Geometry of ODE and Vector Distributions

Symmetry Reduction And The Method of Darboux

I. Introduction

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Many aspects of the geometry of pde originate with the classical problem of solving, in "closed form", the scalar 2nd order equation

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General Solution : $x = A(\alpha)\ldots, \quad y = B(\beta)\ldots, \quad u = \int (A'')^2 \ldots$

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$$\mu^2 F_r + \mu\nu F_s + +\nu^2 F_t = 0$$

$$X = \mu D_x + \nu D_y$$

A function $f = f(x, y, u, p, q, dots)$ on jet space is an internediate integral if

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and conversely ....

Theorem B] Let I be a Darboux integrable Pfaffian system.

- Let $\mathcal{W}_1$ be the pullback of I to a level set of I_1, I_2, ...
- Let $\mathcal{W}_2$ be the pullback of I to a level set of J_1, J_2, ...

Then there is a (algorithmically computed) group action G on M such that

$$I = (\mathcal{W}_1 + \mathcal{W}_2)/G.$$

Example 1

We shall begin by giving the group theoretic derivation of the Liouville's equation

$$u_{xy} = exp(u)$$

The D_x and D_y intermediate integrals for this equation are

$$I_1 = x \quad I_2 = r - \frac{1}{2}p^2 \quad J_1 = y \quad J_2 = t - \frac{1}{2}q^2.$$

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Step 1. Introduce two copies of the jet space $J^3(R, R)$:

$$J \times J = (x, X, X', X'', X''') \times (y, Y, Y', Y'', Y''')$$

and the canonical contact system

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(In other examples we might begin the product of some other differential systems.)

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$$\tilde{X} = \frac{aX + b}{cX + d} \quad \tilde{Y} = \frac{aY + b}{cY + d}$$

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$$U = \frac{X'Y'}{(X - Y)^2}$$

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Step 4. Calculate the derivatives $\{U, U_x, U_y, U_{xx}, U_{xy}, U_{yy}\}$. These are all SL2 invariants. Do a little counting – there are 6 invariants for a free 3-dimensional group action in eight variables.

$$X, X', X'', X''', Y, Y', Y'', Y'''$$

There must be a syzygy between the invariants. We find it to be

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(We shall need a more geometric formulation of this step.)

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Theorem[Goursat]. The quadratic term pq in the equation

$$s + \alpha(x, y, u)pq + \beta(x, y, u)p + \gamma(x, y, u)q + delta(x, y, u)$$

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This leads to the "better" choice of invariant

$$U = log(\frac{X'Y'}{(X-Y)^2})$$

and the equation

$$U_{xy} = \exp(U)$$

Calculate the SL2 differential invariants in just the variables X, X', X'', X''' and $Y, Y' Y''''', Y'''$. These are well-known:

$$I = \frac{2X'X''' - 3(X'')^2}{(X')^4} \quad \text{and} \quad J = \frac{2Y'Y''' - 3(Y'')^2}{(Y')^4}$$

Because these are SL2 invariants we can express then in terms of $U, U_x, U_y, U_{xx}, U_{yy}$.

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$$\boxed{\text{Differential Invariants} \Longleftrightarrow \text{Intermediate Integrals}}$$

This is an important observation. It already suggests that DI is really a group theoretic phenomena.

Summary.

$$
\begin{array}{cccc}
J^3 \times J^3 & \mathcal{C}^3 \times \mathcal{C}^3 & M_1 \times M_2 & \mathcal{W}_1 \times \mathcal{W}_2 \\
\big\downarrow {\scriptstyle q_{\mathrm{SL2}}} & \big\downarrow {\scriptstyle q_{\mathrm{SL2}}} & \big\downarrow {\scriptstyle q_G} & \big\downarrow {\scriptstyle q_G} \\
M & \mathcal{L} & M & \mathcal{I}
\end{array}
$$

Step 1. Form the sum of 2 differential systems.

Step 2. Definite a diagonal symmetry group.

Step 3. Calculate the reduction of the sum by the diagonal action.

Steps 4-5. Try to recognize the reduction as a PDE system. Put into a good form (Optional).

Step 6. Calculate the intermediate integrals for the PDE and hence establish that it DI (Optional).

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What is not always so easy to do is to put these differential systems into a good normal form.

The Liouville equation is encoded on a 7-manifold, with coordinates (x, y, u, p, q, r, t) as a rank 3 Pfaffian system

$$I = \{\, \theta_1 = du - p\,dx - q\,dy, \theta_2 = dp - r\,dx - e^u\,dy, \theta_3 = dq - e^u\,dx - t\,dy \,\}$$

The restriction to $x = 0$, $r = \frac{1}{2}p^2$ gives

$$W = \{\, \omega_1 = du - q\,dy,\ \omega_2 = dp - e^u\,dy, \omega_3 = dq - t\,dy\}$$

We calculate:

$$\dim W = 3 \quad \dim W' = 2, \quad \dim W'' = 1, \quad \dim W''' = 0,$$
$$\dim C(W') = 1 \quad \dim C(W'') = 2.$$

We can conclude that W is the canonical contact system on J^3. The explicit diffeomorphism is

$$y = y, Y = p, Y_1 = e^u, Y_2 = e^u q, Y_3 = e^u(t + q^2).$$

Example 3

Now consider the fully non-linear equation $3rt^3 + 1 = 0$. The corresponding Pfaffian systems is

$$I = \{ \theta_1, \theta_2, \theta_3 \}$$

$$\{ du - p\,dx - q\,dy,\ dp + \frac{1}{3t^3}\,dx - s\,dy,\ dq - s\,dx - t\,dy \}$$

The intermediate integrals are

$$I_1 = s + \frac{1}{t}, \quad I_2 = xI_1 - q, \quad J_1 = s - \frac{1}{t}, \quad J_2 = xJ_1 - q$$

The restriction of I to $s = \dfrac{1}{t}$, $q = 0$ gives

$$W =$$

$$\{ \omega_1 = du - p\,dx,\ \omega_2 = dp + \frac{1}{3t^3}\,dx - \frac{1}{t}\,dy,\ \omega_3 = -\frac{1}{t}dx - t\,dy \}.$$

This time we find that

$$\dim W = 3 \quad \dim W' = 2, \quad \dim W'' = 0,$$

so that W is of generic type.

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An explicit diffeomorphism is

$$x = \frac{2x}{U_2},\ y = -U_1 + \frac{xU_2}{2},\ u = 2U - 2xU_1 + \frac{1}{3}x^2 U_2 + \frac{xV}{U_2},$$
$$p = \frac{1}{2}V - \frac{1}{6}xU_2^2,\ t = -\frac{2}{U_2}$$

Example 4

An interesting class of equations, studied in 2 papers by Goursat, is provided by

$$s = A(x,y)\sqrt{pq}. \tag{1}$$

With $A = \dfrac{2n}{x+y}$, this equation becomes Darboux integrable with intermediate integrals of order $n+1$.
Here are the intermediate integrals I_n for small values of n(where $p = sqrt(u_x)$) :

$$I_2 = \frac{2p}{(x+y)^2} + \frac{4p_x}{(x+y)} + p_{xx}$$

$$I_3 = \frac{6p}{(x+y)^3} + \frac{18p_x}{(x+y)^2} + \frac{9p_{xx}}{(x+y)} + p_{xxx}$$

$$I_4 = \frac{24p}{(x+y)^4} + \frac{96p_x}{(x+y)^3} + \frac{72p_{xx}}{(x+y)^2} + \frac{16p_{xxx}}{(x+y)} + p_{xxxx}$$

$$I_5 = \frac{120p}{(x+y)^5} + \frac{600p_x}{(x+y)^4} + \frac{600p_{xx}}{(x+y)^3} + \frac{200p_{xxx}}{(x+y)^2} + \frac{25p_{xxxx}}{x+y} + p_{xxxxx}$$

We set this equation up as a rank $2n+1$ Pfaffian I system on a manifold M of dimension $2n+5$. Let W be the restriction of I to the level set $x=0$, $I_n=0$. This a rank $2n+1$ Pfaffian system on an manifold Q of dimension $2n+3$.

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We find that we can deprolong these systems n times to arrive at rank $n+1$ Pfaffian systems W^- on manifolds Q^- of dimension $n+3$ – that is, rank 2 distributions. Here are some basis invariants:

n	W^-	W^-	Derived Flag	Sym	Levi
	rk	dm	Growth	dim	ss · rad
2	3	5	[2,1,2]	14	$\mathfrak{g}_2$
3	4	6	[2,1,2 1]	11	$\mathfrak{sl}_2 \cdot [8, 7, 1, 0]$
4	5	7	[2,1,2 1, 1]	13	$\mathfrak{sl}_2 \cdot [10, 9, 1, 0]$
5	6	8	[2,1,2 1, 1, 1]	15	$\mathfrak{sl}_2 \cdot [12, 11, 1, 0]$

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$$\boxed{u_{xy} = \frac{2n}{x+y}\sqrt{pq} \Longleftrightarrow \text{Flat DI ???}}$$

For example, for $n = 3$, the Pfaffian system for the Goursat equation is

$$\theta_1 = du - p^2\,dx - q^2\,dy, \theta_2 = dp - p_x\,dx - \frac{3q}{x+y}\,dy,$$

$$\theta_3 = dq - \frac{3p}{x+y}\,dx - q_y\,dy,$$

$$\theta_4 = dp_x - p_{xx}\,dx - \frac{3(-q+3p)}{x^2+2xy+y^2}\,dy,$$

$$\theta_5 = dq_y + \frac{3(p-3q)}{x^2+2xy+y^2}\,dx - q_{yy}\,dy,$$

$$\theta_6 = dp_{xx} - p_{xxx}\,dx + \frac{3(-3p_x x + 9p - 2q - 3p_x y)}{x^3+3x^2y+3xy^2+y^3}\,dy,$$

$$\theta_7 = dq_{yy} - \frac{3(3q_y x - 9q + 3q_y y + 2p)}{x^3+3x^2y+3xy^2+y^3}\,dx - q_{yyy}\,dy.$$

The restricted system W is

$$\theta_1 = du - q^2\, dy, \quad \theta_2 = dp - \frac{3q}{y}\, dy, \quad \theta_3 = dq - q_y\, dy,$$

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This de-prolongations to $\{\theta_1, \theta_2, \theta_3, \theta_6\}$ which is mapped by the transformation

$$y = -2/x, \quad u = \frac{V}{9}, p = \frac{\sqrt{2}U_2}{2}, \quad q = -\frac{\sqrt{2}}{6}xU_3,$$

$$p_x = \frac{\sqrt{2}}{4}(xU_2 + 8U_1), \quad p_{xx} = \frac{\sqrt{2}}{4}(20U + 16xU_1 + x^2U_2)$$

to the canonical Pfaffian system for $V' = (U''')^2$.

Summary

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- $s = e^u$; $W = C^3$ on $J^3(R, R)$.
- $3rt^3 + 1 = 0$; W is the canonical Pfaffian system for $V' = (U'')^2$.
- $s = \dfrac{2n}{x+y}\sqrt{pq}$; W is the prolongation of the canonical Pfaffian system for $V' = (U^{(n)})^2$.

Exercises

Formulate each of the following PDE as a Pfaffian system I. Check that the given functions are intermediate integrals. Describe the restrictions W of I to the level sets of the intermediate integrals.

1. $s = \dfrac{pq}{u-x}$, with $I_1 = x$, $I_2 = \dfrac{p}{u-x}$, $I_3 = \dfrac{r}{u-x} + \dfrac{p}{(u-x)^2}$, $J_1 = y$, $J_2 = \frac{t}{q} - y$.

2. $s = pu$, with $I_1 = x$, $I_2 = \dfrac{u_{xxx}}{p} - \dfrac{3}{2}\frac{r^2}{p^2}$, $J_1 = y$, $J_2 = q - \dfrac{1}{2}u^2$.

3. $r = \frac{1}{2}s^2$, with $I_1 = y$, $I_2 = t + s$, $J_1 = y - xs$, $J_2 = s$.

4. Consider here the Pfaffian system $I =$

$$\{\, du - p\,dx - q\,dy,\; dp + (\tan\tau - \tau)\,dx - s\,dy,\; dq - s\,dx - (\tau + \cot\tau)\,dy \,\}$$

Find the associated PDE. The invariants are $I_1 = s + \tau$, $J_1 = s - \tau$, $I_2 = -(x+y)I_1 + q + p$, , $J_2 = -(x-y)J_1 + q - p$.

Overview

- I.Introduction
- II. Symmetry Reduction of Exterior Differential Systems.
- III. The Method of Darboux
- VI. Conclusions

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