THE GEOMETRY OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Contents

1. Introduction to second-order differential equations

1.1. **The space of first jets in the plane.** By the plane $\mathbb{R}^2$ we mean a smooth manifold with a fixed coordinate system (x, y).

Definition. The *space of first jets*, denoted $J^1(\mathbb{R}^2)$, is the set of all one-dimensional subspaces (directions) in the tangent spaces to $\mathbb{R}^2$.

In other words,

$$J^1(\mathbb{R}^2) = \{l_p \mid l_p \text{ is a one-dimensional subspace in } T_p\mathbb{R}^2\}.$$

Let $\pi\colon J^1(\mathbb{R}^2) \to \mathbb{R}^2$ denote the natural projection which takes l_p to the corresponding point p of the plane.

We shall now introduce two local coordinate systems (x, y, z_1) and (x, y, z_2) in $J^1(\mathbb{R}^2)$. To this end, we note that every direction l_p at the point $p = (x, y)$ is generated by a nonzero tangent vector $\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$, $\alpha, \beta \in \mathbb{R}$, which is unique up to a constant factor. Define

$$U_1 = \left\{ l_p = \left\langle \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} \right\rangle \,\middle|\, \alpha \neq 0 \right\},$$
$$U_2 = \left\{ l_p = \left\langle \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} \right\rangle \,\middle|\, \beta \neq 0 \right\}.$$

Then $(x, y, z_1 = \frac{\beta}{\alpha})$ and $(x, y, z_2 = \frac{\alpha}{\beta})$ will be coordinates in U_1 and U_2 respectively. If we identify $T_p\mathbb{R}^2$ with $\mathbb{R}^2$, then U_1 will contain all directions which are not parallel to the y-axis, while U_2 will consist of all those which are not parallel to the x-axis. The transition function from U_1 to U_2 has obviously the form

$$\varphi_{12}\colon (x, y, z_1) \mapsto (x, y, \frac{1}{z_1}).$$

These coordinate charts make $J^1(\mathbb{R}^2)$ into a three-dimensional smooth manifold. In the following, unless otherwise stated, we shall use the local coordinate system (x, y, z_1) in $J^1(\mathbb{R}^2)$, denoting it simply by (x, y, z).

Definition. Let N be a one-dimensional submanifold ($\equiv$ a non-parametrized curve) in the plane. The *prolongation* $N^{(1)}$ of N is a curve in $J^1(\mathbb{R}^2)$ which has the form

$$N^{(1)} = \{T_pN \mid p \in N\}.$$

Example. If N is the graph of some function $y(x)$, that is, if $N = \big\{\big(x, y(x)\big) \mid x \in \mathbb{R}\big\}$, then, in local coordinates, $N^{(1)}$ has the form

$$\big\{\big(x, y(x), y'(x)\big) \mid x \in \mathbb{R}\big\}. \tag{1}$$

The tangent space to the curve (1) at the point $(x, y, z) = \big(x, y(x), y'(x)\big)$ is generated by the vector

$$\frac{\partial}{\partial x} + y'(x)\frac{\partial}{\partial y} + y''(x)\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + y''(x)\frac{\partial}{\partial z}. \tag{2}$$

Note that, irrespective of the function $y(x)$, this tangent space always lies in the two-dimensional subspace generated by the vectors $\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

Definition. The *contact distribution* C on $J^1(\mathbb{R}^2)$ is the family

$$C_q = \langle T_qN^{(1)} \mid N^{(1)} \ni q\rangle$$

of two-dimensional subspaces in the tangent spaces to $J^1(\mathbb{R}^2)$.

From (2) it immediately follows that in local coordinates

$$C_q = \left\langle \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

Notice that the field $\frac{\partial}{\partial z}$ is tangent to the fibers of the projection $\pi\colon J^1(\mathbb{R}^2) \to \mathbb{R}^2$, so that $\ker d_q\pi \subset C_q$ for all $q \in J^1(\mathbb{R}^2)$.

By the definition of the contact distribution, the prolongation $N^{(1)}$ of any curve N in the plane is tangent to the contact distribution. It turns out that the converse is also true.

Lemma 1.1. *If M is a curve in $J^1(\mathbb{R}^2)$ tangent to the contact distribution and such that the projection $\pi_{|M}\colon M \to \mathbb{R}^2$ is nondegenerate, then, viewed locally, M is the prolongation of some curve in the plane.*

Proof. Without loss of generality we may assume that the projection of M onto $\mathbb{R}^2$, locally, is the graph of some function $y(x)$. Then

$$M = \big\{\big(x, y(x), z(x)\big) \mid x \in \mathbb{R}\big\}.$$

But the vector $\frac{\partial}{\partial x} + y' \frac{\partial}{\partial x} + z'(x) \frac{\partial}{\partial x}$ lies in $C_{(x,y(x),z(x))}$ if and only if $z = y'$ and M is the prolongation of the curve $N = \big\{\big(x, y(x)\big) \mid x \in \mathbb{R}\big\}$. $\square$

Now let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an arbitrary diffeomorphism of the plane.

Definition. The diffeomorphism $\varphi^{(1)}\colon J^1(\mathbb{R}^2) \to J^1(\mathbb{R}^2)$ defined by

$$\varphi^{(1)}\colon l_p \mapsto (d_p\varphi)(l_p)$$

is said to be the *prolongation* of φ.

It is easily verified that the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^2) & \xrightarrow{\varphi^{(1)}} & J^1(\mathbb{R}^2) \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\ \mathbb{R}^2 & \xrightarrow{\varphi} & \mathbb{R}^2 \end{array}$$

Lemma 1.2. *For any diffeomorphism φ the prolongation $\varphi^{(1)}$ preserves the contact distribution C (i.e., $d_q\varphi^{(1)}(C_q) = C_{\varphi^{(1)}(q)}$ for all $q \in J^1(\mathbb{R}^2)$).*

Proof. The proof is immediate from the definition of the contact distribution and the fact that $\varphi^{(1)}(N^{(1)}) = \varphi(N)^{(1)}$ for any curve $N \subset \mathbb{R}^2$. $\square$

To find the expression for $\varphi^{(1)}$ in local coordinates, we assume that $\varphi(x, y) = (A(x, y), B(x, y))$ for some smooth functions A and B in the plane. Then $\varphi^{(1)}$ has the form

$$\varphi^{(1)}\colon (x, y, z) \mapsto (A(x, y), B(x, y), C(x, y, z)).$$

If we explicitly write down the condition that the contact distribution is invariant under $\varphi^{(1)}$, we get

$$C(x, y, z) = \frac{B_x + B_y z}{A_x + A_y z},$$

where A_x, A_y, B_x, B_y denote the partial derivatives of A and B with respect to x and y.

1.2. **Second-order equations in the plane.** Let

$$y'' = F(x, y, y') \tag{3}$$

be an arbitrary second-order differential equation solved for the highest derivative. Consider the direction field E in the chart U_1 with coordinates (x, y, z) defined by

$$E_{(x,y,z)} = \left\langle \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + F(x, y, z)\frac{\partial}{\partial z} \right\rangle.$$

Lemma 1.3. *If N is the graph of the function $y(x)$, then $y(x)$ is a solution of equation* (3) *if and only if $N^{(1)}$ is an integral curve of the direction field E.*

Proof. Suppose $y(x)$ is a solution of equation (3). Then $N^{(1)} = \{(x, y(x), y'(x)) \mid x \in \mathbb{R}\}$, and the tangent space to $N^{(1)}$ is given by the vector

$$\frac{\partial}{\partial x} + y'\frac{\partial}{\partial x} + y''(x)\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + F(x, y, y')\frac{\partial}{\partial z}.$$

Now since the coordinate z is equal to y' for all points on $N^{(1)}$, we see that $N^{(1)}$ is an integral manifold for E.

The converse is proved in a similar manner. □

Note that E is contained in the contact distribution and that the projection of E onto $\mathbb{R}^2$ by means of $d\pi$ is always nondegenerate. Any integral curve of the direction field E is therefore a lift of the graph of a solution of equation (3).

Thus, **there is a one-to-one correspondence between the solutions of equation (3) and the integral curves of the direction field E: the graphs of the solutions are projections of the integral curves, while the integral curves are of the graphs of the solutions.**

Let us now consider an arbitrary direction field E in $J^1(\mathbb{R}^2)$ such that

(i) E is contained in the contact distribution;
(ii) at each point $q \in J^1(\mathbb{R}^2)$, E_q does not coincide with the *vertical* field $V_q = \ker d_q\pi$.

Then E may be regarded as a second-order equation whose solutions are curves in the plane: a curve $N \subset \mathbb{R}^2$ is by definition a solution of this equation if $N^{(1)}$ is an integral curve of the field E. Note that now solutions need no longer be graphs of functions $y(x)$.

Thus, geometrically, the second-order differential equations (solved for the highest derivative) may be identified with the direction fields in $J^1(\mathbb{R}^2)$ satisfying conditions (i) and (ii).

Consider how (local) diffeomorphisms of the plane (i.e., "changes of variables" x, y) act on these direction fields. Let E be a direction field contained in the contact distribution and corresponding to some second-order differential equation in the plane, and let φ be a local diffeomorphism of the plane. Then φ extends to some local transformation $\varphi^{(1)}$ of $J^1(\mathbb{R}^2)$, which acts on the field E in the following way

$$\left(\varphi^{(1)}.E\right)_{\varphi^{(1)}(q)} = d_q\varphi^{(1)}(E_q)$$

for all $q \in J^1(\mathbb{R}^2)$. It is easy to show that the resulting direction field still satisfies conditions (i) and (ii) and defines a new second-order differential equation.

Let $E_1,\ E_2$ be two directions fields contained in the contact distribution and corresponding to two second-order differential equations. We shall say that these equations are *(locally) equivalent* if there exists a (local) diffeomorphism φ of the plane such that its prolongation $\varphi^{(1)}$ takes E_1 into E_2.

1.3. **Pairs of direction fields in space.** Let V be the vertical direction field in $J^1(\mathbb{R}^2)$ defined by $V_q = \ker d_q\pi$ for all $q \in J^1(\mathbb{R}^2)$. As we mentioned before, the direction field V is contained in the tangent distribution C. In accordance with the previous subsection, a second-order differential equation can be considered as another direction field E, contained in C, such that

$$C_q = V_q \oplus E_q \quad \text{for all } q \in J^1(\mathbb{R}^2).$$

Conversely, suppose we are given two arbitrary direction fields E_1 and E_2 in space that differ at each point of the space. Then E_1 and E_2 define a two-dimensional distribution, denoted $\widetilde{C} = E_1 \oplus E_2$. We say that a pair of direction fields E_1 and E_2 is *nondegenerate* if the corresponding distribution $\widetilde{C}$ is not completely integrable.

We shall now show that the problem of local classification of pairs of direction fields in space is equivalent to local classification of second-order differential equations up to diffeomorphisms of the plane. We shall need the following theorem.

Theorem 1. *If $\widetilde{C}$ is a two-dimensional, not completely integrable distribution in space and $\widetilde{V}$ is an arbitrary direction field contained in $\widetilde{C}$, then the pair $(\widetilde{C}, \widetilde{V})$ is locally equivalent to the pair (C, V) in the space*

$J^1(\mathbb{R}^2)$, that is, there exists a local diffeomorphism $\varphi\colon \mathbb{R}^3 \to J^1(\mathbb{R}^2)$ such that $d\varphi(\widetilde{C}) = C$ and $d\varphi(\widetilde{V}) = V$.

Proof. See, for example, [18, Chapter 11], [8]. □

Now let E_1, E_2 be a nondegenerate pair of direction fields in space and let $\widetilde{C} = E_1 \oplus E_2$. If $\varphi\colon \mathbb{R}^3 \to J^1(\mathbb{R}^2)$ is a local diffeomorphism establishing the equivalence of the pairs $(\widetilde{C}, E_1)$ and (C, V), then the direction field $E = d\varphi(E_2)$ determines a second-order equation in the plane.

Theorem 2. *Two nondegenerate pairs of direction fields are locally equivalent if and only if so are the corresponding second-order differential equations in the plane.*

Before proceeding to the proof of Theorem 2, we establish one auxiliary result.

Definition. A transformation of the space $J^1(\mathbb{R}^2)$ is called *contact* if it preserves the contact distribution.

For example, the prolongation $\varphi^{(1)}$ of any diffeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a contact transformation.

Lemma 1.4. *A contact transformation has the form $\varphi^{(1)}$, where φ is a diffeomorphism of the plane, if and only if it preserves the vertical direction field V.*

Proof. The necessity of this condition is obvious. Assume that ψ is a contact transformation preserving V. Then ψ defines, in the obvious way, a diffeomorphism φ of the plane. Straightforward computation shows that ψ is uniquely determined by φ and hence coincides with $\varphi^{(1)}$. □

Proof of Theorem 2. Let (E_1, E_2) and (E_1', E_2') be two nondegenerate pairs of direction fields in space, and let E and E' be the direction fields, contained in C, that determine the corresponding differential equations. It is clear that these two pairs of direction fields are locally equivalent if and only if so are the pairs (V, E) and (V, E').

If ψ is a local diffeomorphism of the space $J^1(\mathbb{R}^2)$ carrying the pair (V, E) into the pair (V, E'), then ψ preserves the contact distribution $C = V \oplus E = V \oplus E'$ and, at the same time, leaves invariant the direction field V. Hence there exists a local diffeomorphism φ of the plane such that $\psi = \varphi^{(1)}$. Then, by definition, φ establishes the equivalence of the corresponding second-order differential equations.

The converse is obvious. □

1.4. **Duality.** Suppose that a direction field E, contained in the contact distribution of $J^1(\mathbb{R}^2)$, determines a second-order equation in the plane. Then, by Theorem 1, there exists a local diffeomorphism φ of the space $J^1(\mathbb{R}^2)$ that takes the pair of distributions (C, E) into the pair (C, V), so that the direction field E becomes vertical, while the vertical direction field V is transformed in some direction field E' lying in the contact distribution and satisfying conditions (i), (ii) of subsection 1.2.

Definition. The second-order differential equation defined by the field E' is said to be *dual* to the equation corresponding to E.

Example. Consider the differential equation $y'' = 0$. The corresponding direction field is

$$E = \left\langle \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \right\rangle.$$

The Legendre transformation

$$(x, y, z) \mapsto (z, y - xz, -x)$$

is a contact transformation, and it takes the distribution E into the vertical distribution V, while the latter is transformed into E. Thus the equation $y'' = 0$ coincides with its own dual equation.

Note that in general there exist several contact transformations carrying E into V. Moreover, if ψ_1 and ψ_2 are two such transformations, then the mapping $\psi_1 \circ \psi_2^{-1}$ preserves the pair (C, V) and hence has the form $\varphi^{(1)}$ for some local diffeomorphism φ of the plane. Therefore the dual equation is well defined only up to equivalence by local diffeomorphisms of the plane.

In the language of pairs of direction fields, the transition to the dual equation is equivalent to interchanging the positions of the direction fields of the pair. A description of the dual equation can also be given in terms of the solutions of the initial second-order equation when they are written in the form of a two-parameter family $F(x, y, u, v) = 0$, where u, v are parameters (say, $u = y(0)$ and $v = y'(0)$). If we now consider x and y as parameters, u as an independent and v as a dependent variable, we obtain a two-parameter family of curves in the plane, which coincides with the family of solutions of the dual second-order equation. (See [1] for details.)

2. Cartan connections

In this section we give only basic definitions and list the results we shall use later. For more detail see [14].

2.1. **Definitions.** Let $\overline{G}$ be a finite-dimensional Lie group, G a closed subgroup of $\overline{G}$, and $M_0 = \overline{G}/G$ the corresponding homogeneous space of the Lie group $\overline{G}$. Further, let $\bar{\mathfrak{g}}$ be the Lie algebra of $\overline{G}$, and let $\mathfrak{g}$ be the subalgebra of $\bar{\mathfrak{g}}$ corresponding to G. We identify $\bar{\mathfrak{g}}$ with the tangent space $T_e\overline{G}$.

Suppose M is a smooth manifold of dimension $\dim M_0 = \operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g}$, and $\pi\colon P \to M$ is a principal fiber bundle with structural group G over M. For $X \in \mathfrak{g}$, let X^* denote the fundamental vector field on P, corresponding to X; for $g \in G$ let R_g denote the right translation of P by the element g:

$$R_g\colon P \to P,\ p \mapsto pg, \quad p \in P.$$

Definition. A *Cartan connection* in the principal fiber bundle P is a 1-form ω on P with values in $\bar{\mathfrak{g}}$ such that

1°. $\omega(X^*) = X$ for all $X \in \mathfrak{g}$;
2°. $R_g^*\omega = (\operatorname{Ad} g^{-1})\omega$ for all $g \in G$;
3°. $\omega_p\colon T_pP \to \bar{\mathfrak{g}}$ is a vector space isomorphism for all $p \in P$.

Example. Suppose $M = M_0$; then $P = \overline{G}$ may be considered as a principal fiber bundle over M with structural group G. We define ω to be the canonical left-invariant Maurer–Cartan form on $\overline{G}$. It is easily verified that ω satisfies the conditions 1°–3° in the above definition, and hence ω is a Cartan connection. We shall call it the *canonical Cartan connection of the homogeneous space* $(\overline{G}, M_0)$.

2.2. **Coordinate notation.** If s_α is the section of $\pi\colon P \to M$ defined on an open subset $U_\alpha \subset M$, we can identify $\pi^{-1}(U_\alpha)$ with $U_\alpha \times G$ as follows:

$$\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha),\ (x,g) \mapsto s_\alpha(x)g.$$

Given some other section $s_\beta\colon U_\beta \to P$ such that the intersection $U_\alpha \cap U_\beta$ is non-empty, we can consider the *transition function* $\psi_{\alpha\beta}$, which is a function on $U_\alpha \cap U_\beta$ with values in G, uniquely defined by

$$\phi_\beta \circ \phi_\alpha^{-1}\colon \big(U_\alpha \cap U_\beta\big) \times G \to \big(U_\alpha \cap U_\beta\big) \times G,\ (x,g) \mapsto (x, \psi_{\alpha\beta}g).$$

Any principal fiber bundle P is uniquely determined by a covering $\{U_\alpha\}_{\alpha\in I}$ of the manifold M with given transition functions $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ satisfying the obvious condition: $\psi_{\alpha\gamma} = \psi_{\alpha\beta}\psi_{\beta\gamma}$ for all $\alpha, \beta, \gamma \in I$.

Let ω be a Cartan connection on P. The section $s_\alpha\colon U_\alpha \to P$ determines the 1-form $\omega_\alpha = s_\alpha^*\omega$ on U_α with values in $\bar{\mathfrak{g}}$. It is easy to verify that for any other section $s_\beta\colon U_\beta \to P$ and the corresponding form

$\omega_\beta = s_\beta^* \omega$, the following relation holds on the intersection $U_\alpha \cap U_\beta$:

$$\omega_\beta = \mathrm{Ad}\left(\psi_{\alpha\beta}^{-1}\right)\omega_\alpha + \psi_{\alpha\beta}^*\theta, \tag{4}$$

where θ is the canonical left-invariant Maurer–Cartan form on G.

Remark 2.1. Formula (4) is completely identical with the corresponding formula for ordinary connections on principal fiber bundles. The difference is that in the latter case the forms $\omega_\alpha, \omega_\beta$ have their values in $\mathfrak{g}$, not in $\bar{\mathfrak{g}}$.

Conversely, suppose that on every submanifold U_α, $\alpha \in I$, there is given a 1-form ω_α with values in $\bar{\mathfrak{g}}$, and

1°. for any $\alpha, \beta \in I$, we have (4);

2°. for any $\alpha \in I$ and any point $x \in U_\alpha$, the mapping

$$T_x M \to \bar{\mathfrak{g}}/\mathfrak{g}, \ v \mapsto (\omega_\alpha)_x(v) + \mathfrak{g}$$

is a vector space isomorphism.

It is not hard to show that there exists a unique Cartan connection ω on P such that $\omega_\alpha = s_\alpha^*\omega$ for all $\alpha \in I$.

2.3. **Cartan connections and ordinary connections.** Let ω be a Cartan connection on P. Consider the associated fiber bundle $\overline{P} = P \times_G \overline{G}$, where G acts on $\overline{G}$ by left shifts. Denote by $[(p,g)]$ the image of the element $(p,g) \in P \times \overline{G}$ under the natural projection $P \times \overline{G} \to \overline{P}$. The right action

$$[(p,g_1)]g_2 = [(p,g_1g_2)], \quad [(p,g_1)] \in \overline{P},\ g_2 \in \overline{G},$$

of $\overline{G}$ on $\overline{P}$ provides the fiber bundle $\overline{P}$ with a natural structure of principal fiber bundle over M with structural group $\overline{G}$. We shall identify P with a subbundle in $\overline{P}$ by means of the embedding

$$P \hookrightarrow \overline{P},\ p \mapsto [(p,e)].$$

It can be easily shown that there exists a unique connection form $\bar{\omega}$ on $\overline{P}$ such that $\omega = \bar{\omega}|_P$. Conversely, if $\bar{\omega}$ is a connection form on $\overline{P}$ such that

$$\ker \bar{\omega}_p \oplus T_p P = T_p \overline{P} \quad \text{for all } p \in P, \tag{5}$$

then the 1-form $\omega = \bar{\omega}|_P$ with values in $\bar{\mathfrak{g}}$ is a Cartan connection on P. Thus we obtain the following result:

Lemma 2.1. *Cartan connections on P are in one-to-one correspondence with the ordinary connections on $\overline{P}$ whose connection forms satisfy condition (5).*

Remarks.

1. Condition (5) is equivalent to the condition that the form $\omega = \bar{\omega}|_P$ defines an absolute parallelism on P.

2. If $\{U_\alpha\}_{\alpha \in I}$ is a covering of M with given sections $s_\alpha \colon U_\alpha \to P$, then these sections may be extended to sections $\bar{s}_\alpha \colon U_\alpha \to \overline{P}$, $x \mapsto [s_\alpha(x), e]$, for all $\alpha \in I$, $x \in U_\alpha$, and it is clear that $s_\alpha^* \omega = \bar{s}_\alpha^* \bar{\omega}$. Thus, in terms of "coordinates," the connection form $\bar{\omega}$ is defined by the same family $\{\omega_\alpha\}_{\alpha \in I}$ of forms as the Cartan connections ω.

2.4. **Developments.** In this subsection, by L_g and R_g we denote the left and the right shifts of the Lie group $\overline{G}$ by the element $g \in \overline{G}$.

Consider the mapping

$$\gamma \colon \overline{P} \to \overline{G}/G, \quad [p, g] \mapsto g^{-1}G.$$

It is well defined, because for $h \in G$ we have

$$\gamma([ph^{-1}, hg]) = (hg)^{-1}G = g^{-1}G, \quad [p, g] \in \overline{P}.$$

It follows immediately from the definition that $\gamma(\bar{p}g) = g^{-1}.\gamma(\bar{p})$ for all $\bar{p} \in \overline{P}, g \in \overline{G}$.

Let $x(t)$ be an arbitrary curve in M, and $u(t)$ its horizontal lift into the fiber bundle $\overline{P}$ by means of the connection $\bar{\omega}$ (i.e., $\bar{\pi}(u(t)) = x(t)$ and $\bar{\omega}_{u(t)}(\dot{u}(t)) = 0$ for all t). Note that $u(t)$ is unique up to the right action of $\overline{G}$ on $\overline{P}$.

Definition. A *development of the curve $x(t)$ on the manifold M* is a curve of the form $\tilde{x}(t) = \gamma(u(t))$ in the homogeneous space $\overline{G}/G$.

Hence the development of a curve in M is defined uniquely up to the action of $\overline{G}$ on $\overline{G}/G$.

Example. Suppose $(\overline{G}, \overline{G}/G)$ is the projective space. Then a curve $x(t)$ in M is called a geodesic if its development is a segment of a straight line. A similar definition is relevant in any homogeneous space that can be included into the projective space, for example, in affine and Euclidean spaces.

The above definition of development can be easily formulated for one-dimensional submanifolds of M diffeomorphic to the line (or, locally, for any one-dimensional submanifolds). In particular, in the above example one can speak of one-dimensional geodesic submanifolds.

We shall need the following lemma.

Lemma 2.2. *Let $s_\alpha \colon U_\alpha \to P$ be a section of the principal fiber bundle P, and let $\omega_\alpha = s^*\omega$. Assume that a smooth curve $x(t)$ lies in U_α and consider the curve $X(t) = \omega_\alpha(\dot{x}(t))$ in the Lie algebra $\bar{\mathfrak{g}}$. Then up to the* *action of* $\overline{G}$ *on* M_0 *the development of* $x(t)$ *has the form* $\tilde{x}(t) = h(t)G$, *where* $h(t)$ *is the curve in* $\overline{G}$ *completely determined by the differential equation*

$$\dot{h}(t) = d_e L_{h(t)}(X(t)), \quad h(0) = e.$$

Proof. Let $\bar{\omega}$ be the connection form on the principal $\overline{G}$-bundle $\overline{P}$ corresponding to the Cartan connection ω, and let $\bar{s}_\alpha \colon U_\alpha \to P$ be the section generated by the section s_α. Then, as was pointed out before, $\omega_\alpha = s_\alpha^* \omega = \bar{s}_\alpha^* \bar{\omega}$.

Thanks to the section $\bar{s}_\alpha$, we may assume, without loss of generality, that $\overline{P} = U_\alpha \times \overline{G}$ is a trivial fiber bundle, and $\bar{s}_\alpha(x) = (x, e)$ for all $x \in U_\alpha$. Then the horizontal lift $u(t)$ of $x(t)$ with initial condition $u(0) = (x(0), e)$ has the form $u(t) = (x(t), g(t))$, where $g(t)$ is a curve in $\overline{G}$ such that $\bar{\omega}(\dot{u}(t)) = 0$ for all t. But

$$\begin{aligned}\bar{\omega}_{u(t)}(\dot{u}(t)) = \bar{\omega}_{(x,g)}(\dot{x}(t), \dot{g}(t)) &= \\ = \operatorname{Ad} g(t)^{-1} \circ \bar{\omega}_{(x,e)}\bigl(\dot{x}(t), d_e R_g^{-1}(\dot{g}(t))\bigr) &= \\ = \operatorname{Ad} g(t)^{-1}\bigl(\omega_\alpha(\dot{x}(t)) + d_e R_g^{-1}(\dot{g}(t))\bigr) &= 0.\end{aligned}$$

Therefore, $g(t)$ is uniquely determined by the equation

$$\dot{g}(t) = -d_e R_{g(t)}(X(t)), \quad g(0) = e.$$

The development of the curve $x(t)$ has the form $\tilde{x}(t) = h(t)G$, where $h(t) = g(t)^{-1}$. Denote by $\tau \colon \overline{G} \to \overline{G}$ the inversion in the Lie group $\overline{G}$. Then

$$\begin{aligned}\dot{h}(t) = d_{g(t)}\tau(\dot{g}(t)) = d_{g(t)}\tau \circ d_e R_{g(t)}(-X(t)) &= \\ = d_e\bigl(\tau \circ R_{g(t)}\bigr)(-X(t)) = d_e\bigl(L_{g(t)^{-1}} \circ \tau\bigr)(-X(t)) &= \\ = d_e L_{h(t)}(X(t)),&\end{aligned}$$

as was to be proved. □

2.5. **Curvature.** By the *curvature form* of a Cartan connection ω on P we understand the 2-form

$$\Omega = d\omega + 1/2[\omega, \omega]$$

on P with values in $\bar{\mathfrak{g}}$.

If $\bar{\omega}$ is the curvature form on $\overline{P}$ corresponding to a Cartan connection ω and $\overline{\Omega}$ is the curvature of $\bar{\omega}$, then the form Ω is precisely the restriction of $\overline{\Omega}$ to P.

The definitions of Cartan connection and of the form Ω immediately imply that

Lemma 2.3. *The curvature form* Ω *satisfies the following conditions:*

1°. $R_g^*\Omega = (\mathrm{Ad}\, g^{-1})\Omega$ *for all* $g \in G$*;*
2°. $\Omega(X, Y) = 0$ *if at least one of the tangent vectors* X, Y *is vertical.*
3°. (structure equation) $d\Omega = [d\omega, \omega]$*.*

Example. It is known that the canonical Maurer–Cartan form θ on a Lie group satisfies the equation $d\theta + 1/2[\theta, \theta] = 0$. Hence the canonical Cartan connection ω of an arbitrary homogeneous space $(\overline{G}, M_0)$ has zero curvature. Conversely, if some Cartan connection on the principal fiber bundle P has zero curvature, using the Frobenius theorem, it is easy to show that it is locally isomorphic to the canonical Cartan connection.

We say that a Cartan connection ω is a *connection without torsion* (or *connections of zero torsion*), if $\Omega(X, Y) \in \mathfrak{g}$ for all tangent vectors X, Y.

Remark 2.2. The *torsion of a Cartan connection* ω may be defined as the 2-form T on P with values in $\bar{\mathfrak{g}}/\mathfrak{g}$ resulting from the curvature form Ω on passing to the quotient by $\mathfrak{g}$. In this case expression "connection of zero curvature" has the literal meaning.

2.6. **Structure function.** Given an element $X \in \bar{\mathfrak{g}}$, let X^* be the vector field on P defined by

$$(X^*)_p = (\omega_p)^{-1}(X).$$

This definition is in agreement with the definition of the fundamental field X^* for $X \in \mathfrak{g}$. Moreover, the field X^* is vertical if and only if $X^* \in \mathfrak{g}$. Consider how X^* changes along the fibers of the fiber bundle $\pi\colon P \to G$. From the definition of Cartan connection it follows that

$$\omega_{pg} \circ d_p R_g = \mathrm{Ad}\, g^{-1} \circ \omega_p \quad \forall p \in P,\ g \in G.$$

Applying both sides of this equality to the tangent vector X_p^* for some $X \in \bar{\mathfrak{g}}$, we get

$$\omega_{pg}\big(d_p R_g(X_p^*)\big) = \mathrm{Ad}\, g^{-1}(X).$$

Hence

$$\big(\mathrm{Ad}\, g^{-1}(X)\big)^*_{pg} = d_p R_g(X_p^*) \quad \forall p \in P,\ g \in G$$

or, briefly, $dR_g(X^*) = (\mathrm{Ad}\, g^{-1}(X))^*$ $\forall g \in G$. Thus the right shift R_g transforms the fundamental vector field X^* into the fundamental vector field corresponding to the vector $(\mathrm{Ad}\, g^{-1})X$.

Definition. The *structure function* of a Cartan connection ω is defined as a function $c\colon P \to \mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ such that

$$c(p)\colon (X + \mathfrak{g}, Y + \mathfrak{g}) \mapsto \Omega_p(X^*, Y^*).$$

From Lemma 2.3, 2°, we see immediately that $c(p)$ is well defined for all $p \in P$.

Recall that the group G acts on the space $\mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ in the following natural way:

$$(g.\varphi)(X+\mathfrak{g}, Y+\mathfrak{g}) = (\operatorname{Ad} g)\varphi\big(\operatorname{Ad} g^{-1}(X)+\mathfrak{g}, \operatorname{Ad} g^{-1}(Y)+\mathfrak{g}\big)$$

for all $X, Y \in \bar{\mathfrak{g}}$.

Lemma 2.4. *$c(pg) = (g^{-1}).c(p)$ for all $p \in P$, $g \in G$.*

Proof. For $X, Y \in \bar{\mathfrak{g}}$, we have

$$\begin{aligned} c(pg)(X+\mathfrak{g}, Y+\mathfrak{g}) &= \Omega_{pg}(X^*_{pg}, Y^*_{pg}) = \\ &= \Omega_{pg}\big(d_pR_g(\operatorname{Ad} g(X)^*_p), d_pR_g(\operatorname{Ad} g(Y)^*_p)\big) = \\ &= (R_g^*\Omega)_p\big(\operatorname{Ad} g(X)^*_p, \operatorname{Ad} g(Y)^*_p\big) = \\ &= \operatorname{Ad} g^{-1}\big(\Omega_p(\operatorname{Ad} g(X)^*_p, \operatorname{Ad} g(Y)^*_p)\big) = \big((g^{-1}).c(p)\big)(X+\mathfrak{g}, Y+\mathfrak{g}), \end{aligned}$$

as was to be proved. □

We fix a basis $\{e_1, \dots, e_{n+m}\}$ of the Lie algebra $\bar{\mathfrak{g}}$ such that $\{e_{n+1}, \dots, e_{n+m}\}$ form a basis of the subalgebra $\mathfrak{g}$. Then every element $\varphi \in \mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ is defined by a set of structure constants c^k_{ij} ($1 \leqslant i, j \leqslant n$, $1 \leqslant k \leqslant n+m$), $c^k_{ij} = -c^k_{ji}$, where

$$\varphi(e_i+\mathfrak{g}, e_j+\mathfrak{g}) = \sum_{k=1}^{n+m} c^k_{ij} e_k.$$

The structure function $c\colon P \to \mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ defines a set of functions $c^k_{ij}(p)$. These functions may be found from the decomposition of the curvature tensor Ω in terms of the components of the form ω. Indeed, if we write the forms Ω and ω in the form

$$\Omega = \sum_{k=1}^{n+m} \Omega^k e_k, \quad \omega = \sum_{i=1}^{n+m} \omega^i e_i,$$

then the forms $\omega^1, \dots, \omega^{n+m}$ constitute a basis for the module of differential forms on P, and hence the forms Ω^k, ($k = 1, \dots, n+m$) can be uniquely expressed in the form

$$\Omega^k = \sum_{1 \leqslant i < j \leqslant n+m} c^k_{ij} \omega^i \wedge \omega^j \tag{6}$$

for some smooth functions c^k_{ij} on P. By Lemma 2.3, 2°, we have $c^k_{ij} = 0$ whenever $i > n$ or $j < n$. Applying both sides of (6) to the pairs of the form (e^*_i, e^*_j), $i, j = 1, \dots, n$, we get at once that $c^k_{ij} = c^k_{ij}(p)$ are precisely the coordinates of the structure function c.

3. Cartan connections and pairs of direction fields

Suppose $\overline{G} = SL(3,\mathbb{R})$, G is the group of all upper-triangular matrices in $\overline{G}$, and $M_0 = \overline{G}/G$. Then the homogeneous space $(\overline{G}, M_0)$ admits the following two interpretations:

1) $M_0 = J^1(\mathbb{R}P^2)$, the action of $\overline{G}$ is the natural lifting of the standard action of $SL(3,\mathbb{R})$ on $\mathbb{R}P^2$;
2) M_0 is the set of all flags
$$V_1 \subset V_2 \subset \mathbb{R}^3 \quad (\dim V_i = i,\ i = 1,2),$$
the action of $\overline{G}$ is generated by the natural action of $SL(3,\mathbb{R})$ on $\mathbb{R}^3$.

The corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras has the form
$$\bar{\mathfrak{g}} = \mathfrak{sl}(3,\mathbb{R}), \quad \mathfrak{g} = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \middle| \ x_{11} + x_{22} + x_{33} = 0 \right\}.$$
Any differential form ω with values in $\bar{\mathfrak{g}}$ can be uniquely written in the form
$$\omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix}, \quad \omega_{11} + \omega_{22} + \omega_{33} = 0,$$
where ω_{ij} are usual differential forms.

Let $\pi\colon P \to \mathbb{R}^3$ be an arbitrary principal fiber bundle with structural group G, and let ω be a Cartan connection on P. We shall now show that ω determines a pair of direction fields in $\mathbb{R}^3$. Consider an arbitrary (local) section $s\colon \mathbb{R}^3 \to P$ and let
$$E_1 = \langle s^*\omega_{21}, s^*\omega_{31} \rangle^{\perp},$$
$$E_2 = \langle s^*\omega_{31}, s^*\omega_{32} \rangle^{\perp}.$$
Any other section $\tilde{s}$ has the form $\tilde{s} = s\varphi$, where φ is a smooth function from the common domain of s and $\tilde{s}$ to G. Then
$$\tilde{s}^*\omega = (\operatorname{Ad}\varphi^{-1})(s^*\omega) + \varphi^*\theta, \tag{7}$$
where θ is the Maurer–Cartan form on G. Simple calculation shows that the forms $\tilde{s}^*\omega_{21}$ and $\tilde{s}^*\omega_{31}$ are linear combinations of the forms $s^*\omega_{21}$ and $s^*\omega_{31}$. Hence the direction field E_1 is independent of the choice of the section s. The proof that E_2 is also well defined is carried out in a similar manner.

Definition. Let (E_1, E_2) be a pair of direction fields in space. We say that *a Cartan connection* ω *is associated* with the pair (E_1, E_2) if (E_1, E_2) coincides with the pair of direction fields corresponding to ω.

The next theorem is the central result of the present section.

Theorem 3. *If (E_1, E_2) is an arbitrary nondegenerate pair of direction fields in $\mathbb{R}^3$, then up to isomorphism there exist a unique principal fiber bundle $\pi\colon P \to \mathbb{R}^3$ and a unique Cartan connection ω on P satisfying the following conditions:*

(i) ω is associated with the pair (E_1, E_2);

(ii) for all $p \in P$, the structure function c of ω lies in a subspace $W \subset \mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ of the form

$$W = \left\{ e_{21} \wedge e_{31} \mapsto \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{31} \wedge e_{32} \mapsto \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, e_{21} \wedge e_{32} \mapsto 0 \mid a, b, c, d \in \mathbb{R} \right\}.$$

(Here

$$e_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathfrak{g}, \quad e_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \mathfrak{g}, \quad e_{32} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \mathfrak{g}$$

is a basis of the space $\bar{\mathfrak{g}}/\mathfrak{g}$).

Remark 3.1. Condition (ii) is equivalent to the requirement that the curvature form of ω have the form

$$\Omega = \begin{pmatrix} 0 & a\omega_{21} \wedge \omega_{31} & b\omega_{21} \wedge \omega_{31} + c\omega_{31} \wedge \omega_{32} \\ 0 & 0 & d\omega_{31} \wedge \omega_{32} \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 3.2. The space W is invariant under the action of the Lie group G on $\mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$. Indeed, if

$$g = \begin{pmatrix} x & t & v \\ 0 & y & u \\ 0 & 0 & z \end{pmatrix} \in G, \quad xyz = 1,$$

and $\varphi \in W$ has the form

$$\varphi\colon e_{21} \wedge e_{31} \mapsto \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_{31} \wedge e_{32} \mapsto \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, e_{21} \wedge e_{32} \mapsto 0,$$

then, by straightforward computation,

$$\begin{aligned} g.\varphi\colon \quad & e_{21} \wedge e_{31} \mapsto \frac{x^3}{y^2 z^2} \begin{pmatrix} 0 & az & -au+by \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ & e_{31} \wedge e_{32} \mapsto \frac{xy}{z^3} \begin{pmatrix} 0 & 0 & cx+dt \\ 0 & 0 & dy \\ 0 & 0 & 0 \end{pmatrix}, \\ & e_{21} \wedge e_{32} \mapsto 0. \end{aligned} \tag{8}$$

In particular, $g.\varphi \in W$.

Proof. We shall first consider the local case and then the global one. According to Theorem 1, we can always assume that locally $P = \mathbb{R}^3 \times G$

is a trivial principal fiber bundle and that the direction fields E_1 and E_2 have the form

$$E_1 = \left\langle \frac{\partial}{\partial z} \right\rangle, \quad E_2 = \left\langle \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x,y,z)\frac{\partial}{\partial z} \right\rangle,$$

where f is a smooth function on $\mathbb{R}^3$.

Let $s\colon \mathbb{R}^3 \to P$, $a \mapsto (a, e)$ be the trivial section of the fiber bundle π, and $\tilde{\omega} = s^*\omega$ a form on $\mathbb{R}^3$ with values in $\bar{\mathfrak{g}}$. Note that the form $\tilde{\omega}$ uniquely determines the form ω. Indeed, for points of the form $p = (a, e) \in P$ $(a \in \mathbb{R})$, we have

$$(\omega_p)|_{T_a\mathbb{R}^3} = \tilde{\omega}_a, \quad (\omega_p)|_{T_eG} = \mathrm{id}_{\mathfrak{g}},$$

and from the definition of Cartan connection it follows that for any point $p = (a, g)$,

$$\omega_p = (\operatorname{Ad} g) \circ \omega_{(a,e)} \circ d_{(a,g)}R_{g^{-1}}.$$

Let $\widetilde{\Omega} = s^*\Omega$. Then, in just the same way, the form $\widetilde{\Omega}$ uniquely determines the curvature form Ω.

Condition (i) on the Cartan connection ω is equivalent to the following conditions:

$$\begin{aligned}
(\tilde{i}) \qquad & \tilde{\omega}_{21} = \alpha dx + \delta(dy - z\,dx);\\
& \tilde{\omega}_{31} = \gamma(dy - z\,dx);\\
& \tilde{\omega}_{32} = \lambda(dy - z\,dx) + \mu(dz - f\,dx).
\end{aligned}$$

Since the subspace $W \subset \mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ is G-invariant, condition (2) needs to be verified only for the points $p = s(a)$ for all $a \in \mathbb{R}^3$. Thus condition (ii) is equivalent to the condition

$$(\widetilde{ii}) \qquad \widetilde{\Omega} = \begin{pmatrix} 0 & \tilde{a}\omega_{21} \wedge \omega_{31} & \tilde{b}\omega_{21} \wedge \omega_{31} + \tilde{c}\omega_{31} \wedge \omega_{32} \\ 0 & 0 & \tilde{d}\omega_{31} \wedge \omega_{32} \\ 0 & 0 & 0 \end{pmatrix}.$$

Show that there is a unique connection ω on P satisfying these conditions.

Note that the identification $P \equiv \mathbb{R}^3 \times G$ is not canonical and that all such identification are in one-to-one correspondence with the sections $\tilde{s}\colon \mathbb{R}^3 \to P$ of the form $s' = s\varphi$, where $\varphi\colon \mathbb{R}^3 \to G$ is an arbitrary smooth function. Under an identification like this, the form $\tilde{\omega}$ becomes

$$\tilde{\omega}' = (s')^*\omega = (\operatorname{Ad}\varphi^{-1})\tilde{\omega} + \varphi^*\theta,$$

where θ is the canonical Maurer–Cartan form on G.

With a suitable choice of s, we can always ensure that

$$\begin{aligned}\tilde\omega_{21} &= dx,\\ \tilde\omega_{31} &= dy - z\,dx,\\ \tilde\omega_{32} &= k(dz - f\,dx).\end{aligned}$$

This determines s uniquely up to a transformation of the form $s \mapsto s\varphi$, where $\varphi\colon \mathbb{R}^3 \to G_1$ is a transition function with values in the subgroup

$$G \supset G_1 = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle|\; x \in \mathbb{R} \right\}.$$

Using the condition $\widetilde\Omega_{31} = 0$, we find that $k = 1$ and $\tilde\omega_{33} - \tilde\omega_{11} = u(dy - z\,dx)$ for some smooth function u on $\mathbb{R}^3$.

The condition $\widetilde\Omega_{21} = 0$ implies that

$$\begin{aligned}\tilde\omega_{22} - \tilde\omega_{11} =& v\,dx + t(dy - z\,dx),\\ \tilde\omega_{23} =& t\,dx + w(dy - z\,dx),\end{aligned}$$

where $v, w, t \in C^\infty(\mathbb{R}^3)$. By a suitable choice of the section s, the number t can always be made equal to zero, and thus s is uniquely defined.

Now from the condition $\widetilde\Omega_{32} = 0$ we obtain that $v = \frac{\partial f}{\partial z}$ and

$$\tilde\omega_{12} = \frac{\partial f}{\partial y} dx - u(dz - f\,dx) + \lambda(dy - z\,dx), \quad \lambda \in C^\infty(\mathbb{R}^3).$$

Furthermore, the equality $\widetilde\Omega_{22} + \widetilde\Omega_{33} - 2\widetilde\Omega_{11} = 0$ implies that $u = -\frac{1}{2}\frac{\partial^2 f}{\partial z^2}$ and that

$$\tilde\omega_{13} = \lambda dx - \frac{1}{3}\frac{\partial^2 f}{\partial y \partial z} dx - \frac{1}{6} d\left(\frac{\partial^2 f}{\partial z^2}\right) + \mu(dy - z\,dx), \quad \mu \in C^\infty(\mathbb{R}^3).$$

Similarly, from $\widetilde\Omega_{22} - \widetilde\Omega_{11} = 0$ it follows that

$$\begin{aligned} w &= \frac{1}{6}\frac{\partial^3 f}{\partial z^3},\\ \lambda &= \frac{2}{3}\frac{\partial^2 f}{\partial y \partial z} - \frac{1}{6}\frac{d}{dx}\left(\frac{\partial^2 f}{\partial z^2}\right),\end{aligned}$$

where $\frac{d}{dx}$ denotes the vector field $\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f\frac{\partial}{\partial z}$.

Thus, if

$$\widetilde\Omega = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

then

$$\tilde\omega_{21} = dx, \quad \tilde\omega_{31} = dy - z\,dx, \quad \tilde\omega_{32} = dz - f\,dx,$$

$$\tilde\omega_{22} - \tilde\omega_{11} = \frac{\partial f}{\partial z}dx, \quad \tilde\omega_{33} - \tilde\omega_{11} = -\frac{1}{2}\frac{\partial^2 f}{\partial z^2}(dy - z\,dx),$$

$$\tilde\omega_{12} = \frac{\partial f}{\partial y}dx + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(dz - f\,dx) + \left(\frac{2}{3}\frac{\partial^2 f}{\partial y \partial z} - \frac{1}{6}\frac{d}{dx}\frac{\partial^2 f}{\partial z^2}\right)(dy - z\,dx),$$

$$\tilde\omega_{13} = \left(\frac{1}{3}\frac{\partial^2 f}{\partial y \partial z} - \frac{1}{6}\frac{d}{dx}\frac{\partial^2 f}{\partial z^2}\right)dx - \frac{1}{6}d\left(\frac{\partial^2 f}{\partial z^2}\right) + \mu(dy - z\,dx),$$

$$\tilde\omega_{23} = \frac{1}{6}\frac{\partial^3 f}{\partial z^3}(dy - z\,dx).$$

The only arbitrary coefficient here is μ.

From (9), taking into account the structure equation of curvature from Lemma 2.3, we can obtain the following conditions:

$$\tilde\omega_{31} \wedge \widetilde\Omega_{23} = 0, \quad \tilde\omega_{31} \wedge \widetilde\Omega_{12} = 0,$$

$$\tilde\omega_{21} \wedge \widetilde\Omega_{12} + \tilde\omega_{31} \wedge \widetilde\Omega_{13} = 0, \quad \tilde\omega_{32} \wedge \widetilde\Omega_{23} - \tilde\omega_{21} \wedge \widetilde\Omega_{12} = 0,$$

whence

$$\begin{aligned}
\widetilde\Omega_{23} &= \tilde d\tilde\omega_{31} \wedge \tilde\omega_{32} + \tilde e\tilde\omega_{21} \wedge \tilde\omega_{31},\\
\widetilde\Omega_{12} &= \tilde a\tilde\omega_{21} \wedge \tilde\omega_{31} + \tilde e\tilde\omega_{31} \wedge \tilde\omega_{32},\\
\widetilde\Omega_{13} &= \tilde b\tilde\omega_{21} \wedge \tilde\omega_{31} + \tilde c\tilde\omega_{31} \wedge \tilde\omega_{32} + \tilde e\tilde\omega_{21} \wedge \tilde\omega_{32}
\end{aligned}$$

for some $\tilde a, \tilde b, \tilde c, \tilde d, \tilde e \in C^\infty(\mathbb{R}^3)$.

Finally, letting

$$\mu = \frac{1}{6}\frac{\partial^3}{\partial y \partial z^2} - \frac{1}{6}\frac{\partial f}{\partial z} \cdot \frac{\partial^3 f}{\partial z^3} - \frac{1}{6}\frac{d}{dx}\frac{\partial^3 f}{\partial z^3},$$

we can ensure that $e = 0$.

We now show that in the global case the fiber bundle $\pi\colon P \to \mathbb{R}^3$ and the Cartan connection ω are, too, unique up to isomorphism. For this purpose we take a covering $\{U_\alpha\}_{\alpha \in I}$ of the three-dimensional space and, using local consideration from the first part of the proof, construct a Cartan connection ω_α on each trivial fiber bundle $\pi_\alpha\colon U_\alpha \times G \to U_\alpha$.

Let s_α, s_β be the trivial sections of the fiber bundles π_α and π_β, respectively, and let $\tilde\omega_\alpha = s_\alpha^*\omega_\alpha$, $\tilde\omega_\beta = s_\beta^*\omega_\beta$. Show that for any two disjoint domains U_α, U_β there exists a unique function

$$\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$$

such that

$$\tilde\omega_\beta = (\operatorname{Ad}\varphi_{\alpha\beta}^{-1})\tilde\omega_\alpha + \varphi_{\alpha\beta}^*\theta. \tag{10}$$

Without loss of generality we can assume that the form $\tilde{\omega}_\alpha$ has the form stated above. The form $\tilde{\omega}_\beta$ on $U_\alpha \cap U_\beta$ also satisfies conditions (1), (2) and, therefore, can be made $\tilde{\omega}_\alpha$ by a suitable choice of section. But, by the above, this section is uniquely defined. Thus there exists a unique function $\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ satisfying (10).

In its turn, the family of functions $\{\varphi_{\alpha\beta}\}_{\alpha,\beta\in I}$ uniquely (up to isomorphism) determines a principal fiber bundle $\pi\colon P \to \mathbb{R}^3$ with structural group G and a Cartan connection ω on P. $\square$

Theorem 3 implies that there is a one-to-one correspondence between second-order differential equations and the Cartan connections ω on principal fiber bundles $\pi\colon P \to J^1(\mathbb{R}^2)$ with structural group G satisfying condition (ii). In particular, two second-order equations are (locally) equivalent if and only if so are the corresponding Cartan connections.

Explicit calculations show that the functions $\tilde{a}$ and $\tilde{d}$ for the constructed form $\tilde{\omega}$ on $\mathbb{R}^3$ have the form

$$\begin{aligned}
\tilde{a} = {} & -f_{yy} + \frac{1}{2} f f_{yzz} + \frac{1}{2} f_y f_{zz} + \frac{2}{3} f_{xyz} - \frac{1}{6} f_{xxzz} - \\
& - \frac{1}{3} z f_{xyzz} - \frac{1}{6} f_x f_{zzz} - \frac{1}{3} f f_{xzzz} + \frac{2}{3} z f_{yyz} - \\
& - \frac{1}{6} z^2 f_{yyzz} - \frac{1}{6} z f_y f_{zzz} - \frac{1}{3} z f f_{yzzz} - \frac{2}{3} f_z f_{yz} + \\
& + \frac{1}{6} f_z f_{xzz} + \frac{1}{6} z f_z f_{yzz} - \frac{1}{6} f^2 f_{zzzz}; \\
\tilde{d} = {} & -\frac{1}{6} f_{zzzz}.
\end{aligned}$$

Further, from the structure equation $d\Omega = [\Omega, \omega]$ it follows that

$$\tilde{b} = \frac{\partial \tilde{a}}{\partial z}, \quad \tilde{c} = -\frac{d}{dx}(\tilde{d}) - 2 f_z \tilde{d}.$$

In particular, the conditions $\tilde{a} = 0$ and $\tilde{d} = 0$ imply that $\tilde{b} = 0$ and $\tilde{c} = 0$, respectively.

From formulas (8) of the action of the Lie group G on the space W, in which the structure function takes its values, it follows that the equalities $\tilde{a} = 0$ and $\tilde{d} = 0$ have invariant nature. To be precise, the subspaces W_1 and W_2 in W defined by the equalities $a = b = 0$ and $c = d = 0$, respectively, are also G-invariant. In particular, the classes of second-order equations described by these equalities are stable under the diffeomorphism group of the plane.

The condition $\tilde{c} = \tilde{d} = 0$ is equivalent to the condition that the corresponding differential equation has the form

$$y'' = A(y')^3 + B(y')^2 + Cy' + D, \tag{11}$$

where A, B, C, D are functions of x, y.

The condition $\tilde{a} = \tilde{b} = 0$ implies that the dual differential equation has the form (11). Later on we shall show that the whole structure function is zero if and only if that the corresponding differential equation is locally equivalent to the equation $y'' = 0$. Thus,

Theorem 4. *A second-order differential equation is equivalent to the equation $y'' = 0$ if and only if both the equation itself and its dual have the form (11).*

The main results of this section are due to È. Cartan [4]. Modern versions of these results from slightly different points of view can be found in [7, 9, 10, 11, 18, 19].

4. Projective connections in the plane and equations of degree 3 with respect to y'

As shown in the previous section, the class of second-order equations of the form

$$y'' = A(y')^3 + B(y')^2 + Cy' + D \tag{12}$$

is stable under the diffeomorphism group of the plane. In this section we show that every equation of this form may be naturally associated with some projective Cartan connection on the plane.

Let $M_0 = \mathbb{R}P^3$, and let $\overline{G} = SL(3, \mathbb{R})$ be the group of projective transformations of M_0. Fix the point $o = [1 : 0 : 0]$ in M_0. The stationary subgroup $G = \overline{G}_o$ has the form

$$G = \left\{ \begin{pmatrix} (\det A)^{-1} & B \\ 0 & A \end{pmatrix} \middle| \; A \in GL(2, \mathbb{R}), B \in \mathrm{Mat}_{1\times 2}(\mathbb{R}) \right\},$$

and the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras has the form

$$\bar{\mathfrak{g}} = \mathfrak{sl}(3, \mathbb{R}), \mathfrak{g} = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \middle| \; A \in \mathfrak{gl}(2, \mathbb{R}), B \in \mathrm{Mat}_{1\times 2}(\mathbb{R}) \right\}.$$

Definition. A *projective connection* in the plane is a Cartan connection with the model $(\overline{G}, M_0)$.

Any differential form ω with values in the Lie algebra $\bar{\mathfrak{g}}$ may be expressed uniquely in the form

$$\omega = \begin{pmatrix} -\omega_1^1 - \omega_2^2 & \omega_1 & \omega_2 \\ \omega^1 & \omega_1^1 & \omega_2^1 \\ \omega^2 & \omega_1^2 & \omega_2^2 \end{pmatrix}.$$

Suppose $\pi\colon P \to \mathbb{R}^2$ is a principal fiber bundle with structural group G, and ω is a Cartan connections on P, and let (x, y) be a coordinate system in the plane.

Lemma 4.1. *Locally, in the neighborhood of any point $(x, y) \in \mathbb{R}^2$ there exists a unique section $s\colon \mathbb{R}^2 \to P$ such that*

$$s^*\omega = \begin{pmatrix} 0 & * & * \\ dx & * & * \\ dy & * & * \end{pmatrix}.$$

Proof. Let s be an arbitrary local section. Then s is defined up to a transformation $s \to s\varphi$, where $\varphi\colon \mathbb{R}^2 \to G$ is a smooth transition function. If

$$(s^*\omega^1) = \alpha\, dx + \beta\, dy,$$

$$(s^*\omega^2) = \gamma\, dx + \delta\, dy,$$

then, choosing φ to have the form

$$\varphi\colon (x, y) \mapsto \begin{pmatrix} \lambda^{-2}(\det A)^{-1} & 0 \\ 0 & \lambda A \end{pmatrix},$$

where

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \lambda = (\det A)^{-1/3},$$

we get

$$s^*\omega = \begin{pmatrix} * & * & * \\ dx & * & * \\ dy & * & * \end{pmatrix}.$$

Then the section s is defined uniquely up to transformations of the form $s \mapsto s\varphi$, where $\varphi\colon \mathbb{R}^2 \to G_1$ takes values in the subgroup

$$G \supset G_1 = \left\{ \begin{pmatrix} 1 & B \\ 0 & E_2 \end{pmatrix} \middle| \; B \in \mathrm{Mat}_{1\times 2}(\mathbb{R}) \right\}.$$

Similarly, it is easy to show that there is a unique function φ such that the section $s\varphi$ has the desired form. $\square$

Recall that a *geodesic* of a projective connection ω is a curve in the plane whose development is a segment of a straight line in $\mathbb{R}P^2$. Similarly, a *geodesic submanifold* is a one-dimensional submanifold in the plane whose development is a segment of a straight line in $\mathbb{R}P^2$ irrespective of the parametrization. Suppose that a submanifold L in the plane is the graph of some function $y(x)$. Then L may be parametrized like this:

$$t \mapsto x(t) = (t, y(t)), \quad t \in \mathbb{R}.$$

We shall now determine when L is a geodesic submanifold.

Consider the section $s\colon \mathbb{R}^2 \to P$ satisfying the conditions of the above lemma, and let $\tilde\omega = s^*\omega$. Define a curve $X(t)$ in the Lie algebra $\bar{\mathfrak{g}}$ as

$$X(t) = \tilde\omega(\dot x(t)) = \begin{pmatrix} 0 & \tilde\omega_1(\dot x) & \tilde\omega_2(\dot x) \\ 1 & \tilde\omega_1^1(\dot x) & \tilde\omega_2^1(\dot x) \\ y' & \tilde\omega_1^2(\dot x) & \tilde\omega_2^2(\dot x) \end{pmatrix}.$$

Then the development of the curve $x(t)$ has the form $\tilde x(t) = h(t).o$, where $h(t)$ is the curve in $SL(3,\mathbb{R})$ such that

$$\dot h(t) = h(t)X(t), \quad h(0) = E_3.$$

Passing to the section $s' = s\varphi$, where

$$\varphi\colon (x,y) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y'(x) & 1 \end{pmatrix},$$

we get

$$\tilde\omega' = (s')^*\omega = (\operatorname{Ad}\varphi^{-1})\tilde\omega + \varphi^{-1}\,d\varphi.$$

Straightforward computation shows that

$$X'(t) = \tilde\omega'(\dot x(t)) = \begin{pmatrix} 0 & * & * \\ 1 & * & * \\ 0 & y'' + (\tilde\omega_1^2 + y'(\tilde\omega_2^2 - \tilde\omega_1^1) - (y')^2\tilde\omega_2^1) & * \end{pmatrix}.$$

It follows that the tangent vector to the development $\tilde x(t)$ at the point o in non-homogeneous coordinates is equal to $(1,0)$. But in $\mathbb{R}P^2$ there exists a unique straight line l through o with tangent vector $(1,0)$, namely $l = \{[x : y : 0]\}$. Hence a necessary and sufficient condition for L to be a geodesic submanifold is that $\tilde x(t) \in l$ for all $t \in \mathbb{R}$. This condition is equivalent to the requirement that $h(t)$ lie in the subgroup $H \subset \overline{G}$ preserving the straight line l:

$$H = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \;\middle|\; g(ae - bd) = 1 \right\}.$$

But this is possible if and only if the curve $X'(t)$ lies in the Lie algebra of H, which has the form

$$\mathfrak{h} = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} \,\middle|\, a + e + g = 0 \right\}.$$

Thus every geodesic submanifold satisfies the equation

$$y'' = -(\tilde{\omega}_1^2 + y'(\tilde{\omega}_2^2 - \tilde{\omega}_1^1) - (y')^2 \tilde{\omega}_2^1)(\dot{x}). \tag{13}$$

Assume that

$$\begin{aligned} \tilde{\omega}_1^1 &= -\tilde{\omega}_2^2 = \alpha_1^1\, dx + \beta_1^1\, dy, \\ \tilde{\omega}_2^1 &= \alpha_2^1\, dx + \beta_2^1\, dy, \\ \tilde{\omega}_1^2 &= \alpha_1^2\, dx + \beta_1^2\, dy. \end{aligned}$$

Then equation (13) can be written explicitly as

$$y'' = \beta_2^1 (y')^3 + (2\beta_1^1 + \alpha_2^1)(y')^2 + (2\alpha_1^1 - \beta_1^2) y' - \alpha_1^2.$$

Any second-order equation of the form (12) can therefore be interpreted as an equation for geodesic submanifolds of some projective connection in the plane.

We now formulate the main result of this section.

Theorem 5. *Given a second-order differential equation of the form (12), there exists a unique (up to isomorphism) principal fiber bundle $\pi\colon P \to \mathbb{R}^2$ with structural group G and a unique Cartan connection ω on P satisfying the following conditions:*

(i) the geodesics of ω satisfy the equation (12);
(ii) the structure function $c\colon P \to \mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$ takes values in the subspace

$$W = \left\{ e_1 \wedge e_2 \mapsto \left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right) a, b \in \mathbb{R} \right\}.$$

(Here $e_1 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right) + \mathfrak{g}$, $e_2 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right) + \mathfrak{g}$ is a basis of the quotient space $\bar{\mathfrak{g}}/\mathfrak{g}$).

Remark 4.1. It is easy to show that the subspace W is invariant under the action of G on $\mathcal{L}(\bar{\mathfrak{g}}/\mathfrak{g} \wedge \bar{\mathfrak{g}}/\mathfrak{g}, \bar{\mathfrak{g}})$.

Proof. The proof of this theorem is very similar to that of Theorem 3, Section 3. We shall only construct the $\bar{\mathfrak{g}}$-valued form $\tilde{\omega} = s^*\omega$ on $\mathbb{R}^2$.

We can assume, without loss of generality, that

$$\begin{aligned} \tilde\omega^1 &= dx,\\ \tilde\omega^2 &= dy,\\ \tilde\omega^1_1 + \tilde\omega^2_2 &= 0. \end{aligned}$$

Suppose

$$\begin{aligned} \tilde\omega^1_1 &= -\tilde\omega^2_2 = \alpha^1_1\,dx + \beta^1_1\,dy,\\ \tilde\omega^1_2 &= \alpha^1_2\,dx + \beta^1_2\,dy,\\ \tilde\omega^2_1 &= \alpha^2_1\,dx + \beta^2_1\,dy. \end{aligned}$$

Then condition (i) together with the equations $\widetilde\Omega^1 = \widetilde\Omega^2 = 0$ gives the following system of equations for the coefficients α^i_j, β^i_j $(i,j = 1,2)$:

$$\begin{cases} \beta^1_2 = A,\\ 2\beta^1_1 + \alpha^1_2 = B,\\ 2\alpha^1_1 - \beta^2_1 = C,\\ -\alpha^2_1 = D; \end{cases} \qquad \begin{cases} \beta^1_1 = \alpha^1_2,\\ \beta^2_1 = -\alpha^1_1. \end{cases}$$

It immediately follows that

$$\begin{aligned} \tilde\omega^1_1 &= -\tilde\omega^2_2 = \frac{1}{3}(C\,dx + B\,dy),\\ \tilde\omega^1_2 &= \frac{1}{3}B\,dx + A\,dy,\\ \tilde\omega^2_1 &= -D\,dx - \frac{1}{3}C\,dy. \end{aligned}$$

The conditions $\widetilde\Omega^i_j = 0$ $(i,j = 1,2)$ uniquely determine the forms

$$\begin{aligned} \tilde\omega_1 &= \left(\frac{\partial D}{\partial y} - \frac{1}{3}\frac{\partial C}{\partial x} - \frac{2}{3}BD + \frac{2}{9}C^2\right)dx + \left(\frac{1}{3}\frac{\partial C}{\partial y} - \frac{1}{3}\frac{\partial B}{\partial x} + \frac{1}{9}B^2 - AD\right)dy,\\ \tilde\omega_2 &= \left(\frac{1}{3}\frac{\partial C}{\partial y} - \frac{1}{3}\frac{\partial B}{\partial x} + \frac{1}{9}BC - AD\right)dx + \left(\frac{1}{3}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial x} + \frac{2}{9}B^2 - \frac{2}{3}AC\right)dy. \end{aligned}$$

□

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