Differential Equations
&
Conformal Structures

Paweł Nurowski

Uniwersytet Warszawski

Instytut Fizyki Teoretycznej

Praha, September 2004

VolkswagenStiftung

Motivation

Numerous examples of relations between differential equations and conformal geometry:

- Most recent and intriguing (in particular in General Relativity theory) see:

  Fritelli S, Kozameh C, Newman E T, (2001) "Differential geometry from differential equations" *Comm. Math. Phys.* **223** 383-408

Further references:

- Cartan E (1941) "La geometria de las ecuaciones diferenciales de tercer orden" *Rev. Mat. Hispano-Amer.* **4** 1-31

- Cartan E (1910) "Les systemes de Pfaff a cinq variables et les equations aux derivees partielles du second ordre" *Ann. Sc. Norm. Sup.* **27** 109-192

- Chern S S (1940) "The geometry of the differential equations $y''' = F(x, y, y', y'')$" *Sci. Rep. Nat. Tsing Hua Univ.* **4** 97-111

- Hilbert D (1912) "Ueber den Begriff der Klasse von Differentialgleichungen" *Mathem. Annalen Bd.* **73**, 95-108

- Nurowski P, Sparling G A J (2003) "Three dimensional Cauchy-Riemann structures and second order ordinary differential equations" *Class. Q. Grav.* **20**, 4995-5016

- Wuenschmann K, (1905) "*Ueber Beruhrungsbedingungen bei Differentialgleichungen*", Dissertation, Greifswald

- For a review including
  - FKN's system of two PDEs corresponding to 4-dimensional conformal Lorentzian geometries
  - K Wuenschmann's relations between 3rd order ODEs considered modulo conatct transformations and 3-dimensional Lorentzian geometries
  - E Cartan's relations between 3rd order ODEs considered modulo point transformations and 3-dimensional Einstein-Weyl geometries
  - relations between 2nd order ODEs considered modulo point transformations and 4-dimensional Feffermann-like geometries of signature $(++--)$

relations between equations $z' = F(x, y, y', y'', z)$ and 5-dimensional conformal geometry of signature $(+++--)$

see: Nurowski P, (2004) "Differential equations and conformal structues" math.DG/0406400

Equations with integral-free solutions

Equation $z' = F(x, y, y', y'', z)$ is a special case of an equation

$$\boxed{G(x, y, y', ..., y^{(m)}, z, z', ..., z^{(k)}) = 0 \qquad \text{(H)}}$$

for two real functions $y = y(x)$ and $z = z(x)$ of one variable x.

Definition

Equation (H) has integral-free solutions iff its general solution can be written as

$$\boxed{\begin{aligned} x &= x(t, w(t), w'(t), ..., w^{(r)}(t)) \\ y &= y(t, w(t), w'(t), ..., w^{(r)}(t)) \\ z &= z(t, w(t), w'(t), ..., w^{(r)}(t)) \end{aligned}}$$

where $w = w(t)$ is an arbitrary sufficiently smooth function of one variable.

Example

$$y - z' = 0 \quad \Longrightarrow \quad x = t, \quad y = w'(t), \quad z = w(t).$$

Equations (H) of the first order

$$\boxed{z' = F(x, y, y', z) \qquad (M)}$$

Cartan's treatment:

Let $p = y'$. Then on the space J parametrized by (x, y, p, z) consider two 1-forms

$$\boxed{\begin{aligned} \omega^1 &= \mathrm{d}z - F(x, y, p, z)\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x. \end{aligned}}$$

Clearly, every solution of (M) is a curve

$$\boxed{c(t) = (x(t), y(t), p(t), z(t))}$$

in J on which ω^1 and ω^2 vanish.

Suppose that there exists a (local) diffeomorphism $\phi : (x, y, p, z) \to (\bar{x}, \bar{y}, \bar{p}, \bar{z})$ such that

$$\boxed{\begin{aligned} \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x} &= \alpha\omega^1 + \beta\omega^2 \\ \mathrm{d}\bar{p} - \bar{z}\mathrm{d}\bar{x} &= \gamma\omega^1 + \delta\omega^2 \end{aligned}}$$

with α, β, γ, δ functions on J satisfying $\Delta = \alpha\delta - \beta\gamma \neq 0$. In such case

$$\boxed{\begin{aligned} \omega^1 &= \Delta^{-1} \left[\ \delta(\mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x}) - \beta(\mathrm{d}\bar{p} - \bar{z}\mathrm{d}\bar{x})\ \right] \\ \omega^2 &= \Delta^{-1} \left[-\gamma(\mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x}) + \alpha(\mathrm{d}\bar{p} - \bar{z}\mathrm{d}\bar{x})\ \right], \end{aligned}}$$

Thus, taking

$$\boxed{\bar{x} = t, \quad \bar{y} = w(t), \quad \bar{p} = w'(t), \quad \bar{z} = w''(t)}$$

we construct a curve in J on which the forms ω^1 and ω^2 identically vanish. Now, the inverse of ϕ which gives $x = x(\bar{x}, \bar{y}, \bar{p}, \bar{z})$, etc., provides

$$\boxed{\begin{aligned} x &= x(t, w(t), w'(t), w''(t)) \\ y &= y(t, w(t), w'(t), w''(t)) \\ z &= z(t, w(t), w'(t), w''(t)), \end{aligned}}$$

which is an integral-free solution of equation (M).

Consider equation

$$\boxed{z' = (y')^2}$$

Its corrgesponding forms are

$$\begin{aligned} \omega^1 &= \mathrm{d}z - p^2\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x \end{aligned}$$

The change of variables

$$\begin{aligned} x &= \tfrac{1}{2}\bar{z}, \\ y &= \tfrac{1}{2}(\bar{z}\bar{x} - \bar{p}), \\ z &= \tfrac{1}{2}\bar{z}\bar{x}^2 - \bar{p}\bar{x} + \bar{y}, \\ p &= \bar{x} \end{aligned}$$

brings them to the form

$$\begin{aligned} \omega^1 &= \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x} - \bar{x}(\mathrm{d}\bar{p} - \bar{z}\mathrm{d}\bar{x}), \\ \omega^2 &= -\tfrac{1}{2}(\mathrm{d}\bar{p} - \bar{z}\mathrm{d}\bar{x}). \end{aligned}$$

The integral-free solution:

$$\begin{aligned} x &= \tfrac{1}{2}w''(t) \\ y &= \tfrac{1}{2}tw''(t) - \tfrac{1}{2}w'(t) \\ z &= \tfrac{1}{2}t^2w''(t) - tw'(t) + w(t) \end{aligned}$$

Equivalence of equations (M)

Definition

Two equations

$$\boxed{z' = F(x, y, y', z) \qquad \text{and} \qquad \bar{z}' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{z})}$$

represented by the respective forms

$$\boxed{\begin{aligned} \omega^1 &= \mathrm{d}z - F(x, y, p, z)\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x. \end{aligned}}$$

$$\boxed{\begin{aligned} \bar{\omega}^1 &= \mathrm{d}\bar{z} - \bar{F}(\bar{x}, \bar{y}, \bar{p}, \bar{z})\mathrm{d}\bar{x} \\ \bar{\omega}^2 &= \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x}. \end{aligned}}$$

are (locally) equivalent iff there exists a (local) diffeomorphism $\phi : (x, y, p, z) \to (\bar{x}, \bar{y}, \bar{p}, \bar{z})$ such that

$$\boxed{\begin{aligned} \phi^*\bar{\omega}^1 &= \alpha\omega^1 + \beta\omega^2 \\ \phi^*\bar{\omega}^2 &= \gamma\omega^1 + \delta\omega^2. \end{aligned}}$$

Theorem (Monge)

All equations $z' = F(x, y, y', z)$ split onto two nonequivalent classes. All the equations within each of the two classes are locally equivalent. In the first class their forms ω^1 and ω^2 can be always brought to the form

$$\boxed{\bar{\omega}^1 = \mathrm{d}\bar{z}, \qquad \bar{\omega}^2 = \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x}}$$

in the second class one can always achieve

$$\boxed{\bar{\omega}^1 = \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x}, \qquad \bar{\omega}^2 = \mathrm{d}\bar{p} - \bar{z}\mathrm{d}\bar{x}} \quad .$$

Corollary (Monge)

All equations $z' = F(x, y, y', z)$ have integral-free solutions.

In the first case take:

$$\boxed{\bar{z} = \text{const},\ \bar{x} = t,\ \bar{y} = w(t),\ \bar{p} = w'(t).}$$

This brings the general solution to the form

$$\boxed{x = x(t, w, w'),\ y = y(t, w, w'),\ z = z(t, w, w').}$$

In the second case the solution depends also on w''.

Hilbert equation

In 1912 Hilbert obsereved that equation

$$\boxed{z' = (y'')^2}$$

is not in the class of equations (H) which have integral-free solutions. A bit earlier Cartan in his famous '5-variables' paper implicitely solved the equivalence problem for more general equations

$$\boxed{z' = F(x, y, y', y'', z) \qquad \text{(2M)}}$$

Each equation $(2M)$ may be represented by forms

$$\boxed{\begin{aligned} \omega^1 &= \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x \\ \omega^3 &= \mathrm{d}p - q\mathrm{d}x \end{aligned}}$$

on a 5-dimensional manifold parametrized by $(x, y, p = y', q = y'', z)$.

Equivalence of equations $(2M)$

Definition

Two equations

$$z' = F(x, y, y', y'', z) \qquad \text{and} \qquad \bar{z}' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'', \bar{z})$$

represented by the respective forms

$$\begin{aligned} \omega^1 &= \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x \\ \omega^2 &= \mathrm{d}y - p\mathrm{d}x \\ \omega^3 &= \mathrm{d}p - q\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \bar{\omega}^1 &= \mathrm{d}\bar{z} - \bar{F}(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{z})\mathrm{d}\bar{x} \\ \bar{\omega}^2 &= \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x} \\ \bar{\omega}^3 &= \mathrm{d}\bar{p} - \bar{q}\mathrm{d}\bar{x} \end{aligned}$$

are (locally) equivalent iff there exists a (local) diffeomorphism $\phi : (x, y, p, q, z) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{z})$ such that

$$\phi^* \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \lambda \\ \kappa & \mu & \nu \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}$$

Solution for equivalence problem for eqs. $z' = F(x, y, y', y'', z)$

Theorem (Cartan)

- There are two main branches of nonequivalent equations $z' = F(x, y, y', y'', z)$. They are distinguished by vanishing or not of the relative invariant F_{qq}, $q = y''$.

- If $F_{qq} \equiv 0$ then such equations have integral-free solutions.

- There are nonequivalent equations among the equations having $F_{qq} \neq 0$. All these equations are beyond the class of equations with integral-free solutions.

Equations $z' = F(x, y, y', y'', z)$ with $F_{y''y''} \neq 0$

Theorem (Cartan)

An equivalence class of equations $z' = F(x, y, y', y'', z)$ with $F_{y''y''} \neq 0$ uniquely defines a 14-dimensional manifold P and a preferred coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_9)$ on it such that

$$
\boxed{
\begin{aligned}
\mathrm{d}\theta^1 &= \theta^1 \wedge (2\Omega_1 + \Omega_4) + \theta^2 \wedge \Omega_2 + \theta^3 \wedge \theta^4 \\
\mathrm{d}\theta^2 &= \theta^1 \wedge \Omega_3 + \theta^2 \wedge (\Omega_1 + 2\Omega_4) + \theta^3 \wedge \theta^5 \\
\mathrm{d}\theta^3 &= \theta^1 \wedge \Omega_5 + \theta^2 \wedge \Omega_6 + \theta^3 \wedge (\Omega_1 + \Omega_4) + \theta^4 \wedge \theta^5 \\
\mathrm{d}\theta^4 &= \theta^1 \wedge \Omega_7 + \tfrac{4}{3}\theta^3 \wedge \Omega_6 + \theta^4 \wedge \Omega_1 + \theta^5 \wedge \Omega_2 \\
\mathrm{d}\theta^5 &= \theta^2 \wedge \Omega_7 - \tfrac{4}{3}\theta^3 \wedge \Omega_5 + \theta^4 \wedge \Omega_3 + \theta^5 \wedge \Omega_4.
\end{aligned}
}
$$

The system provides all the local invariants for the equivalence class of equations satisfying $F_{qq} \neq 0$.

Note that the above theorem implies formulae for the differentials of the forms Ω_μ, $\mu = 1, 2, \dots, 9$.

For example, we have

$$\mathrm{d}\Omega_1 = \Omega_3 \wedge \Omega_2 + \tfrac{1}{3}\theta^3 \wedge \Omega_7 - \tfrac{2}{3}\theta^4 \wedge \Omega_5 + \\ \tfrac{1}{3}\theta^5 \wedge \Omega_6 + \theta^1 \wedge \Omega_8 + \tfrac{3}{8}c_2\theta^1 \wedge \theta^2 + \\ b_2\theta^1 \wedge \theta^3 + b_3\theta^2 \wedge \theta^3 + \\ a_2\theta^1 \wedge \theta^4 + a_3\theta^1 \wedge \theta^5 + a_3\theta^2 \wedge \theta^4 + a_4\theta^2 \wedge \theta^5.$$

where a_2, a_3, a_4, b_2, b_3, c_2 are functions on P uniquely defined by the equivalence class of equations $(2M)$. The other differentials, when decomposed on the basis θ^i, Ω_μ, define more functions, which Cartan denoted by a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, c_1, c_2, c_3, δ_1, δ_2, e, h_1, h_2, h_3, h_4, h_5, h_6, k_1, k_2, k_3.

If one is given two equations $z' = F(x, y, y', y'', z)$ and $\bar{z}' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'', \bar{z})$ then there exists a local diffeomorphism $\phi : (x, y, p, q, z) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{z})$ realizing

$$\boxed{\begin{array}{rcl} \phi^*(\bar{\omega}^1) & = & \alpha\omega^1 + \beta\omega^2 + \gamma\omega^3 \\ \phi^*(\bar{\omega}^2) & = & \delta\omega^1 + \epsilon\omega^2 + \lambda\omega^3 \\ \phi^*(\bar{\omega}^3) & = & \kappa\omega^1 + \mu\omega^2 + \nu\omega^3 \end{array}}$$

iff there exists a diffeomorphism $\Phi : P \to \bar{P}$ between the associated 14-dimensional manifolds P and $\bar{P}$ such that

$$\boxed{\Phi^*(\bar{\theta}^i) = \theta^i, \qquad \Phi^*(\bar{\Omega}_\mu) = \Omega_\mu}$$

for all $i = 1, 2, 3, 4, 5$ and $\mu = 1, 2, 3, ..., 9$. This, in particular means that to realize the equivalence between the equationss, the diffeomorphism Φ must also satisfy

$$\boxed{\Phi^*(\bar{a}_1) = a_1, \quad \Phi^*(\bar{b}_1) = b_1, \quad \Phi^*(\bar{c}_1) = c_1, \quad \text{etc.}}$$

This gives severe algebraic (i.e. non-differential) constraints on Φ and, in generic cases, quickly leads to the answer if the two equations are equivalent.

We ask for those equivalence classes of equations $z' = F(x, y, y', y'', z)$ which correspond to systems $\boxed{\text{with all the scalar invariants}}$ a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, c_1, c_2, c_3, δ_1, δ_2, e, h_1, h_2, h_3, h_4, h_5, h_6, k_1, k_2, k_3 $\boxed{\text{being constants}}$.

It follows that it is possible if and only if $\boxed{\text{all of them}}$ are identically $\boxed{\text{equal to zero.}}$

In this well defined case the system of the Theorem can be understood as a system consisting of right invariants forms (θ^i, Ω_μ) on a 14-dimensional Lie group. This group is simple and has indefinite Killing form.

This identifies this group as a noncompact real form $\tilde{G}_2$ of the $\boxed{\text{exceptional group } G_2}$.

It follows that there is only one equivalence class of equations corresponding to the system with all the scalar invariants vanishing. It can be defined by the function $F = q^2$ associated with the Hilbert equation

$$\boxed{z' = (y'')^2.}$$

The system defines a curvature of a certain Cartan $\underline{\tilde{\mathfrak{g}}_2}$-valued connection.

P is a principal fibre bundle over J with the 9-dimensional parabolic subgroup H of $\tilde{G}_2$ as its structure group.

On this fibre bundle the following matrix of 1-forms:

$$\boxed{\begin{pmatrix} -\Omega_1-\Omega_4 & -\Omega_8 & -\Omega_9 & -\frac{1}{\sqrt{3}}\Omega_7 & \frac13\Omega_5 & \frac13\Omega_6 & 0 \\ \theta^1 & \Omega_1 & \Omega_2 & \frac{1}{\sqrt{3}}\theta^4 & -\frac13\theta^3 & 0 & \frac13\Omega_6 \\ \theta^2 & \Omega_3 & \Omega_4 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac13\theta^3 & -\frac13\Omega_5 \\ \frac{2}{\sqrt{3}}\theta^3 & \frac{2}{\sqrt{3}}\Omega_5 & \frac{2}{\sqrt{3}}\Omega_6 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{\sqrt{3}}\Omega_7 \\ \theta^4 & \Omega_7 & 0 & \frac{2}{\sqrt{3}}\Omega_6 & -\Omega_4 & \Omega_2 & \Omega_9 \\ \theta^5 & 0 & \Omega_7 & -\frac{2}{\sqrt{3}}\Omega_5 & \Omega_3 & -\Omega_1 & -\Omega_8 \\ 0 & \theta^5 & -\theta^4 & \frac{2}{\sqrt{3}}\theta^3 & -\theta^2 & \theta^1 & \Omega_1+\Omega_4 \end{pmatrix},}$$

becomes a Cartan connection ω with values in the Lie algebra of $\tilde{G}_2$.

The curvature of this connection $\boxed{R = \mathrm{d}\omega + \omega \wedge \omega}$ 'measures' how much the equivalence class of equations $(2M)$ is 'distorted' from the flat Hilbert case corresponding to $F = q^2$.

$(3,2)$-signature conformal metric

Given equivalence class of equation $z' = F(x,y,y',y'',z)$ consider its corresponding bundle P with the coframe $(\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\Omega_1,\Omega_2,\Omega_3,\Omega_4,\Omega_5,\Omega_6,\Omega_7,\Omega_8,\Omega_9)$.

Define a bilinear form

$$\boxed{\tilde{g} = 2\theta^1\theta^5 - 2\theta^2\theta^4 + \tfrac{4}{3}\theta^3\theta^3}$$

This form is degenerate on P and has signature $(3,2,0,0,0,0,0,0,0,0,0)$.

The 9 degenerate directions generate the vertical space of P.

Theorem

- The bilinear forms $\tilde{g}$ $\boxed{\text{transforms conformally}}$ when Lie transported along any of the vertical directions.

- It descends to a well defined $\boxed{\text{conformal } (3,2)\text{-}}$ $\boxed{\text{signature metric}}$ on the 5-dimensional space J on which the equation $z' = F(x, y, y', y'', z)$ is defined

- The $\boxed{\text{Cartan normal conformal connection}}$ associated with this conformal metric yields all the invariant information about the equivalence class of the equation

- This $\underline{\mathbf{so}(4,3)}$-valued connection is reducible and, after reduction, can be identified with the $\underline{\tilde{\mathfrak{g}}_2}$ Cartan connection ω on P.

It follows that the Hilbert equation has $\tilde{G}_2$ as its symmetry group.

Cartan knew that $z' = F(x, y, y', y'', z)$ is either equivalent to the Hilbert equation or its group of transitive symmetries is at most 7-dimensional.

The equations with 7-dimensional group of transitive symmetries are among those equivalent to

$$z' = F(y''), \qquad \text{with} \qquad F_{y''y''} \neq 0.$$

For such F's the $(3,2)$-signature conformal metric reads

$$\begin{aligned}
g = {} & 30(F'')^4 \,[\, \mathrm{d}q\mathrm{d}y - p\mathrm{d}q\mathrm{d}x \,] \; + \; [\, 4F^{(3)2} - 3F''F^{(4)} \,] \,\mathrm{d}z^2 + \\
& 2\,[-5(F'')^2F^{(3)} - 4F'F^{(3)2} + 3F'F''F^{(4)} \,]\, \mathrm{d}p\mathrm{d}z + \\
& 2\,[15(F'')^3 + 5q(F'')^2F^{(3)} - 4FF^{(3)2} + 4qF'F^{(3)2} + \\
& 3FF''F^{(4)} - 3qF'F''F^{(4)} \,]\, \mathrm{d}x\mathrm{d}z + \\
& [-20(F'')^4 + 10F'(F'')^2F^{(3)} + 4(F')^2F^{(3)2} - 3(F')^2F''F^{(4)} \,]\, \mathrm{d}p^2 + \\
& 2\,[-15F'(F'')^3 + 20q(F'')^4 + 5F(F'')^2F^{(3)} - 10qF'(F'')^2F^{(3)} + \\
& 4FF'F^{(3)2} - 4q(F')^2F^{(3)2} - 3FF'F''F^{(4)} + 3q(F')^2F''F^{(4)} \,]\, \mathrm{d}p\mathrm{d}x + \\
& [-30F(F'')^3 + 30qF'(F'')^3 - 20q^2(F'')^4 - 10qF(F'')^2F^{(3)} + \\
& 10q^2F'(F'')^2F^{(3)} + 4F^2F^{(3)2} - 8qFF'F^{(3)2} + 4q^2(F')^2F^{(3)2} - \\
& 3F^2F''F^{(4)} + 6qFF'F''F^{(4)} - 3q^2(F')^2F''F^{(4)} \,]\, \mathrm{d}x^2.
\end{aligned}$$

It is always conformal to an Einstein metric $\hat{g} = \mathrm{e}^{2\Upsilon} g$ with the conformal factor $\Upsilon = \Upsilon(q)$ satisfying

$$10(F'')^2 \,[\, \Upsilon'' - (\Upsilon')^2 \,] - 40F''F^{(3)}\Upsilon' + 17F''F^{(4)} - 56F^{(3)2} = 0.$$

Cartan classified various types of nonequivalent equations $(2M)$ according to the $\boxed{\text{roots}}$ of

$$\boxed{\Psi(z) = a_1 z^4 + 4a_2 z^3 + 6a_3 z^2 + 4a_4 z + a_5,}$$

where $(a_1, a_2, a_3, a_4, a_5)$ are the scalar invariants of the equation. This polynomial encodes partial information of the $\boxed{\text{Weyl tensor}}$ of the associated conformal metric. In particular, the well known invariant $I_\Psi = 6a_3^2 - 8a_2 a_4 + 2a_1 a_5$ of this polynomial is, modulo a numerical factor, proportional to the $\boxed{\text{square of the Weyl tensor}}$ $C^2 = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$ of the conformal metric. Vanishing of I_Ψ means that $\Psi = \Psi(z)$ has a root with $\boxed{\text{multiplicity no smaller than } 3.}$ Our example above corresponds to the situation when this multiplicity is equal to 4. According to Cartan, all nonequivalent equations for which Ψ has quartic root are covered by this example. In this example nonequivalent equations are distinguished by the only nonvanishing scalar invariant a_5 to which the Weyl tensor of the metric g is proportional. If $a_5 =$const the equation has 7-dimensional group of symmetries.