $\mathfrak{a}+\mathfrak{p}=\mathfrak{g}$. Hence $\mathfrak{a}\cap\mathfrak{p}$ is a *seaweed* or *biparabolic* Lie algebra. This class of Lie algebras was introduced by Dergachev and Kirilov [13] in 2000 and has since then been intensively studied. Our proposed class of geometric structures could therefore be called *seaweed geometries* or *biparabolic geometries*.

Example 2.(ii) is of a different nature. It is interesting, for at least two reasons. First, it naturally arises from the twistor construction of rolling bodies discussed in Section 1.6.1, and second, it is an example of a refinement of a parabolic geometry that itself can be understood as a subclass of a parabolic geometry (of a different type) with the property that a certain harmonic curvature component is *nonvanishing*. In that sense the study of these structures is also related to the problems discussed in Section 2.2.

- **Objective 12.** Develop a suitable calculus, with invariants in terms of the curvature of a suitable Cartan connection, for a structure $(M,\mathscr{D},\ell)$, consisting of a generic rank 2 distribution $\mathscr{D}$ on a 5-manifold M, with a tranversal to $\mathscr{D}$ line field ℓ in the derived rank 3-distribution $\mathscr{D}_{-2}=[\mathscr{D},\mathscr{D}]=\ell\oplus\mathscr{D}$.
- **Objective 13.** Establish curvature conditions for the structure $(M,\mathscr{D},\ell)$ from problem the Objective 12, which are necessary and sufficient for $\mathscr{D}$ to be the twistor distribution $\mathscr{D}_{\mathbb{T}}$ on the circle twistor bundle $\mathbb{S}^1 \to \mathbb{T}(N) \to N$ over a certain conformal split-signature structure $(N,[g])$ on a four-manifold N.
- **Objective 14.** In the special case, when the distribution $\mathscr{D}$ of the structure $(M,\mathscr{D},\ell)$ from the Objective 12 is locally equivalent to the $(2,3,5)$ distribution having the simple exceptional Lie group G_2 as a symmetry, determine all possible line fields ℓ, such that the curvature conditions from the Objective 13 for the structure $(M,\mathscr{D},\ell)$ are satisfied. Find the corresponding conformal classes $(N,[g])$ and answer the question which conformal split signature geometries $(N,[g])$ have twistor distributions with local symmetry G_2.

A study of these structures is ongoing work of Dennis The with Michael Eastwood and Katja Sagerschnig as well as of the PI and Katja Sagerschnig.

We already saw that irreducible $GL(2,\mathbb{R})$ structures appear in the theory of integrable hierarchies. It is also true that other $GL(2,\mathbb{R})$ structures lead to integrable geometries. (An instance related to quaternionic structures is an un-published work of D.Calderbank presented at the 2019 Abel Symposium.) The G_2 contact structure is an example of $GL(2,\mathbb{R})$ structure on the contact distribution, i.e. a reducible structure on the tangent space $TM \simeq S^3\mathbb{R}^2 \oplus \mathbb{R}$ (real version). This is a parabolic geometry.

Reduction of the structure group on general distributions lead to more complicated geometries. These appear in the theory of integrable systems. In fact, by the result of David Calderbank and Boris Kruglikov, with an input by Eugene Ferapontov, a second order PDE admitting a nontrivial dispersionless Lax pair on a manifold of dimension $d > 4$ is necessarily degenerate. Thus every solution u of this PDE possesses a canonical degenerate conformal structure on T^*M_u. This yields a distribution and a sub-conformal structure on it.

- **Objective 15.** Setup the equivalence problem for such non-holonomic tensorial structures. Derive maximal symmetry models and sub-maximal symmetry bounds. Find fundamental invariants responsible for the equivalence to models and to the symmetry breaking.
- **Objective 16.** Compute invariants of these non-holonomic tensorial structures that are responsible for integrability. Define involutive structures in the spirit of [20] and parametrize all involutive structures by an involutive PDE system. Investigate its integrability.

2.4. **Differential invariants of parabolic and subordinated structures.** The two approaches to differential invariants, i.e. the method of Lie-Tresse and the method of Cartan, co-existed independently in the literature. The practitioners seldom compared the outputs. For example, one of the most classical problems in geometry of differential equations is that of point equivalence for the second order ODEs.

The problem was first solved via Lie's approach in scalar differential invariants by A.Tresse. He computed all relative invariants, from which it is not difficult to derive all absolute differential invariants, see [32]. E. Cartan solved the same problem by constructing the Cartan bundle $B \to \mathscr{G} \to M$ with $M = J^1(\mathbb{R},\mathbb{R})$ and $B \subset G = SL(3)$ the Borel subgroup. The Cartan connection gives all absolute invariants as structure functions of the absolute parallelism on the total space $\mathscr{G}$, yet to push the invariants down to M one has to take a subalgebra of those that are B-invariant. Since B is a solvable group, the quotient problem is non-trivial.

- **Objective 17.** Detail this construction for second order ODEs mod point transformation. Derive all differential invariants as Cartan invariants, and derive invariant derivations and syzygies. Deduce the count of differential invariants in the jet-formalism from that of the Cartan method.
- **Objective 18.** Apply the same for a different real version: 3-dimensional CR structures. This is also a classical subject, with the invariants described by the two methods of Chern and Moser. Find an explicit relation between the different approaches. As a consequence, obtain the asymptotics of the Bergman kernel via absolute differential invariants.
- **Objective 19.** Generalize this to other parabolic geometries. Make a count of relative and absolute differential invariants, and compute functional dimensions of the local moduli spaces.

We plan to attack this by the approach of Kruglikov-Lychagin via rational differential invariants and invariant derivations on the Zariski open sets in the space of jets. We expect a fruitful interplay of the symbolic computations with the representation theory of parabolic subgroup $P \subset G$ acting on the space of invariants in the Cartan bundle $\mathscr{G}$.

This idea can be also applied to geometric structures subordinated to parabolic geometries. For instance, we can consider a conformal structure with a differential invariant condition. The Einstein-Weyl condition is a typical example. The conformally Einstein condition is yet another example.

Next we plan to study the class of parabolic geometries equipped with additional structures, like complex or para-complex structures on the distribution. Such structures arise in various contexts, e.g. for 3-*edge snakes* from Section 1.6.3. It turns out that the geometric behaviour of such a snake is easier to analyze if one observes that the $(2,3,5)$ distribution of the snake's velocity space has an additional structure.

- **Objective 20.** Find the set of all local differential invariants of generic rank 2 distributions in dimension 5 split into a direct sum of two rank 1 subdistributions, i.e. develop a theory of para-CR structures of type $(3,1,1)$.
- **Objective 21.** Find the set of all local differential invariants of generic rank 2 distributions in dimension 5 equipped with a complex structure, i.e. develop a theory of a class of CR manifolds of CR-dimension 1 and CR-codimension 3.

2.5. **Parabolic contact geometries, contactifications, and generalizations.** As we have already noticed, *contact distributions* do *not* possess local invariants. That is, they are all locally equivalent. Geometric structures with local invariants may be obtained by endowing the contact distribution with additional structure. The simplest example is of course an inner product, i.e. a sub-Riemannian structure. (Note that the contact distribution $\mathscr{D}$ is already equipped with a pseudo-symplectic structure, i.e. a section of $\Lambda^2\mathscr{D}^* \otimes TM/\mathscr{D}$ induced by the Lie bracket.) Another familiar case is a CR-structure, i.e. an almost complex structure on $\mathscr{D}$ satisfying a suitable integrability condition.

Parabolic contact geometries are parabolic geometries whose homogeneous models correspond infinitesimally to a *contact grading* on a simple Lie algebra:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

with $\dim(\mathfrak{g}_{-2}) = 1$, $\mathfrak{g}_{-1}$ being even-dimensional, and the Lie bracket map $[\cdot,\cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ being nondegenerate.

Contact gradings exist and on most noncompact real forms (and all split forms) of simple Lie algebras, and if they exist they are unique. In particular, we may refer to a parabolic contact geometry of type (G,P) as a (parabolic) *G-contact geometry*. A unified approach to *exceptional parabolic contact geometries*, that is for exceptional simple Lie algebras $\mathfrak{g}$, was proposed by the PI in his lectures in Canberra in October 2013. In recent work [55], Dennis The, has studied parabolic contact structures in connection with PDEs, generalizing the Cartan-Engel models in the G_2 case.

- **Objective 22.** Observing that all the exceptional parabolic contact geometries are defined by tensors of the same kind, give a unifying description in terms of an explicit differential system. Compare to other parabolic contact geometries in different dimensions (in particular Lie contact structures [44]).
- **Objective 23.** In [41] the PI together with Thomas Leistner and Katja Sagerschnig produced interesting $SO(3,4)$ Lie contact structures in dimension 7 with *Cartan holonomy* reduced from $SO(3,4)$ to G_2. These are associated with $(2,3,5)$ distributions $\mathscr{D} \subset TM$, and are defined on the affine plane bundle $\mathbb{P}[\mathscr{D},\mathscr{D}] \setminus \mathbb{P}\mathscr{D}$. Find a 'completion' of this structure that would exhibit the five-dimensional 'boundary' carrying a G_2 contact structure.
- **Objective 24.** Develop a *spinorial calculus* for the 4-dimensional geometry with exotic $GL(2,\mathbb{R})$ holonomy [5, 49] as an analog of the spinorial calculus for the 4-dimensional (pseudo) Riemannian geometry.
- **Objective 25.** Investigate relations between $GL(2,\mathbb{R})$ geometry in dimension four and parabolic G_2 contact geometry in dimension five. Is there a way of contactifying a *nonflat* 4-dimensional $GL(2,\mathbb{R})$ geometry to a G_2 contact geometry in dimension five in a fashion similar to the flat model as explained in Section 1.6.4? If yes, develop a theory of such a contactifications, and characterize those G_2 contact geometries in dimension five that come from the contactifications of 4-dimensional $GL(2,\mathbb{R})$ geometries in terms of the curvature invariants of the G_2 contact geometry. See 1.6.4 for more details.
- **Objective 26.** Using results of Objectives 19 and 20, develop a tensorial and spinorial calculus for the G_2 contact geometry in dimension five, where G_2 is the split real form of the simple exceptional Lie group G_2. Find a tensorial/spinorial expression for the septic defining the harmonic curvature of this geometry.

2.6. **Applications in Cosmology.** This section concerns applications of methods and ideas of this proposal to recent ideas of Roger Penrose [54], called Conformal Cyclic Cosmology (CCC).

In CCC, the metric $\check{g}$ of the Universe is conformally flat at the surface $t = 0$ of the initial singularity. Consider a conformal class $[\check{g}]$ of metrics conformal to $\check{g}$. Assume that the conformal class $\check{g}$ is regular in a strip $t \in]-\varepsilon,\varepsilon[$. In particular, this means that there exists a Lorentzian metric g in the class $[\check{g}]$ that is regular for all $t \in]-\varepsilon,\varepsilon[$. Penrose calls g the intermediate metric and relates it to two physical metrics: (i) the metric $\check{g}$ describing the Universe close to the singularity, when $t \in]0,\varepsilon[$, and (ii) the metric $\hat{g}$, which is interpreted as the physical metric of the previous Universe (previous eon), when $t \in]-\varepsilon,0[$. Formally, having chosen the intermediate metric g, one gets three metrics: $\hat{g}$, g, and $\check{g}$ in the entire 'wounded' region of the Universe described by $t \in]-\varepsilon,\varepsilon[$. This is called bandage region of the Universe. If one only considers spatially homogeneous Universes, the three metrics are related via $\hat{g} = \frac{1}{\Omega^2}g$ and $\check{g} = \Omega^2 g$, where $\Omega = \Omega(t)$ is chosen in such a way that $\check{g}$ coincides with the metric $\check{g}$ of the current Universe (current eon), when $t \in [0,\varepsilon[$, and $\hat{g}$ coincides with the physical metric $\hat{g}$ of the previous eon, when $t \in]-\varepsilon,0]$. In Penrose's proposal for the CCC, it is the conformal geometry $[g]$ of the metric g that is relevant for the cosmology of the Universe in the bandage region $t \in]-\varepsilon,\varepsilon[$. According to the paradigm of CCC, around the end of an old eon ($t \to 0^-$) and the beginning of the new eon ($t \to 0^+$), the Universe loses a part of the information about its (pseudo-)Riemannian physical metrics ($\hat{g}$ in $]-\varepsilon,0[$ and $\check{g}$ in $]0,\varepsilon[$). The physical remnant of these (pseudo)-Riemannian geometries around the hypersurface $t = 0$ is the *conformal geometry* $[g]$ of g. The question of what kind of dynamics this conformal geometry obeys is not stated by the CCC. One can for example require that the physical metrics $\hat{g}$ and $\check{g}$ satisfy

Einstein equations in their domains, and try to deduce what kind of *energy momentum tensors*, on the both sides of the wound of the Universe, allow for a *conformal transitions* from $\hat{g}$ to $\check{g}$.

The energy momentum tensor of General Relativity is a certain rank two ($T^\mu{}_\nu$) symmetric ($g_{\mu\alpha}T^\alpha{}_\nu = g_{\nu\alpha}T^\alpha{}_\mu$) tensor T on a 4-dimensional manifold M (*spacetime*) with a Lorentzian metric $g_{\mu\nu}$, which carries information about the matter/energy content of the spacetime. At every point x of M, T is an endomorphism of the Minkowski vector space and as such it can be considered modulo the action of the local *Lorentz group*. The equivalence classes, obtained as the orbits of this action, provide the *Churchill-Plebański classification of the energy momentum tensors* at a point x of M. In the following we will assume that in the considered spacetimes M the Churchill-Plebański class of their energy momentum tensor is constant along M.

Via the *Einstein field equations*, $E = T$, the energy momentum tensor T of a spacetime M coincides with its *Einstein tensor* E. This in turn, is purely defined by means of the Lorentzian geometry (M,g) of the spacetime: it is a certain trace-corrected version of the classical *Ricci tensor*, $E = Ricci - \frac{1}{2}Sg$, where S is the trace of *Ricci*.

These preparations are enough to state the next few Objectives of this proposal.

- **Objective 27.** Given a conformal structure $(M,[g])$, describe the space of conformal scales giving rise to metrics whose *Einstein tensor* lies in a fixed Churchill-Plebański class. Find obstructions to the existence of such scales.
- **Objective 28.** Study the singularities of such Einstein-tensor-algebraically-special scales. Describe the space of pairs of Einstein-tensor-algebraically-special scales on $\hat{M}$ and $\check{M}$, considered as consecutive eons in the bandage region of CCC, with the prescribed singular behaviour along the wound hypersurface Σ, and interpret the corresponding energy-momentum tensors on $\hat{M}$ and $\check{M}$ in terms of suitable matter models.

The simplest case is of course that of Einstein metrics, i.e. Ricci tensor in the class proportional to the metric [26]. The methods of parabolic geometry give a powerful description of Einstein scales in a given conformal class: these correspond to the kernel of a particular *curved BGG operator*, one of a sequence of natural differential operators on certain natural bundles associated with every type of parabolic geometry [11, 8].

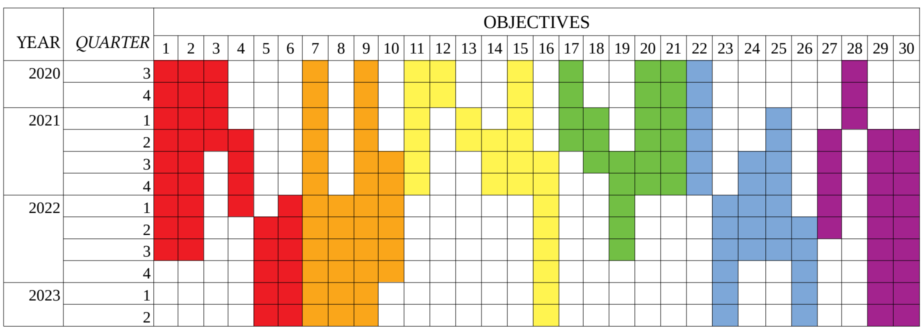
- **Objective 29.** Find a description of the Churchill-Plebański class of the Einstein tensor of a conformal scale in terms of the BGG machinery of parabolic geometry. Characterise obstructions.
- **Objective 30.** Use the above to give a tractor/parabolic description of the data inducing the pair $(\hat{g},\check{g})$ on a conformal manifold $M = \hat{M}\cup\Sigma\cup\check{M}$.

A part of Objective 27 has been carried out in the paper [40] in collaboration with Thomas Leistner.

3. Work Programme including proposed research methods, role of the participating research team members, the added value of international cooperation

3.1. **Work Programme including proposed research methods.**

3.1.1. *Project implementation plan.* The research programme is divided into six sub-projects related to the objectives grouped in six Subsections 2.1 – 2.6 of Section 2. In the table below, the expected time span for carrying out each of the objectives is depicted with the color assigned to the sub-problem to which it belongs.



The time allocation for the meetings, conferences, workshops and outreach activities is presented, with an accuracy of 3 months, in the table below.

Year	2020		2021				2022				2023	
Quarter of the year	3	4	1	2	3	4	1	2	3	4	1	2
Conferences												
Conference in Norway				■								
Conference in Poland												■
Orientation Conference with experts in Poland		■										
Workshops												
Internal miniworkshops	■											
Teams meeting in Norway			■					■				
Teams meeting in Poland						■				■		
Outreach activities												
Science Festival					■				■			

3.2. **Role of the participating research team members.** In the below table we show *the allocation of tasks of the project*, described in the Objectives 1-30, of Sections 2.1 – 2.6, *to team members*. Note that every scientist team member has allocated Objectives, and that all Objectives have their executors. The table also shows that the members of both teams, i.e. the Polish and the Norwegian ones, will significantly collaborate to realize the Objectives.

	OBJECTIVES																													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
PI, Paweł Nurowski			■	■	■		■							■						■	■	■	■		■		■	■	■	■
Co-Investigator in Polish team										■	■	■	■	■									■	■	■	■				
Post-doc in Polish team																											■	■	■	■
Post-doc in Polish team	■	■							■						■	■				■	■									
Student in Polish team			■	■	■	■			■																					
Student in Polish team									■													■		■		■				
Partner PI, Boris Kruglikov				■		■				■					■	■	■	■	■						■					
Co-Investigator in Norwegian team					■		■	■		■	■	■	■	■					■										■	■
Post-doc in Norwegian team	■	■															■	■	■											
Post-doc in Norwegian team							■	■	■	■													■							

Note that both Students will always have a Senior member from either team (PI, PI Partner, Coinvestigator, or a Post-doc) who will supervise them within a given task.

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