Contact geometry of hyperbolic equations of generic type

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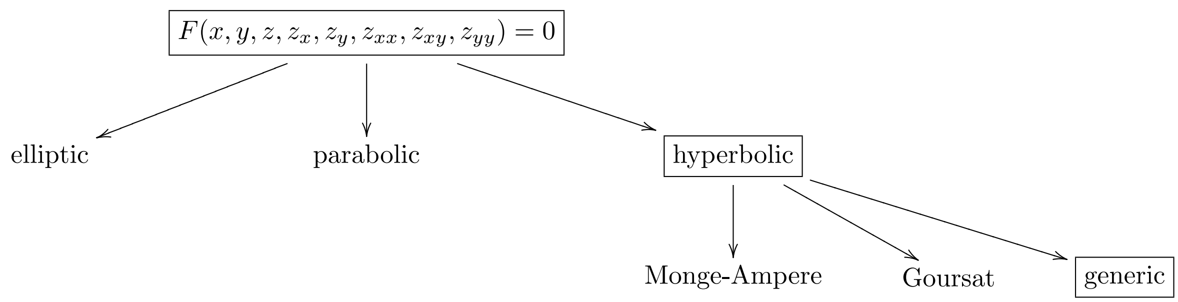
Contact–invariant classification of 2nd order scalar PDE in the plane:



WARNING: For this lecture, drop whatever preconceived notion you have of "hyperbolic Goursat" or "hyperbolic generic": there is unfortunately an abuse of terminology in the literature. Refer to page 3 for defn here.

1 Contact equivalence

We'll work locally and in C^∞ category. Consider

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0$$

Work in $J^2(\mathbb{R}^2, \mathbb{R}) : (x, y, z, p, q, r, s, t)$ with $\mathcal{C}^{(2)} = \{\theta^1, \theta^2, \theta^3\}$.

$$\theta^1 = dz - pdx - qdy, \quad \theta^2 = dp - rdx - sdy, \quad \theta^3 = dq - sdx - tdy.$$

Parametrization of $F = 0$: $i_F : \Sigma^7 \to J^2(\mathbb{R}^2, \mathbb{R})$. Assume: i_F is maximal rank & $(F_r, F_s, F_t) \neq 0 \Rightarrow$ can loc. solve $F = 0$ for one of r, s, t. Define:

$$I_F = i_F^*(\mathcal{C}^{(2)}) = \{\omega^1, \omega^2, \omega^3\}.$$

Fact: There is a 1-1 correspondence between local solutions of $F = 0$ and local integral manifolds of I_F (satisfying an independence condition).

Definition 1.1. *$F = 0$ and $\bar{F} = 0$ (with $i_{\bar{F}} : \bar{\Sigma} \to J^2(\mathbb{R}^2, \mathbb{R})$) are* contact-equivalent *if $\exists$ local diffeo. $\phi : \Sigma \to \bar{\Sigma}$ such that $\phi^* I_{\bar{F}} = I_F$. A* contact symmetry *is a self-equivalence.*

Remark 1.2. *More precisely, this is* internal *contact-equivalence.* External *contact-equivalence refers to $\rho \in \mathrm{Diff}_{loc.}(J^2(\mathbb{R}^2, \mathbb{R}))$ preserving $\mathcal{C}^{(2)}$ which restricts to a local diffeomorphism $\tilde{\rho} : i_F(\Sigma) \to i_{\bar{F}}(\bar{\Sigma})$. Under our assumptions on i_F, $i_{\bar{F}}$, these notions are in fact equivalent. [Anderson, Kamran, Olver]*

Define a symmetric $C^\infty(\Sigma)$-bilinear form $\langle\cdot,\cdot\rangle$ (Gardner tensor) on I_F, namely

$$\langle\varphi,\psi\rangle Vol_\Sigma := d\varphi\wedge d\psi\wedge\omega^1\wedge\omega^2\wedge\omega^3, \qquad \forall\varphi,\psi\in I_F.$$

Since $ker(i_F^*)=\{dF\}$, this definition is equivalent to

$$\langle\varphi,\psi\rangle_p (Vol_{J^2})_{i_F(p)} := (d\tilde{\varphi}\wedge d\tilde{\psi}\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge dF)_{i_F(p)}$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are *any* forms such that $\varphi=i_F^*\tilde{\varphi}$ and $\psi=i_F^*\tilde{\psi}$.

e.g. For $\langle\omega^2,\omega^2\rangle$, note $d\theta^2=dx\wedge dr+dy\wedge ds$, and

$$d\theta^2\wedge d\theta^2\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge dF = 2dx\wedge dr\wedge dy\wedge ds\wedge dz\wedge dp\wedge dq\wedge F_t dt = F_t Vol_{J^2}$$

Wrt some Vol_{J^2}, we have

$$(\langle\omega^\alpha,\omega^\beta\rangle)_p = \left(\begin{array}{ccc} 0 & 0 & 0\\ 0 & F_t & -\frac{1}{2}F_s\\ 0 & -\frac{1}{2}F_s & F_r \end{array}\right)_{i_F(p)}.$$

Since $(F_r,F_s,F_t)\neq 0$, $\langle\cdot,\cdot\rangle$ has either rank 1 or 2. Defining

$$\Delta = i_F^*\left(F_rF_t-\frac{1}{4}{F_s}^2\right),$$

we have the following (pointwise) mutually exclusive cases:

elliptic	parabolic	hyperbolic
$\Delta(p)>0$	$\Delta(p)=0$	$\Delta(p)<0$

Since d and pullbacks commute, this classification is *contact-invariant.*

2 Hyperbolic eqns: Monge–Ampère, Goursat, generic

Hyperbolic case: $\exists$ pair of rank 2 maximally isotropic subsystems

$$M_1=\{\omega^1,\omega^2\},\quad M_2=\{\omega^1,\omega^3\}\quad\text{of}\quad I_F=\{\omega^1,\omega^2,\omega^3\}$$

By a choice of volume form,

$$(\langle\omega^\alpha,\omega^\beta\rangle)_p = \left(\begin{array}{ccc} 0&0&0\\0&0&1\\0&1&0\end{array}\right),$$

which is equivalent to

$$\begin{array}{lcl} d\omega^1 & \equiv & 0\\ d\omega^2 & \equiv & \omega^4\wedge\omega^5 \\ d\omega^3 & \equiv & \omega^6\wedge\omega^7 \end{array}\quad \text{mod } I_F,\quad \text{with}\quad \omega^1\wedge...\wedge\omega^7\neq 0$$

Theorem 2.1 (Hyperbolic structure equations)**.** *Given any hyperbolic equation $F=0$, there is an associated coframe $\boldsymbol{\omega}=\{\omega^i\}_{i=1}^7$ on Σ such that*

1. $I_F=\{\omega^1,\omega^2,\omega^3\},\quad M_1=\{\omega^1,\omega^2\},\quad M_2=\{\omega^1,\omega^3\}$
2. *We have the structure equations*

$$\begin{array}{lcll} d\omega^1 & \equiv & \omega^3\wedge\omega^6+\omega^2\wedge\omega^4 & \text{mod } \{\omega^1\}\\ d\omega^2 & \equiv & \omega^4\wedge\omega^5+U_1\omega^3\wedge\omega^7 & \text{mod } \{\omega^1,\omega^2\}\\ d\omega^3 & \equiv & \omega^6\wedge\omega^7+U_2\omega^2\wedge\omega^5 & \text{mod } \{\omega^1,\omega^3\} \end{array}$$

Let's examine Cauchy characteristics of M_1 and M_2. Let $\{e_i\}$ be dual to $\{\omega^i\}$. Then

$$C(M_1) = \{X \in \mathfrak{X}(\Sigma) : X \in M_1^\perp,\ X \lrcorner dM_1 \subset M_1\}^\perp \supset \{e_7\}^\perp, \qquad C(M_2) \supset \{e_5\}^\perp.$$

Define $class(M_i) = rank(C(M_i))$.

Lemma 2.2. *For any hyperbolic eqn, $class(M_i) = 6$ or 7; moreover, $class(M_i) = 6$ iff $U_i = 0$.*

Name	$\{class(M_1), class(M_2)\}$
MA	{ 6 }
Goursat	{ 6, 7 }
generic	{7 }

Table 1: Contact-invariant subclassification of hyperbolic eqns

In the generic case, neither M_1 nor M_2 have Cauchy characteristics!

Example 2.3.

- *all MA: $a(rt - s^2) + br + cs + dt + e = 0$ where a, b, c, d, e are functions of x, y, z, p, q. (Includes wave, Liouville, Klein–Gordon eqns.)*
- *Goursat: $r = f(s)$ where $f'' \neq 0$ (admits $g(y)\frac{\partial}{\partial z}$, so an infinite-dimensional symmetry group)*
- *generic: $s = \frac{1}{2}\sin(r)\cos(t)$, $\qquad 3rt^3 + 1 = 0$.*

Vranceanu (1940): relative invariants for hyperbolic eqns of the form $r = f(x, y, z, p, q, s, t)$.
Juras (1997): relative invariants for general hyperbolic eqns $F = 0$.

Theorem 2.4. *(T. 2008) Suppose $F = 0$ is a hyp. eqn with $F_s \geq 0$ (at a point σ on $F = 0$). Let*

$$I_1 = i_F^* det \begin{pmatrix} F_r & F_s & F_t \\ \lambda_+ & F_t & 0 \\ \left(\frac{F_t}{\lambda_+}\right)_r & \left(\frac{F_t}{\lambda_+}\right)_s & \left(\frac{F_t}{\lambda_+}\right)_t \end{pmatrix}, \quad I_2 = i_F^* det \begin{pmatrix} 0 & F_r & \lambda_+ \\ F_r & F_s & F_t \\ \left(\frac{F_r}{\lambda_+}\right)_r & \left(\frac{F_r}{\lambda_+}\right)_s & \left(\frac{F_r}{\lambda_+}\right)_t \end{pmatrix}.$$

where $\lambda_+ > 0$ satisfies $\lambda^2 - F_s\lambda + F_rF_t = 0$. Then we have the following classification of $F = 0$ (at σ):

Type	*Contact-invariant classification*
Monge–Ampère	$I_1 = I_2 = 0$
Goursat	*exactly one of I_1 or I_2 is zero*
generic	$I_1 I_2 \neq 0$

(Remark: I_i only depend on the locus.)

More General Examples:

- $F(x, y, z, p, q, r, t) = 0$: 6-6 or 7-7
- $G(x, y, z, p, q, r, s) = 0$: 6-6 or 6-7
- $rt = f(x, y, z, p, q, s)$: 6-6 or 7-7 (iff $f_{ss} \neq 2$)

Definition 2.5. *For hyperbolic equations: Wrt M_i, define*

$$C(I_F, dM_i) = \{X \in \mathfrak{X}(\Sigma) : X \in I_F^\perp,\ X \lrcorner dM_i \subset I_F\}^\perp.$$

We can use these contact-invariants to distinguish equivalence classes.

e.g. (Complete) derived flag of $C(I_F, dM_1)$ (and $C(I_F, dM_2)$) for:

- wave eqn $z_{xy} = 0$: 5, 4, 3
- Liouville eqn $z_{xy} = e^z$: 5, 4, 3, 1
- Klein–Gordon eqn $z_{xy} = z$: 5, 4, 3, 2, 1

Hence, these 3 equations are inequivalent.

3 Hyperbolic equations of generic type

Vranceanu (1937): initiated the study of this class of eqns

Lemma 3.1. *Given a generic hyp. eqn $F=0$, $\exists$ coframe $\boldsymbol{\omega}=\{\omega^i\}_{i=1}^7$ on Σ s.t.*

$$\begin{array}{lcll} d\omega^1 & \equiv & \omega^3\wedge\omega^6+\omega^2\wedge\omega^4 & \mod \{\omega^1\} \\ d\omega^2 & \equiv & \omega^4\wedge\omega^5+\omega^3\wedge\omega^7 & \mod \{\omega^1,\omega^2\} \\ d\omega^3 & \equiv & \omega^6\wedge\omega^7+\epsilon\omega^2\wedge\omega^5 & \mod \{\omega^1,\omega^3\} \\ d\omega^4 & \equiv & \epsilon\omega^5\wedge\omega^6 & \mod \{\omega^1,\omega^2,\omega^4\} \\ d\omega^5 & \equiv & 0 & \mod \{\omega^1,\omega^2,\omega^4,\omega^5\} \\ d\omega^6 & \equiv & -\omega^4\wedge\omega^7 & \mod \{\omega^1,\omega^3,\omega^6\} \\ d\omega^7 & \equiv & 0 & \mod \{\omega^1,\omega^3,\omega^6,\omega^7\} \end{array}$$

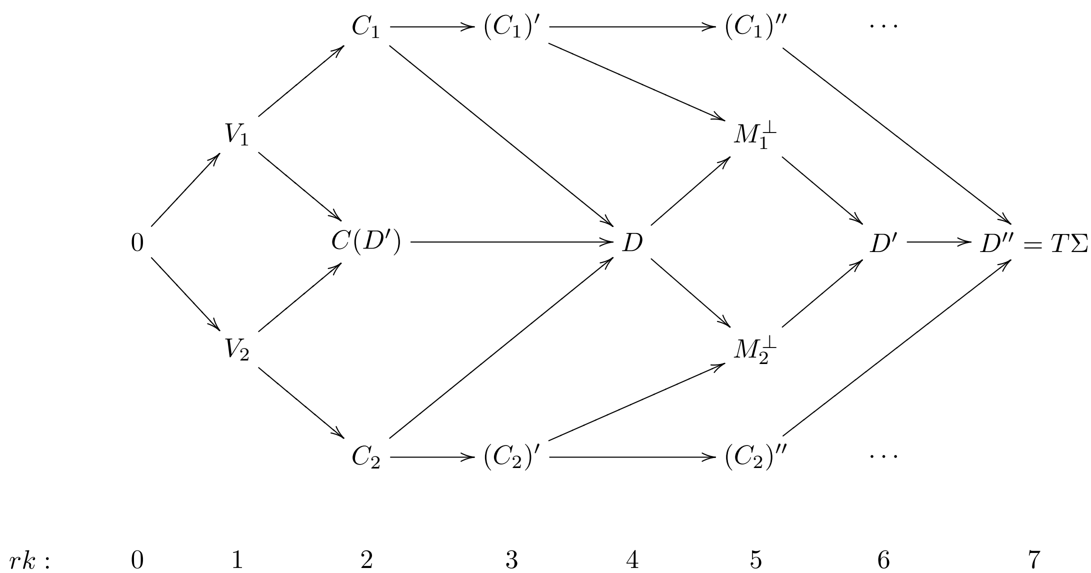
where $\epsilon = sgn(I_1 I_2)=\pm 1$ is a contact-invariant, and

$$I_F=\{\omega^1,\omega^2,\omega^3\},\quad I_F^{(1)}=\{\omega^1\},\quad M_1=\{\omega^1,\omega^2\},\quad M_2=\{\omega^1,\omega^3\},$$

$$\begin{array}{ll} C(I_F,dM_1) = \{\omega^1,\omega^2,\omega^3,\omega^4,\omega^5\} & C(I_F,dM_2) = \{\omega^1,\omega^2,\omega^3,\omega^6,\omega^7\} \\ C(I_F,dM_1)^{(1)} = \{\omega^1,\omega^2,\omega^4,\omega^5\} & C(I_F,dM_2)^{(1)} = \{\omega^1,\omega^3,\omega^6,\omega^7\} \\ C(I_F,dM_1)^{(2)} = \{\omega^4,\omega^5\} & C(I_F,dM_2)^{(2)} = \{\omega^6,\omega^7\} \end{array}$$

Remark 3.2. *$C(I_F,dM_i)^{(2)}$ may or may not be Frobenius.*

We have the following picture for the vector distributions. A (path of) directed arrow(s) indicates inclusion. Write $C_i=\{X\in D : [X,D]\subset M_i^\perp\}$.



Cartan equivalence problem: $\phi\in Contact(\Sigma,\bar\Sigma)$ iff $\phi^*\bar{\boldsymbol{\omega}}=g\boldsymbol{\omega}$ for some $g:\Sigma\to G$.

- Structure group: $G=G^0\rtimes D_8$, where

$$G^0=\left\{\begin{pmatrix} {a_1}^2 & 0 & 0 & & & & \\ a_1a_2 & a_1 & 0 & & & & \\ \epsilon a_1a_3 & 0 & a_1 & & & & \\ & & & a_1 & 0 & & \\ & & & a_3 & 1 & & \\ & & & & & a_1 & 0 \\ & & & & & a_2 & 1 \end{pmatrix} : a_1>0; a_2,a_3\in\mathbb{R}\right\}$$

where $S=diag(-1,1,-1,-1,-1,1,-1)$ and $D_8=\langle R,S : R^4=S^2=SRSR=1\rangle$

- Can always reduce the structure group dimension by at least one.
- Either we arrive at an $\{e\}$-structure on Σ or (since $\mathfrak{g}^{(1)} = 0$) we arrive at an $\{e\}$-structure on $\Sigma \times G_\Gamma$, where $dim(G_\Gamma) \leq 2$.
- Since $dim(Aut(\mathbf{\Theta})) = dim(M) - rank(\mathbf{\Theta}) \geq 0$, the maximally symmetric model has $\leq$ 9-dim. symmetry.

Theorem 3.3. *The (contact) sym. grp of any generic hyp. eqn has dim. ≤ 9.*

FACT: Upper bound is realized, but not uniquely.

OPEN QUESTION: For these max. sym. models, it turns out that $C(I_F, dM_i)^{(2)}$ are both Frobenius (hence, the eqns are Darboux integrable). However, this is NOT sufficient to characterize these equations! (I have examples of generic hyperbolic structures with 7 and 8-dim. symmetry with the same property.) What conditions suffice?

4 Maximally symmetric models

All torsion coefficients must be *constant*. This leads to (inequivalent) structures which depend on parameters $(\epsilon, m) \in \{\pm 1\} \times (0, 1]$. These structure eqns can be integrated and this leads to a parametrization of the surfaces: $\Sigma : (x, y, z, p, q, v, w) \mapsto J^2$.

$$r = -\frac{1}{3}(\epsilon m w^3 + v^3), \qquad s = -\frac{1}{2}(\epsilon m^2 w^2 - v^2), \qquad t = -(\epsilon m^3 w + v).$$

Write $\alpha = 1 - \epsilon a$, where $a = m^4$.

Theorem 4.1. *Normal forms for contact-equiv. classes of max. sym. generic hyperbolic PDE are:*

$$\alpha = 0 : \qquad rt - s^2 - \frac{t^4}{12} = 0. \qquad (\textit{contact equivalent to } 3rt^3 + 1 = 0)$$

$$\alpha \in (0, 1) \cup (1, 2] : \qquad (2 - \alpha)^2 (2s - t^2)^3 + (1 - \alpha)(3r - 6st + 2t^3)^2 = 0$$

Example 4.2. *Any equation of the form $F(r, s, t) = 0$ admits the symmetries*

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial z}, \qquad X_4 = x\frac{\partial}{\partial z} \qquad X_5 = y\frac{\partial}{\partial z}, \qquad X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}.$$

The equation $3rt^3 + 1 = 0$ admits

$$X_7 = xy\frac{\partial}{\partial z}, \qquad X_8 = 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}, \qquad X_9 = x^2\frac{\partial}{\partial x} + xz\frac{\partial}{\partial z},$$

For $\alpha \neq 0$, we have the symmetries

$$X_7 = y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}, \qquad X_8 = x\frac{\partial}{\partial y} - \frac{1}{2}y^2\frac{\partial}{\partial z}, \qquad X_9 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + \left(xz - \frac{1}{6}y^3\right)\frac{\partial}{\partial z}.$$

NOTE: Independent of α!

Theorem 4.3. *There are exactly two (isomorphism classes of) contact symmetry algebras for maximally symmetric generic hyperbolic equations. These are both contact-equivalent to projectable point symmetry algebras. Abstractly,*

$$\mathfrak{h}_\delta \cong \mathfrak{gl}(2, \mathbb{R}) \ltimes \mathfrak{n}_\delta, \quad \delta = 0, 1$$

where the brackets on the nilradical $\mathfrak{n}_\delta$ are:

$\mathfrak{n}_\delta$:

	e_1	e_2	e_3	e_4	e_5
e_1	$\cdot$	$\cdot$	$\cdot$	e_2	δe_4
e_2		$\cdot$	$\cdot$	$\cdot$	$\cdot$
e_3			$\cdot$	$\cdot$	$\cdot$
e_4				$\cdot$	e_3
e_5					$\cdot$

5 Degenerations

The 9-dim. symmetry algebra $\mathfrak{h}_1$ is in fact a maximal parabolic subalgebra of (the non-compact real form) $\mathfrak{g}_2$. (Delete the long root in the Dynkin diagram.) $\mathfrak{h}_0$ is *not* the other parabolic.

Recall the symmetry algebra in the cases $\alpha \neq 0$ are independent of α. Hence, equation degenerations automatically inherit $\mathfrak{h}_1$ as a symmetry subalgebra. (The symmetry vector fields do not degenerate.)

- $\alpha \to 0$: Get $F = 4(2s - t^2)^3 + (3r - 6st + 2t^3)^2 = 9r^2 + 12t^2(rt - s^2) + 32s^3 - 36rst = 0$. (*) This is a *parabolic* equation and has G_2 symmetry. [Yamaguchi; also alluded to in Cartan's 5-variables paper] The additional 5 symmetries appear to be genuine contact symmetries.

- $\alpha \to 2$: $F = 3r - 6st + 2t^3 = 0$. Here, $\Delta = F_r F_t - \frac{1}{4}F_s^2 = -9(2s - t^2)$ on the equation $F = 0$. It is elliptic / hyperbolic / parabolic if the sign of $2s - t^2$ is $-/+/0$ respectively. In the elliptic / hyperbolic cases (i.e. when $2s - t^2 \neq 0$), the symmetry algebra stays 9-dimensional. In the parabolic case, this yields the system

$$r = \frac{t^3}{3}, \quad s = \frac{t^2}{2}$$

  which has G_2 symmetry (5-variables paper). This is a proper invariant submanifold of (*) under the G_2-action.

- $\alpha \to 1$: get $s = \frac{t^2}{2}$, which is hyperbolic 6-7; the symmetry algebra is now infinite-dimensional since it possesses the symmetry $f(x)\partial_z$.

QUESTIONS:

1. Is the subalgebra $\mathfrak{h}_0$ distinguished in any natural way? (Is it a subalgebra of $\mathfrak{g}_2$? Is it the parabolic of some other semisimple Lie algebra?)
2. Is there a simple, direct proof that the above degenerations have G_2 symmetry? Deeper reason for why this is happening?

For further details, see:

The, D., Contact geometry of hyperbolic equations of generic type. *SIGMA 4* (2008), 058, 52 pp.
http://arxiv.org/abs/0804.1559