Differential invariants and finiteness theorem. Classification of 2nd order ODEs and other examples

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Plan

Integrable pseudogroup

A pseudogroup $G \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ acting on a manifold M consists of a collection of local diffeomorphisms φ, each bearing own domain of definition $\mathrm{dom}(\varphi)$ and range $\mathrm{im}(\varphi)$, with properties:

- $\mathrm{id}_M \in G$ and $\mathrm{dom}(\mathrm{id}_M) = \mathrm{im}(\mathrm{id}_M) = M$,
- If $\varphi, \psi \in G$, then $\varphi \circ \psi \in G$ whenever $\mathrm{dom}(\varphi) \subset \mathrm{im}(\psi)$,
- If $\varphi \in G$, then $\varphi^{-1} \in G$ and $\mathrm{dom}(\varphi^{-1}) = \mathrm{im}(\varphi)$,
- $\varphi \in G$ iff for every open subset $U \in \mathrm{dom}(\varphi)$ the restriction $\varphi|_U \in G$,
- The pseudogroup is of order k if this is the minimal number such that $\varphi \in G \Leftrightarrow \forall a \in \mathrm{dom}(\varphi) : [\varphi]_a^k \in G^k$.

General pseudogroup

A more general approach is to consider $G\subset J^k_{\text{reg}}(M,M)$, where the latter space consists of the jets of local diffeomorphisms, such that G satisfies the above mentioned properties, but is not required to be integrable.

This finite-jet pseudogroup is usually considered with the action ρ on the jets of submanifolds of codimension r

$$G\ni\varphi_k\mapsto\rho(\varphi_k):J^k_r(M)\to J^k_r(M).$$

The map ρ obeys the following property:

$$\rho(\varphi_k\circ\psi_k^{-1})=\rho(\varphi_k)\circ\rho(\psi_k)^{-1},$$

whenever the composition on one side is defined.

General pseudogroup

Generalization of this concerns pseudogroup actions on equations $\mathcal{E} \subset J^k_r(M)$, which is the same kind of representation ρ, but such that $\varphi_k \equiv \rho(\varphi_k)$ action preserves the equation on submanifolds.

Notice that since in general both G and $\mathcal{E}$ are not assumed integrable from the very beginning, prolongation-projection scheme may change the action substantially.

In this scheme both G and $\mathcal{E}$ are prolonged simultaneously together with the action.
For instance, order 1 action of $G^1 = \mathrm{SL}(m)$ $[m = \dim M]$ prolongs to representation of volume-preserving diffeomorphisms, while for $G^1 = \mathrm{O}(m)$ the group does not change $G = \mathrm{O}(m)$ (in the perfect case, when the pseudogroup structure is integrable).

Differential invariants

A function $I \in \mathrm{C}^\infty_{\text{loc}}(\mathcal{E}_s)$ constant on the orbits of the above action is called a differential invariant. Here for $s \geq k = \operatorname{ord}(\mathcal{E})$ we define $\mathcal{E}_s = \mathcal{E}^{(s-k)}$ to be the prolongation of the given equation $\mathcal{E} = \mathcal{E}_k$ and $\mathcal{E}_s = J^s_r(M)$ for smaller s.

The action above is the prolonged action of the pseudogroup G, but the concept is better defined in terms of the Lie sheaf $\mathfrak{g}$ corresponding to pseudogroup G, since then the defining equation is expressed via Lie derivative as follows: $L_X(I) = 0$.

In addition if the pseudogroup is formal ($k = \infty$), it can be identified with the formal sheaf $\mathfrak{g}_\infty$, determining L_X, while integration to G is usually a more complicated problem. The algebra of differential invariants of order k will be denoted by $\mathcal{I}_k$. The algebra of all differential invariants is $\mathcal{I} = \varinjlim \mathcal{I}_k$.

Invariant differentiations

Vector field $v \in C^\infty(\mathcal{E}^\infty) \otimes_{C^\infty(M)} D(M)$ invariant under the action of pseudogroup G is called an invariant differentiation. It acts as follows:

$$v : \mathcal{I}_{k-1} \to \mathcal{I}_k.$$

An important case of invariant differentiations constitute Tresse derivatives, defined as follows. Suppose we have $n = m - r$ differential invariants $f_1, \dots, f_n$ on $\mathcal{E}$ of order $\leq k$, which are functionally independent on almost all finite jet-solutions of $\mathcal{E}$: $df_1 \wedge \dots \wedge df_n|_{L(a_{k+1})} \neq 0$. Then for any $f \in \mathcal{I}_k$ we get

$$df|_{L(a_{k+1})} = \sum_{i=1}^{n} \hat{\partial}_i(f)(a_{k+1})\, df_i|_{L(a_{k+1})},$$

which uniquely defines the function $\hat{\partial}_i(f)$. $v = \hat{\partial}_i = \hat{\partial}/\hat{\partial} f_i$ are invariant differentiations, $v : \mathcal{I} \to \mathcal{I}$.
Examples are Levi-Civita derivative in Riemannian geometry, Study derivative in projective geometry etc.

Invariant differentiations

The space of differential invariants is an algebra with respect to linear combinations over $\mathbb{R}$, operation of multiplication and the composition $I_1, \dots, I_s \mapsto I = F(I_1, \dots, I_s)$ for $F \in C^\infty_{\text{loc}}(\mathbb{R}^s, \mathbb{R})$.

However even with these operations the algebra $\mathcal{I}$ is usually not locally finitely generated (though the subalgebras $\mathcal{I}_k \subset \mathcal{I}$ are finitely generated on non-singular strata). To obtain this property one must add invariant differentiations or Tresse derivatives.

It was established by Valentin Lychagin & B.K. that this finite-dimensionality is equivalent to vanishing of certain cohomology of covariants in stable range. These are certain Spencer-like cohomology with dual graded components of differential invariants instead of usual symbols. With this approach the finiteness theorem for PDEs is equivalent to vanishing theorems in algebraic geometry.

Finiteness theorems

Sophus Lie in particular case of vertical actions and later his French student Arthur Tresse in general suggested the following

Theorem

There is a finite set of invariant differentiations $\nabla_1, \dots, \nabla_n$ and a finite set of differential invariants $I_{n+1}, \dots, I_{n+s}$ such that the algebra of differential invariants $\mathcal{I}$ is generated by them.

There is another formulation:

Theorem

There is a finite set of differential invariants $I_1, \dots, I_n, \dots, I_{n+s}$ such that if the first n are considered as basic, then all differential invariants are obtained from these by (higher) Tresse derivatives of the second part by the first and combinations.

History

It is not known if the statement holds true without additional assumptions. Most likely it holds if all the involved objects are analytic (like Cartan-Kähler theorem or Malgrange's proof of Cartan-Kuranishi theorem).
But usually the regularity requirement is imposed.
The story of proof is shortly this:

- S.Lie & A.Tresse - general idea.
- L.Ovsyannikov - action of Lie groups (finite-dimensional).
- A.Kumpera - action of a pseudogroup on spaces of jets.
- V.Lychagin & B.K. - action of a pseudogroup on equations.

More

Remark

In addition to finiteness of differential invariants, there is a finiteness theorem for differential syzygy. Its meaning is that the space $\mathcal{I}$ of absolute differential invariants behaves like an infinitely prolonged differential equation: given by a limiting construction, though with a finite number of generators and relations.

Cartan-Kuranishi theorem does not directly apply to the system generated by differential invariants, since invariant derivations do not commute and we don't get the usual jet-calculus. But it still holds due to the above vanishing theorem.

Some references

- S. Lie, *Ueber Differentialinvarianten*, Math. Ann. (1884); *Verwertung des Gruppenbegriftes für Differentialgleichungen*, Leipzig Ber. (1895).
- A. Tresse, *Sur les invariants differentiels des groupes continus de transformations*, Acta Math. (1894).
- A. Kumpera, *Invariants differentiels d'un pseudogroupe de Lie. I-II.*, J. Diff. Geometry (1975).
- L. V. Ovsiannikov, *Group analysis of differential equations*, Nauka-Moscow (1978).
- B.K. & V.Lychagin, *Invariants of pseudogroup actions: Homological methods and Finiteness theorem*, Int. J. Geomet. Meth. Mod. Phys. (2006).
- P. Olver, J.Pohjanpelto, *Moving frames for Lie pseudo-groups*, Canadian J. Math. (2008).

Relative differential invariants

The point transformation LAS $\mathfrak{D}_{\text{loc}}(J^0\mathbb{R})$, with $J^0\mathbb{R}(x) = \mathbb{R}^2(x, y)$, equals $\mathfrak{g} = \{\xi_0 = a\partial_x + b\partial_y : a = a(x, y), b = b(x, y)\}$ and it prolongs to the subalgebra

$$\mathfrak{g}_2 = \{\xi = a\partial_x + b\partial_y + A\partial_p + B\partial_u\} \subset \mathfrak{D}_{\text{loc}}(J^2\mathbb{R}), \ J^2\mathbb{R} = \mathbb{R}^4(x, y, p, u)$$

$$A = D_x(\varphi), \ B = D_x^2(\varphi) + u(\partial_y(\varphi) - 2D_x(a)),$$

where $p = y'$, $u = y''$, $D_x = \partial_x + p\,\partial_y$,
$\varphi = (dy - p\,dx)(a\partial_x + b\partial_y) = b - p\,a$.

Thus the LAS $\mathfrak{h} = \mathfrak{g}_2 \subset \mathfrak{D}_{\text{loc}}(J^0\mathbb{R}^3(x, y, p))$ being given we represent a second order ODE as a surface $u = f(x, y, p)$ in $J^0\mathbb{R}^3(x, y, p) = \mathbb{R}^4(x, y, p, u)$ and

k^{th} order differential invariants of this ODE are invariant functions $I \in C^\infty_{\text{loc}}(J^k\mathbb{R}^3)$ of the prolongation

$$\mathfrak{h}_k = \{\hat{\xi} = a\,\mathcal{D}_x + b\,\mathcal{D}_y + A\,\mathcal{D}_p + \sum_{|\sigma|\le k} \mathcal{D}_\sigma^{(k)}(f)\,\partial_{u_\sigma}\} \subset \mathfrak{D}(J^k\mathbb{R}^3),$$

$$f = B - a\,u_x - b\,u_y - A\,u_p \;:\qquad \hat{\xi}(I) = 0.$$

Here $\mathcal{D}_\sigma^{(k)} = \mathcal{D}_\sigma = \mathcal{D}_x^l \mathcal{D}_y^m \mathcal{D}_p^n|_{J^k}$ for $\sigma = (l\cdot 1_x + m\cdot 1_y + n\cdot 1_p)$.

It is more convenient, following Tresse, to use the operator $D_x = \partial_x + p\,\partial_y$ on the base instead and to form the corresponding total derivative $\hat{\mathcal{D}}_x = \mathcal{D}_x + p\mathcal{D}_y$. These operators will no longer commute and we use the following notation for non-holonomic partial derivatives: $u^k_{lm} = \hat{\mathcal{D}}_x^l \mathcal{D}_y^m \mathcal{D}_p^k(u)$,
which equals u_{lmk} mod (lower order terms).

The first relative invariants calculated by Tresse have order 4:

$$I = u^4, \quad H = u^2_{20} - 4u^1_{11} + 6u_{02} + u^2_{20} - 4u^1_{11} + 6u_{02} +$$
$$+ u(2u^3_{10} - 3u^2_{01}) - u^1(u^2_{10} - 4u^1_{01}) + u^3 u_{10} - 3u^2 u_{01} + u \cdot u \cdot u^4.$$

In this case the relative invariants are

$$\mathcal{R}^{r,s} = \{\psi \in C^\infty(J^\infty\mathbb{R}^3) : \hat{\xi}(\psi) = (-r\, C^0_\xi + (r-s)\, C^1_\xi)\psi\},$$

so that the weights form a 2D lattice with the basis

$$C^0_\xi = a_x + b_y = \mathrm{div}_{\omega_0}(\xi_0) \text{ and } C^1_\xi = \partial_y(\varphi) = \tfrac{1}{2}\,\mathrm{div}_{\Omega_0}(\xi_1)$$
$$\text{for } \xi_0 = \hat{\xi}|_{J^0} = a\partial_x + b\partial_y,\ \xi_1 = \hat{\xi}|_{J^1} = a\partial_x + b\partial_y + A\partial_p,$$

with $\omega_0 = dx \wedge dy$ the volume on $J^0\mathbb{R}$ and $\Omega_0 = -\omega \wedge d\omega$ on $J^1\mathbb{R}$, $\omega = dy - p\,dx$ being the standard contact form of $J^1\mathbb{R}$.

There are relative invariant differentiations (differential parameters):

$$\Delta_p = \mathcal{D}_p + (r-s)\frac{u^5}{5u^4} : \mathcal{R}^{r,s} \to \mathcal{R}^{r-1,s+1}$$

$$\Delta_x = \hat{\mathcal{D}}_x + u\,\Delta_p + \Bigl((3r+2s)\Bigl(u^1 + \frac{3u\,u^5}{5u^4}\Bigr) + (2r+s)\frac{u^4_{10}}{u^4}\Bigr) : \mathcal{R}^{r,s} \to \mathcal{R}^{r+1,s}$$

$$\Delta_y = \mathcal{D}_y + \frac{u^5}{5u^4}\,\Delta_x + \Bigl(2\,u^1 + \frac{u^4_{10} + u\,u^5}{u^4}\Bigr)\Delta_p + \Bigl((r+2s)\frac{u^4_{01}}{4u^4} + \\ +(3r+2s)\Bigl(\frac{u^2}{8} + \frac{3}{20}\,\frac{u^5(u^4_{10} + u\,u^5 + 2u^1u^4)}{u^4u^4}\Bigr)\Bigr) : \mathcal{R}^{r,s} \to \mathcal{R}^{r,s+1}.$$

Theorem

The space of relative differential invariants $\mathcal{R}$ is generated by the invariant H and differentiations $\Delta_x, \Delta_y, \Delta_p$ on the generic stratum.

Specifications

The number of basic relative differential invariants of pure order k:

$k:$	0	1	2	3	4	5	6	7	8	$\dots$	k	$\dots$
$\#:$	0	0	0	0	2	3	11	17	24	$\dots$	$\frac{1}{2}(k^2-k-8)$	

The generators are the following.

In order 4: $I \in \mathcal{R}^{-2,3}$ and $H \in \mathcal{R}^{2,1}$;

In order 5: $H_{10} = \Delta_x(H) \in \mathcal{R}^{3,1}$, $H_{01} = \Delta_y(H) \in \mathcal{R}^{2,2}$ and $K = \Delta_p(H) \in \mathcal{R}^{1,2}$;

In order 6: $(H_{20}, H_{11}, H_{02}) \in \mathcal{R}^{4,1} \oplus \mathcal{R}^{3,2} \oplus \mathcal{R}^{2,3}$, $(K_{10}, K_{01}) \in \mathcal{R}^{2,2} \oplus \mathcal{R}^{1,3}$ and $\Omega^l_{ij} = u^l_{ij} +$ (lower "order" terms) $\in \mathcal{R}^{i+2-l,j+l-1}$, $\deg \Omega^l_{ij} = i+j+l = 6$, $l > 3$ etc.

Summarizing we have the table:

order k	basic relative differential invariants
4	$I,\ H$
5	$H_{10},\ H_{01},\ K$
6	$H_{20},\ H_{11},\ H_{02},\ K_{10},\ K_{01},\ \Omega^4_{20},\ \Omega^4_{11},\ \Omega^4_{02},\ \Omega^5_{10},\ \Omega^5_{01},\ \Omega^6$

Thus in ascending order k, we must add the generators I, H and then Ω^{6-i-j}_{ij}, $i+j \le 2$ (one encounters the relations $\Delta_x(I) = \Delta_y(I) = \Delta_p(I) = 0$). Invariants of order $k > 6$ are obtained via invariant derivations from the lower order.

Other generators on finer strata... differential syzygies... etc
On $I \neq 0$ the following is a generating set: $I, H, \Omega^6 = u^6 - \frac{6}{5}\frac{u^5 \cdot u^5}{u^4}$, and $\Delta_x, \Delta_y, \Delta_p$.

Absolute differential invariants

There are two ways of adjusting a basis on the lattice $\mathfrak{M}$ of weights via relative invariants. The basic invariants can be taken

$$J_1 = I^{-1/8} H^{3/8} \in \mathcal{R}^{1,0}, \quad J_2 = I^{1/4} H^{1/4} \in \mathcal{R}^{0,1}.$$

or (to avoid branching but increasing the order)

$$\tilde{J}_1 = \frac{H_{10}}{H} \in \mathcal{R}^{1,0}, \quad \tilde{J}_2 = \frac{H_{01}}{H} \in \mathcal{R}^{0,1}.$$

Then (choosing J_i or $\tilde{J}_i$) we get the isomorphism for $k > 4$:

$$\mathcal{R}^{r,s}_k / \mathcal{R}^{r,s}_{k-1} \simeq \mathcal{I}_k / \mathcal{I}_{k-1}, \qquad F \mapsto F/(J_1^r J_2^s).$$

With any choice the list of basic differential invariants in order 5 is

$$\bar{H}_{10} = H_{10}/(J_1^3 J_2), \; \bar{H}_{01} = H_{01}/(J_1^2 J_2^2), \; \bar{K} = K/(J_1 J_2^2)$$

and in pure order 6 is

$$\bar{H}_{20} = H_{20}/(J_1^4 J_2),\ \bar{H}_{11} = H_{11}/(J_1^3 J_2^2),\ \bar{H}_{02} = H_{02}/(J_1^2 J_2^3),$$
$$\bar{K}_{10} = K_{10}/(J_1^2 J_2^2),\ \bar{K}_{01} = K_{01}/(J_1 J_2^3),$$
$$\bar{\Omega}^4_{20} = \Omega^4_{20}/(J_2^3),\ \bar{\Omega}^4_{11} = \Omega^4_{11}/(J_1^{-1} J_2^4),\ \bar{\Omega}^4_{02} = \Omega^4_{02}/(J_1^{-2} J_2^5),$$
$$\bar{\Omega}^5_{10} = \Omega^5_{10}/(J_1^{-2} J_2^4),\ \bar{\Omega}^5_{01} = \Omega^5_{01}/(J_1^{-3} J_2^5),\ \bar{\Omega}^6 = \Omega^6/(J_1^{-4} J_2^5).$$

Higher order differential invariants can be obtained in a similar way from the basic relative invariants, but alternatively we can adjust invariant derivations by letting $\nabla_j = J_1^{\rho_j} J_2^{\sigma_j} \cdot \Delta_j|_{r=s=0}$ with a proper choice of the weights ρ_j, σ_j. Namely we let

$$\nabla_p = \frac{J_1}{J_2}\mathcal{D}_p, \qquad \nabla_x = \frac{1}{J_1}\big(\hat{\mathcal{D}}_x + u\mathcal{D}_p\big),$$
$$\nabla_y = \frac{1}{J_2}\Big(\mathcal{D}_y + \frac{u^5}{5u^4}\hat{\mathcal{D}}_x + \big(\frac{u^4_{10}}{u^4} + \frac{6u\,u^5}{5u^4} + 2u^1\big)\mathcal{D}_p\Big).$$

These form a basis of invariant derivations over $\mathcal{I}$ and we have

The syzygy:

$$[\nabla_p, \nabla_x] = -\tfrac{1}{8}\bar{H}_{10}\nabla_p - \tfrac{3}{8}\bar{K}\nabla_x + \nabla_y,$$
$$[\nabla_p, \nabla_y] = (\bar{\Omega}^5_{10} - \tfrac{1}{8}\bar{H}_{01})\nabla_p + \tfrac{1}{5}\bar{\Omega}^6\nabla_x - \tfrac{1}{4}\bar{K}\nabla_y,$$
$$[\nabla_x, \nabla_y] = \bar{\Omega}^4_{20}\nabla_p + (\tfrac{1}{5}\bar{\Omega}^5_{10} + \tfrac{3}{8}\bar{H}_{01})\nabla_x - \tfrac{1}{4}\bar{H}_{10}\nabla_y.$$

The derivations and coefficients can be also expressed in terms of non-branching invariants $\tilde{J}_1 = \frac{8}{3}\nabla_x J_1$ and $\tilde{J}_2 = 4\nabla_y J_2$.

Theorem

The space $\mathcal{I}$ of differential invariants is generated by the invariant derivations $\nabla_x, \nabla_y, \nabla_p$ on the generic stratum.

Indeed, we mean here that taking coefficients of the commutators, adding their derivatives etc leads to a complete list of basic differential invariants.

Equivalence problem

2nd order ODEs $\mathcal{E}$ can be considered as sections $\mathfrak{s}_{\mathcal{E}}$ of the bundle π, whence we can restrict any differential invariant $J \in \mathcal{I}_k$ to the equation via pull-back of the prolongation:

$$J^{\mathcal{E}} := (\mathfrak{s}_{\mathcal{E}}^{(k)})^*(J) \in C^\infty_{\text{loc}}(\mathbb{R}^3(x,y,p)).$$

In this way we obtain the invariants

$$\bar{H}_{10}^{\mathcal{E}},\ \bar{H}_{01}^{\mathcal{E}},\ \bar{K}^{\mathcal{E}},\ \bar{H}_{20}^{\mathcal{E}},\ \bar{H}_{11}^{\mathcal{E}},\ \bar{H}_{02}^{\mathcal{E}},\ \bar{K}_{10}^{\mathcal{E}},\ \bar{K}_{01}^{\mathcal{E}},\ \bar{\Omega}^{6\,\mathcal{E}},\ \bar{\Omega}_{10}^{5\,\mathcal{E}}, \bar{\Omega}_{20}^{4\,\mathcal{E}}.$$

These are functions of 3 variables, so we get the map $\mathbb{R}^3 \simeq \mathcal{E} \to \mathbb{R}^{11}$. Its image (of dim $= 3$, 2, 1 or 0) is an invariant.

Theorem

Two 2nd order regular differential equations $\mathcal{E}_1$, $\mathcal{E}_2$ are point equivalent iff the corresponding submanifolds in the space of differential invariants $\mathbb{R}^{11}$ coincide.

Projective connections

On the space $J^3\mathbb{R}^3(x,y,p)$ the lifted action of the pseudogroup $\mathfrak{h}$ is transitive. But its lift to the space of 4-jets is not longer such: There are singular strata, given by the equations $I=0, H=0$. Moreover they have a singular substratum $I=H=0$ in itself, on which the pseudogroup action is transitive, so that any equation from it is point equivalent to trivial ODE $y''=0$.

We will restrict here only to the singular stratum $I=0$ (the other stratum $H=0$ can be treated similarly. Indeed, though the invariants I,H look quite unlike, they are proportional to self-dual and anti-self-dual components of the Fefferman metric...), i.e.

$$y''=\alpha_0(x,y)+\alpha_1(x,y)p+\alpha_2(x,y)p^2+\alpha_3(x,y)p^3, \quad p=y'.$$

This class of equations is invariant under point transformations. Moreover it has very important geometric interpretation, namely such ODEs correspond to projective connections on 2D manifolds. There are 3 different approaches to the equivalence problem.

The original approach of Tresse

Following S.Lie's method, Tresse studied lift of the action of point transformation to the space $J^k(\mathbb{R}^2,\mathbb{R}^4)$ and investigated the algebra of differential invariants. The number of basic invariants of pure order k is

$$\begin{array}{lccccccccccc} k: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots & k & \dots \\ \#: & 0 & 0 & 0 & 0 & 6 & 8 & 10 & 12 & 14 & \dots & 2(k-1) & \end{array}$$

The action of $\mathfrak{g}$ is transitive on J^1 and its lift is transitive on $J^2(\mathbb{R}^2,\mathbb{R}^4)$ outside the singular orbit $L_1 = L_2 = 0$, where

$$L_1 = -\alpha_{2xx} + 2\alpha_{1xy} - 3\alpha_{0yy} - 3\alpha_3\alpha_{0x} + \alpha_1\alpha_{2x} - 6\alpha_0\alpha_{3x} + 3\alpha_2\alpha_{0y} - 2\alpha_1\alpha_{1y} + 3\alpha_0\alpha_{2y}$$

$$L_2 = -3\alpha_{3xx} + 2\alpha_{2xy} - \alpha_{1yy} - 3\alpha_3\alpha_{1x} + 2\alpha_2\alpha_{2x} - 3\alpha_1\alpha_{3x} + 6\alpha_3\alpha_{0y} - \alpha_2\alpha_{1y} + 3\alpha_0\alpha_{3y}$$

These second order operators were found by S.Lie, who showed that vanishing $L_1 = L_2 = 0$ characterizes trivial (=linearizable) ODEs. R.Liouville proved that the tensor

$$L = (L_1 dx + L_2 dy) \otimes (dx \wedge dy),$$

responsible for this, is an absolute differential invariant.
Further on Tresse claimed that all absolute differential invariants can be expressed via L_1, L_2 ... finished recently by V.Yumaguzhin.
The action of $\mathfrak{g}$ in $J^3(\mathbb{R}^2, \mathbb{R}^4)$ is transitive outside the stratum $F_3 = 0$, where

$$\begin{aligned} F_3 = (L_1)^2 \mathcal{D}_y(L_2) - L_1 L_2 (\mathcal{D}_x(L_2) + \mathcal{D}_y(L_1)) + (L_2)^2 \mathcal{D}_x(L_1) \\ - (L_1)^3 \alpha_3 + (L_1)^2 L_2 \alpha_2 - L_1 (L_2)^2 \alpha_1 + (L_2)^2 \alpha_0 \end{aligned}$$

is the Liouville relative differential invariant. The other tensor invariants can be expressed through these.

The invariant derivations are

$$\nabla_1 = \frac{L_2}{(F_3)^{2/5}}\mathcal{D}_x - \frac{L_1}{(F_3)^{2/5}}\mathcal{D}_y, \quad \nabla_2 = \frac{\Psi_2}{(F_3)^{4/5}}\mathcal{D}_x - \frac{\Psi_1}{(F_3)^{2/5}}\mathcal{D}_y,$$

where

$$\Psi_1 = -L_1(L_1)_y + 4L_1(L_2)_x - 3L_2(L_1)_x - (L_1)^2\alpha_2 + 2L_1L_2\alpha_1 - 3(L_2)^2\alpha_0,$$
$$\Psi_2 = 3L_1(L_2)_y - 4L_2(L_1)_y + L_2(L_2)x - 3(L_1)^2\alpha_3 + 2L_1L_2\alpha_2 - (L_2)^2\alpha_1.$$

Now we can get two differential invariants of order 4 as the coefficients of the commutator

$$[\nabla_1, \nabla_2] = I_1\nabla_1 + I_2\nabla_2.$$

Four more invariants $I_3, \dots, I_6$ of order 4, finish the story.

The second Tresse approach

The invariants of the general theory are not defined on the stratum $I=0$. However the relative invariants I,H are on equal footing. And in fact Tresse constructed another basis of relative invariants with H in denominator.

Thus if we restrict this set to the stratum $I=0$ minus the trivial equation $\{I=H=0\}$, we get differential invariants of the ODEs cubic in p. For instance H is proportional to $L_1+L_2p=L/(dx\otimes dx\wedge dy)$. The other invariants are rational functions in p on the cubics. The proposed idea can be viewed as a change of coordinates in the algebra $\mathcal{I}$.

Yet, another non-local approach was sketched by Tresse: straighten L to $\lambda\,dx$. The pseudogroup is reduced to triangular $x\mapsto X(x)$, $y\mapsto Y(x,y)$, and the invariants are generated by the invariant derivatives Δ_x,Δ_y and the invariants B,C,D of orders 1, 2, 2.

Lie equations

Denote by $\mathfrak{s}_{\mathcal{E}}:\mathbb{R}^2\to\mathbb{R}^4$ the maps $(x,y)\mapsto(a_0,a_1,a_2,a_3)$ corresponding to cubic 2nd order ODEs $\mathcal{E}$. The Lie equation on the equivalence between them is

$$\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)=\{[\varphi]^2_z\in J^2(\mathbb{R}^2,\mathbb{R}^2):\hat{\varphi}\bigl(\mathfrak{s}_{\mathcal{E}_1}(z)\bigr)=\mathfrak{s}_{\mathcal{E}_2}(\varphi(z))\},$$

where $\hat{\varphi}:\mathbb{R}^2\times\mathbb{R}^4\to\mathbb{R}^2\times\mathbb{R}^4$ is the lift of $\varphi:\mathbb{R}^2\to\mathbb{R}^2$. On infinitesimal level, the lift of a vector field $X=a\,\partial_x+b\,\partial_y$ is

$$\hat{X}=a\,\partial_x+b\,\partial_y+(b_{xx}+\alpha_0(b_y-2a_x)-\alpha_1 b_x)\,\partial_{\alpha_0}$$
$$+(2b_{xy}-a_{xx}-3\alpha_0 a_y-\alpha_1 a_x-2\alpha_2 b_x)\partial_{\alpha_1}+(b_{yy}-2a_{xy}-2\alpha_1 a_y-\alpha_2 b_y$$
$$-3\alpha_3 b_x)\partial_{\alpha_2}+(-a_{yy}-\alpha_2 a_y+\alpha_3(a_x-2b_y))\partial_{\alpha_3}.$$

For $\mathcal{E}_1=\mathcal{E}_2$ infinitesimal version of the finite Lie equation $\mathfrak{Lie}(\mathcal{E},\mathcal{E})$ describes the equation $\mathfrak{lie}(\mathcal{E})\subset J^2(\mathbb{R}^2,\mathbb{R}^2)$ for the symmetry algebra $\mathrm{sym}(\mathcal{E})$.

The basic differential invariants of the pseudogroup $\mathrm{Diff}_{\mathsf{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ action on cubic ODEs arise as the obstruction to formal integrability of the equation $\mathfrak{lie}(\mathcal{E})$. In coordinates it has the form

$$\begin{aligned} b_{xx}+\alpha_0(b_y-2a_x)-\alpha_1 b_x &= a\,\alpha_{0x}+b\,\alpha_{0y}\\ 2b_{xy}-a_{xx}-3\alpha_0 a_y-\alpha_1 a_x-2\alpha_2 b_x &= a\,\alpha_{1x}+b\,\alpha_{1y}\\ b_{yy}-2a_{xy}-2\alpha_1 a_y-\alpha_2 b_y-3\alpha_3 b_x &= a\,\alpha_{2x}+b\,\alpha_{2y}\\ -a_{yy}-\alpha_2 a_y+\alpha_3(a_x-2b_y) &= a\,\alpha_{3x}+b\,\alpha_{3y} \end{aligned}$$

Application of prolongation-projection method and Spencer theory gives all basic invariants.

Remark

Other ways of getting differential invariants arise from problems which with projectively invariant answers. For instance the following system arose in 3 independent problems:

$$u_y = P_0[u,v,w], \quad u_x + 2v_y = P_1[u,v,w],$$
$$2v_x + w_y = P_2[u,v,w], \quad w_x = P_3[u,v,w],$$

where $P_i[u,v,w]$ are linear in u,v,w and smooth in x,y. It is obtained similar to $\mathfrak{lie}$ as the existence condition for Killing tensors.

B.K. solved this system to characterize invariantly Liouville metrics, R. Bryant, G. Manno, V. Matveev to give normal forms of metrics with transitive group of projective transformations and R. Bryant, M. Dunajski, M. Eastwood to decide local metrisability of projective structures on surfaces.

References

- S. Lie, *Klassifikation und Integration von gewöhnlichen Differentialgl. zw. x, y, die eine Gruppe von Transf. gestatten. III*, Archiv (1883); Gesam. Abh. Bd. **5** (1924), paper XIV.
- R. Liouville, *Sur les invariants de certaines équations différentielles et sur leurs applications*, Journal de l'École Polytechnique (1889).
- A. Tresse, *Détermination des invariants ponctuels de l'équation différentielle ordinaire du second ordre* $y'' = \omega(x, y, y')$, Leipzig (1896).
- E. Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France (1924).
- (many others) N. Kamran, W. Shadwick, P. Nurowski, Yu. Romanovskii, M. Dunajski, V. Yumaguzhin . . .
- B.K. *Point classification of 2nd order ODEs: Tresse classification revisited and beyond*, ArXiv (2008).

An interesting overdetermined system

When is the following system on $u \in \mathbb{C}^\infty(\mathbb{R}^n)$ solvable:

$$\mathcal{E}: \qquad |\nabla u| = 1, \qquad \Delta u = f(u).$$

- Compatible $\Leftrightarrow n = 2$ & solvable.
- Solvable $\Rightarrow f \equiv 0$ or $f(o) = \infty$, we normalize $o = 0$.
- Solvable $\Leftrightarrow uf(u) = \kappa \in \mathbb{Z} \cap [0, n-1]$.
- $Sol(\mathcal{E}) = T^\perp Gr(n-1-\kappa, n) = \{(v, \Pi) : v \perp \Pi \subset \mathbb{R}^n\}$.

This system has symmetry group $G = O(n) \ltimes \mathbb{R}^n$, but the standard differential invariants approach does not work.
Indeed, the 2nd order invariants are precisely eigenvalues of the Hesse matrix $H_u = D^2 u$, or equivalently $\text{Trace}(H_u^j)$, $j = 1, \dots, n$.
But the Hamilton-Jacobi equation relates them since $\det H_u = 0$ on $\mathcal{E}$. Thus pseudogroup G action on equation should be used (singular orbits!).

Criterion in differential invariants

Classical question of surface metric geometry is to recognize when a geodesic flow has quadratic integral (Darboux problem). Then metric is Liouville:

$$ds^2 = \bigl(f(x) + g(y)\bigr)(dx^2 + dy^2).$$

Denote by $\mathcal{J}_2$ the space of quadratic integrals. $\dim \mathcal{J}_2$ (it is 6,4,3,2 or 1) can be expressed via differential invariants. For instance,

$$\dim \mathcal{J}_2 = 3 \;\Leftrightarrow\; J_{6a} = J_{6b} = J_{6c} = J_{6d} = 0,$$

where J_{6i} are some differential invariants of order 6.
But the r.h.s. invariant overdetermined system is not compatible!
Applying invariant derivatives $\nabla_1 = \mathfrak{L}_{\operatorname{grad} K}$ and $\nabla_2 = \mathfrak{L}_{\operatorname{sgrad} K}$ we get new conditions $J_{6e} = 0$ and $\tilde{J}_5 = 0$.
The solution to the general problem $\dim \mathcal{J}_2 = 2$ has similar but much more complicated answer.

Linearization of webs etc

Similar problem arises in resolution of Blaschke conjecture by V.Goldberg & V.Lychagin (there're 1024 invariants of order 9!).

The problem is reduced to a "simple" solvability of an overdetermined system of 2 scalar PDEs. The method is prolongation-projection, i.e. writing successively compatibility conditions, starting with Mayer bracket (introduced by Lychagin and B.K.)

Another problem in webs solved in a similar way (but via a multi-bracket) is a counting of Abelian relations. The answer is formulated via differential invariants of webs.

References

- C.B. Collins, *Complex potential equations*, Math. Proc. Cambridge Philos. Soc. (1976).
- W.Fushchich, R.Zhdanov, I.Yegorchenko, *On the reduction of the nonlinear wave equations*, J. Math.Anal.Appl. (1991).
- B.K. & V.Lychagin, Differential invariants of the motion group actions, NOVA Sci. Publ. (2008).
- G. Kœnigs, *Sur les géodésiques à intégrales quadratiques*, in: G. Darboux, *Leçons sur la théorie générale des surfaces* (1896).
- B.K., *Invariant characterization of Liouville metrics and polynomial integrals* J. Geometry and Physics (2008).
- V. Goldberg & V.Lychagin, *On the Blaschke conjecture for 3-webs*, J. Geom. Anal. (2006).