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Differential Equations and Para-CR Structures

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Dedicated to the memory of Professor Aldo Andreotti
on the 30th anniversary of his death.

Abstract. – *We study the local geometry of n dimensional manifolds which are equipped with two integrable distributions, one of dimension r and one of dimension s, where r and s are allowed to be unequal. We call them para-CR structures of type* (k, r, s), *with* $k = n - r - s \geq 0$ *being the para-CR codimension. When* $r = s$ *they are the real analogues of CR structures. In the general case these structures are the natural geometric setting in which to discuss the geometry of systems of ODE's, as well as the geometry of systems of PDE's of finite type. For particular small values of* k, r, s *we determine the basic local invariants of such structures.*

Contents

1. – Introduction.

Aldo Andreotti liked simple ideas best. He often said "The more simple an idea is, the better it is". He also liked explicit provocative examples which begged for the development of a new general theory. We think he would have enjoyed hearing the story we tell here.

A para-CR structure is the real analogue of a CR structure (see Definition 2.1). The main point is that $K^2 = I$, instead of $J^2 = -I$, and one does not insist that $\dim H^+ = \dim H^-$, as in the situation of CR structures (where $\dim H^{1,0} = \dim H^{0,1}$ happens accidentally). Here $H^{\pm}$ are the ± 1 eigenspaces of K. Assuming that one is already familiar with CR structures, then here is the simple idea: "Change the sign and allow the dimensions of the eigenspaces to differ".

What are the provocative examples? One of the goals of this paper is to provide a few of them.

Rather than overburden this introduction with a lengthy description of what is contained here, we refer the reader to the detailed table of contents. If we were to highlight the Sections of the paper that in our opinion are the most interesting, we would indicate Sections 7 and 8.

2. – To para-CR structures via ODEs.

2.1 – *Geometry of general solutions of ODEs modulo point transformations.*

The abstract notion of a *para-CR manifold* [1] appears naturally in the context of systems of differential equations considered modulo point transformations of variables [14, 16]. In the simplest case of a single ordinary differential equation of nth order,

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}), \tag{2.1}$$

for a real function $\mathbb{R} \ni x \mapsto y(x) \in \mathbb{R}$, such an equation has a *general solution*

$$y = \psi(x, a_0, a_1, \ldots, a_{n-1}), \tag{2.2}$$

depending on n arbitrary real parameters $(a_0, a_1, \ldots, a_{n-1})$. Thus the general solution of such an equation may be considered as a *hypersurface* Σ in $\mathbb{R}^2 \times \mathbb{R}^n$ defined by

$$\Sigma = \{\mathbb{R}^2 \times \mathbb{R}^n \ni (y, x, a_0, a_1, \ldots, a_{n-1}) \mid \Psi(y, x, a_0, a_1, \ldots, a_{n-1}) = 0\}, \tag{2.3}$$

where $\Psi(y, x, a_0, a_1, \ldots, a_{n-1}) = y - \psi(x, a_0, a_1, \ldots, a_{n-1})$. Now consider a diffeomorphism of $\mathbb{R}^2 \times \mathbb{R}^n$, which preserves the split of $\mathbb{R}^{(2+n)}$ onto $\mathbb{R}^2$ and $\mathbb{R}^n$. This may mix the variables y and x, and, *separately*, may mix the variables a_0, $a_1, \ldots, a_{n-1}$; it cannot however mix y and x with the a_is. Explicitly it is given by

$$\mathbb{R}^2 \times \mathbb{R}^n \ni (y, x, a_0, a_1, \ldots, a_{n-1}) \mapsto (\bar{y}, \bar{x}, \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n-1}) \in \mathbb{R}^2 \times \mathbb{R}^n,$$

where

$$\begin{aligned} \bar{y} &= \bar{y}(y, x), \\ \bar{x} &= \bar{x}(y, x), \\ \bar{a}_i &= \bar{a}_i(a_0, a_1, \ldots, a_{n-1}), \qquad i = 0, 1, \ldots, n-1. \end{aligned} \tag{2.4}$$

This diffeomorphism transforms Σ to another hypersurface in $\mathbb{R}^2 \times \mathbb{R}^n$, which defines the general solution to an ODE which is locally *point equivalent* to the ODE (2.1).

To understand the geometry of general solutions of such ODEs (2.1) modulo point transformations better, it is convenient to pass to a bit more general setting. Thus, without referring to any ODE, we consider $\mathbb{R}^{(2+n)}$ equipped with a *linear* operator

$$\kappa : \mathbb{R}^{(2+n)} \to \mathbb{R}^{(2+n)}, \qquad \text{such that} \qquad \kappa^2 = \mathrm{id}.$$

The operator κ has two eigenvalues: $+1$ and -1, and we *assume* that the corresponding eigenspaces are, respectively, $\chi_+ = \mathbb{R}^2$, with eigenvalue $+1$, and $\chi_- = \mathbb{R}^n$, with eigenvalue -1. We adapt a coordinate system $(y, x, a_0, \ldots, a_{n-1})$ in $\mathbb{R}^{(2+n)}$, so that $\chi_+ = \mathrm{Span}(\partial_y, \partial_x)$ and $\chi_- = \mathrm{Span}(\partial_{a_0}, \ldots, \partial_{a_{n-1}})$.

Given $\mathbb{R}^{(2+n)}$ with such a κ, we consider a smooth real function

$$\Psi : \mathbb{R}^{(2+n)} \to \mathbb{R}.$$

This function is supposed to have *zero* as a *regular value*. With this assumption the set Σ as in (2.3) is a codimension one submanifold of $\mathbb{R}^{(2+n)}$. In addition we assume that Σ is *generically* embedded, which means that its tangent space at each point, $\mathrm{T}_p\Sigma$, is spanned by the *linearly independent* vectors

$$\begin{aligned} X_1 =&\Psi_x\partial_y - \Psi_y\partial_x\\ Y_1 =&\Psi_1\partial_0 - \Psi_0\partial_1\\ Y_2 =&\Psi_2\partial_1 - \Psi_1\partial_2\\ &\dots\\ Y_{n-1} =&\Psi_{n-1}\partial_{n-2} - \Psi_{n-2}\partial_{n-1}\\ Z =&\Psi_0\partial_y - \Psi_y\partial_0. \end{aligned}$$

Here $\partial_i = \dfrac{\partial}{\partial a_i}$, $i = 0, \dots, (n-1)$, and $\Psi_x = \partial_x(\Psi)$, $\Psi_y = \partial_y(\Psi)$, $\Psi_i = \partial_i(\Psi)$.

Note that the operator κ from the ambient space $\mathbb{R}^{(2+n)}$ defines a vector subspace H_p of $\mathrm{T}_p\Sigma$ by

$$H_p = \kappa(\mathrm{T}_p\Sigma) \cap \mathrm{T}_p\Sigma.$$

In the above basis of $\mathrm{T}_p\Sigma$ we have

$$H_p = \mathrm{Span}\,(X_1, Y_1, \dots, Y_{n-1}).$$

Moreover, κ restricts to H_p, defining an operator $K_p : H_p \to H_p$, $K_p = \kappa_{|H_p}$. Since $K_p^2 = id$, it splits H_p onto $H_p = H_p^+ \oplus H_p^-$; the spaces $H_p^\pm$ correspond to the $\pm$ eigenvalues of K_p. We have

$$H_p^+ = \mathrm{Span}(X_1), \qquad H_p^- = \mathrm{Span}(Y_1, \dots, Y_{n-1}).$$

It further follows that the distributions $H^+ = \bigcup_{p\in\Sigma} H_p^+$ and $H^- = \bigcup_{p\in\Sigma} H_p^-$ are *integrable*. They define *two foliations* on Σ, one of which has 1-dimensional leaves tangent to X_1, and the other has $(n-1)$-dimensional leaves tangent to all the Y_is. These two foliations are obtained by the intersections of Σ with the leaves $\pi_\pm^{-1}(v_\mp)$, $v_\mp \in \chi_\mp$, of the respective foliations $\pi_+ : \mathbb{R}^{(2+n)} \to \chi_-$ and $\pi_- : \mathbb{R}^{(2+n)} \to \chi_+$.

Note also that although both distributions H^+ and H^- are automatically integrable, the distribution H is *in general not* integrable. For H to be integrable the defining function Ψ would have to satisfy the $\dfrac{n(n-1)}{2}$ conditions:

$$\Psi_y\Psi_{x[i}\Psi_{j]} - \Psi_x\Psi_{y[i}\Psi_{j]} = 0,$$

for all $i,j = 0, 1, \dots, (n-1)$. Here $\Psi_{yi} = \dfrac{\partial^2\Psi}{\partial y\partial a_i}$, and $\Psi_{x[i}\Psi_{j]} = \dfrac{1}{2}(\Psi_{xi}\Psi_j - \Psi_{xj}\Psi_i)$, etc.

2.2 – *Abstract para-CR manifolds.*

The structure on Σ consisting of K and $H = H^+ \oplus H^-$ is precisely the structure of a *para-CR manifold*, which abstractly can be defined, somewhat more generally, as follows:

DEFINITION 2.1. – *A $(k+n)$-dimensional manifold M equipped with an n-dimensional distribution H together with a linear operator $K : H \to H$, such that $K^2 = id$, is called an almost para-CR manifold. If in addition both eigenspaces of K, $H^+ = \{X \in H, KX = X\}$ and $H^- = \{X \in H, KX = -X\}$, are integrable, $[H^\pm, H^\pm] \subset H^\pm$, then an almost para-CR manifold (M, H, K) is called an abstract para-CR manifold. The type of the abstract para-CR manifold will be denoted by (k, r, s) where k is the para-CR codimension, and $r = \dim H^+$, $s = \dim H^-$*

In the following we will only consider *smooth* para-CR structures, i.e. smooth manifolds M, with both H and K being smooth.

In the case of the hypersurfaces Σ considered above, Σ has type $(1, 1, n-1)$. What is more important, in this case the para-CR structure (H, K) was *induced on Σ from the ambient space* $(\mathbb{R}^{(2+n)}, \kappa)$. A natural question arises if an abstractly defined para-CR manifold (M, K, H), as in Definition 2.1, can be (locally) generically embedded as a submanifold Σ in some $\mathbb{R}^{(m+n)}$ equipped with a linear operator $\kappa : \mathbb{R}^{(m+n)} \to \mathbb{R}^{(m+n)}$, $\kappa^2 = id$, having $\mathbb{R}^m$ as its $+1$ eigenspace, and $\mathbb{R}^n$ as its -1 eigenspace, so that the induced para-CR structure on Σ coincides with that of (M, K, H).

To answer this question we need some preparations.

DEFINITION 2.2. – *Two abstract para-CR structures (M_1, H_1, K_1) and (M_2, H_2, K_2) are (locally) equivalent iff there exists a (local) diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi_* H_1 = H_2$ and $\Phi_* \circ K_1 = K_2 \circ \Phi_*$. Such a Φ is called a para-CR diffeomorphism.*

A *dual formulation* of the para-CR definition is very useful:

DEFINITION 2.3. – *An almost para-CR structure (of type (k, r, s)) is a $(k+n)$-dimensional manifold M equipped with an equivalence class of $(k+r+s)$ one-forms $(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)$ such that*

- *$r + s = n$,*
- *$\lambda_1 \wedge \dots \lambda_k \wedge \mu_1 \dots \mu_r \wedge \nu_1 \dots \nu_s \neq 0$ at each point of M,*
- *two choices of 1-forms $(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)$ and $(\lambda'_1, \dots, \lambda'_k, \mu'_1, \dots, \mu'_r, \nu'_1, \dots, \nu'_s)$ are in an equivalence relation iff there exist real functions $a^i_{\ j}$, $b^j_{\ A}$, $c^j_{\ a}$, f^A_B, h^a_β, with $i, j = 1, \dots k$; $A, B = 1, \dots, r$; $a, \beta = 1, \dots, s$, on M such that:*

$$\lambda'_i = a^j_{\ i} \lambda_j, \quad \mu'_A = f^B_A \mu_B + b^j_{\ A} \lambda_j, \quad \nu'_a = h^\beta_{\ a} \nu_\beta + c^j_{\ a} \lambda_j, \tag{2.5}$$

and $\det(a^i_j) \det(f^A_B) \det(h^a_\beta) \neq 0$.

An almost para-CR structure is an *integrable* para-CR structure iff, in addition, the following equations

$$\begin{aligned} &\mathrm{d}\lambda_i \wedge \lambda_1 \wedge \ldots \wedge \lambda_k \wedge \mu_1 \wedge \ldots \wedge \mu_r = 0 \\ &\mathrm{d}\mu_A \wedge \lambda_1 \wedge \ldots \wedge \lambda_k \wedge \mu_1 \wedge \ldots \wedge \mu_r = 0 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\mathrm{d}\lambda_i \wedge \lambda_1 \wedge \ldots \wedge \lambda_k \wedge \nu_1 \wedge \ldots \wedge \nu_s = 0 \\ &\mathrm{d}\nu_a \wedge \lambda_1 \wedge \ldots \wedge \lambda_k \wedge \nu_1 \wedge \ldots \wedge \nu_s = 0 \end{aligned} \tag{2.7}$$

are simultaneously satisfied, for all $i = 1, \ldots, k$, $A = 1, \ldots, r$, $a = 1, \ldots, s$, and for one (therefore all) representatives $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s)$ of an equivalence class $[(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s)]$.

One observes that Definition 2.3 is the dual version of Definition 2.1 identifying H^- with the anihilator of $(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_r)$ and H^+ with the anihilator of $(\lambda_1, \ldots, \lambda_k, \nu_1, \ldots, \nu_s)$. Thus H^+ is r-dimensional, and H^- is s-dimensional, with $H = H^+ \oplus H^-$ being $r + s = n$-dimensional. In particular H is integrable iff $\mathrm{d}\lambda_i \wedge \lambda_1 \wedge \ldots \wedge \lambda_k = 0$ for all $i = 1, \ldots, k$.

EXAMPLE 2.4. – Given an n-th order ODE (2.1) we introduce a canonical para-CR structure on the space $\mathcal{J}$ of the $(n-1)$ jets. Parametrizing this space by $(x, y, y^1, \ldots, y^{n-1})$ we introduce

$$\begin{aligned} &\lambda = \mathrm{d}y - y^1 \mathrm{d}x, \\ &\mu = \mathrm{d}x, \\ &\nu_i = \mathrm{d}y^i - y^{i+1}\,\mathrm{d}x, \qquad \forall i = 1, \ldots, n-2, \\ &\nu_{n-1} = \mathrm{d}y^{n-1} - F(x, y, y^1, \ldots, y^{n-1})\mathrm{d}x, \end{aligned} \tag{2.8}$$

and define the class $[\lambda, \mu, \nu_a]$ on $\mathcal{J}$ via:

$$(\lambda, \mu, \nu_a) \sim (\lambda', \mu', \nu'_a) \qquad \text{iff} \qquad \lambda' = a\lambda,\ \mu' = f\mu + b\lambda,\ \text{and}\ \nu'_a = h^\beta_a \nu_\beta + c_a\lambda,$$

with functions a, b, c, h^β_a, c_a on $\mathcal{J}$, such that $af \det(h^\alpha_\beta) \neq 0$. Obviously $\lambda \wedge \mu \wedge \nu_1 \wedge \ldots \wedge \nu_{n-1} \neq 0$, $\mathrm{d}\lambda \wedge \lambda \wedge \mu \equiv 0 \equiv \mathrm{d}\mu \wedge \lambda \wedge \mu$, and for dimensional reasons $\mathrm{d}\lambda \wedge \lambda \wedge \nu_1 \wedge \ldots \wedge \nu_{n-1} \equiv 0 \equiv \mathrm{d}\nu_a \wedge \lambda \wedge \nu_1 \wedge \ldots \wedge \nu_{n-1}$ for all $a = 1, \ldots, n-1$. This shows that $(\mathcal{J}, [\lambda, \mu, \nu_a])$ is an abstract para-CR structure of type $(1, 1, n-1)$. This para-CR structure is called the *canonical para-CR structure of an ODE* $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$.

Returning to the general discussion we have the following Proposition.

PROPOSITION 2.5. – *Every abstract para-CR manifold* $(M, [\lambda_i, \mu_A, \nu_a])$ *of type* (k, r, s) *locally admits two overlaping coordinate systems* (y_i, x_A, a_a) *and*

$(\bar{y}_i, x_A, a_a)$ in which the forms $(\lambda_i, \mu_A, \nu_a)$ can be written either as:

$$\lambda_i = \mathrm{d}y_i + L^A_i \mathrm{d}x_A, \qquad \mu_A = \mathrm{d}x_A, \qquad \nu_a = \mathrm{d}a_a, \tag{2.9}$$

or by

$$\lambda_i = \mathrm{d}\bar{y}_i + \bar{L}^a_i \mathrm{d}a_a, \qquad \mu_A = \mathrm{d}x_A, \qquad \nu_a = \mathrm{d}a_a, \tag{2.10}$$

where $L^A_i = L^A_i(y, x, a)$ and $\bar{L}^a_i = \bar{L}^a_i(\bar{y}, x, a)$, $i = 1, \ldots, k$, $A = 1, \ldots, r$, $a = 1, \ldots, s$, are appropriate real functions of the respective variables (y_i, x_A, a_a) and $(\bar{y}_i, x_A, a_a)$.

PROOF. – The proof is a simple application of the Frobenius theorem:

On one hand, the Frobenius theorem applied to the integrability conditions (2.6), together with the use of transformations (2.5), imply the existence of functions (y_i, x_A, L^A_i) for which $\lambda_i = \mathrm{d}y_i + L^A_i \mathrm{d}x_A$ and $\mu_A = \mathrm{d}x_A$ holds. On the other hand, the same argument applied to the integrability conditions (2.7), imply the existence of functions $(\bar{y}_i, a_a, \bar{L}^a_i)$ for which $\lambda_i = \mathrm{d}\bar{y}_i + \bar{L}^a_i \mathrm{d}a_a$ and $\nu_a = \mathrm{d}a_a$ holds. But since $\lambda_1 \wedge \ldots \lambda_k \wedge \mu_1 \wedge \ldots \wedge \mu_r \wedge \nu_1 \ldots \wedge \nu_s \neq 0$, then taking λs and μs from the first representation, and νs from the second we get $\mathrm{d}y_1 \wedge \ldots \mathrm{d}y_k \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_s \wedge \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_s \neq 0$. Similarly, taking λs and νs from the second representation, and μs from the first we get $\mathrm{d}\bar{y}_1 \wedge \ldots \mathrm{d}\bar{y}_k \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_s \wedge \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_s \neq 0$. This shows that both sets of functions (y_i, x_A, a_a) and $(\bar{y}_i, x_A, a_a)$ form local coordinates on M. In these coordinates the para-CR forms have the respective desired representation (2.9) and (2.10).

2.3 – *The embedding problem.*

Once an integrable para-CR structure is defined in terms of $[(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s)]$ it is easy to solve the embedding problem, at least locally.

We have the following embedding theorem.

THEOREM 2.6. – *Every smooth $(k + r + s)$-dimensional abstract para-CR manifold (M, H, K) with $\dim H^+ = r$ and $\dim H^- = s$ is locally embeddable in $\mathbb{R}^{(k+r)+(k+s)}$, with the embedding $\imath : M \to \mathbb{R}^{(k+r)+(k+s)}$ being a para-CR diffeomorphism between (M, H, K) and the para-CR structure which $\imath(M)$ aquires from the ambient space $(\mathbb{R}^{(k+r)+(k+s)}, \kappa)$. Here κ is the canonical linear map $\kappa : \mathbb{R}^{(k+r)+(k+s)} \to \mathbb{R}^{(k+r)+(k+s)}$, $\kappa^2 = id$, having $\mathbb{R}^{k+r}$ and $\mathbb{R}^{k+s}$ as its respective $+1$ and -1 eigenspaces.*

PROOF. – Choosing a representative $(\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_s)$ we consider vector fields $(Z_1,\dots,Z_k,X_1,\dots,X_r,Y_1,\dots,Y_s)$ which are the respective duals of $(\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_s)$. This in particular means that $H^+=\mathrm{Span}(X_1,\dots,X_r)$ and $H^-=\mathrm{Span}(Y_1,\dots,Y_s)$. Also any differentiable function $f: M\to\mathbb{R}$ has

$$\mathrm{d}f = Z_i(f)\lambda_i + X_A(f)\mu_A + Y_a(f)\nu_a$$

as its differential. Now, one looks for all functions f and h on M which satsify

$$\mathrm{d}f\wedge\lambda_1\wedge\dots\wedge\lambda_k\wedge\mu_1\wedge\dots\wedge\mu_r=0,\quad\text{and}\tag{2.11}$$

$$\mathrm{d}h\wedge\lambda_1\wedge\dots\wedge\lambda_k\wedge\nu_1\wedge\dots\wedge\nu_s=0,\tag{2.12}$$

or, what is the same,

$$Y_a(f)=0,\quad \forall a=1,\dots,s,\qquad\text{and}\quad X_A(h)=0,\quad\forall A=1,\dots,r.$$

If, for example, we choose $(\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_r,\nu_1,\dots,\nu_s)$ in the local representation (2.9), then equations (2.11)-(2.12) are, respectively,

$$\frac{\partial f}{\partial a_a}=0,\quad\forall a=1,\dots,s,\tag{2.13}$$

$$\frac{\partial h}{\partial x_A}-L^A_{\ i}\frac{\partial h}{\partial y^i}=0,\quad\forall A=1,\dots,r.\tag{2.14}$$

Thus in this coordinate system equations (2.13) for the function f are trivial to solve: they obviously have $k+r$ independent solutions given by $f_1=y_1,\dots,f_k=y_k,\tilde f_1=x_1,\dots,\tilde f_r=x_r$. The equations (2.14) for the function h do not look very nice in this coordinate system. To analyse them it is convenient to use the other coordinate system, $(\bar y_i,x_A,a_a)$, in which equations (2.11)-(2.12) are, respectively:

$$\frac{\partial f}{\partial a_a}-\bar L^a_{\ i}\frac{\partial f}{\partial \bar y^i}=0,\quad\forall a=1,\dots,s,\tag{2.15}$$

$$\frac{\partial h}{\partial x_A}=0,\quad\forall A=1,\dots,r.\tag{2.16}$$

In this coordinate system equation (2.16) for the function h is trivial: it has $k+s$ independent solutions, $h_1=\bar y_1,\dots,h_k=\bar y_k,\tilde h_1=a_1,\dots,\tilde h_s=a_s$. Now since both coordinate systems (y_i,x_A,a_a) and $(\bar y_i,x_A,a_a)$ are defined over the same region of M, and because the coordinates (x,a) are the same in both systems, we have:

$$\bar y_i=\bar y_i(y,x,a),\qquad\text{and}\qquad y_i=y_i(\bar y,x,a).$$

This shows that the two maps:

$$M\ni(y_i,x_A,a_a)\overset{\iota}{\mapsto}(f_i,\tilde f_A,h_j,\tilde h_a)=(y_i,x_A,\bar y_j(y,x,a),a_a)\in\mathbb{R}^{(k+r)+(k+s)}$$

and

$$M \ni (\bar{y}_i, x_A, a_a) \stackrel{\bar{\imath}}{\mapsto} (f_i, \tilde{f}_A, h_j, \tilde{h}_a) = (y_i(\bar{y}, x, a), x_A, \bar{y}_j, a_a) \in \mathbb{R}^{(k+r)+(k+s)}$$

give two local embeddings of the para-CR structure $(M, [(\lambda, \mu, \nu)])$ in $\mathbb{R}^{(k+r)+(k+s)}$ with coordinates $(f_i, \tilde{f}_A, h_j, \tilde{h}_a)$. It follows that the κ operator in $\mathbb{R}^{(k+r)+(k+s)}$, splitting it onto $\mathbb{R}^{(k+r)+(k+s)} = \mathbb{R}^{k+r} \times \mathbb{R}^{k+s}$, induces two para-CR structures on the repective images of $\imath$ and $\bar{\imath}$. These two para-CR structures are locally equivalent, and are locally equivalent to the original structure from M. □

Para-CR structures with $k = 1$, for obvious reasons, are called para-CR structures of *hypersurface type*.

2.4 – *Para-CR equivalence a'la Cartan.*

In the following a reformulation of the (local) equivalence of two para-CR manifolds, in the language of the differential forms $(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)$, will be useful. It can be seen that Definition 2.2 is equivalent to

DEFINITION 2.7. – *Two para-CR structures* $(M, [(\lambda_i, \mu_A, \nu_a)])$ *and* $(M', (\lambda_i{}', \mu_A{}', \nu_a{}')])$, $i = 1, \dots, k$, $A = 1, \dots, r$, $a = 1, \dots, s$, *on* $k + r + s$ *dimensional manifolds* M *and* M' *are (locally) equivalent iff there exists a (local) diffeomorphism* $\Phi : M \to M'$ *and real functions* a^i_j, b^j_A, c^j_a, f^A_B, h^a_β *on* M *such that:*

$$\begin{aligned} \Phi^*(\lambda'_i) &= a^j{}_i \lambda_j, \\ \Phi^*(\mu'_A) &= f^B{}_A \mu_B + b^j{}_A \lambda_j, \\ \Phi^*(\nu'_a) &= h^\beta{}_a \nu_\beta + c^j{}_a \lambda_j, \end{aligned} \tag{2.17}$$

and

$$\det(a^i{}_j) \det(f^A{}_B) \det(h^a{}_\beta) \neq 0$$

for all $i,j = 1, \dots k$; $A, B = 1, \dots, r$; $a, \beta = 1, \dots, s$.

3. – Para-CR structures of type $(1, 1, n-1)$.

In Example 2.4 we associated a para-CR structure of type $(1, 1, n-1)$ with every n-th order ODE in the form (2.1). A natural question arises: is every para-

CR structure of type $(1,1,n-1)$, at least locally, para-CR equivalent to a canonical type $(1,1,n-1)$ para-CR structure of some n-th order ODE (2.1)? Since all canonical para-CR structures of n-th order ODEs, as in Example 2.4, satisfy $\mathrm{d}\lambda\wedge\lambda=\mathrm{d}x\wedge\mathrm{d}y^1\wedge\mathrm{d}y\neq 0$, and since nonvanishing of $\mathrm{d}\lambda\wedge\lambda$ is invariant under any para-CR map $\lambda\to\lambda'=a\lambda$, then we have

PROPOSITION 3.1. – *A type* $(1,1,n-1)$ *para-CR structure* $[\lambda,\mu,v_a]$ *which is locally equivalent to the canonical para-CR structure of an n-th order ODE* $y^{(n)}=F(x,y,y',\dots,y^{(n-1)})$ *has* $\mathrm{d}\lambda\wedge\lambda\neq 0$.

In view of this proposition, we now ask if every type $(1,1,n-1)$ para-CR structure with $\mathrm{d}\lambda\wedge\lambda\neq 0$ is locally equivalent to a structure from Example 2.4. To illustrate the problems associated with this question we consider low dimensions first.

3.1 – *Para-CR structures of type* $(1,1,1)$.

This case, in a bit different context, was studied by one of us in [16]. We have the following proposition.

PROPOSITION 3.2. – *Every type* $(1,1,1)$ *para-CR structure* $(M,[\lambda,\mu,v])$ *with* $\mathrm{d}\lambda\wedge\lambda\neq 0$ *is locally para-CR equivalent to a type* $(1,1,1)$ *para-CR structure associated with a point equivalence class of second order ODEs.*

PROOF. – This Proposition was proved in [16]. For completness we present this proof also here.

Choosing any representative (λ,μ,v) of $[\lambda,\mu,v]$, due to the low dimension of M, we have $\mathrm{d}\lambda\wedge\lambda\wedge\mu\equiv 0$ and $\mathrm{d}\mu\wedge\lambda\wedge\mu\equiv 0$. Thus, by the Frobenius theorem, we have functions (x,y,A,B,C,E) on M such that $\lambda=A\mathrm{d}x+B\mathrm{d}y$, and $\mu=C\mathrm{d}x+E\mathrm{d}y$. Considering the allowed para-CR gauge of λ and μ, we can rescale λ to the form $\lambda=\mathrm{d}y-p\,\mathrm{d}x$, with some function p on M, and shift and rescale μ to the form $\mu=\mathrm{d}x$. Now our assumption $0\neq\mathrm{d}\lambda\wedge\lambda$ shows that $0\neq\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}p$ and, thus, (x,y,p) can be considered a coordinate system on M. In this coordinate system the form μ is locally $\mu=a\mathrm{d}x+\beta\mathrm{d}y+\gamma\mathrm{d}p$, where a,β,γ are some functions on M. Because of the allowed para-CR transformations for μ, we can, without loss of generality, take $\mu=\mathrm{d}p-Q(x,y,p)\mathrm{d}x$, with $Q=Q(x,y,p)$ being some function on M. Thus our type $(1,1,1)$ para-CR structure $(M,[\lambda,\mu,v])$ with $\mathrm{d}\lambda\wedge\lambda\neq 0$ is locally para-CR equivalent to $(M,[\lambda=\mathrm{d}y-p\,\mathrm{d}x,\mu=\mathrm{d}x,v=\mathrm{d}p-Q\mathrm{d}x])$. Therefore M can be locally identified with the first jet space of the equation $y''=Q(x,y,y')$. The (x,y,p) are canonical coordinates (x, y, p) on this jet space and the contact forms are given by the para-CR forms $\lambda = \mathrm{d}y - p\,\mathrm{d}x$, $\nu = \mathrm{d}p - Q\mathrm{d}x$. The para-CR structure associated with the point equivalent class of ODEs represented by $y'' = Q(x, y, y')$ is locally para-CR equivalent to the para-CR structure we started with. □

Further details about this case, including relations to the Fefferman construction, can be found in [16].

3.2 – *Para-CR structures of type* $(1, 1, 2)$.

Let $(M, [\lambda, \mu, \nu_1, \nu_2])$ be a general para-CR manifold of type $(1, 1, 2)$ with

$$\mathrm{d}\lambda \wedge \lambda \neq 0. \tag{3.1}$$

By Proposition 2.5, we can introduce a coordinate system (x, y, a_1, a_2) on M in which

$$\lambda = \mathrm{d}y - p(x, y, a_1, a_2)\mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}a_1, \qquad \nu_2 = \mathrm{d}a_2,$$

with some function p of the variables (x, y, a_1, a_2).

Our key question is if we can find new coordinates (x, y, y^1, y^2) on M, and functions h^α_β, c_a, F on M, so that the form $\nu'_1 = h^1_1\nu_1 + h^2_1\nu_2 + c_1\lambda$ is equal to

$$\nu'_1 = \mathrm{d}y^1 - y^2\mathrm{d}x$$

and the form $\nu'_2 = h^1_2\nu_1 + h^2_2\nu_2 + c_2\lambda$ is equal to

$$\nu'_2 = \mathrm{d}y^2 - F(x, y, y^1, y^2)\mathrm{d}x.$$

If this were possible, we could bring this para-CR structure, by a para-CR transformation, to the canonical form corresponding to the third order ODE $y''' = F(x, y, y', y'')$.

When looking for the desired coordinates (x, y, y^1, y^2) we proceed as follows: We set

$$y^1 = p(x, y, a_1, a_2),$$

and notice that (3.1) implies $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}y^1 \neq 0$. Thus the functions (x, y, y^1) can serve as three independent coordinates on M. The condition $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}y^1 \neq 0$ also means that at least one of the derivatives $\frac{\partial y^1}{\partial a_1}$ or $\frac{\partial y^1}{\partial a_2}$ is not equal to zero. Assuming, without loss of generality, that $\frac{\partial y^1}{\partial a_1} \neq 0$, we can solve $y^1 = p(x, y, a_1, a_2)$ for a_1 obtaining

$$a_1 = a_1(x, y, y^1, a_2).$$

This enables us to parametrize M by (x, y, y^1, a_2). In this new parametrization we have

$$\lambda = \mathrm{d}y - y^1\mathrm{d}x, \qquad \mu = \mathrm{d}x, \quad \nu_1 = \mathrm{d}[a_1(x, y, y^1, a_2)], \qquad \nu_2 = \mathrm{d}a_2.$$

We note that since

$$\nu_1 = \mathrm{d}[a_1(x, y, y^1, a_2)] = \frac{\partial a_1}{\partial x}\,\mathrm{d}x + \frac{\partial a_1}{\partial y}\,\mathrm{d}y + \frac{\partial a_1}{\partial y^1}\,\mathrm{d}y^1 + \frac{\partial a_1}{\partial a_2}\,\mathrm{d}a_2,$$

and $\lambda \wedge \mu \wedge \nu_1 \wedge \nu_2 \neq 0$, then $\mathrm{d}y \wedge \mathrm{d}x \wedge \frac{\partial a_1}{\partial y^1}\,\mathrm{d}y^2 \wedge \mathrm{d}a_2 \neq 0$, and hence $\frac{\partial a_1}{\partial y^1} \neq 0$. Thus we may replace the para-CR form ν_1 by the form

$$\nu_1' = \Big(\frac{\partial a_1}{\partial y^1}\Big)^{-1}\Big(\nu_1 - \frac{\partial a_1}{\partial a_2}\,\nu_2 - \frac{\partial a_1}{\partial y}\,\lambda\Big)$$

from the same para-CR class, obtaining

$$\nu_1' = \mathrm{d}y^1 - y^2\mathrm{d}x.$$

Here the function y^2 is given by

$$y^2 = -\Big(\frac{\partial a_1}{\partial x} + y^1\frac{\partial a_1}{\partial y}\Big)\Big(\frac{\partial a_1}{\partial y^1}\Big)^{-1}. \tag{3.2}$$

Summarizing, starting with an arbitrary type $(1, 1, 2)$ para-CR structure $(M, [\lambda, \mu, \nu_1, \nu_2])$, with $\mathrm{d}\lambda \wedge \lambda \neq 0$, we can always choose the coordinate system (x, y, y^1, a_2) and the representatives of the basis 1-forms, so that the para-CR structure is represented by

$$\lambda = \mathrm{d}y - y^1\mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - y^2\mathrm{d}x, \qquad \nu_2 = \mathrm{d}a_2,$$

with a function $y^2 = q(x, y, y^1, a_2)$ given by (3.2).

Now, two cases may occur:

- the general case, when $\frac{\partial y^2}{\partial a_2} \neq 0$, or
- the degenerate case, when $\frac{\partial y^2}{\partial a_2} = 0$.

In the general case, i.e. in the case when

$$\frac{\partial}{\partial a_2}\Big(\Big(\frac{\partial a_1}{\partial x} + y^1\frac{\partial a_1}{\partial y}\Big)\Big(\frac{\partial a_1}{\partial y^1}\Big)^{-1}\Big) \neq 0, \tag{3.3}$$

we can solve $y^2 = q(x, y, y^1, a_2)$ for a_2 obtaining

$$a_2 = a_2(x, y, y^1, y^2),$$

and a system of coordinates (x, y, y^1, y^2) on M, in which

$$\lambda = \mathrm{d}y - y^1\mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - y^2\mathrm{d}x, \qquad \nu_2 = \mathrm{d}[a_2(x, y, y^1, y^2)].$$

Now we have

$$\nu_2 = \frac{\partial a_2}{\partial x}\,\mathrm{d}x + \frac{\partial a_2}{\partial y}\,\mathrm{d}y + \frac{\partial a_2}{\partial y^1}\,\mathrm{d}y^1 + \frac{\partial a_2}{\partial y^2}\,\mathrm{d}y^2,$$

and since $\lambda \wedge \mu \wedge \nu_1 \wedge \nu_2 \neq 0$, we get $\frac{\partial a_2}{\partial y^2} \neq 0$. This enables us to replace ν_2 by another representative

$$\nu_2' = \Big(\frac{\partial a_2}{\partial y^2}\Big)^{-1}\Big(\nu_2 - \frac{\partial a_2}{\partial y^1}\,\nu_1 - \frac{\partial a_2}{\partial y}\,\lambda\Big),$$

which can be written as:

$$\nu_2' = \mathrm{d}y^2 - F(x, y, y^1, y^2)\mathrm{d}x,$$

with

$$F(x, y, y^1, y^2) = -\Big(\frac{\partial a_2}{\partial x} + y^1\frac{\partial a_2}{\partial y} + y^2\frac{\partial a_2}{\partial y^1}\Big)\Big(\frac{\partial a_2}{\partial y^2}\Big)^{-1}.$$

Summarizing we have the following proposition.

PROPOSITION 3.3. – *Every type* $(1, 1, 2)$ *para-CR structure* $(M, [\lambda, \mu, \nu_1, \nu_2])$ *with* $\mathrm{d}\lambda \wedge \lambda \neq 0$ *can be locally represented by* 1*-forms*

$$\lambda = \mathrm{d}y - y^1\mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - y^2\mathrm{d}x, \qquad \nu_2 = \mathrm{d}a_2,$$

with a function $y^2 = q(x, y, y^1, a_2)$ *of coordinates* (x, y, y^1, a_2) *on* M*. If, in addition, the function* y^2 *satisfies* $\frac{\partial y^2}{\partial a_2} \neq 0$ *in* $\mathcal{U} \subset \mathcal{M}$*, one can introduce a coordinate system* (x, y, y^1, y^2) *in* $\mathcal{U}$ *such that the para-CR structure can be represented by*

$$\lambda = \mathrm{d}y - y^1\mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - y^2\mathrm{d}x, \qquad \nu_2 = \mathrm{d}y^2 - F(x, y, y^1, y^2)\mathrm{d}x.$$

In such case the para-CR structure is locally para-CR equivalent to the canonical para-CR structure associated with a third order ODE $y''' = F(x, y, y', y'')$*.*

The nongeneric case in which (3.3) is not satisfied can be realized in several ways. The simplest of them is if $\frac{\partial y^2}{\partial a_2} \equiv 0$ in the neighbourhood $\mathcal{U} \subset M$. In such a case we have

$$\lambda = \mathrm{d}y - y^1\mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - q(x, y, y^1)\mathrm{d}x, \qquad \nu_2 = \mathrm{d}a_2,$$

and locally $\mathcal{U} = \mathcal{U}_3 \times \mathbb{R}$, where $\mathcal{U}_3$, parametrized by (x, y, y^1), is equipped with a canonical $(1,1,1)$ type para-CR structure of the second order ODE $y''=q(x,y,y')$. Thus in such a case the type $(1,1,2)$ para-CR structure is obtained by extending the canonical $(1,1,1)$ type para-CR structure of the equation $y''=q(x,y,y')$, from the first jet space $\mathcal{J}$ with the canonical forms $\lambda=\mathrm{d}y-y^1\mathrm{d}x$, $\mu=\mathrm{d}x$, $\nu_1=\mathrm{d}y^1-q(x,y,y^1)\mathrm{d}x$ to the Cartesian product $\mathcal{J}\times\mathbb{R}\xrightarrow{\pi}\mathcal{J}$. If $\mathbb{R}$ in $\mathcal{J}\times\mathbb{R}$ is parametrized by a_2, then the type $(1,1,2)$ para-CR structure on $\mathcal{J}\times\mathbb{R}$ is given by the class of para-CR forms $[\pi^*(\lambda),\pi^*(\mu),\pi^*(\nu_1),\nu_2=\mathrm{d}a_2]$. So also in this nongeneric case the para-CR structure $(M,[\lambda,\mu,\nu_1,\nu_2])$ is related to the canonical para-CR structure of an ODE, the only difference with the generic case is that now, the ODE is of lower order.

This discussion shows that, the structure of type $(1,1,2)$ para-CR manifolds may change from point to point: in some regions it is locally equivalent to a para-CR structure of a third order ODE, in some regions, to a para-CR which is Cartesian product of a para-CR structure of second order ODE and a real line.

To illustrate the discussion of this section we consider the following example.

EXAMPLE 3.4. – Consider $\mathbb{R}^4$ parametrized by (x,y,a_1,a_2) and a type $(1,1,2)$ para-CR structure on it given in terms of a function

$$p(x,y,a_1,a_2)=xa_1+ya_2.$$

By this we mean that the para-CR structure is defined in terms of the class of para-CR 1-forms $[\lambda,\mu,\nu_1,\nu_2]$ with representatives

$$\lambda=\mathrm{d}y-(xa_1+ya_2)\mathrm{d}x,\qquad \mu=\mathrm{d}x,\qquad \nu_1=\mathrm{d}a_1,\qquad \nu_2=\mathrm{d}a_2. \tag{3.4}$$

Proceeding as in our discussion above we define $y^1=xa_1+ya_2$, solve it for a_1,

$$a_1=\frac{y^1-ya_2}{x},$$

and use (x,y,y^1,a_2) as new coordinates, in which

$$\lambda=\mathrm{d}y-y^1\mathrm{d}x,\qquad \mu=\mathrm{d}x,\qquad \nu_1=\mathrm{d}\left[\frac{y^1-ya_2}{x}\right],\qquad \nu_2=\mathrm{d}a_2.$$

Now because $\nu_1=\frac{a_2y-y^1}{x^2}\mathrm{d}x-\frac{a_2}{x}\mathrm{d}y+\frac{\mathrm{d}y^1}{x}-\frac{y}{x}\mathrm{d}a_2$, we can replace ν_1 by a new form

$$\nu_1=\mathrm{d}y^1-\frac{a_2y^1x+y^1-a_2y}{x}\mathrm{d}x.$$

Introducing the function

$$y^2=a_2y^1+\frac{y^1-a_2y}{x}, \tag{3.5}$$

we see that we are in the situation $\frac{\partial y^2}{\partial a_2} = y^1 - \frac{y}{x} \neq 0$. So we can solve for a_2 obtaining:

$$a_2 = \frac{y^2 x - y^1}{y^1 x - y}.$$

Using the coordinates (x, y, y^1, y^2) we get

$$\lambda = \mathrm{d}y - y^1 \mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - y^2 \mathrm{d}x, \qquad \nu_2 = \mathrm{d}\left[\frac{y^2 x - y^1}{y^1 x - y}\right].$$

Expanding the differential we have

$$\nu_2 = \frac{x}{y^1 x - y}\,\mathrm{d}y^2 + \frac{y - y^2 x^2}{(y^1 x - y)^2}\,\mathrm{d}y^1 + \frac{(y^1)^2 - y^2 y}{(y^1 x - y)^2}\,\mathrm{d}x + \frac{y^2 x - y^1}{(y^1 x - y)^2}\,\mathrm{d}y \sim$$
$$\frac{x}{y^1 x - y}\left(\mathrm{d}y^2 + \frac{y^2(y^1 - y^2 x)}{y^1 x - y}\,\mathrm{d}x\right).$$

This means that locally the starting para-CR structure is equivalent to

$$\lambda = \mathrm{d}y - y^1 \mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - y^2 \mathrm{d}x, \qquad \nu_2 = \mathrm{d}y^2 - \frac{y^2(y^2 x - y^1)}{y^1 x - y}\,\mathrm{d}x,$$

and thus it comes from the third order ODE

$$y''' = \frac{y''(y'' x - y')}{y' x - y}. \tag{3.6}$$

To solve this equation we may use our result on local embeddability. We can start with any representation of the class $[\lambda, \mu, \nu_1, \nu_2]$, then find an embedding, and finally interpret it as a general solution to (3.6). It turns out that the simplest calculations are in the representation (3.4):

Obviously the two independent solutions $(f_1, \tilde{f}_1)$ of the embedding equations $\mathrm{d}f \wedge \lambda \wedge \mu \equiv 0$ are $f_1 = x$ and $\tilde{f}_1 = y$. Also, two independent solutions of the embedding equation $\mathrm{d}h \wedge \lambda \wedge \nu_1 \wedge \nu_2 \equiv \mathrm{d}h \wedge (\mathrm{d}y - (x a_1 + y a_2)\mathrm{d}x) \wedge \mathrm{d}a_1 \wedge \mathrm{d}a_2 \equiv 0$ are obviously $h_1 = a_2$ and $h_2 = a_2$. The third independent solution of this equation can be taken as: $\tilde{h}_1 = \mathrm{e}^{-a_2 x}\left(y + \frac{a_1}{a_2} x + \frac{a_1}{a_2^2}\right)$. Thus the embedding is given by:

$$\mathbb{R}^4 \ni (x, y, a_1, a_2) \to (x, y, a_0, a_1, a_2) = \left(x, y, \mathrm{e}^{-a_2 x}\left(y + \frac{a_1}{a_2} x + \frac{a_1}{a_2^2}\right), a_1, a_2\right) \in \mathbb{R}^{2+3},$$

which is a hypersurface in $\mathbb{R}^5$ with coordinates (x, y, a_0, a_1, a_2), given by

$a_0 \mathrm{e}^{a_2 x} = y + \frac{a_1}{a_2} x + \frac{a_1}{a_2^2}$. It is easy to check that, magically,

$$y = a_0 \mathrm{e}^{a_2 x} - \frac{a_1}{a_2} x - \frac{a_1}{a_2^2}$$

is the general solution to (3.6).

We end this example with a comment that if we had started with a function $p(x, y, a_1, a_2) = x a_1$, then our procedure would change after equation (3.5). In such case, the function y^2 would be independent of a_2 everywhere, and we would end up with

$$\lambda = \mathrm{d}y - y^1 \mathrm{d}x, \qquad \mu = \mathrm{d}x, \qquad \nu_1 = \mathrm{d}y^1 - \frac{y^1}{x} \mathrm{d}x, \qquad \nu_2 = \mathrm{d}a_2.$$

Thus the $(1,1,2)$ type para-CR structure $[\lambda = \mathrm{d}y - x a_1 \mathrm{d}x, \mu = \mathrm{d}x, \nu_1 = \mathrm{d}a_1, \nu_2 = \mathrm{d}a_2]$ would be equivalent to a Cartesian product of the canonical type $(1,1,1)$ para-CR structure of the second order ODE $y'' = \frac{y'}{x}$, and the real line represented by a_2.

4. – Para-CR structures of type $(n-1, 1, 1)$.

Returning to Example 2.4, and using the contact 1-forms (2.8) defining the canonical para-CR structure of type $(1, 1, n-1)$ corresponding to an ODE $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$, we can define another para-CR structure on the space $\mathcal{J}$ of $(n-1)$ jets. This para-CR structure is of type $(n-1, 1, 1)$, and is obtained from the contact forms $(l_1 = \lambda, l_2 = \nu_1, \dots, l_{n-1} = \nu_{n-2}, m = \mu, n = \nu_{n-1})$ as in (2.8) by extending them to a class $[l_1, \dots, l_{n-1}, m, n]$ via

$$\begin{aligned} &l_i \to l'_i = a_{ij} l_j, \\ &m \to m' = fm + b_i l_i, \\ &n \to n' = hn + c_i l_i \qquad i, j = 1, \dots, n-1 \end{aligned}$$

where the functions a_{ij}, b_i, c_i, f, h on $\mathcal{J}$ satisfy $\det(a) fh \neq 0$. Since this para-CR structure has $\dim H^+ = \dim H^- = 1$, the integrability conditions $[H^\pm, H^\pm] \subset H^\pm$ are automatically satisfied here.

EXAMPLE 4.1. – It is instructive to examine this para-CR structure in case of $n = 3$. In such case we have

(4.1) $\quad l_1 = \mathrm{d}y - y^1 \mathrm{d}x, \quad l_2 = \mathrm{d}y^1 - y^2 \mathrm{d}x, \quad n = \mathrm{d}y^2 - F(x, y, y^1, y^2) \mathrm{d}x, \quad m = \mathrm{d}x$

and

$$
\begin{aligned}
l'_1 &= a_{11} l_1 + a_{12} l_2, \\
l'_2 &= a_{21} l_1 + a_{22} l_2, \\
n' &= hn + c_1 l_1 + c_2 l_2, \\
m' &= fm + b_1 l_1 + b_2 l_2,
\end{aligned}
\tag{4.2}
$$

We now consider a *contact* transformation $(x, y, y^1, y^2) \to (\bar{x}, \bar{y}, \bar{y}^1, \bar{y}^2)$ of the variables of the corresponding third order ODE $y^{(3)} = F(x, y, y', y'')$. This changes the ODE to a new form $\bar{y}^{(3)} = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')$. It follows that, if we started with this equation and calculated the corresponding forms $(\bar{l}_1, \bar{l}_2, \bar{m}, \bar{n})$ as in (4.1), then these forms would be expressible in terms of forms (4.1) via

$$
\begin{aligned}
\bar{l}_1 &= a_{11} l_1, \\
\bar{l}_2 &= a_{21} l_1 + a_{22} l_2, \\
\bar{n} &= hn + c_1 l_1 + c_2 l_2, \\
\bar{m} &= fm + b_1 l_1 + b_2 l_2,
\end{aligned}
\tag{4.3}
$$

with functions a_{ij}, b_i, c_i, f and h which would depend on the particular form of the contact transformation we considered, and which would satisfy $\det(a) fh \neq 0$. Although transformation (4.3) seems to be more restrictive than the one in (4.2), it turns out that they are equivalent. Actually, it follows that starting with a general transformation (4.2) and forms (4.1) there is unique way of killing a_{12} in (4.2). This is done by observing that the most general forms (l'_1, l'_2, m', n') from (4.2) satisfy

$$
\mathrm{d}l'_1 \wedge l'_1 \wedge l'_2 = \frac{a_{12}}{fh}\, m' \wedge n' \wedge l'_1 \wedge l'_2 \,.
$$

Thus we can alsways normalize the transformation (4.2) to one in which $a_{12} = 0$. This proves the following proposition.

PROPOSITION 4.2. – *The local geometry of the type* $(2, 1, 1)$ *para-CR structure defined in* (4.1)-(4.2) *is identical to the local geometry of a general third order ODE* $y^{(3)} = F(x, y, y', y'')$ *considered modulo contact transformation of variables.*

The geometry described by the above proposition was studied by Chern [4] in the context of ODEs, and by Tanaka [19] in the context of para-CR structures. Actually Tanaka in [19] showed that the natural geometry associated with an n-th order ODE $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$, considered modulo *contact* transformations, is the geometry of type $(n-1, 1, 1)$ para-CR structures, which he called *pseudo-product* structures.

REMARK 4.3. – It is interesting to note that the passage from a $(1, 1, n-1)$ para-CR structure to a type $(n-1, 1, 1)$ para-CR structure, in the context of para-

CR structures associated with an ODE $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$, corresponds to the passage from the geometry of an ODE given modulo *point* transformations to the geometry of an ODE given modulo *contact* transformations. This is a first instance of a more general phenomenon, which will be discussed in Section 8.1.

5. – Invariants.

We are interested in objects naturally associated with a given para-CR manifold which are not changed under (local) para-CR diffeomorphisms. We call such objects (local) *invariants*. Clearly the simplest invariants of a para-CR manifold $(M, [(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)])$ are the integers (k, r, s). If $k = 1$, we have also another obvious invariant. This is defined as follows:

REMARK 5.1. – Note that the canonical $(1, 1, n-1)$ type para-CR structures corresponding to n-th order ODEs satisfy $\mathrm{d}\lambda \wedge \lambda \neq 0$ and $\mathrm{d}\lambda \wedge \mathrm{d}\lambda \wedge \lambda \equiv 0$. These conditions are invariant under para-CR transformations, since any such transformation brings $\lambda \to \lambda' = a\lambda$, with some $a \neq 0$. If we have a general $(1, r, s)$ type para-CR structure, a simple local invariant, is the *rank* of the para-CR form λ, i.e. the integer t, such that

$$\underbrace{\mathrm{d}\lambda \wedge \dots \wedge \mathrm{d}\lambda}_{t \text{ times}} \wedge \lambda \neq 0 \qquad \text{and} \qquad \underbrace{\mathrm{d}\lambda \wedge \dots \wedge \mathrm{d}\lambda}_{(t+1) \text{ times}} \wedge \lambda \equiv 0.$$

Immediately there are two questions:

- Are the numbers (k, r, s), (or t when $k = 1$), the *only* local invariants of $(M, [(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)])$?
- And if the answer to the above question is negative, how does one construct the system of *all* local invariants of $(M, [(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_r, \nu_1, \dots, \nu_s)])$?

It is rather obvious that the answer for the first question above is ‘no’. We are thus led to discuss how to construct the invariants. We do not make an exhaustive discussion in the following. Instead we concentrate on low dimensional cases, producing invariants for structures of type $(1, 1, 2)$ and $(1, 2, 3)$. These examples are complicated enough to illustrate the basic features that the general case can have.

5.1 – *Local invariants for para-CR structures of type* $(1, 1, 2)$.

In Section 3.1 we proved that every para-CR structure of type $(1, 1, 1)$ for which $\mathrm{d}\lambda \wedge \lambda \neq 0$, is locally para-CR equivalent to a second order ODE considered modulo point transformation of variables. Thus all local invariants for such para-CR structures are in one-to-one correspondence with the local invariants of second oder ODEs considered modulo point transformations. All such invariants are known since the times of the classical papers of Lie [11], Tresse [20] and Cartan [3]. We refer an interested reader to the para-CR treatment of these invariants in [16]. Since we will need some results about the $(1,1,1)$ case in the following, we quote them here for completeness.

5.1.1 – Brief summary of the $(1,1,1)$ case.

As we know (see Proposition 2.5, or the proof of Proposition 3.2) every para-CR structure $(M,[\lambda,\mu,\nu])$ of type $(1,1,1)$ with $\mathrm{d}\lambda\wedge\lambda\neq 0$ can be locally represented by

$$\lambda=\mathrm{d}y-p(x,y,a_1)\mathrm{d}x,\qquad \mu=\mathrm{d}x,\qquad \nu=\mathrm{d}a_1,$$

with a function $p=p(x,y,a_1)$ of variables (x,y,a_1) on M such that $p_1=\dfrac{\partial p}{\partial a_1}\neq 0$. Consider now the most general forms $(\theta^0,\theta^1,\theta^3)\in[\lambda,\nu,\mu]$ in the class $[\lambda,\nu,\mu]$. They are given on M by:

$$\theta^0=a\lambda,\qquad \theta^1=c_1\lambda+h_{11}\nu,\qquad \theta^3=b\lambda+f\mu,$$

with some functions a, h_{11}, c_1, f and b such that $ah_{11}f\neq 0$ (the strange numbering of the forms will become clear in the next section). Extending the manifold M to $M\times G$, where G is parametrized by (a,h_{11},c_1,f,b), we can apply Cartan's equivalence method to find the invariants of such structures. This was done by Cartan in [3]. His result adapted to our situation is summarized in the following proposition.

PROPOSITION 5.2. – *Every para-CR manifold $(M,[\lambda,\mu,\nu])$ of type $(1,1,1)$ with $\mathrm{d}\lambda\wedge\lambda\neq 0$ uniquely defines an 8-dimensional manifold P with a unique coframe $(\theta^0,\theta^1,\theta^3,\Omega_1,\Omega_2,\Omega_3,\Omega_7,\Omega_8)$ on it, which satisfies the following equations*

$$\begin{aligned}
\mathrm{d}\theta^0&=\Omega_1\wedge\theta^0+\theta^3\wedge\theta^1\\
\mathrm{d}\theta^1&=\Omega_2\wedge\theta^0+\Omega_3\wedge\theta^1\\
\mathrm{d}\theta^3&=(\Omega_1-\Omega_3)\wedge\theta^3+\Omega_7\wedge\theta^0\\
\mathrm{d}\Omega_1&=2\Omega_8\wedge\theta^0+\Omega_7\wedge\theta^1-\Omega_2\wedge\theta^3\\
\mathrm{d}\Omega_2&=(\Omega_3-\Omega_1)\wedge\Omega_2+\Omega_8\wedge\theta^1+K\theta^0\wedge\theta^3\\
\mathrm{d}\Omega_3&=\Omega_8\wedge\theta^0+2\Omega_7\wedge\theta^1+\Omega_2\wedge\theta^3\\
\mathrm{d}\Omega_7&=\Omega_7\wedge\Omega_3+\Omega_8\wedge\theta^3+J\theta^0\wedge\theta^1\\
\mathrm{d}\Omega_8&=\Omega_8\wedge\Omega_1+\Omega_7\wedge\Omega_2+\frac{\partial J}{\partial\theta^3}\theta^1\wedge\theta^0+\frac{\partial K}{\partial\theta^1}\theta^3\wedge\theta^0.
\end{aligned}\tag{5.1}$$

Here the functions J and K are given by:

$$
\begin{aligned}
&(6fh_{11}^3p_1^4)\ J = \\
&-15p_{11}^3p_{x1}+10p_1p_{11}p_{111}p_{x1}p_{x1}+15p_1p_{11}^2p_{x11}-4p_1^2p_{111}p_{x11}\\
&+12p_1^2p_{11}^2p_{y1}-15pp_{11}^3p_{y1}-4p_1^3p_{111}p_{y1}+10pp_1p_{11}p_{111}p_{y1}\\
&-12p_1^3p_{11}p_{y11}+15pp_1p_{11}^2p_{y11}-4pp_1^2p_{111}p_{y11}-6p_1^2p_{11}p_{x111}\\
&+4p_1^2(p_1^2-\frac{3}{2}pp_{11})p_{y111}-p_1^2(1+pp_{y1})p_{1111}+p_1^3(p_{x1111}+pp_{y1111})
\end{aligned}
$$

and

$$
\begin{aligned}
&(6f^3h_{11}p_1^4)\ K = \\
&-15p_{11}p_{x1}^3+15p_1p_{x1}^2p_{x11}+10p_1p_{11}p_{x1}p_{xx1}-4p_1^2p_{x11}p_{xx1}\\
&-6p_1^2p_{x1}p_{xx11}-p_1^2p_{11}p_{xxx1}+p_1^3p_{xxx11}-2p_1^4p_{xxy1}-3pp_1^2p_{11}p_{xxy1}\\
&+3pp_1^3p_{xxy11}-p_1^2p_{11}p_{x1}p_{xy}+p_1^3p_{x11}p_{xy}-3p_1^2p_{11}p_xp_{xy1}+6p_1^3p_{x1}p_{xy1}\\
&+20pp_1p_{11}p_{x1}p_{xy1}-8pp_1^2p_{x11}p_{xy1}+3p_1^3p_xp_{xy11}-12pp_1^2p_{x1}p_{xy11}+2p_1^5p_{xyy}\\
&-4pp_1^4p_{xyy1}-3p^2p_1^2p_{11}p_{xyy1}+3p^2p_1^3p_{xyy11}+10p_1p_{11}p_{x1}^2p_y-10p_1^2p_{x1}p_{x11}p_y\\
&-3p_1^2p_{11}p_{xx1}p_y+3p_1^3p_{xx11}p_y-6p_1^4p_{xy1}p_y-9pp_1^2p_{11}p_{xy1}p_y+9pp_1^3p_{xy11}p_y\\
&-2p_1^2p_{11}p_{x1}p_y^2+2p_1^3p_{x11}p_y^2+10p_1p_{11}p_xp_{x1}p_{y1}-6p_1^2p_{x1}^2p_{y1}-45pp_{11}p_{x1}^2p_{y1}\\
&-4p_1^2p_xp_{x11}p_{y1}+30pp_1p_{x1}p_{x11}p_{y1}-p_1^2p_{11}p_{xx}p_{y1}+2p_1^3p_{xx1}p_{y1}\\
&+10pp_1p_{11}p_{xx1}p_{y1}-6pp_1^2p_{xx11}p_{y1}-2p_1^4p_{xy}p_{y1}-3pp_1^2p_{11}p_{xy}p_{y1}\\
&+10pp_1^3p_{xy1}p_{y1}+20p^2p_1p_{11}p_{xy1}p_{y1}-12p^2p_1^2p_{xy11}p_{y1}-4p_1^2p_{11}p_xp_yp_{y1}\\
&+8p_1^3p_{x1}p_yp_{y1}+30pp_1p_{11}p_{x1}p_yp_{y1}-14pp_1^2p_{x11}p_yp_{y1}-4p_1^4p_y^2p_{y1}\\
&-6pp_1^2p_{11}p_y^2p_{y1}+2p_1^3p_xp_{y1}^2+10pp_1p_{11}p_xp_{y1}^2-12pp_1^2p_{x1}p_{y1}^2\\
&-45p^2p_{11}p_{x1}p_{y1}^2+15p^2p_1p_{x11}p_{y1}^2+10pp_1^3p_yp_{y1}^2+20p^2p_1p_{11}p_yp_{y1}^2-6p^2p_1^2p_{y1}^3\\
&-15p^3p_{11}p_{y1}^3-6p_1^2p_xp_{x1}p_{y11}+15pp_1p_{x1}^2p_{y11}+p_1^3p_{xx}p_{y11}-4pp_1^2p_{xx1}p_{y11}\\
&+3pp_1^3p_{xy}p_{y11}-8p^2p_1^2p_{xy1}p_{y11}+4p_1^3p_xp_yp_{y11}-16pp_1^2p_{x1}p_yp_{y11}+6pp_1^3p_y^2p_{y11}\\
&-10pp_1^2p_xp_{y1}p_{y11}+30p^2p_1p_{x1}p_{y1}p_{y11}-20p^2p_1^2p_yp_{y1}p_{y11}+15p^3p_1p_{y1}^2p_{y11}\\
&-2p_1^4p_{x1}p_{yy}-pp_1^2p_{11}p_{x1}p_{yy}+pp_1^3p_{x11}p_{yy}+4p_1^5p_yp_{yy}-4pp_1^4p_{y1}p_{yy}\\
&-2p^2p_1^2p_{11}p_{y1}p_{yy}+2p^2p_1^3p_{y11}p_{yy}-2p_1^4p_xp_{yy1}-3pp_1^2p_{11}p_xp_{yy1}+6pp_1^3p_{x1}p_{yy1}\\
&+10p^2p_1p_{11}p_{x1}p_{yy1}-4p^2p_1^2p_{x11}p_{yy1}-8pp_1^4p_yp_{yy1}-6p^2p_1^2p_{11}p_yp_{yy1}\\
&+8p^2p_1^3p_{y1}p_{yy1}+10p^3p_1p_{11}p_{y1}p_{yy1}-4p^3p_1^2p_{y11}p_{yy1}+3pp_1^3p_xp_{yy11}\\
&-6p^2p_1^2p_{x1}p_{yy11}+6p^2p_1^3p_yp_{yy11}-6p^3p_1^2p_{y1}p_{yy11}+2pp_1^5p_{yyy}-2p^2p_1^4p_{yyy1}\\
&-p^3p_1^2p_{11}p_{yyy1}+p^3p_1^3p_{yyy11},
\end{aligned}
$$

and $\frac{\partial J}{\partial \theta^3}$ *and* $\frac{\partial K}{\partial \theta^1}$ *denote the coframe derivatives of functions* J *and* K *with respect to the coframe element* θ^3 *and* θ^1*, respectively.*

Two type $(1,1,1)$ *para-CR manifolds* $(M,[\lambda,\mu,\nu])$ *and* $(M',[\lambda',\mu',\nu'])$*, with* $\mathrm{d}\lambda\wedge\lambda\neq 0$ *and* $\mathrm{d}\lambda'\wedge\lambda'\neq 0$ *are locally para-CR equivalent iff there exists a local diffeomorphism* $\phi : P \to P'$*, of the corresponding* 8*-manifolds* P *and* P'*, which pulls back the coframe* $(\theta'^0,\theta'^1,\theta'^3,\ \Omega'_1,\Omega'_2,\Omega'_3,\Omega'_7,\Omega'_8)$ *to* $(\theta^0,\theta^1,\theta^3,$ $\Omega_1,\Omega_2,\Omega_3,\Omega_7,\Omega_8)$*.*

In particular the vanishing of each of the functions J and K is a para-CR invariant property. These functions are para-CR versions of the classical two point invariants w_1 and w_2 (see [16]) of the corresponding second order ODE, which were known to Lie and Tresse [11, 20]. This proposition solves the local equivalence problem for type $(1,1,1)$ para-CR structures: they are either locally equivalent to $[\lambda=\mathrm{d}y,\mu=\mathrm{d}x,\nu=\mathrm{d}a_2]$, or they are described by the above proposition.

5.1.2 – The simplest relative invariant for type $(1,1,2)$.

Passing to the $(1,1,2)$ case we consider a para-CR structure $(M,[\lambda,\mu,\nu_1,\nu_2])$, and since all para-CR structures with $\mathrm{d}\lambda\wedge\lambda\equiv 0$ are locally equivalent to $(\mathbb{R}^{(1+1+2)},[\lambda=\mathrm{d}y,\mu=\mathrm{d}x,\nu_1=\mathrm{d}a_1,\nu_2=\mathrm{d}a_2])$, we will assume $\mathrm{d}\lambda\wedge\lambda\neq 0$ in the following. As at the begining of Section 3.2 we may introduce a local coordinate system (x,y,a_1,a_2) on M so that the para-CR structure is represented by

$$\lambda=\mathrm{d}y-p(x,y,a_1,a_2)\mathrm{d}x,\qquad \mu=\mathrm{d}x,\qquad \nu_1=\mathrm{d}a_1,\qquad \nu_2=\mathrm{d}a_2.$$

Here p is an appropriate function $p=p(x,y,a_1,a_2)$ on M which satisfies $\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}p\neq 0$. Without loss of generality we can assume in the following that

$$p_1=\frac{\partial p}{\partial a_1}\neq 0.$$

Now we introduce the most general forms $(\lambda',\mu',\nu'_1,\nu'_2)$ from the class $[\lambda,\mu,\nu_1,\nu_2]$. These are:

$$(5.2)\qquad \begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \mu\end{pmatrix}\to\begin{pmatrix}\lambda'\\ \nu'_1\\ \nu'_2\\ \mu'\end{pmatrix}=\begin{pmatrix}a&0&0&0\\ c_1&h_{11}&h_{12}&0\\ c_2&h_{21}&h_{22}&0\\ b&0&0&f\end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \mu\end{pmatrix}\stackrel{\mathrm{def}}{=}\begin{pmatrix}\theta^0\\ \theta^1\\ \theta^2\\ \theta^3\end{pmatrix}.$$

Now we are in a position to determine the first relative invariant. We do it using Cartan's equivalence method (see e.g. [17]) in the following steps:

(1) We first calculate the invariant form $\mathrm{d}\theta^0\wedge\theta^0$. This is given by

$$\mathrm{d}\theta^0\wedge\theta^0=\frac{a(h_{22}p_1-h_{21}p_2)}{f(h_{12}h_{21}-h_{11}h_{22})}\theta^0\wedge\theta^1\wedge\theta^3+\frac{a(h_{11}p_2-h_{12}p_1)}{f(h_{12}h_{21}-h_{11}h_{22})}\theta^0\wedge\theta^2\wedge\theta^3.$$

(Here and in the following the partial derivatives with respect to a_i are denoted by a subscript i at the differentiated function; derivatives with respect to x and y are denoted by the respective subscript x or y.)

(2) Then we impose the invariant condition $\mathrm{d}\theta^0\wedge\theta^0=-\theta^0\wedge\theta^1\wedge\theta^3$. This is achieved by taking

$$a=\frac{h_{11}f}{p_1}\qquad\text{and}\qquad h_{12}=\frac{h_{11}p_2}{p_1}. \tag{5.3}$$

(3) Then, on an 11-dimensional manifold $M^{(1)}$ parametrized by $(x,y,a_1,a_2,c_1,c_2,h_{11},h_{21},h_{22},b,f)$, we introduce a 1-form Ω_1 so that we have

$$\mathrm{d}\theta^0=\Omega_1\wedge\theta^0+\theta^3\wedge\theta^1.$$

The form Ω_1 is given by

$$\Omega_1=\frac{\mathrm{d}f}{f}+\frac{\mathrm{d}h_{11}}{h_{11}}-\frac{\mathrm{d}p_1}{p_1}+\frac{bp_1}{h_{11}f}\theta^1+\frac{h_{11}p_y-c_1p_1}{h_{11}f}\theta^3+f_0\theta^0, \tag{5.4}$$

where f_0 is an additional function on $M^{(1)}$.

(4) It is easy to check that at this stage we have

$$\mathrm{d}\theta^1\wedge\theta^0\wedge\theta^1=-I\frac{h_{11}}{p_1(h_{22}p_1-h_{21}p_2)f}\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3,$$

where

$$I=p_1(p_{x2}+pp_{y2})-p_2(p_{x1}+pp_{y1}).$$

Comparing this with

$$\det\begin{pmatrix}a&0&0&0\\c_1&h_{11}&h_{12}&0\\c_2&h_{21}&h_{22}&0\\b&0&0&f\end{pmatrix}=(h_{22}p_1-h_{21}p_2)f^2h_{11}^2p_1^{-1}\neq0,$$

we see that the condition that I vanishes or not is a para-CR invariant property of the class $[\lambda,\mu,\nu_1,\nu_2]$. This shows that I is a *relative invariant* for the considered para-CR structure.

(5) For example if $I\neq0$ in the considered neighborhood, we can normalize $\mathrm{d}\theta^1\wedge\theta^0\wedge\theta^1$ to $\mathrm{d}\theta^1\wedge\theta^0\wedge\theta^1=-\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3$, by choosing

$$h_{11}=\frac{(h_{22}p_1-h_{21}p_2)p_1f}{I}.$$

Such a normalization is obviously impossible if $I=0$ in the considered region.

(6) It can be checked that it is this invariant that distinguishes between the $(1,1,2)$ para-CR structures that correspond to the extension of $(1,1,1)$ type structures by $\mathbb{R}$ and the type $(1,1,2)$ para-CR structures equivalent to the canonical para-CR structures corresponding to third order ODEs.

5.1.3 – Branch $I \neq 0$.

Actually, further application of Cartan's equivalence method proves the following theorem.

THEOREM 5.3. – *Every type* $(1,1,2)$ *para-CR structure* $(M,[\lambda,\mu,\nu_1,\nu_2])$ *with* $\mathrm{d}\lambda \wedge \lambda \neq 0$ *for which the invariant* I *is non vanishing, is locally para-CR equivalent to a canonical para-CR structure of a certain point equivalence class of* 3*-rd order ODEs* $y''' = F(x,y,y',y'')$.

In particular, if $I \neq 0$, all the local invariants of such para-CR structures are identical with the local *point* invariants of the corresponding point equivalence classes of 3rd order ODEs. For example the lowest order relative invariant, next after I, is the Wünschmann invariant [21] of the corresponding class of ODEs. This can be written explicitly in terms of the function $p = p(x,y,a_1,a_2)$ used above. Although we calculated this invariant in terms of p we do not display it here. It is given by quite a lengthy and complicated expression in terms of p and its derivatives up to the 5th order.

The above proposition enables us to find the para-CR structures with $I \neq 0$ and large symmetry groups. Since third order ODEs with large symmetry groups of point symmetries are classified in [8, 9], we know that such para-CR manifolds have a maximal group of para-CR symmetries of dimension *seven*. They are locally para-CR equivalent to the para-CR structure corresponding to the point equivalent class of the simple equation $y''' = 0$. The $I \neq 0$ para-CR structures with a group of symmetries of dimension 6, 5 and 4 are also easily obtained from the results of [8, 9]. We have the following proposition.

PROPOSITION 5.4. – *All homogeneous type* $(1,1,2)$ *para-CR structures are locally para-CR equivalent to the canonical para-CR structure of the following point equivalent classes of 3rd order ODEs:*

- $y''' = 0$*; in this case the symmetry algebra is* $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$ *of dimension* 7;
- $y''' = \frac{3}{2}\frac{(y'')^2}{y'}$*; symmetry algebra* $\mathfrak{o}(2,2)$ *of dimension* 6;
- $y''' = \frac{3(y'')^2 y'}{1+(y')^2}$*; symmetry algebra* $\mathfrak{o}(4)$ *of dimension* 6;

- $y''' = -2\mu y' + y$; *each* $\mu \in \mathbb{R}$ *defines a nonequivalent para-CR structure with a* 5*-dimensional symmetry algebra, with generators* V_i *satisfying* $[V_1, V_4] = -\mu V_2 + V_3$, $[V_1, V_5] = V_1$, $[V_2, V_4] = V_1 - \mu V_3$, $[V_2, V_5] = V_2$ $[V_3, V_4] = V_2$, $[V_3, V_5] = V_3$;
- $y''' = (y'')^3$; *symmetry algebra of dimension 4 with generators* V_i *satisfying* $[V_1, V_4] = 2V_1$, $[V_2, V_4] = \frac{4}{3}V_2$, $[V_2, V_3] = V_1$, $[V_3, V_4] = -\frac{2}{3}V_3$;
- $y''' = \mu\frac{(y'')^2}{y'}$; *here each* $\mu > \frac{3}{2}$ *such that* $\mu \neq 3$, *defines a nonequivalent para-CR structure having a 4-dimensional symmetry algebra, with generators* V_i *satisfying* $[V_1, V_2] = V_1$, $[V_3, V_4] = V_3$;
- $y''' = \frac{3y' + \mu}{1 + (y')^2}$; *for each* $\mu > 0$ *we have a nonequivalent para-CR structure with a* 4*-dimensional symmetry algebra; its generators* V_i *satisfy* $[V_1, V_2] = V_3$, $[V_3, V_1] = V_2$, $[V_3, V_4] = V_3$, $[V_2, V_4] = V_2$.

5.1.4 – Branch $I \equiv 0$.

This case is a bit easier to describe explicitly than the above $I \neq 0$ case. Thus we choose this case to present all the details of constructing invariants for such para-CR structures, rather then those with $I \neq 0$.

When constructing these invariants we proceed as follows:

Starting with the defining forms ($l = \mathrm{d}y - p\,\mathrm{d}x, n_1 = \mathrm{d}a_1, n_2 = \mathrm{d}a_2, m = \mathrm{d}x$) as in (4.1), for which the function $p = p(x, y, a_1, a_2)$ satisfies

$$p_1 \neq 0 \qquad \text{and} \qquad I \equiv 0,$$

we consider the most general forms $(\theta^0, \theta^1, \theta^2, \theta^3)$ from the class $[l, n_1, n_2, m]$ as in (5.2). Then we repeat the entire Cartan's procedure for these forms we performed in Section 5.1.2 from item (1) up to item (4). After this we have forms θ^0 and θ^1 normalized so that

$$\mathrm{d}\theta^0 = \Omega_1 \wedge \theta^0 + \theta^3 \wedge \theta^1$$

and

$$\mathrm{d}\theta^1 \wedge \theta^0 \wedge \theta^1 = 0.$$

This second equation holds since we assumed that

$$I \equiv 0.$$

The form Ω_1 is given by (5.4), and the normalizations for a and h_{12} are as in (5.3). Continuing with Cartan's equivalence method we now make the following steps:

- First we introduce forms Ω_2 and Ω_3 so that the form θ^1 satisfies:

$$\mathrm{d}\theta^1 = \Omega_2 \wedge \theta^0 + \Omega_3 \wedge \theta^1.$$

This defines forms Ω_2 and Ω_3 to be:

$$\Omega_2 = \frac{p_1 \mathrm{d}c_1}{f h_{11}} - \frac{c_1 p_1 \mathrm{d}h_{11}}{f h_{11}^2} + \frac{c_1 p(p_1 p_{12} - p_{11} p_2) + h_{11}(p_1 p_{x2} - p_2 p_{x1})}{f h_{11} p(h_{22} p_1 - h_{21} p_2)} \theta^2$$
$$- \frac{c_1 p_1 (c_1 p_1 - h_{11} p_y)}{f^2 h_{11}^2} \theta^3 + c_{10}\theta^0 + c_{11}\theta^1,$$

$$\Omega_3 = \mathrm{d}\log(h_{11})$$
$$+ \Big(\frac{c_{11} f^2 h_{11}^2 - b c_1 p_1^2}{f^2 h_{11}^2} + \frac{c_2 p(p_1 p_{12} - p_{11} p_2) + h_{21}(p_1 p_{x2} - p_2 p_{x1})}{f h_{11} p(h_{22} p_1 - h_{21} p_2)} \Big) \theta^0$$
$$+ \frac{p_{11} p_2 - p_{12} p_1}{p_1(h_{22} p_1 - h_{21} p_2)} \theta^2 + \frac{c_1 p_1}{f h_{11}} \theta^3 + h_{111}\theta^1.$$

As we see the forms Ω_2 and Ω_3 are defined modulo the terms θ^0 and θ^1 (the form Ω_2), and θ^1 (the form Ω_3), respectively. Thus to write them down in full generality one has to introduce additional parameters c_{10}, c_{11} and h_{111}.

- At the next step we introduce forms Ω_4, Ω_5 and Ω_6 such that the form θ^2 satisfies

$$\mathrm{d}\theta^2 = \Omega_6 \wedge \theta^0 - \Omega_5 \wedge \theta^1 + \Omega_4 \wedge \theta^2.$$

These forms are defined as follows:

$$\Omega_4 = \frac{p_2 \mathrm{d}h_{21}}{h_{21} p_2 - h_{22} p_1} + \frac{p_1 \mathrm{d}h_{22}}{h_{22} p_1 - h_{21} p_2} + h_{220}\theta^0 + h_{221}\theta^1 + h_{222}\theta^2$$

$$\Omega_5 = -\frac{\mathrm{d}h_{21}}{h_{11}} + \frac{h_{21}}{h_{11}}\Omega_4 - \frac{(h_{11} h_{221} + h_{21} h_{222})}{h_{11}} \theta^2 - \frac{c_2 p_1}{f h_{11}} \theta^3 + h_{210}\theta^0 + h_{211}\theta^1$$

$$\Omega_6 = \frac{p_1}{f h_{11}} \mathrm{d}c_2 - \frac{c_2 p_1}{f h_{11}} \Omega_4 + \frac{c_1 p_1}{f h_{11}} \Omega_5$$
$$- \frac{f^2 h_{11}(h_{11} h_{210} + h_{21} h_{220}) + p_1 \big(c_1 f (h_{11} h_{211} + h_{21} h_{221}) - c_2 (b p_1 + f h_{11} h_{221}) \big)}{f^2 h_{11}^2} \theta^1$$
$$+ \Big(h_{220} + \frac{p_1 (c_1 h_{221} + c_2 h_{222})}{f h_{11}} \Big) \theta^2 + \frac{c_2 p_1 p_y}{f^2 h_{11}} \theta^3 - c_{20}\theta^0.$$

Here we had to introduce new parameters h_{220}, h_{221}, h_{222}, h_{210}, h_{211} and c_{20}, which take care of the undefined terms in the expressions for Ω_4, Ω_5, and Ω_6.

- Analysing $\mathrm{d}\theta^3$ we first observe that

$$\mathrm{d}\theta^3\wedge\theta^0=(\Omega_3-\Omega_1)\wedge\theta^0\wedge\theta^3$$
$$+\frac{fh_{11}h_{111}(h_{22}p_1-h_{21}p_2)+h_{22}(fp_{11}-2bp_1^2)+h_{21}(2bp_1p_2-fp_{12})}{fh_{11}(h_{22}p_1-h_{21}p_2)}\theta^0\wedge\theta^1\wedge\theta^3.$$

  This enables us to fix h_{111}:

$$h_{111}=\frac{h_{22}(2bp_1^2-fp_{11})+h_{21}(fp_{12}-2bp_1p_2)}{fh_{11}(h_{22}p_1-h_{21}p_2)}.$$

- After this normalization an introduction of a form

$$\Omega_7=\frac{p_1\mathrm{d}b}{fh_{11}}+\frac{bp_1}{fh_{11}}(\Omega_3-\Omega_1)+b_0\theta^0-\Big(c_{11}-f_0+\frac{2bc_1p_1^2}{f^2h_{11}^2}$$
$$+\frac{b}{f^2h_{11}}(p_{x1}-2p_1p_y+pp_{y1})+\frac{c_2p(p_1p_{12}-p_2p_{11})+h_{21}(p_1p_{x2}-p_2p_{x1})}{fh_{11}p(h_{22}p_1-h_{21}p_2)}\Big)\theta^3,$$

  brings $\mathrm{d}\theta^3$ into the form:

$$\mathrm{d}\theta^3=\Omega_7\wedge\theta^0+(\Omega_1-\Omega_3)\wedge\theta^3.$$

  Again we had to introduce a new parameter which we denoted by b_0 here.

Summarizing our efforts in this section so far, we conclude that the invariant forms $\theta^0,\theta^1,\theta^2,\theta^3$ of a para-CR structure with $I\equiv 0$ can be gauged in such a way that they have the following differentials:

$$\begin{aligned}\mathrm{d}\theta^0&=\Omega_1\wedge\theta^0+\theta^3\wedge\theta^1\\ \mathrm{d}\theta^1&=\Omega_2\wedge\theta^0+\Omega_3\wedge\theta^1\\ \mathrm{d}\theta^2&=\Omega_6\wedge\theta^0-\Omega_5\wedge\theta^1+\Omega_4\wedge\theta^2\\ \mathrm{d}\theta^3&=\Omega_7\wedge\theta^0+(\Omega_1-\Omega_3)\wedge\theta^3.\end{aligned}\tag{5.5}$$

Now we pass to the analysis of this system in terms of Cartan's characters and Cartan's test for the involutivity (see [17], pp. 350-355 for definitions; for a one page description of the procedure see e.g. [15], pp. 4066-4067). Since the forms θ^i are given up to the action of the residual ($r=7$)-dimensional group parametrized by $f,b,c_1,c_2,h_{11},h_{21},h_{22}$, we easily calculate the four Cartan characters associated to this system. They are $s_1'=4$, $s_2'=2$, $s_3'=1$, $s_4'=0$. Moreover, since the new forms $\Omega_1,\Omega_2,\Omega_3,\Omega_4,\Omega_5,\Omega_6,\Omega_7$ transversal to the respective residual group directions ∂_f, ∂_{c_1}, $\partial_{h_{11}}$, $\partial_{h_{22}}$, $\partial_{h_{21}}$, ∂_{c_2}, ∂_b, are determined modulo $r^{(1)}=10$ parameters f_0, c_{10}, c_{11}, h_{220}, h_{221}, h_{222}, h_{210}, h_{211}, c_{20}, b_0, we have

$$1s_1'+2s_2'+3s_3'+4s_4'=11\neq 10=r^{(1)}.$$

Thus the system (5.5) is *not* involutive, and has to be prolonged. Calculating $d\Omega_1$, $d\Omega_2$, $d\Omega_3$, $d\Omega_7$ we fix c_{10}, c_{11} and b_0 in such a way that the forms Ω_1, Ω_2, Ω_3 and Ω_7 satisfy:

$$\begin{aligned} d\Omega_1 &= 2\Omega_8\wedge\theta^0+\Omega_7\wedge\theta^1-\Omega_2\wedge\theta^3\\ d\Omega_2 &= \Omega_2\wedge(\Omega_1-\Omega_3)+\Omega_8\wedge\theta^1+K\theta^0\wedge\theta^3\\ d\Omega_3 &= \Omega_8\wedge\theta^0+2\Omega_7\wedge\theta^1+\Omega_2\wedge\theta^3\\ d\Omega_7 &= \Omega_7\wedge\Omega_3+\Omega_8\wedge\theta^3+J\theta^0\wedge\theta^1. \end{aligned}\tag{5.6}$$

Here the form Ω_8 and functions J and K are totally determined by the above equations. The form Ω_8 is given by:

$$2\Omega_8 = df_0+f_0\Omega_1+\Big(\frac{bp_1}{fh_{11}}+\frac{h_{21}p_{12}-h_{22}p_{11}}{h_{11}(h_{22}p_1-h_{21}p_2)}\Big)\Omega_2-\frac{p_1p_{12}-p_2p_{11}}{p_1(h_{22}p_1-h_{21}p_2)}\Omega_6$$

$$-\frac{c_1p_1^2+h_{11}p_{x1}-h_{11}p_1p_y+h_{11}pp_{y1}}{fh_{11}p_1}\Omega_7+(\dots)\theta^0+(\dots)\theta^1+(\dots)\theta^2+(\dots)\theta^3,$$

where we skip writing down very compliceted, yet still *totally determined*, coefficients at the terms θ^0, θ^1, θ^2 and θ^3. It turns out, *and this is the result of our calculations*, that the functions J and K are given by *the same formulae* as in Proposition 5.2. This is not surprising, if one notices the identical forms of the systems (5.5)-(5.6) and (5.1) with the equation for $d\theta^2$ and $d\Omega_8$ removed. Actually, after calculating $d\Omega_8$ in the present situation, we get

$$d\Omega_8 = \Omega_8\wedge\Omega_1+\Omega_7\wedge\Omega_2+\frac{\partial J}{\partial\theta^3}\theta^1\wedge\theta^0+\frac{\partial K}{\partial\theta^1}\theta^3\wedge\theta^0,\tag{5.7}$$

which again agrees with the system (5.1). Now we are ready to perform the Cartan analysis of the the composed system (5.5)-(5.7). We have here $m=4+5$ differentials $d\theta^0$, $d\theta^1$, $d\theta^2$, $d\theta^3$, $d\Omega_1$, $d\Omega_2$, $d\Omega_3$, $d\Omega_7$, $d\Omega_8$, of the forms θ^0, θ^1, θ^2, θ^3, Ω_1, Ω_2, Ω_3, Ω_4, Ω_5, which are given modulo the ($r=3$)-dimensional residual group parametrized by c_2, h_{21} and h_{22}. The new forms Ω_4, Ω_5 and Ω_6, transversal to the respective vector fields $\partial_{h_{22}}$, $\partial_{h_{21}}$ and ∂_{c_2} are given up to $r^{(1)}=6$ parameters h_{220}, h_{221}, h_{222} (Ω_4), h_{210}, h_{211} (Ω_5), and c_{20} (Ω_6). Simple linear algebra gives the following Cartan's characters of the system (5.5)-(5.7): $s'_1=s'_2=s'_3=1$, $s'_i=0$ for all $i=4,\dots 9$. Thus for this system we have

$$1s'_1+2s'_2+3s_3+4s'_4+5s'_5+6s'_6+7s'_7+8s'_8+9s'_9=6=r^{(1)},$$

and, hence, the system *is* involutive. This result together with Cartan's Theorem 11.16, [17], p. 367, tells us that there is no para-CR invariant information encoded in the forms Ω_4, Ω_5 and Ω_6. Hence we can take them in the most simple representation $\Omega_4=\Omega_5=\Omega_6\equiv 0$. (Note that this can be achieved by setting $c_2=h_{21}=h_{22}=c_{20}=h_{210}=h_{211}=h_{220}=h_{221}=h_{222}=0,\ \ \theta^2=da_2$. Cartan's theorem says also that we can do it in many ways. Since we are in the involutive case, the local group of para-CR symmetries is infinite dimensional; it depends on $s'_{k=3} = 1$ arbitrary real function of $k = 3$ variables.) Concluding we have the following theorem.

THEOREM 5.5. – *All type* $(1,1,2)$ *para-CR structures* $(M, [\lambda, \mu, \nu_1, \nu_2])$ *with* $\mathrm{d}\lambda \wedge \lambda \neq 0$*, and with the invariant* $I \equiv 0$*, are locally equivalent to one of the para-CR structures* $(M, [\lambda = \mathrm{d}y - p\,\mathrm{d}x, \nu_1 = \mathrm{d}p - Q(x,y,p)\mathrm{d}x, \nu_2 = \mathrm{d}a_2, \mu = \mathrm{d}x])$*. Thus they are obtained by extending by* $\nu_2 = \mathrm{d}a_2$ *the type* $(1,1,1)$ *para-CR structure defined by* $[\lambda = \mathrm{d}y - p\,\mathrm{d}x, \nu_1 = \mathrm{d}p - Q(x,y,p)\mathrm{d}x, \mu = \mathrm{d}x]$*. All local invariants of such* $(M, [\lambda, \mu, \nu_1, \nu_2])$ *are given by the point invariants of the corresponding point equivalence class of second order ODEs represented by* $y'' = Q(x, y, y'')$*.*

It is convenient to introduce the following definition.

DEFINITION 5.6. – *A type* $(1,1,2)$ *para-CR manifold* $(M, [\lambda, \mu, \nu_1, \nu_2])$ *with* $\mathrm{d}\lambda \wedge \lambda \neq 0$ *is regular if the invariant I is either not equal to zero in M or it is zero everywhere in M.*

Now comparing Theorems 5.3 and 5.5 we obtain:

COROLLARY 5.7. – *All regular type* $(1,1,2)$ *para-CR manifolds* $(M, [\lambda, \mu, \nu_1, \nu_2])$ *with* $\mathrm{d}\lambda \wedge \lambda \neq 0$ *are locally equivalent either to canonical para-CR structures of point equivalence classes of* 3*rd order ODEs (if* $I \neq 0$*), or to the trivial extensions of the canonical para-CR structures of point equivalent classes of* 2*nd order ODEs (if* $I \equiv 0$*).*

5.2 – *Local invariants for para-CR structures of type* $(1,1,n-1)$.

We believe that the situation described in Cotrollary 5.7 is typical for any regular type $(1,1,n-1)$ para-CR structures $(M, [\lambda, \mu, \nu_a])$ with $\mathrm{d}\lambda \wedge \lambda \neq 0$ and any $n > 3$. By this we mean the following. In the *generic* case, such para-CR structures should be locally equivalent to the canonical para-CR structures associated with point equivalent classes of nth order ODEs. This generic case should be distinguished by the simultaneous *non*vanishing of a finite number t of relative invariants $(I_1, \dots, I_t)$, generalizing our invariant I. These invariants should have some hierarchical structure, so that if all invariants above some level, say n_0, in the hierarchy identically vanish, then the para-CR structure is a trivial extension of a canonical para-CR structure of type $(1,1,n-n_0-1)$, by adding n_0 forms $\nu_{n-1} = \mathrm{d}a_{n-1}, \dots, \nu_{n-n_0} = \mathrm{d}a_{n-n_0}$ to the canonical contact forms $[\lambda, \mu, \nu_1, \dots, \nu_{n-n_0-1}]$. Proving or disproving our belief goes beyond this article.

6. – Relations with other differential equations.

Given a para-CR structure of type (k, r, s) we consider its local embedding in $\mathbb{R}^{(k+r)+(k+s)}$, as in Theorem 2.6. The obtained codimension-k submanifold Σ we intend to interprete as a general solution of a certain system of differential equations. We know how to do it in the case of para-CR structures of type $(1, 1, n-1)$: in this case Σ describes the general solution of an nth order ODE considered modulo point transformations of variables. In the case of a general (k, r, s) we expect that Σ corresponds to the general solution of a *system of ODEs*, or more generally, to the general solution of a *system of PDEs of finite type*.

6.1 – *Systems of ODEs.*

Given a system of first order ODEs

$$\frac{\mathrm{d}y^i}{\mathrm{d}x} = F^i(x, y^1, \ldots, y^n), \qquad i = 1, 2, \ldots n, \tag{6.1}$$

we consider its general solution

$$y^i = \psi^i(x, a_0, a_1, \ldots, a_{n-1}), \qquad i = 1, 2, \ldots n,$$

where the constants a_μ, $\mu = 0, 1, \ldots, n-1$, are the constants of integration. This defines a codimension n submanifold

$$\Sigma = \{\mathbb{R}^{(n+1,n)} \ni (x, y^1, \ldots, y^n, a_0, \ldots, a_{n-1}) \mid y^i = \psi^i\}$$

in $\mathbb{R}^{(n+1,n)}$, which aquires a para-CR structure from the split $2n+1 = (n+1)+n$ in the ambient $\mathbb{R}^{(n+1,n)}$, given by the linear operator $\kappa(\partial_\mu) = -\partial_\mu$, $\kappa(\partial_x) = \partial_x$, $\kappa(\partial_{y^i}) = \partial_{y^i}$. Interestingly this para-CR structure is of type $(n, 1, 0)$.

Indeed, the tangent space $\mathrm{T}\Sigma$ to Σ is spanned by

$$\begin{aligned} X &= \partial_x + \psi^i_x \partial_{y^i} \\ Z_\mu &= \partial_\mu + \psi^i_\mu \partial_{y^i}. \end{aligned}$$

Since $\kappa(Z_\mu) \cap \mathrm{T}\Sigma = \{0\}$, for all $\mu = 0, \ldots, n-1$, and $\kappa(X) = X$, then $\kappa(\mathrm{T}\Sigma) \cap \mathrm{T}\Sigma = H^+ = \mathrm{Span}(X)$, and the $k(=n)$ codimensions of the $(n, 1, 0)$-type para-CR structure on Σ are spanned by the n vectors Z_μ.

Hence a typical representantive of para-CR structures of type $(n, 1, 0)$ is a *system of n first order ODEs for n scalar functions of one variable, considered modulo point transformations of the variables*. The study of invariants of such para-CR structures, as well as para-CR structures representing systems of ODEs of higher orders, will be performed elsewhere.

6.2 – *PDEs of finite type.*

Recall that the finite type property of a system of PDEs means that its most general solution depends on a finite number of parameters. Instead of studying the para-CR structures associated with the most general PDEs of finite type, in the next few sections we will study the para-CR structures of type $(1,2,3)$ and $(3,2,1)$. They include, as the simplest example, the para-CR structure corresponding to $z_{xx}=0$ & $z_{yy}=0$, i.e. a system of two PDEs for one real function $z=z(x,y)$ of two real variables x and y, with the general solution $z=a_0+a_1x+a_2y+a_3xy$, depending on four real parameters a_0, a_1, a_2 and a_3. Generalization of this example to the finite type PDEs of the form $z_{xx}=R(x,y,z,z_x,z_y,z_{xy})$ & $z_{yy}=T(x,y,z,z_x,z_y,z_{xy})$, provides examples of $(1,2,3)$ and $(3,2,1)$ type para-CR structures with very nice properties.

7. – Para-CR structures of type $(1,2,3)$.

7.1 – *The flat model.*

Consider a pair of second order PDEs

$$z_{xx}=0 \qquad \& \qquad z_{yy}=0, \tag{7.1}$$

for a real function $z=z(x,y)$ of two real variables x and y. The general solution for this system is clearly

$$z=a_0+a_1x+a_2y+a_3xy. \tag{7.2}$$

This means that the solution space of this system is 4-dimensional, and that its points are parametrized by $\boldsymbol{a}=(a_0,a_1,a_2,a_3)\in\mathbb{R}^4$. Thus we have here a generically embedded hypersurface

$$\Sigma=\{\mathbb{R}^7\ni(x,y,z,a_0,a_1,a_2,a_3)\mid z=a_0+a_1x+a_2y+a_3xy\},$$

in the 'correspondence space' $\mathbb{R}^7=\mathbb{R}^3\times\mathbb{R}^4$, with the respective coordinates (x,y,z) and (a_0,a_1,a_2,a_3). The linear map $\kappa:\mathbb{R}^7\to\mathbb{R}^7$, such that

$$\kappa(x,y,z,a_0,a_1,a_2,a_3)=(x,y,z,-a_0,-a_1,-a_2,-a_3),$$

induces a para-CR-structure of type $(1,2,3)$ on Σ. Indeed, the tangent space to Σ is spanned by

$$\begin{aligned}
&X_1=a_1\partial_y-a_2\partial_x, \qquad X_2=a_2\partial_z+\partial_y,\\
&Y_1=x\partial_0-\partial_1, \qquad Y_2=y\partial_1-x\partial_2, \qquad Y_3=x\partial_2-\partial_3,\\
&Z=\partial_z+\partial_0,
\end{aligned}$$

and we have a $(k,r,s)=(1,2,3)$-type para-CR structure, with $k=1$ corresponding to Span(Z), $r=2$ corresponding to the eigenspace $H^+=\text{Span}(X_1,X_2)$, and $s=3$ corresponding to the eigenspace $H^-=\text{Span}(Y_1,Y_2,Y_3)$. Obviously $H=H^+\oplus H^-$. Any diffeomorphism of $\mathbb{R}^7$ of the form

$$\Phi(x,y,z,a_i)=(\bar{x}(x,y,z),\bar{y}(x,y,z),\bar{z}(x,y,z),\bar{a}_i(a_j))$$

is, on the one hand, a para-CR diffeomorphism of the para-CR manifold Σ, and on the other hand, can be interpreted as coming from a point transformation of the variables of the system (7.1).

Dually this para-CR manifold is defined on Σ in terms of the forms

$$\begin{aligned}
\lambda &= \mathrm{d}a_0 + x\mathrm{d}a_1 + y\mathrm{d}a_2 + xy\mathrm{d}a_3\\
\mu_1 &= \mathrm{d}x\\
\mu_2 &= \mathrm{d}y,\\
\nu_1 &= \mathrm{d}a_1\\
\nu_2 &= \mathrm{d}a_2\\
\nu_3 &= \mathrm{d}a_3
\end{aligned} \tag{7.3}$$

given up to the transformation

$$\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix} \to \begin{pmatrix}\lambda'\\ \nu_1'\\ \nu_2'\\ \nu_3'\\ \mu_1'\\ \mu_2'\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ b_1 & f_{11} & f_{12} & f_{13} & 0 & 0\\ b_2 & f_{21} & f_{22} & f_{23} & 0 & 0\\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0\\ c_1 & 0 & 0 & 0 & h_{11} & h_{12}\\ c_2 & 0 & 0 & 0 & h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix}\theta^4\\ \theta^1\\ \theta^2\\ \theta^3\\ \Omega_3\\ \Omega_2\end{pmatrix}. \tag{7.4}$$

In this formulation the question of local equivalence of a given para-CR structure of type $(1,2,3)$ to the one defined by (7.3)-(7.4) can be solved by using *Cartan's equivalence method*, see e.g. [17]. Using it we get the following theorem.

THEOREM 7.1. – *The para-CR structure* (7.3)-(7.4) *defines a unique* 11*-dimensional manifold P on which the forms* $(\theta^1,\theta^2,\theta^3,\theta^4,\Omega_2,\Omega_3)$, *as defined in* (7.4), *can be supplemented by the unique* 1*-forms* $(\Omega_1,\Omega_4,\Omega_5,\Omega_6,A)$ *in such a way that the eleven* 1*-forms* (θ^i,Ω_μ,A), $i=1,2,3,4$, $\mu=1,2,3,4,5,6$, *constitute a coframe on P, and that they satisfy the exterior differential system*

$$\mathrm{d}\theta^i + \Gamma^i_{\ j}\wedge\theta^j = 0 \tag{7.5}$$

$$\mathrm{d}\Gamma^i_{\ j} + \Gamma^i_{\ k}\wedge\Gamma^k_{\ j} = 0, \tag{7.6}$$

with

$$\Gamma^i_{\ j} = g^{ik}\Gamma_{kj}, \qquad \Gamma_{ij} = \Gamma_{[ij]} + \frac{1}{2}Ag_{ij}, \tag{7.7}$$

where

$$g^{ik}=g_{ik}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}, \tag{7.8}$$

and

$$\Gamma_{[ij]}=\begin{pmatrix}0&\Omega_1&\Omega_2&\Omega_4\\-\Omega_1&0&\Omega_3&\Omega_5\\-\Omega_2&-\Omega_3&0&\Omega_6\\-\Omega_4&-\Omega_5&-\Omega_6&0\end{pmatrix}. \tag{7.9}$$

Moreover, if $(\bar\Sigma,[(\bar\lambda,\bar\mu_1,\bar\mu_2,\bar\nu_1,\bar\nu_2,\bar\nu_3)])$ *is an arbitrary* 6*-dimensional para-CR structure of type* $(1,2,3)$*, then it is locally para-CR-equivalent to the para-CR structure* (7.3)-(7.4) *if and only if its corresponding forms*

$$\begin{pmatrix}\theta^4\\\theta^1\\\theta^2\\\theta^3\\\Omega_3\\\Omega_2\end{pmatrix}=\begin{pmatrix}\bar a&0&0&0&0&0\\\bar b_1&\bar f_{11}&\bar f_{12}&\bar f_{13}&0&0\\\bar b_2&\bar f_{21}&\bar f_{22}&\bar f_{23}&0&0\\\bar b_3&\bar f_{31}&\bar f_{32}&\bar f_{33}&0&0\\\bar c_1&0&0&0&\bar h_{11}&\bar h_{12}\\\bar c_2&0&0&0&\bar h_{21}&\bar h_{22}\end{pmatrix}=\begin{pmatrix}\bar\lambda\\\bar\nu_1\\\bar\nu_2\\\bar\nu_3\\\bar\mu_1\\\bar\mu_2\end{pmatrix}$$

can be suplemented by five 1*-forms* $(\Omega_1,\Omega_4,\Omega_5,\Omega_6,A)$ *in such a way that on some* 11*-dimensional manifold* $\bar P$ *they satisfy the exterior differential system* (7.5)-(7.9).

PROOF. – The proof of this fact is a standard application of Cartan's method of equivalence. It requires massive calculations to show that the 1-forms (7.3)-(7.4) can be *uniquely* brought to the form, in which they satisfy (7.5)-(7.9) with *unique* $(\Omega_1,\Omega_4,\Omega_5,\Omega_6,A)$. Actually Cartan's method of equivalence constructs the manifold P with a natural parametrization of P by $(x,y,a_0,a_1,a_2,a_3,a,f_{11},f_{22},f_{31},f_{32})$, and gives, in an algorithmic way, the explicit formulae for the coframe 1-forms (θ^i,Ω_μ,A), $i=1,2,3,4$, $\mu=1,2,3,4,5,6$, which correspond to (7.3)-(7.4) on P. These coframe 1-forms read:

$$\begin{aligned}
\theta^1&=-\frac{af_{32}}{f_{22}}(\mathrm{d}a_0+y\mathrm{d}a_2)+\frac{f_{11}f_{22}-xaf_{32}}{f_{22}}(\mathrm{d}a_1+y\mathrm{d}a_3),\\
\theta^2&=-\frac{af_{31}}{f_{11}}(\mathrm{d}a_0+x\mathrm{d}a_1)+\frac{f_{11}f_{22}-yaf_{31}}{f_{11}}(\mathrm{d}a_2+x\mathrm{d}a_3),\\
\theta^3&=-\frac{af_{31}f_{32}}{f_{11}f_{22}}\,\mathrm{d}a_0+\frac{f_{31}(f_{11}f_{22}-xaf_{32})}{f_{11}f_{22}}\,\mathrm{d}a_1+\frac{f_{32}(f_{11}f_{22}-yaf_{31})}{f_{11}f_{22}}\,\mathrm{d}a_2\\
&\quad-\frac{(f_{11}f_{22}-xaf_{32})(f_{11}f_{22}-yaf_{31})}{af_{11}f_{22}}\,\mathrm{d}a_3,\\
\theta^4&=a(\mathrm{d}a_0+x\mathrm{d}a_1+y\mathrm{d}a_2+xy\mathrm{d}a_3),\qquad \Omega_2=\frac{a}{f_{11}}\,\mathrm{d}x,\qquad \Omega_3=\frac{a}{f_{22}}\,\mathrm{d}y,
\end{aligned}$$

$$\begin{aligned}\Omega_1 &= \frac{1}{2}\,\mathrm{d}\log\left(\frac{f_{11}}{f_{22}}\right) + \frac{a}{f_{11}f_{22}}(f_{31}\,\mathrm{d}y - f_{32}\,\mathrm{d}x),\\ \Omega_4 &= \frac{f_{31}}{f_{11}}\,\mathrm{d}\log\left(\frac{af_{31}}{f_{11}f_{22}}\right) + \frac{af_{31}^2}{f_{11}^2 f_{22}}\,\mathrm{d}y,\\ \Omega_5 &= \frac{f_{32}}{f_{22}}\,\mathrm{d}\log\left(\frac{af_{32}}{f_{11}f_{22}}\right) + \frac{af_{32}^2}{f_{11}f_{22}^2}\,\mathrm{d}x,\\ \Omega_6 &= \frac{1}{2}\,\mathrm{d}\log\left(\frac{f_{11}f_{22}}{a^2}\right) - \frac{a}{f_{11}f_{22}}(f_{31}\,\mathrm{d}y + f_{32}\,\mathrm{d}x),\\ A &= -\,\mathrm{d}\log(f_{11}f_{22}).\end{aligned}$$

It can be checked by a direct calculation that these forms satisfy (7.5)-(7.9). □

7.2 – *Newman's construction..*

After E. Ted Newman [6, 7] we recall that the system (7.1) has the interesting property that its solution space $\mathbb{R}^4$ is naturally equipped with a conformal metric of split signature. This is defined as follows.

Consider two neighboring solutions of (7.1) corresponding to two points $\boldsymbol{a}$ and $\boldsymbol{a} + \mathrm{d}\boldsymbol{a}$ in $\mathbb{R}^4$. These two solutions can be considered as two surfaces, the graphs of two functions,

$$z(x,y) = a_0 + a_1x + a_2y + a_3xy \qquad \&$$

$$(z + \mathrm{d}z)(x,y) = (a_0 + \mathrm{d}a_0) + (a_1 + \mathrm{d}a_1)x + (a_2 + \mathrm{d}a_2)y + (a_3 + \mathrm{d}a_3)xy,$$

in $\mathbb{R}^3$ with coordinates (x,y,z). One can ask what conditions the two points $\boldsymbol{a}$ and $\boldsymbol{a} + \mathrm{d}\boldsymbol{a}$ in the solution space $\mathbb{R}^4$ must satisfy for these two surfaces to be tangent at some point (x,y,z) in $\mathbb{R}^3$. An elementary argument shows that the point (x,y,z) at which the two surfaces are tangent satisfies the following equations:

$$\mathrm{d}z = \mathrm{d}a_0 + \mathrm{d}a_1x + \mathrm{d}a_2y + \mathrm{d}a_3xy = 0,$$

$$(\mathrm{d}z)_x = \mathrm{d}a_1 + \mathrm{d}a_3y = 0 \quad \& \quad (\mathrm{d}z)_y = \mathrm{d}a_2 + \mathrm{d}a_3x = 0.$$

The first of the above equations says that the two surfaces intersect at a point $(x,y,z(x,y))$, and the second two equations say that they are tangent at the same point $(x,y,z(x,y))$. These three equations for the two unknowns (x,y) have a solution if and only if $\mathrm{d}\boldsymbol{a}$ satisfies a compatibility condition, which is obtained by eliminating x and y from the two second equations, and by inserting the so determined x and y in the first equation. This compatibility condition is:

$$\mathrm{d}a_0\mathrm{d}a_3 - \mathrm{d}a_1\mathrm{d}a_2 = 0.$$

Thus: two neighboring solutions $\boldsymbol{a}$ and $\boldsymbol{a} + \mathrm{d}\boldsymbol{a}$ of (7.1) are tangent to each other at some point (x, y, z) in $\mathbb{R}^3$ if and only if they are *null separated* in the flat split signature metric

$$g = 2(\mathrm{d}a_0\mathrm{d}a_3 - \mathrm{d}a_1\mathrm{d}a_2) \tag{7.10}$$

in $\mathbb{R}^4$. This shows that the solution space of (7.1) is naturally equipped with a conformal structure. This gives a correspondence between the incidence relations between two solutions of (7.1) treated as surfaces in $\mathbb{R}^3$ and conformal properties of points in the solution space $\mathbb{R}^4$. This description is very similar to the well known correspondences in the *Lie sphere geometry*, or more generally, in *Penrose's twistor theory*.

A new view of Newman's construction, stressing the *Weyl geometric* aspect of it, follows from our Theorem 7.1, and is included in the following theorem.

THEOREM 7.2. – *Every para-CR structure of type* $(1, 2, 3)$ *which is para-CR equivalent to the para-CR structure* (7.3)-(7.4) *uniquely defines an* 11-*dimensional principal fiber bundle* $\mathbf{CO}(2, 2) \to P \to \mathcal{S}$, *with the* 7-*dimensional homothetic structure group* $\mathbf{CO}(2, 2)$, *over a* 4-*dimensional manifold* $\mathcal{S}$, *which can be identified with the solution space of a pair of PDEs on the plane:* $z_{xx} = 0 = z_{yy}$. *It also defines a flat Weyl geometry* $[g, A]$ *on* $\mathcal{S}$, *in which the* $(2, 2)$-*signature metric* g *and the* 1-*form* A *change conformally,* $g \to \mathrm{e}^{2\phi} g$, $A \to A - 2\mathrm{d}\phi$, *when the system* $z_{xx} = 0 = z_{yy}$ *undergoes a point transformation of the variables* (x, y, z).

PROOF. – Given a para-CR manifold locally equivalent to (7.3)-(7.4) we use the previous theorem and construct an 11-dimensional manifold P with the coframe $(\theta^i, \Omega_\mu, A)$ satisfying (7.5)-(7.8). It is convenient to write down these equations explicitly. Equations (7.5), when written in the coframe $(\theta^i, \Omega_\mu, A)$ read:

$$\begin{aligned}
\mathrm{d}\theta^1 &= \Big(\Omega_1 - \frac{1}{2}A\Big) \wedge \theta^1 - \Omega_3 \wedge \theta^3 - \Omega_5 \wedge \theta^4\\
\mathrm{d}\theta^2 &= \Big(-\Omega_1 - \frac{1}{2}A\Big) \wedge \theta^2 - \Omega_2 \wedge \theta^3 - \Omega_4 \wedge \theta^4\\
\mathrm{d}\theta^3 &= \Omega_4 \wedge \theta^1 + \Omega_5 \wedge \theta^2 + \Big(\Omega_6 - \frac{1}{2}A\Big) \wedge \theta^3\\
\mathrm{d}\theta^4 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \Big(-\Omega_6 - \frac{1}{2}A\Big) \wedge \theta^4,
\end{aligned} \tag{7.11}$$

whereas equations (7.6) read:

$$
\begin{aligned}
\mathrm{d}\Omega_1 &= \Omega_2 \wedge \Omega_5 - \Omega_3 \wedge \Omega_4 \\
\mathrm{d}\Omega_2 &= \Omega_2 \wedge (\Omega_1 + \Omega_6) \\
\mathrm{d}\Omega_3 &= (\Omega_1 - \Omega_6) \wedge \Omega_3 \\
\mathrm{d}\Omega_4 &= \Omega_4 \wedge (\Omega_1 - \Omega_6) \\
\mathrm{d}\Omega_5 &= (\Omega_1 + \Omega_6) \wedge \Omega_5 \\
\mathrm{d}\Omega_6 &= \Omega_2 \wedge \Omega_5 + \Omega_3 \wedge \Omega_4 \\
\mathrm{d}A &= 0.
\end{aligned} \tag{7.12}
$$

The appearance of only constant coefficients in front of the 2-forms on the right hand sides of equations (7.11)-(7.12) enables us to identify the coframe forms $(\theta^i, \Omega_\mu, A)$ with the Maurer-Cartan forms on an 11-dimensional Lie group with a Lie algebra having structure constants equal to these coeficients. This shows that P is a Lie group. A look at the structure constants of the corresponding Lie algebra given by (7.11)-(7.12), shows that this Lie group is $P = \mathbb{R}^4 \rtimes \mathbf{CO}(2,2)$. The $\mathbf{CO}(2,2)$ principal fibre bundle structure on $P = \mathbb{R}^4 \rtimes \mathbf{CO}(2,2)$ corresponds to the fibration $\mathbf{CO}(2,2) \to \mathbb{R}^4 \rtimes \mathbf{CO}(2,2) \to \mathbb{R}^4$, i.e. to the natural principal $\mathbf{CO}(2,2)$ fibration over the homogeneous space $\mathbb{R}^4 \simeq (\mathbb{R}^4 \rtimes \mathbf{CO}(2,2))/\mathbf{CO}(2,2)$. Existence of this fibration on P is guaranteed by the equations (7.5) (or what is the same (7.11)). They say that the 1-forms $(\theta^1, \theta^2, \theta^3, \theta^4)$ form a closed differential ideal, so that their annihilator defines a foliation of P by 7-dimensional manifolds. On each of these 7-dimensional manifolds the forms $(\theta^1, \theta^2, \theta^3, \theta^4)$ vanish identically, and the additional seven 1-forms $(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, A)$ form a coframe. The differentials of this coframe, on each leaf of the foliation, satisfies a closed exterior differential system with constant coefficients (7.12). Thus each leaf can be identified with the same Lie group, whose Lie algebra has structure constants determined by (7.12). It is easy to see that this 7-dimensional Lie algebra is the homothetic Lie algebra $\mathfrak{co}(2,2)$ of homothetic motions in 4-dimensions associated with a metric of signature $(+,+,-,-)$.

The appearance of the Lie group $\mathbf{CO}(2,2)$ as a subgroup of P suggests that $\mathcal{S} = \mathbb{R}^4 \simeq (\mathbb{R}^4 \rtimes \mathbf{CO}(2,2))/\mathbf{CO}(2,2)$ is naturally equipped with a conformal metric of signature $(+,+,-,-)$. This is indeed the case. The metric is obtained as follows: consider the bilinear form G on P defined by:

$$G = 2(\theta^1\theta^2 + \theta^3\theta^4).$$

This form is highly degenerate on P, but its degenerate directions are precisely along the fiber directions of the foliation $\mathbf{CO}(2,2) \to P \to M$; actually G

has signature $(+,+,-,-,0,0,0,0,0,0,0)$. Morever, using the sytem (7.5) it can be easily checked that the Lie derivatives of G along all the directions tangent to the fibres are just multiples of G. In particular, if Z is any vector field on P tangent to the fibres, we have $Z \lrcorner\, \theta^i \equiv 0$, and as a consequence of (7.11) we get

$$\mathcal{L}_Z G = -(Z \lrcorner\, A)G.$$

Thus G descends to a conformal metric $[g]$ of signature $(+,+,-,-)$ on the quotient space $\mathcal{S} = P/\mathbf{CO}(2,2)$.

Using the last equation (7.12) we also get

$$\mathcal{L}_Z A = \mathrm{d}(Z \lrcorner\, A),$$

so we see that the pair (G,A), changes as $(G,A) \to (G',A') = (\mathrm{e}^{2\phi}G, A - 2\mathrm{d}\phi)$ when it is Lie dragged along the fibres of $\mathbf{CO}(2,2) \to P \to \mathcal{S}$. Thus it descends to a split signature Weyl geometry $[g,A]$ on $\mathcal{S}$. The equations (7.12), when pulled back to $\mathcal{S}$, show that this Weyl geometry is flat.

To interpret the quotient $\mathcal{S} = P/\mathbf{CO}(2,2)$ as the solution space of the pair of equations $z_{xx} = 0 = z_{yy}$ we use the corresponding para-CR forms (7.3), together with the explicit expressions for the invariant forms $(\theta^1,\theta^2,\theta^3,\theta^4)$ and A in coordinates $(x,y,a_0,a_1,a_2,a_3,a,f_{11},f_{22},f_{31},f_{32})$ on P, as in the proof of Theorem 7.1. A short calculation shows that

$$G = 2(\theta^1\theta^2 + \theta^3\theta^4) = -2f_{11}f_{22}(\mathrm{d}a_0\mathrm{d}a_3 - \mathrm{d}a_1\mathrm{d}a_2).$$

This, together with $A = -\mathrm{d}\log(f_{11}f_{22})$, shows that the representative $(g,A) \in [g,A]$ can be taken as

$$g = 2(\mathrm{d}a_0\mathrm{d}a_3 - \mathrm{d}a_1\mathrm{d}a_2), \qquad A = 0,$$

and that $\mathcal{S}$ is parametrized by (a_0,a_1,a_2,a_3). Since these parameters constitute all the integration constants of the equations $z_{xx} = 0 = z_{yy}$, the quotient $\mathcal{S}$ can be naturally identified with the solution space of these equations. □

7.3 – *The principal bundle point of view and Weyl geometry.*

In the previous section we have shown how to associate an 11-dimensional principal fiber bundle $\mathbf{CO}(2,2) \to P \to \mathcal{S}$ to any flat para-CR structure of type $(1,2,3)$. Here we reverse this construction.

PROPOSITION 7.3. – *Every 11-dimensional manifold P with a coframe $(\theta^i, \Omega_\mu, A)$, $i = 1,2,3,4$, $\mu = 1,2,3,4,5,6$, satisfying the differential system*

$$
\begin{aligned}
\mathrm{d}\theta^1 &= (\Omega_1 - \frac{1}{2}A)\wedge\theta^1 - \Omega_3\wedge\theta^3 - \Omega_5\wedge\theta^4\\
\mathrm{d}\theta^2 &= \Big(-\Omega_1 - \frac{1}{2}A\Big)\wedge\theta^2 - \Omega_2\wedge\theta^3 - \Omega_4\wedge\theta^4\\
\mathrm{d}\theta^3 &= \Omega_4\wedge\theta^1 + \Omega_5\wedge\theta^2 + \Big(\Omega_6 - \frac{1}{2}A\Big)\wedge\theta^3\\
\mathrm{d}\theta^4 &= \Omega_2\wedge\theta^1 + \Omega_3\wedge\theta^2 + \Big(-\Omega_6 - \frac{1}{2}A\Big)\wedge\theta^4,\\
\mathrm{d}\Omega_1 &= \Omega_2\wedge\Omega_5 - \Omega_3\wedge\Omega_4 + \frac{1}{2}\kappa_{1ij}\theta^i\wedge\theta^j\\
\mathrm{d}\Omega_2 &= \Omega_2\wedge(\Omega_1+\Omega_6) + \frac{1}{2}\kappa_{2ij}\theta^i\wedge\theta^j\\
\mathrm{d}\Omega_3 &= (\Omega_1-\Omega_6)\wedge\Omega_3 + \frac{1}{2}\kappa_{3ij}\theta^i\wedge\theta^j\\
\mathrm{d}\Omega_4 &= \Omega_4\wedge(\Omega_1-\Omega_6) + \frac{1}{2}\kappa_{4ij}\theta^i\wedge\theta^j\\
\mathrm{d}\Omega_5 &= (\Omega_1+\Omega_6)\wedge\Omega_5 + \frac{1}{2}\kappa_{5ij}\theta^i\wedge\theta^j\\
\mathrm{d}\Omega_6 &= \Omega_2\wedge\Omega_5 + \Omega_3\wedge\Omega_4 + \frac{1}{2}\kappa_{6ij}\theta^i\wedge\theta^j\\
\mathrm{d}A &= \frac{1}{2}F_{ij}\theta^i\wedge\theta^j,
\end{aligned}
\tag{7.13}
$$

with κ_{aij}, F_{ij} being functions on P, is locally a principal fiber bundle $\mathbf{CO}(2,2)\to P\to\mathcal{S}$ over a 4-dimensional manifold $\mathcal{S}$ naturally equipped with a Weyl geometry $[g,A]$, in which the split signature conformal metric g is determined by a bilinear form $G = 2(\theta^1\theta^2+\theta^3\theta^4)$ on P, and the Weyl potential 1-form is determined by the 1-form A on P. The curvature of this Weyl geometry is given by

$$
\mathcal{R} = \begin{pmatrix}
0 & \kappa_1 + \frac{1}{2}\mathcal{F} & \kappa_2 & \kappa_4\\
-\kappa_1 + \frac{1}{2}\mathcal{F} & 0 & \kappa_3 & \kappa_5\\
-\kappa_2 & -\kappa_3 & 0 & \kappa_6 + \frac{1}{2}\mathcal{F}\\
-\kappa_4 & -\kappa_5 & -\kappa_6 + \frac{1}{2}\mathcal{F} & 0
\end{pmatrix},
$$

where $\kappa_a = \frac{1}{2}\kappa_{aij}\theta^i\wedge\theta^j$ and $\mathcal{F} = \frac{1}{2}F_{ij}\theta^i\wedge\theta^j$.

PROOF. – As in the proof of Theorem 7.2 we easily see that the 7-dimensional distribution annihilating $(\theta^1, \theta^2, \theta^3, \theta^4)$ is integrable, and hence we have a local projection $\pi : P \to \mathcal{S}$, identifying points along the same leaves of the corresponding foliation. Since on the leaves the forms θ^i vanish, and since the differentials $\mathrm{d}\Omega_\mu$s differ from those in (7.12) by terms that vanish on the leaves, every leave is a local Lie group isomorphic to $\mathbf{CO}(2,2)$. This proves that the manifold P is locally a principal fiber bundle $\mathbf{CO}(2,2) \to P \to \mathcal{S}$.

To prove that $\mathcal{S}$ has a natural Weyl structure $[g, A]$, one repeats the argument from the previous proof. Although in (7.13), when compared to (7.11)-(7.12), the new terms κ_a and $\mathcal{F}$ appear, the argument from the previous proof is not altered. This is because (1) the new terms do not appear in the 'conformal metricity/torsion' part of the equations (i.e. $\mathrm{d}\theta^i$ equations) and (2) they appear in $\mathrm{d}A$ only in harmless terms which are annihilated by any vertical direction.

The curvature of this Weyl structure can be calculated, by observing that on any section $\sigma(\mathcal{S})$ of P the Weyl connection is given by $\Gamma^i{}_j = g^{ik}\sigma^*(\Gamma_{[kj]} + \frac{1}{2} A g_{kj})$, where Γ_{ij} is expressed in terms of the forms Ω_μ appearing in (7.13) via formula (7.9), and g_{ij}, g^{ji} are as in (7.8). The rest of the proof consists in calculating $\mathcal{R}^i{}_j = \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j$ using (7.13) and lowering one index. □

This proposition is crucial for the remaining sections. In particular it can be used to prove the theorem, which gives the converse of Newman's construction:

THEOREM 7.4. – *Every* 11*-dimensional manifold P which is equipped with a coframe* $(\theta^i, \Omega_\mu, A)$, $i = 1,2,3,4$, $\mu = 1,2,3,4,5,6$, *satisfying the differential system* (7.11)-(7.12), *is locally a principal fiber bundle* $\mathbf{CO}(2,2) \to P \to \mathcal{S}$, *originating from a flat para-CR manifold* $(\Sigma, [\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3])$ *of type* $(1,2,3)$, *via the procedure described by Theorem* 7.1.

PROOF. – That P with a system (7.11)-(7.12) is locally a principal fiber bundle $\mathbf{CO}(2,2) \to P \to \mathcal{S}$ is an immediate consequence of Proposition 7.3 with $\kappa_a \equiv 0$ and $\mathcal{F} \equiv 0$. Here we show that apart from the foliation $\mathbf{CO}(2,2) \to P$, the system (7.11)-(7.12) defines another foliation of the manifold P, whose leaf space can be identified with a 6-dimensional flat para-CR structure Σ.

To see this consider the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$, and observe that the system (7.11) and the second and the third equations from system (7.12) guarantee that these six forms constitute a closed differential ideal. Therefore their annihilator is a 5-dimensional integrable distribution on P, whose integral manifolds define a 6-parameter foliation of P. Putting $\Omega_2 \equiv 0 \equiv \Omega_3$ in equations (7.12) we see that the coframe $(\Omega_1, \Omega_4, \Omega_5, \Omega_6, A)$ on these integral manifolds satsifies a closed differential system with all the coefficients being constants. Thus all these integral manifolds can be identified with a unique Lie group K, which turns out to be a direct product $K = \mathbf{Aff}(1) \times \mathbf{Aff}(1) \times \mathbb{R}^*$ of two independent groups of affine transformations of the real line, **Aff**(1), and the multiplicative group of the real numbers $\mathbb{R}^*$. This shows that the manifold P, with the system of 1-forms (7.11)-(7.12), can be also locally viewed as a principal fibre bundle $K \to P \to \Sigma$. Here Σ is the 6-dimensional leaf space of the foliation whose leaves are identified with K. Any manifold $\bar{\Sigma}$ transversal to the fibres of these fibration is equipped with a coframe $(\bar{\theta}^i, \bar{\Omega}_2, \bar{\Omega}_3) = (\theta^i, \Omega_2, \Omega_3)_{|\bar{\Sigma}}$, $i = 1, 2, 3, 4$, which satisfies the system

$$
\begin{aligned}
&\mathrm{d}\bar{\theta}^1 = (\bar{\Omega}_1 - \frac{1}{2}\bar{A}) \wedge \bar{\theta}^1 - \bar{\Omega}_3 \wedge \bar{\theta}^3 - \bar{\Omega}_5 \wedge \bar{\theta}^4\\
&\mathrm{d}\bar{\theta}^2 = \Big(-\bar{\Omega}_1 - \frac{1}{2}\bar{A}\Big) \wedge \bar{\theta}^2 - \bar{\Omega}_2 \wedge \bar{\theta}^3 - \bar{\Omega}_4 \wedge \bar{\theta}^4\\
&\mathrm{d}\bar{\theta}^3 = \bar{\Omega}_4 \wedge \bar{\theta}^1 + \bar{\Omega}_5 \wedge \bar{\theta}^2 + \Big(\bar{\Omega}_6 - \frac{1}{2}\bar{A}\Big) \wedge \bar{\theta}^3\\
&\mathrm{d}\bar{\theta}^4 = \bar{\Omega}_2 \wedge \bar{\theta}^1 + \bar{\Omega}_3 \wedge \bar{\theta}^2 + \Big(-\bar{\Omega}_6 - \frac{1}{2}\bar{A}\Big) \wedge \bar{\theta}^4,\\
&\mathrm{d}\bar{\Omega}_2 = \bar{\Omega}_2 \wedge (\bar{\Omega}_1 + \bar{\Omega}_6)\\
&\mathrm{d}\bar{\Omega}_3 = (\bar{\Omega}_1 - \bar{\Omega}_6) \wedge \bar{\Omega}_3,
\end{aligned}
\tag{7.14}
$$

with forms $\bar{\Omega}_1$, $\bar{\Omega}_4$, $\bar{\Omega}_5$, $\bar{\Omega}_6$ and $\bar{A}$ on $\bar{\Sigma}$. That these forms are the restrictions of Ω_1, Ω_4, Ω_5, Ω_6 and A to $\bar{\Sigma}$ is not important in the following. What is important, is that the system (7.14) on $\bar{\Sigma}$ is satisfied by a *coframe* $(\bar{\theta}^i, \bar{\Omega}_2, \bar{\Omega}_3)$, and that it implies that the quartet of forms $(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, \bar{\theta}^4)$, as well as the triplet of forms $(\bar{\theta}^4, \bar{\Omega}_2, \bar{\Omega}_3)$, *both* form closed differential ideals of 1-forms on $\bar{\Sigma}$. Thus the 2-dimensional anihilator $\bar{H}^+$ of $(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, \bar{\theta}^4)$, as well as the 3-dimensional anihilator $\bar{H}^-$ of $(\bar{\theta}^4, \bar{\Omega}_2, \bar{\Omega}_3)$, define foliations of $\bar{\Sigma}$ by, respectively, a 4-parameter family of 2-dimensional leaves, and a 3-parameter family of 3-dimensional leaves. The integrable distributions $\bar{H}^+$ and $\bar{H}^-$ obviously have $\bar{H}^+ \cap \bar{H}^- = \{0\}$, equipping each $\bar{\Sigma}$ with a para-CR structure $(\bar{\Sigma}, \bar{H}^+, \bar{H}^-)$. It is matter of checking that the $(1,2,3)$-type para-CR structures on each $\bar{\Sigma}$ are localy equivalent to each other, and that they descend to the unique $(1,2,3)$-type para-CR structure (Σ, H^+, H^-) on the quotient $\Sigma = P/K$. Obviously this para-CR structure is the flat one of Theorem 7.1. □

7.4 – *Non flat case.*

Now we generalize the flat example of Sections 7.1-7.3 to systems of PDEs on the plane of the form

$$
z_{xx} = R(x, y, z, z_x, z_y, z_{xy}) \qquad \& \qquad z_{yy} = T(x, y, z, z_x, z_y, z_{xy}).
\tag{7.15}
$$

We assume that they are finite type, or, what is the same, we assume that their general solution can be written as

$$z = \psi(x, y, a_0, a_1, a_2, a_3).$$

This is always the case [7], when the functions $R = R(x, y, z, p, q, s)$ and $T = T(x, y, z, p, q, s)$ satisfy

$$D_x^2 T = D_y^2 R, \tag{7.16}$$

where the differential operators D_x and D_y are implicitly given by

$$D_x = \partial_x + p\partial_z + R\partial_p + s\partial_q + D_y R\partial_s \quad \& \quad D_y = \partial_y + q\partial_z + s\partial_p + T\partial_q + D_x T\partial_s. \tag{7.17}$$

To make this implicit definition of D_x and D_y explicit one has to solve for $D_x T$ and $D_y R$ in $D_x T = T_x + pT_z + RT_p + sT_q + (D_y R)T_s$ and $D_y R = R_y + qR_z + sR_p + TR_q + (D_x T)R_s$. This is only possible if

$$T_s R_s \neq 1, \tag{7.18}$$

which when assumed, defines $D_x T$ and $D_y R$ uniquely, and in turn after insertion in (7.17), makes the operators D_x and D_y explicit. Thus we assume (7.18) from now on.

To define a type $(1, 2, 3)$ para-CR structure associated with the system (7.15), (7.16), (7.18) we do as follows. First, using the general solution $z = \psi(x, y, a_0, a_1, a_2, a_3)$, we define the forms

$$\begin{aligned}
\lambda &= \psi_0 \mathrm{d}a_0 + \psi_1 \mathrm{d}a_1 + \psi_2 \mathrm{d}a_2 + \psi_3 \mathrm{d}a_3\\
\mu_1 &= \mathrm{d}x\\
\mu_2 &= \mathrm{d}y\\
\nu_1 &= \mathrm{d}a_1\\
\nu_2 &= \mathrm{d}a_2\\
\nu_3 &= \mathrm{d}a_3,
\end{aligned} \tag{7.19}$$

Then we extend these forms to the class $[\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3]$ via (7.4). This equips the 6-dimensional hypersurface

$$\Sigma = \{(x, y, z, a_0, a_1, a_2, a_3, a_4) \in \mathbb{R}^7 |\ z = \psi(x, y, a_0, a_1, a_2, a_3)\}$$

in $\mathbb{R}^3 \times \mathbb{R}^4$ with the $(1, 2, 3)$-para-CR structure $[\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3]$. Alternatively, a *para-CR equivalent* structure may be defined on the second jets $\mathcal{J}^2$ of the system (7.15)-(7.16). Parametrizing this space by (x, y, z, p, q, s) we use the contact forms

$$\begin{aligned}\lambda &= \mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y\\ \mu_1 &= \mathrm{d}x\\ \mu_2 &= \mathrm{d}y\\ \nu_1 &= \mathrm{d}p - R\mathrm{d}x - s\mathrm{d}y\\ \nu_2 &= \mathrm{d}q - s\mathrm{d}x - T\mathrm{d}y\\ \nu_3 &= \mathrm{d}s - D_yR\mathrm{d}x - D_xT\mathrm{d}y,\end{aligned} \tag{7.20}$$

and define the type $(1,2,3)$ para-CR structure by extending these forms to a class of para-CR forms $[\lambda,\mu_1,\mu_2,\nu_1,\nu_2,\nu_3]$ on $\mathcal{J}^2$ via:

$$\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix} \to \begin{pmatrix}\lambda'\\ \nu_1'\\ \nu_2'\\ \nu_3'\\ \mu_1'\\ \mu_2'\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ b_1 & f_{11} & f_{12} & f_{13} & 0 & 0\\ b_2 & f_{21} & f_{22} & f_{23} & 0 & 0\\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0\\ c_1 & 0 & 0 & 0 & h_{11} & h_{12}\\ c_2 & 0 & 0 & 0 & h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix}, \tag{7.21}$$

where $a, b_A, c_a, f^A_B, h^a_\beta$ are arbitrary parameters such that $a\det(f^A_b)\det(h^a_\beta) \neq 0$. Let us now define, as before, the lifted coframe

$$\begin{pmatrix}\theta^4\\ \theta^1\\ \theta^2\\ \theta^3\\ \Omega_3\\ \Omega_2\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ b_1 & f_{11} & f_{12} & f_{13} & 0 & 0\\ b_2 & f_{21} & f_{22} & f_{23} & 0 & 0\\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0\\ c_1 & 0 & 0 & 0 & h_{11} & h_{12}\\ c_2 & 0 & 0 & 0 & h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix}. \tag{7.22}$$

We ask which conditions the functions R and T must satsify so that the forms $(\theta^1,\theta^2,\theta^3,\theta^4,\Omega_2,\Omega_3)$ are forced to satisfy the system (7.13) with some auxiliary forms $(\Omega_1,\Omega_4,\Omega_5,\Omega_6,A)$, on a certain 11-dimensional manifold P, where (θ^i,Ω_μ,A) would serve as a coframe. As a first result in this respect we have the following theorem.

THEOREM 7.5. – *A neccessary condition for the equations* $z_{xx} = R(x,y,z,z_x,z_y,z_{xy})$ *&* $z_{yy} = T(x,y,z,z_x,z_y,z_{xy})$ *satisfying* $D^2_xT = D^2_yR$, $1 - R_sT_s > 0$ *to admit forms* (7.21)-(7.22) *with*

$$\mathrm{d}\theta^4\wedge\theta^4 = (\Omega_2\wedge\theta^1 + \Omega_3\wedge\theta^2)\wedge\theta^4 \tag{7.23}$$

$$\mathrm{d}\theta^1\wedge\theta^1\wedge\theta^2\wedge\theta^4\wedge\Omega_3 = 0 \tag{7.24}$$

$$\mathrm{d}\theta^2\wedge\theta^1\wedge\theta^2\wedge\theta^4\wedge\Omega_2 = 0 \tag{7.25}$$

$$\mathrm{d}\theta^1\wedge\theta^1\wedge\theta^3\wedge\theta^4\wedge\Omega_3 = 0 \tag{7.26}$$

$$\mathrm{d}\theta^2\wedge\theta^2\wedge\theta^3\wedge\theta^4\wedge\Omega_2 = 0 \tag{7.27}$$

is that the functions $R = R(x, y, z, z_x, z_y, z_{xy})$ *and* $T = T(x, y, z, z_x, z_y, z_{xy})$ *satisfy*

$$J_1 \equiv 0, \qquad \& \qquad J_2 \equiv 0,$$

where

$$J_1 = (R_sT_s - 4)D_xR_s + R_s(2D_yR_s - R_sD_xT_s) \\ + 8R_q - 6R_qR_sT_s + 4R_pR_s + 2R_s^2T_q - 2R_pR_s^2T_s + 2R_s^3T_p$$

$$J_2 = (R_sT_s - 4)D_yT_s + T_s(2D_xT_s - T_sD_yR_s) \\ + 8T_p - 6R_sT_pT_s + 4T_qT_s + 2R_pT_s^2 - 2R_sT_qT_s^2 + 2R_qT_s^3.$$

PROOF. – We force the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$ to satisfy (7.23)-(7.27) in the following steps:

First we fix coefficients f_{23}, f_{33}, h_{11}, h_{12}, h_{21} and h_{22} by forcing $d\theta^4$ to satsify (7.23). For this to be satisfied we must take:

$$f_{13} = f_{23} = 0, \tag{7.28}$$

and

$$h_{11} = \frac{af_{12}}{f_{12}f_{21} - f_{11}f_{22}}, \quad h_{12} = -\frac{af_{11}}{f_{12}f_{21} - f_{11}f_{22}}, \\ h_{21} = -\frac{af_{22}}{f_{12}f_{21} - f_{11}f_{22}}, \quad h_{22} = \frac{af_{21}}{f_{12}f_{21} - f_{11}f_{12}}. \tag{7.29}$$

After these normalizations we have

$$d\theta^1 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 \wedge \Omega_3 = \frac{2f_{11}f_{12} + R_sf_{11}^2 + T_sf_{12}^2}{af_{33}} \Omega_2 \wedge \Omega_3 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$$

and

$$d\theta^2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 \wedge \Omega_2 = \frac{2f_{21}f_{22} + R_sf_{21}^2 + T_sf_{22}^2}{af_{33}} \Omega_3 \wedge \Omega_2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4.$$

Thus to satisfy (7.24) and (7.25) we must equate to zero the right hand sides of these equations. It is the moment, when we need the assumption

$$1 - R_sT_s > 0.$$

When this is assumed we achieve (7.24) and (7.25) by normalizing:

$$f_{21} = \frac{-1 \pm w}{R_s} f_{22}, \qquad f_{11} = \frac{-1 \mp w}{R_s} f_{12}, \qquad w = \sqrt{1 - R_sT_s}. \tag{7.30}$$

With these normalizations we now have

$$\begin{aligned}
&\mathrm{d}\theta^1\wedge\theta^1\wedge\theta^3\wedge\theta^4\wedge\Omega_3\\
&=f_{12}^2\,\frac{(1+3w^2\pm 3w\pm w^3)J_1-R_s^3J_2}{4af_{22}R_s^2w^2}\,\Omega_2\wedge\Omega_3\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4\\
&\mathrm{d}\theta^2\wedge\theta^2\wedge\theta^3\wedge\theta^4\wedge\Omega_2\\
&=f_{22}^2\,\frac{(1+3w^2\mp 3w\mp w^3)J_1-R_s^3J_2}{4af_{12}R_s^2w^2}\,\Omega_2\wedge\Omega_3\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4.
\end{aligned}\tag{7.31}$$

The right hand sides of these equations vanish identically if and only if

$$\begin{aligned}
(1+3w^2\pm 3w\pm w^3)J_1-R_s^3J_2\equiv 0\\
(1+3w^2\mp 3w\mp w^3)J_1-R_s^3J_2\equiv 0.
\end{aligned}$$

Since

$$\det\begin{pmatrix}1+3w^2\pm 3w\pm w^3 & -R_s^3\\ 1+3w^2\mp 3w\mp w^3 & -R_s^3\end{pmatrix}=\mp 2R_s^3w(3+w^2)\neq 0$$

this is only possible if and only if $J_1\equiv J_2\equiv 0$, which finishes the proof. □

The meaning of vanishing of both J_1 and J_2, known as *Newman's metricity conditions* [6, 7], is given in the following theorem.

THEOREM 7.6. – *If conditions*

$$J_1\equiv 0\qquad\&\qquad J_2\equiv 0$$

are satisfied then one can normalize the forms $(\theta^1,\theta^2,\theta^3,\theta^4,\Omega_2,\Omega_3)$ *in such a way that they, together with the auxiliary forms* $(\Omega_1,\Omega_2,\Omega_5,\Omega_6,A)$*, satisfy*

$$\begin{aligned}
\mathrm{d}\theta^1&=\Big(\Omega_1-\frac{1}{2}A\Big)\wedge\theta^1-\Omega_3\wedge\theta^3-\Omega_5\wedge\theta^4+t^1_{23}\theta^2\wedge\theta^3\\
\mathrm{d}\theta^2&=\Big(-\Omega_1-\frac{1}{2}A\Big)\wedge\theta^2-\Omega_2\wedge\theta^3-\Omega_4\wedge\theta^4+t^2_{13}\theta^1\wedge\theta^3\\
\mathrm{d}\theta^3&=\Omega_4\wedge\theta^1+\Omega_5\wedge\theta^2+\Big(\Omega_6-\frac{1}{2}A\Big)\wedge\theta^3\\
\mathrm{d}\theta^4&=\Omega_2\wedge\theta^1+\Omega_3\wedge\theta^2+\Big(-\Omega_6-\frac{1}{2}A\Big)\wedge\theta^4,
\end{aligned}\tag{7.32}$$

where

$$\begin{aligned}
t^1_{23}&=-\frac{a}{8f_{22}^2w^4}\big(R_{ss}(1\pm w)^2+T_{ss}R_s^2\big),\\
t^2_{13}&=-\frac{a}{8f_{32}^2w^4}\big(R_{ss}(1\mp w)^2+T_{ss}R_s^2\big).
\end{aligned}\tag{7.33}$$

With this normalization the bilinear form $G = 2(\theta^1\theta^2 + \theta^3\theta^4)$ *descends to a conformal* $(+,+,-,-)$ *signature metric* $[g]$ *on the 4-dimensional leaf space* $\mathcal{S}$ *of the foliation defined by the integrable distribution anihilating* $(\theta^1,\theta^2,\theta^3,\theta^4)$.

Modulo a discrete point transformation, interchanging θ^1 *with* θ^2*, the vanishing or not of at least one of*

$$K_1 = R_{ss}(1-\sqrt{1-R_sT_s})^2 + T_{ss}R_s^2, \qquad K_2 = R_{ss}(1+\sqrt{1-R_sT_s})^2 + T_{ss}R_s^2,$$

is a point invariant property of the corresponding system $z_{xx} = R(x,y,z,z_x,z_y,z_{xy})$ *&* $z_{yy} = T(x,y,z,z_x,z_y,z_{xy})$*. In particular, the simultaneous vanishing of* R_{ss} *and* T_{ss}*,* $R_{ss}\equiv T_{ss}\equiv 0$*, is a point invariant property of the system.*

PROOF. – If we prove that the forms $(\theta^1,\theta^2,\theta^3,\theta^4)$ can be forced to satisfy the system (7.32) on some 11-dimensional manifold P, where the forms (θ^i,Ω_μ,A) are linearly independent, then similarly as in the proof of Proposition 7.3, we will have a foliation of P by the integral leaves of a 7-dimensional *integrable* distribution annihilated by $(\theta^1,\theta^2,\theta^3,\theta^4)$. Moreover because (7.32) differs from (7.11) by only the appearence of $\theta^i\wedge\theta^k$ terms, the Lie derivtives of G with respect to the vectors tangent to the foliation, will be given by the same expressions as in the proof of Theorem 7.2. Thus, if we prove (7.32), we will get the conclusion that the leaf space $\mathcal{S}$ is equipped with the conformal split signature metrics $[g]$ to which G descends.

The procedure of bringing the forms (θ^i) to the form in which they satisfy (7.32) is based on Cartan's equivalence method. The Cartan process of normalizing the group coefficients $a, b_i, c_i, f_{ij}, h_{ij}$ has two loops, the first of which ends after normalization of the coeficient b_1.

THE FIRST LOOP. – We first impose the conditions (7.23)-(7.27), as in the previous proof, and as before reduce the possible freedom in the choice of $(\theta^1,\theta^2,\theta^3,\theta^4,\Omega_2,\Omega_3)$ to

$$\begin{pmatrix}\theta^4\\ \theta^1\\ \theta^2\\ \theta^3\\ \Omega_3\\ \Omega_2\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ b_1 & \frac{-1\mp w}{R_s}f_{12} & f_{12} & 0 & 0 & 0\\ b_2 & \frac{-1\pm w}{R_s}f_{22} & f_{22} & 0 & 0 & 0\\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0\\ c_1 & 0 & 0 & 0 & \pm\frac{aR_s}{2wf_{22}} & \pm\frac{a(1\pm w)}{2wf_{22}}\\ c_2 & 0 & 0 & 0 & \mp\frac{aR_s}{2wf_{12}} & \pm\frac{a(-1\pm w)}{2wf_{12}} \end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix} \tag{7.34}$$

In the next step we impose the condition $d\theta^1 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = -\Omega_3 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$. This gives the normalization

$$f_{33} = \frac{2w^2 f_{12} f_{22}}{aR_s} \tag{7.35}$$

and implies also that $d\theta^2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = -\Omega_2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$.

Then we require that $d\theta^2 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0$. This determines b_2 as:

$$\begin{aligned} b_2 = &\pm \frac{a(f_{32} + f_{31}R_s \mp f_{32}w)}{2f_{12}w} \pm \frac{f_{22}}{R_s^2 w(3+w^2)} \\ &\times \big((1 \pm w)(1-w^2)^2 R_q + (1 \mp w)^2 R_s(D_y R_s + T_p R_s^2 + R_s T_q) \\ &+ (1 \pm w) R_s (R_s D_x T_s + R_p(1-w^2))\big). \end{aligned} \tag{7.36}$$

Similarly the condition $d\theta^1 \wedge \theta^1 \wedge \theta^3 \wedge \theta^4 = 0$ determines b_1 as:

$$\begin{aligned} b_1 = &\mp \frac{a(f_{32} + f_{31}R_s \pm f_{32}w)}{2f_{22}w} \mp \frac{f_{12}}{R_s^2 w(3+w^2)} \\ &\times \big((1 \mp w)(1-w^2)^2 R_q + (1 \pm w)^2 R_s(D_y R_s + T_p R_s^2 + R_s T_q) \\ &+ (1 \mp w) R_s (R_s D_x T_s + R_p(1-w^2))\big). \end{aligned} \tag{7.37}$$

After these normalizations have been imposed, we have to associate the remaining undetermined parameters $a, f_{12}, f_{22}, f_{31}, f_{32}, b_3, c_1$ and c_2 with the auxiliary forms $\Omega_1, \Omega_4, \Omega_5, \Omega_6$ and A.

This is done by first observing that the equation

$$d\theta^4 = \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 - \left(\Omega_6 + \frac{1}{2}A\right) \wedge \theta^4. \tag{7.38}$$

is equivalent to

$$-\Omega_6 - \frac{1}{2}A = \frac{da}{a} - \frac{b_1}{a}\Omega_2 - \frac{b_2}{a}\Omega_3 + \frac{c_2}{a}\theta^1 + \frac{c_1}{a}\theta^2 + u_{111}\theta^4, \tag{7.39}$$

with b_1 and b_2 as above, and an unspecified new parameter u_{111}.

From now on we only sketch the proof, which is based on massive computer calculations using Mathematica.

After relating da to $\Omega_6 + \frac{1}{2}A$ we pass to the condition

$$d\theta^3 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = 0. \tag{7.40}$$

It follows that this can only be satisfied if the differential db_3 is

$$\begin{aligned} db_3 = &\, b_{302}\Omega_2 + b_{303}\Omega_3 + b_{306}\left(\Omega_6 + \frac{1}{2}A\right) \\ &+ b_{322}\, df_{22} + b_{312}\, df_{12} + b_{331}\, df_{31} + b_{332}\, df_{32} + b_{31}\theta^1 + b_{32}\theta^2 + b_{33}\theta^3 + b_{34}\theta^4. \end{aligned} \tag{7.41}$$

The functions b_{302}, b_{303}, b_{306}, b_{322}, b_{312}, b_{331} and b_{332} are *uniquely* determined by (7.40), and are expressible in terms of R, T, their derivatives up to order two, and the free parameters $a, f_{12}, f_{22}, f_{31}, f_{32}, b_3$. The parameters b_{31}, b_{32}, b_{33} and b_{34} are arbitrary. Using Mathematica we found explicit expressions for this differential up to the undetermined θ^i terms. Due to the enormous size of this formula we do not quote it here. We note, however, that the free parameters c_1 and c_2 are not present in $\mathrm{d}b_3$.

Now, using all the normalizations obtained so far, and $\mathrm{d}b_3$ as above, we impose the condition

$$\mathrm{d}\theta^1 \wedge \theta^3 \wedge \theta^4 = \left(\Omega_1 - \frac{1}{2}A\right) \wedge \theta^1 \wedge \theta^3 \wedge \theta^4. \tag{7.42}$$

This gives

$$\Omega_1 - \frac{1}{2}A = \mathrm{d}\log f_{12} + f_{122}\Omega_2 + f_{123}\Omega_3 + f_{121}\theta^2 + \dots. \tag{7.43}$$

The dots here denote the undetermined $(\theta^1, \theta^3, \theta^4)$ terms. The functions f_{122}, f_{123} and f_{121} are *uniquely* and *explicitly* determined by (7.42). Similarly, imposition of

$$\mathrm{d}\theta^2 \wedge \theta^3 \wedge \theta^4 = \left(-\Omega_1 - \frac{1}{2}A\right) \wedge \theta^2 \wedge \theta^3 \wedge \theta^4. \tag{7.44}$$

gives

$$-\Omega_1 - \frac{1}{2}A = \mathrm{d}\log f_{22} + f_{222}\Omega_2 + f_{223}\Omega_3 + f_{221}\theta^1 + \dots \tag{7.45}$$

with *uniquely* determined functions f_{222}, f_{223} and f_{221}, and dots denoting the undetermined $(\theta^2, \theta^3, \theta^4)$ terms.

Now the condition

$$\mathrm{d}\theta^3 \wedge \theta^1 \wedge \theta^2 = \left(\Omega_6 - \frac{1}{2}A\right) \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \tag{7.46}$$

is used to reduce the freedom in the choice of the undetermined θ^4 terms in (7.39), (7.43), (7.45) and the undetermined θ^3 term in (7.41). This *one scalar* condition gives a linear relation between the coefficient u_{111}, the coefficient b_{33} at the θ^3 term in (7.41), and the two coefficients at θ^4 in (7.43) and (7.45). Denoting the last two coefficients by f_{124} and f_{224} respectively, we use (7.46) to obtain b_{33} as a linear combination (with coefficients depending on R, T, their derivatives, and the free parameters such as a, etc.) of u_{111}, f_{124} and f_{224}.

At this stage we have associated the forms Ω_6, Ω_1, and A to nonsingular linear combinations of the differentials $\mathrm{d}a$, $\mathrm{d}f_{22}$ and $\mathrm{d}f_{12}$. The still unknown forms Ω_4 and Ω_5 can now be related to $\mathrm{d}f_{31}$ and $\mathrm{d}f_{32}$ by imposing the condition

$$\mathrm{d}\theta^3 = \Omega_4 \wedge \theta^1 + \Omega_5 \wedge \theta^2 + \left(\Omega_6 - \frac{1}{2}A\right) \wedge \theta^3. \tag{7.47}$$

The imposition of this condition results in

$$\Omega_4 = \mp \frac{R_s}{2f_{12}w}\,\mathrm{d}f_{31} \mp \frac{1\mp w}{2f_{12}w}\,\mathrm{d}f_{32} + \ldots + \alpha\theta^1 + \beta\theta^2 \tag{7.48}$$

$$\Omega_5 = \pm \frac{R_s}{2f_{22}w}\,\mathrm{d}f_{31} \pm \frac{1\pm w}{2f_{22}w}\,\mathrm{d}f_{32} + \ldots + \beta\theta^1 + \gamma\theta^2, \tag{7.49}$$

where the doted terms are totally and *uniquely* determined by R, T, their derivatives, and the previous choices. Here α, β, γ are new free parameters.

We stress that we calculated explicitly the right hand sides of equations (7.41), (7.43), (7.45), (7.48) and (7.49). We do not quote them here in full generality due to the lack of space. But now, having these right hand sides calculated, we can calculate $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$. It follows from these calculations that

$$\mathrm{d}\theta^1\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\Omega_5 = H\Omega_3\wedge\Omega_5\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4,$$

and

$$\mathrm{d}\theta^2\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\Omega_4 = H\Omega_2\wedge\Omega_4\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4.$$

The function H appearing in these equations has the form

$$H = Ab_3 + B,$$

where $A \neq 0$ and B are functions of R, T, their derivatives up to order *three*, and only *five* free parameters $\alpha, f_{12}, f_{22}, f_{31}$ and f_{32}. To satisfy the first two of the equations (7.32) we need $H \equiv 0$. This gives the normalization of the parameter b_3 as

$$b_3 = -\frac{B}{A}.$$

This, when compared with $\mathrm{d}b_3$ given by (7.41), and everything after this equation, might bring compatibility conditions. Thus we are at the end of the first loop: we have to return to the formula (7.41) with $b_3 = -B/A$ and repeat all the steps after this formula, inserting this b_3 everywhere.

Note that as the result of the first loop we have forms $(\theta^1, \theta^2, \theta^3, \theta^4)$ satisfying the last two equations (7.32).

THE SECOND LOOP. – Now we start with the forms (7.34), in which we use f_{33}, b_1, b_2 and b_3 determined in the first loop. Then, as before, (7.39) guarantees that (7.38) is valid, and (7.40) is satisfied *automatically*. This means that we do not need equation (7.41) anymore. Equations (7.42) and (7.44) as before determine $\Omega_1 - \frac{1}{2}A$ and $-\Omega_2 - \frac{1}{2}A$, so that (7.43) and (7.45) are satisfied, with new but still explicitly determined $f_{122}, f_{123}, f_{121}, f_{222}, f_{223}, f_{221}$. Since now we do not have (7.41), we use (7.46) to determine u_{111}. After this, we calculate $\mathrm{d}\theta^3$. This satisfies (7.47) provided that Ω_4 and Ω_5 are as in (7.48) and (7.49), with everything determined except the parameters a, β, γ. Choosing these Ω_4 and Ω_5 we have also have

$$\mathrm{d}\theta^1 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \Omega_5 = \mathrm{d}\theta^2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \Omega_4 = 0.$$

It turns out that out of the *nine* undetermined parameters: a, β, γ and the ones in the dotted terms in (7.45) and (7.43), *eight* are totally determined by the requirement that $\mathrm{d}\theta^1 = \Big(\Omega_1 - \frac{1}{2}A\Big) \wedge \theta^1 - \Omega_3 \wedge \theta^3 - \Omega_5 \wedge \theta^4 + t^1_{23}\theta^2 \wedge \theta^3$ & $\mathrm{d}\theta^2 = \Big(-\Omega_1 - \frac{1}{2}A\Big) \wedge \theta^2 - \Omega_2 \wedge \theta^3 - \Omega_4 \wedge \theta^4 + t^2_{13}\theta^1 \wedge \theta^3$. If this condition is imposed the remaining free parameters are $a, f_{12}, f_{22}, f_{31}, f_{32}, c_1, c_2, \beta$. It also follows that this condition forces the coefficients t^1_{23} and t^2_{13} to be given by (7.33). This finishes the proof. $\square$

REMARK 7.7. – Further conditions

$$\mathrm{d}\Omega_2 \wedge \theta^1 \wedge \theta^3 \wedge \theta^4 \wedge \Omega_2 = 0$$
$$\mathrm{d}\Omega_3 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \Omega_3 = 0,$$

imposed on the system $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$ uniquely determine parameters c_1 and c_2. To fix the parameter β we use the requirement that the differential $\mathrm{d}\Omega_2$ does not involve a $\Omega_2 \wedge \theta^4$ term. After imposing this, the remaining free parameters in the definitions of $(\theta^i, \Omega_\mu, A)$ are *only*: $a, f_{12}, f_{22}, f_{31}, f_{32}$. This shows that the system for a $(1,2,3)$ type para-CR structure with $J_1 \equiv J_2 \equiv 0$ naturally *closes* on P, and that P can be locally parametrized by (x, y, z, p, q, s) (the base) and $(a, f_{12}, f_{22}, f_{31}, f_{32})$ (fibers).

REMARK 7.8. – Theorem 7.6 assures that the solution space of a pair of PDEs $z_{xx} = R(x, y, z, z_x, z_y, z_{xy})$ & $z_{yy} = T(x, y, z, z_x, z_y, z_{xy})$ satisfying $D^2_x T \equiv D^2_y R$ and $J_1 \equiv J_2 \equiv 0$ is naturally equipped with a $(+,+,-,-)$ signature conformal structure, and that this conformal structure is a *point invariant* of the corresponding pair of PDEs. However the appearence of the *torsion* terms t^1_{23} and t^2_{13} in (7.32), as well as the *nonhorizontal* terms, such as e.g. $\Omega_2 \wedge \theta^1$ in $\mathrm{d}\Omega_2$, show, that there might be *many point nonequivalent* PDEs $z_{xx} = R(x, y, z, z_x, z_y, z_{xy})$ & $z_{yy} = T(x, y, z, z_x, z_y, z_{xy})$ with $D^2_x T \equiv D^2_y R$ and $J_1 \equiv J_2 \equiv 0$, which correspond to *the same* conformal class of metrics. Although the forms $(\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, A)$ together with $(\theta^1, \theta^2, \theta^3, \theta^4)$, as constructed in the proof of Theorem 7.6 and in the Remark 7.7, *solve the equivalence problem* for the $(1,2,3)$ type para-CR structures in question, they in general do *not* define a Weyl connection on $\mathcal{S}$. For this to be possible the torsion coefficients t^1_{23}, t^2_{13}, as well as the nonhorizontal terms in $\mathrm{d}\Omega_2$ and $\mathrm{d}\Omega_3$ must vanish. In the rest of this section we will find those point nonequivalent classes of equations $z_{xx} = R(x, y, z, z_x, z_y, z_{xy})$ & $z_{yy} = T(x, y, z, z_x, z_y, z_{xy})$ for which this is the case.

LEMMA 7.9. – *The forms* (7.20)-(7.21)-(7.22) *satisfy the differential system* (7.13) *if and only if they can be brought to the form in which they satisfy:*

$$\begin{aligned}
&\mathrm{d}\theta^1 = \left(\Omega_1 - \frac{1}{2}A\right)\wedge\theta^1 - \Omega_3\wedge\theta^3 - \Omega_5\wedge\theta^4\\
&\mathrm{d}\theta^2 = \left(-\Omega_1 - \frac{1}{2}A\right)\wedge\theta^2 - \Omega_2\wedge\theta^3 - \Omega_4\wedge\theta^4\\
&\mathrm{d}\theta^3 = \Omega_4\wedge\theta^1 + \Omega_5\wedge\theta^2 + \left(\Omega_6 - \frac{1}{2}A\right)\wedge\theta^3\\
&\mathrm{d}\theta^4 = \Omega_2\wedge\theta^1 + \Omega_3\wedge\theta^2 + \left(-\Omega_6 - \frac{1}{2}A\right)\wedge\theta^4,\\
&\mathrm{d}\Omega_1 = \Omega_2\wedge\Omega_5 - \Omega_3\wedge\Omega_4 - \varkappa\theta^1\wedge\theta^2\\
&\mathrm{d}\Omega_2 = \Omega_2\wedge(\Omega_1+\Omega_6) + \varkappa\theta^2\wedge\theta^4\\
&\mathrm{d}\Omega_3 = (\Omega_1-\Omega_6)\wedge\Omega_3 + \varkappa\theta^1\wedge\theta^4\\
&\mathrm{d}\Omega_4 = \Omega_4\wedge(\Omega_1-\Omega_6) + \varkappa\theta^2\wedge\theta^3\\
&\mathrm{d}\Omega_5 = (\Omega_1+\Omega_6)\wedge\Omega_5 + \varkappa\theta^1\wedge\theta^3\\
&\mathrm{d}\Omega_6 = \Omega_2\wedge\Omega_5 + \Omega_3\wedge\Omega_4 - \varkappa\theta^3\wedge\theta^4\\
&\mathrm{d}A = 0,\\
&\mathrm{d}\varkappa = \varkappa A.
\end{aligned} \tag{7.50}$$

PROOF. – As we noticed in Theorem 7.6 the forms (7.20)-(7.21)-(7.22) may satsify the first two of equations (7.13) if and only if $K_1\equiv K_2\equiv 0$, or what is the same, if and only if $R_{ss}\equiv T_{ss}\equiv 0$. Moreover, because the forms $(\theta^1,\Omega_2,\Omega_3,\theta^2,\theta^3,\theta^4)$ are in the class of forms $(\lambda,\mu_1,\mu_2,\nu_1,\nu_2,\nu_3)$ defining the $(1,2,3)$ para-CR structure, the forms $(\theta^4,\Omega_2,\Omega_3)$ form a closed differential ideal corresponding to the integrable distribution H^-. Thus, since $\mathrm{d}\Omega_2\wedge\Omega_2\wedge\Omega_3\wedge\theta^4\equiv 0$ and $\mathrm{d}\Omega_3\wedge\Omega_2\wedge\Omega_3\wedge\theta^4\equiv 0$, the only possibility of satisfaction of the sixth and seventh equations in (7.13) is that:

$$\begin{aligned}
&\mathrm{d}\Omega_2 = \Omega_2\wedge(\Omega_1+\Omega_6) + \Gamma_1\wedge\theta^4\\
&\mathrm{d}\Omega_3 = (\Omega_1-\Omega_6)\wedge\Omega_3 + \Gamma_2\wedge\theta^4,
\end{aligned} \tag{7.51}$$

with two 1-forms Γ_1,Γ_2 on P, which can be chosen such that $\Gamma_1=\gamma_{11}\theta^1+\gamma_{12}\theta^2+\gamma_{13}\theta^3$ and $\Gamma_2=\gamma_{21}\theta^1+\gamma_{22}\theta^2+\gamma_{23}\theta^3$. Here γ_{ij} are some functions on P. Now, one successively imposes the condition that the differentials of the right hand sides of the first four of equations (7.50), the differentials of the right hand sides of equations (7.51), and the differentials of the right hand sides of the last five of equations (7.13) are zero (they must be, as they are differentials of the coframe forms (θ^i, Ω_μ)). This straightforwardly leads to the conclusion that it is possible if and only if (7.50) is satisfied. This finishes the proof. $\square$

THEOREM 7.10. – *All finite type systems of PDEs on the plane*

$$z_{xx} = R(x, y, z, z_x, z_y, z_{xy}) \quad \& \quad z_{yy} = T(x, y, z, z_x, z_y, z_{xy}),$$

which in a natural way define a split signature Weyl geometry $[g, A]$ *on their* 4*-dimensional solution space, are locally point equivalent to the system:*

$$\begin{aligned} z_{xx} &= -\frac{2yz_xz_{xy}}{z + xz_x - yz_y} \quad \& \\ z_{yy} &= -\frac{2\kappa}{y}\frac{z_{xy}}{z + xz_x - yz_y} - \frac{2x}{y}\frac{(z - yz_y)z_{xy}}{z + xz_x - yz_y}, \end{aligned} \tag{7.52}$$

with κ *being a real number. All such systems with* $\kappa \neq 0$ *are locally point equivalent to the system with* $\kappa = 1$*. They are point nonequivalent with the system with* $\kappa \equiv 0$*. For each* κ *system* (7.52) *has*

$$z = \frac{\kappa(a_0a_1 + a_2a_3)y + \kappa a_1 - y - a_0y^2 - a_3xy}{a_2y - a_1x}$$

as its general solution. The Weyl geometry $[g_\kappa, A_\kappa]$ *on the* 4*-dimensional solution space, with points parametrized by* (a_0, a_1, a_2, a_3)*, is represented by*

$$g_\kappa = \frac{2\big(\mathrm{d}a_0\mathrm{d}a_1 + \mathrm{d}a_2\mathrm{d}a_3\big)}{\big(1 + \kappa(a_0a_1 + a_2a_3)\big)^2}, \qquad A_\kappa = 0.$$

The type $(1, 2, 3)$ *para-CR structures corresponding to the two different values* 1 *or* 0 *are locally nonequivalent. If* $\kappa = 0$*, then the corresponding* $(1, 2, 3)$ *type para-CR structure has an* 11*-dimensional group of symmetries* $\mathbf{CO}(2, 2)$*, and is equivalent to the* $(1, 2, 3)$ *para-CR structure corresponding to the system* $z_{xx} = z_{yy} = 0$*. If* $\kappa \neq 0$*, the corresponding type* $(1, 2, 3)$ *para-CR structures have a* 10*-dimensional group of symmetries isomorphic to* $\mathbf{SO}(2, 3)$*. This group acts naturally as the group of motions on the solution space, which is equipped with a metric of constant curvature* g_κ*.*

PROOF. – We first show that the $(1, 2, 3)$ type para-CR structure associated with the system (7.52) defines forms $(\theta^i, \Omega_\mu, A)$ satisfying (7.50). Since we have the general solution of the system (7.52), it is convenient to use the representation (7.19), rather than (7.20), for the defining forms $(\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3)$. Thus, inserting

$$\psi = \frac{\kappa(a_0a_1 + a_2a_3)y + \kappa a_1 - y - a_0y^2 - a_3xy}{a_2y - a_1x}$$

in (7.19), we have

$$\lambda = -\frac{(a_1\kappa - y)y}{a_1x - a_2y}\,\mathrm{d}a_0 + \frac{(a_2\kappa - x)y(1 + a_3x + a_0y)}{(a_1x - a_2y)^2}\,\mathrm{d}a_1$$
$$- \frac{(a_1\kappa - y)y(1 + a_3x + a_0y)}{(a_1x - a_2y)^2}\,\mathrm{d}a_2 - \frac{(a_2\kappa - x)y}{a_1x - a_2y}\,\mathrm{d}a_3,$$
$$\nu_1 = \mathrm{d}a_1, \quad \nu_2 = \mathrm{d}a_2, \quad \nu_3 = \mathrm{d}a_3$$
$$\mu_1 = \mathrm{d}x, \quad \mu_2 = \mathrm{d}y.$$

We now take the forms (7.22) with these $(\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3)$ and apply the procedure of fixing the gauge as in the proof of Theorems 7.5, 7.6 and Remark 7.7. This procedure leads to the following choices for the free parameters $b_1, b_2, b_3, c_1, c_2, f_{13}, f_{22}, f_{23}, f_{31}, f_{33}, h_{11}, h_{12}, h_{21}, h_{22}$:

$$b_1 = \frac{f_{12}u}{f_{21}f_{32}(a_1\kappa - y)y(1 + a_3x + a_0y)}$$
$$b_2 = -\frac{af_{12}}{f_{32}}$$
$$b_3 = \frac{u}{f_{21}(a_1\kappa - y)y(1 + a_3x + a_0y)}$$
$$c_1 = -\frac{a\kappa(1 + a_3x + a_0y)}{f_{21}(1 + \kappa(a_0a_1 + a_2a_3))(a_1\kappa - y)}$$
$$c_2 = -\frac{a(a_1\kappa - y)}{f_{32}(1 + \kappa(a_0a_1 + a_2a_3))(a_1x - a_2y)}$$
$$f_{13} = \frac{(a_1x - a_2y)u}{a(a_1\kappa - y)y(1 + a_3x + a_0y)^2}$$
$$f_{22} = 0$$
$$f_{23} = -\frac{(a_1x - a_2y)f_{21}}{1 + a_3x + a_0y}$$
$$f_{31} = -\frac{(a_2\kappa - x)f_{32}}{a_1\kappa - y}$$
$$f_{33} = 0$$
$$h_{11} = -\frac{ay(1 + a_3x + a_0y)}{f_{21}(a_1x - a_2y)^2}$$
$$h_{12} = \frac{a(a_2\kappa - x)y(1 + a_3x + a_0y)}{f_{21}(a_1\kappa - y)(a_1x - a_2y)^2}$$
$$h_{21} = \frac{a(a_1\kappa - y)y(a_1 + y(a_0a_1 + a_2a_3))}{f_{32}(a_1x - a_2y)^3}$$
$$h_{22} = -\frac{a(a_1\kappa - y)y(a_2 + x(a_0a_1 + a_2a_3))}{f_{32}(a_1x - a_2y)^3}.$$

Here

$$u = a_1^2 f_{21} f_{32} x^2 - a_1 y\big(2a_2 f_{21} f_{32} x + a f_{11}\kappa(1 + a_3 x + a_0 y)\big) + y\big(a_2^2 f_{21} f_{32} y + a(1 + a_3 x + a_0 y)\big(f_{12}(x - a_2\kappa) + f_{11} y\big)\big).$$

It follows from the construction that these normalizations force the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$ to satisfy the system (7.32) and the three conditions from remark 7.7. Because of the choice of $z = z(x, y, a_0, a_1, a_2, a_3)$ as the general solution to (7.52), it turns out that in these normalizations the forms (7.22) satisfy, in addition (7.50), with

$$\varkappa = -\frac{2\kappa(1 + a_3 x + a_0 y)}{f_{21} f_{32}(1 + \kappa(a_0 a_1 + a_2 a_3))^2 (a_1 x - a_2 y)}.$$

If $\kappa \equiv 0$, we get $\varkappa \equiv 0$, and the system (7.50) becomes (7.11)-(7.12). This proves that if $\kappa \equiv 0$, then the system (7.52) is point equivalent to $z_{xx} = z_{yy} = 0$, or what is the same, that the corresponding $(1, 2, 3)$ type para-CR structure is locally equivalent to the flat one described by Theorem 7.2.

If $\kappa \neq 0$ we normalize $\varkappa$ to $\varkappa = 1$ by choosing

$$f_{32} = -\frac{2\kappa(1 + a_3 x + a_0 y)}{f_{21}(1 + \kappa(a_0 a_1 + a_2 a_3))^2 (a_1 x - a_2 y)}.$$

This choice reduces P to a 10-dimensional manifold P_0, with coordinates $(x, y, z, p, q, s, a, f_{11}, f_{12}, f_{21}, f_{22})$, on which $A = 0$ and the ten linearly independent 1-forms (θ^i, Ω_μ) satisfy the system

$$\begin{aligned}
\mathrm{d}\theta^1 &= \Omega_1 \wedge \theta^1 - \Omega_3 \wedge \theta^3 - \Omega_5 \wedge \theta^4\\
\mathrm{d}\theta^2 &= -\Omega_1 \wedge \theta^2 - \Omega_2 \wedge \theta^3 - \Omega_4 \wedge \theta^4\\
\mathrm{d}\theta^3 &= \Omega_4 \wedge \theta^1 + \Omega_5 \wedge \theta^2 + \Omega_6 \wedge \theta^3\\
\mathrm{d}\theta^4 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 - \Omega_6 \wedge \theta^4,\\
\mathrm{d}\Omega_1 &= \Omega_2 \wedge \Omega_5 - \Omega_3 \wedge \Omega_4 - \theta^1 \wedge \theta^2\\
\mathrm{d}\Omega_2 &= \Omega_2 \wedge (\Omega_1 + \Omega_6) + \theta^2 \wedge \theta^4\\
\mathrm{d}\Omega_3 &= (\Omega_1 - \Omega_6) \wedge \Omega_3 + \theta^1 \wedge \theta^4\\
\mathrm{d}\Omega_4 &= \Omega_4 \wedge (\Omega_1 - \Omega_6) + \theta^2 \wedge \theta^3\\
\mathrm{d}\Omega_5 &= (\Omega_1 + \Omega_6) \wedge \Omega_5 + \theta^1 \wedge \theta^3\\
\mathrm{d}\Omega_6 &= \Omega_2 \wedge \Omega_5 + \Omega_3 \wedge \Omega_4 - \theta^3 \wedge \theta^4.
\end{aligned} \tag{7.53}$$

Since in these relations only constant coefficients appear on the right hand sides, P_0 is locally a Lie group, with the forms (θ^i, Ω_μ) as its left invariant forms. This group is isomorphic to $\mathbf{SO}(2, 3)$ and, it follows from the Cartan equivalence method, that it is the *full* symmetry group of the type $(1, 2, 3)$ para-CR structure corresponding to (7.52) with $\kappa \neq 0$. Accordingly it is also the full group of local point symmetries of the system (7.52) with $\kappa \neq 0$. The appearence of the group

$\mathbf{SO}(2,3)$ is not accidental, since one can check that the so normalized forms $(\theta^1,\theta^2,\theta^3,\theta^4)$ satisfy

$$G=2(\theta^1\theta^2+\theta^3\theta^4)=\frac{4\kappa(\mathrm{d}a_0\mathrm{d}a_1+\mathrm{d}a_2\mathrm{d}a_3)}{(1+\kappa(a_0a_1+a_2a_3))^2}.$$

This means that the 4-dimensional solution space $\mathcal{S}$ of the system (7.52) with $\kappa\neq 0$ is naturally equipped with a split-signature *constant curvature* metric G. The symmetry group of the pseudoriemannian structure $(\mathcal{S},G)$ is obvioulsy $\mathbf{SO}(2,3)$.

Since the parameter κ does not appear in the equations (7.53), we conclude that $\kappa\neq 0$ can always be brought to $\kappa=1$ by a point transformation of (7.52), or what is the same, by a para-CR diffeomorphism of the corresponding para-CR structure. This proves that among type $(1,2,,3)$ para-CR structures associated with (7.52) there are only two para-CR nonequivalent ones: the one with $\kappa=0$, and those with $\kappa\neq 0$, which are all locally equivalent to the one with $\kappa=1$.

To prove that these two structures, modulo para-CR equivalence, are the only ones that satisfy Lemma 7.9, we proceed as follows:

Suppose that we have a finite type system of PDEs $z_{xx}=R(x,y,z,z_z,z_y,z_{xy})$ & $z_{yy}=T(x,y,z,z_z,z_y,z_{xy})$, which via the procedure described in Theorems 7.5, 7.6 and Remark 7.7, leads to the differential system (7.50), as in Lemma 7.9. If we have $\varkappa\equiv 0$, then our PDEs are point equivalent to $z_{xx}=z_{yy}=0$. If $\varkappa\neq 0$ then the last equation (7.50) says that $A=\frac{\mathrm{d}\varkappa}{\varkappa}$. Then putting $\varepsilon=\operatorname{sign}\varkappa$ we rescale the forms $(\theta^1,\theta^2,\theta^3,\theta^4)$ to

$$(\bar\theta^1,\bar\theta^2,\bar\theta^3,\bar\theta^4)=\big(\varepsilon\varkappa\big)^{\frac12}(\theta^1,\theta^2,\theta^3,\theta^4).$$

Obviously this rescaling is a para-CR transformation. The advantage of this rescaling is that, after it, the form A disappears from the first ten equations (7.50). Explicitly, after the rescaling, the system (7.50) becomes:

$$\begin{aligned}
&\mathrm{d}\bar\theta^1=\Omega_1\wedge\bar\theta^1-\Omega_3\wedge\bar\theta^3-\Omega_5\wedge\bar\theta^4\\
&\mathrm{d}\bar\theta^2=-\Omega_1\wedge\bar\theta^2-\Omega_2\wedge\bar\theta^3-\Omega_4\wedge\bar\theta^4\\
&\mathrm{d}\bar\theta^3=\Omega_4\wedge\bar\theta^1+\Omega_5\wedge\bar\theta^2+\Omega_6\wedge\bar\theta^3\\
&\mathrm{d}\bar\theta^4=\Omega_2\wedge\bar\theta^1+\Omega_3\wedge\bar\theta^2-\Omega_6\wedge\bar\theta^4,\\
&\mathrm{d}\Omega_1=\Omega_2\wedge\Omega_5-\Omega_3\wedge\Omega_4-\varepsilon\bar\theta^1\wedge\bar\theta^2\\
&\mathrm{d}\Omega_2=\Omega_2\wedge(\Omega_1+\Omega_6)+\varepsilon\bar\theta^2\wedge\bar\theta^4\\
&\mathrm{d}\Omega_3=(\Omega_1-\Omega_6)\wedge\Omega_3+\varepsilon\bar\theta^1\wedge\bar\theta^4\\
&\mathrm{d}\Omega_4=\Omega_4\wedge(\Omega_1-\Omega_6)+\varepsilon\bar\theta^2\wedge\bar\theta^3\\
&\mathrm{d}\Omega_5=(\Omega_1+\Omega_6)\wedge\Omega_5+\varepsilon\bar\theta^1\wedge\bar\theta^3\\
&\mathrm{d}\Omega_6=\Omega_2\wedge\Omega_5+\Omega_3\wedge\Omega_4-\varepsilon\bar\theta^3\wedge\bar\theta^4\\
&A=\frac{\mathrm{d}\varkappa}{\varkappa}.
\end{aligned}\tag{7.54}$$

This shows that if $\varkappa \neq 0$ we can always reduce the system to 10 dimensions, and that there are *at most* two different para-CR structures with such $\varkappa$, corresponding to the different signs of ε. However, a discrete para-CR transformation on this system, transforming

$$(\bar{\theta}^1, \bar{\theta}^3, \Omega_2, \Omega_5) \to (-\bar{\theta}^1, -\bar{\theta}^3, -\Omega_2, -\Omega_5),$$

and being the identity on the rest of the coframe forms, brings the system (7.54) into the form (7.53), in which $\varepsilon = +1$. This shows that the para-CR structures with different values of ε are equivalent, and that there are only two, locally nonequivalent type $(1,2,3)$ para-CR structures satisfying system (7.50). We found the representatives of both of them, as the para-CR structures corresponding to $\kappa = 0$ or $\kappa = 1$ in (7.52). This finishes the proof. □

8. – Para-CR structures of type $(3,2,1)$.

8.1 – *Type* $(3,2,1)$ *versus* $(1,2,3)$.

As noted in Section 4, the flip $(1,1,n-1) \to (n-1,1,1)$, changes a para-CR structure corresponding to an nth order ODE considered modulo *point* transformations, to a para-CR structure corresponding to an nth order ODE considered modulo *contact* transformations. In this section we further investigate the meaning of the flip

$$(k,r,s) \to (s,r,k),$$

on an example of type $(k=1, r=2, s=3)$ para-CR structures corresponding to PDEs (7.15). We expect that the passage $(1,2,3) \to (3,2,1)$ will again change the geometric setting in such a way that the type $(1,2,3)$ para-CR structure corresponding to PDEs (7.15) considered modulo *point* transformations will become a para-CR structure corresponding to the same pair of PDEs but considered modulo *contact* transformations.

That this is really the case can be seen from the following:

Given a pair of equations

$$z_{xx} = R(x,y,z,z_x,z_y,z_{xy}) \qquad \& \qquad z_{yy} = T(x,y,z,z_x,z_y,z_{xy})$$

we use the contact forms $\lambda = \mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y$, $\nu_1 = \mathrm{d}p - R\mathrm{d}x - s\mathrm{d}y$, $\nu_2 = \mathrm{d}q - s\mathrm{d}x - T\mathrm{d}y$, $\nu_3 = \mathrm{d}s - D_yR\mathrm{d}x - D_xT\mathrm{d}y$, $\mu_1 = \mathrm{d}x$, $\mu_2 = \mathrm{d}y$ on the 6-dimensional jet space $\mathcal{J}$ parametrized by (x,y,z,p,q,s). It is easy to see that when the equations undergo a *point* transformation of variables, then the forms change according to:

$$(8.1)\qquad \begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix} \to \begin{pmatrix}\lambda'\\ \nu_1'\\ \nu_2'\\ \nu_3'\\ \mu_1'\\ \mu_2'\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ b_1 & f_{11} & f_{12} & 0 & 0 & 0\\ b_2 & f_{21} & f_{22} & 0 & 0 & 0\\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0\\ c_1 & 0 & 0 & 0 & h_{11} & h_{12}\\ c_2 & 0 & 0 & 0 & h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix},$$

and when the equations undergo a *contact* transformation of variables the forms change as:

$$(8.2)\qquad \begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix} \to \begin{pmatrix}\lambda'\\ \nu_1'\\ \nu_2'\\ \nu_3'\\ \mu_1'\\ \mu_2'\end{pmatrix} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0\\ b_1 & f_{11} & f_{12} & 0 & 0 & 0\\ b_2 & f_{21} & f_{22} & 0 & 0 & 0\\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0\\ c_1 & u_{11} & u_{12} & 0 & h_{11} & h_{12}\\ c_2 & u_{21} & u_{22} & 0 & h_{21} & h_{22}\end{pmatrix}\begin{pmatrix}\lambda\\ \nu_1\\ \nu_2\\ \nu_3\\ \mu_1\\ \mu_2\end{pmatrix}.$$

Introducing vector fields $(Z, X_1, X_2, Y_1, Y_2, Y_3)$, which are respective duals to the coframe $(\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3)$, we easily see that under the *point* transformations they transform according to:

$$\begin{pmatrix}Z\\ Y_1\\ Y_2\\ Y_3\\ X_1\\ X_2\end{pmatrix} \to \begin{pmatrix}Z'\\ Y_1'\\ Y_2'\\ Y_3'\\ X_1'\\ X_2'\end{pmatrix} = \begin{pmatrix} * & * & * & * & * & *\\ 0 & * & * & * & 0 & 0\\ 0 & * & * & * & 0 & 0\\ 0 & 0 & 0 & * & 0 & 0\\ 0 & 0 & 0 & 0 & * & *\\ 0 & 0 & 0 & 0 & * & *\end{pmatrix}\begin{pmatrix}Z\\ Y_1\\ Y_2\\ Y_3\\ X_1\\ X_2\end{pmatrix},$$

and under the *contact* transformations they transform according to:

$$\begin{pmatrix}Z\\ Y_1\\ Y_2\\ Y_3\\ X_1\\ X_2\end{pmatrix} \to \begin{pmatrix}Z'\\ Y_1'\\ Y_2'\\ Y_3'\\ X_1'\\ X_2'\end{pmatrix} = \begin{pmatrix} * & * & * & * & * & *\\ 0 & * & * & * & * & *\\ 0 & * & * & * & * & *\\ 0 & 0 & 0 & * & 0 & 0\\ 0 & 0 & 0 & 0 & * & *\\ 0 & 0 & 0 & 0 & * & *\end{pmatrix}\begin{pmatrix}Z\\ Y_1\\ Y_2\\ Y_3\\ X_1\\ X_2\end{pmatrix},$$

where by $*$ we denoted the matrix entries that are nonzero. This shows that the *point* transformations preserve the two vector spaces: 2-dimensional $H^+ = \mathrm{Span}(X_1, X_2)$ and 3-dimensional $H^-_{\rm point} = \mathrm{Span}(Y_1, Y_2, Y_3)$, while the contact transformations preserve H^+ and only a 1-dimensional $H^-_{\rm contact} = \mathrm{Span}(Y_3)$. We have the following proposition:

PROPOSITION 8.1. – *Assume that a pair of equations*

$$z_{xx} = R(x, y, z, z_x, z_y, z_{xy}) \qquad \& \qquad z_{yy} = T(x, y, z, z_x, z_y, z_{xy})$$

satisfies the compatibility conditions $D_x^2T = D_y^2R$, *where* $D_x = \partial_x + p\partial_z + R\partial_p + s\partial_q + D_yR\partial_s$, $D_y = \partial_y + q\partial_z + T\partial_q + s\partial_p + D_xT\partial_s$ *and* $p = z_x, q = z_y$, $s = z_{xy}$. *Let* $H^+ = \mathrm{Span}(D_x, D_y)$, $H^-_{\mathrm{point}} = \mathrm{Span}(Y_1, Y_2, Y_3)$, *and* $H^-_{\mathrm{contact}} = \mathrm{Span}(Y_3)$, *with* $Y_1 = \partial_p$, $Y_2 = \partial_q$, $Y_3 = \partial_s$, *be three distributions, with respective dimensions* 2, 3, 1, *on the* 6*-dimensional jet space* $\mathcal{J}$ *parametrized by* (x, y, z, p, q, s). *Then:*

If this pair of equations is considered modulo point transformation of variables, it defines a type $(1, 2, 3)$ *para-CR structure* $(\mathcal{J}, H^+, H^-_{\mathrm{point}})$ *on* $\mathcal{J}$.

If this pair of equations is considered modulo contact transformations, it defines a type $(3, 2, 1)$ *para-CR structure* $(\mathcal{J}, H^+, H^-_{\mathrm{contact}})$ *on* $\mathcal{J}$.

PROOF. – In view of the discussion preceeding the Proposition, the only thing to be proven is that the distributions H^+ and H^-_{point} are integrable on $\mathcal{J}$. Using the local coordinates (x, y, z, p, q, s) we see that the duals to a coframe $(\lambda, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3)$ are $(Z = \partial_z, X_1 = D_x, X_2 = D_y, Y_1 = \partial_p, Y_2 = \partial_q, Y_3 = \partial_s)$. Hence, obviously, H^-_{point} is integrable. Calculating the commutator $[D_x, D_y]$ we get $[D_x, D_y] = (D_x^2T - D_y^2R)\partial_s$, which vanishes due to our assumptions. Thus also H^+ is integrable. □

8.2 – *Towards invariants for type* $(3, 2, 1)$.

The contact transformations (8.2) are *more restrictive* than the most general para-CR transformations

$$(8.3)\qquad \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ n \\ m_1 \\ m_2 \end{pmatrix} \to \begin{pmatrix} l'_1 \\ l'_2 \\ l'_3 \\ n' \\ m'_1 \\ m'_2 \end{pmatrix} = \begin{pmatrix} a & a_{11} & a_{12} & 0 & 0 & 0 \\ b_1 & f_{11} & f_{12} & 0 & 0 & 0 \\ b_2 & f_{21} & f_{22} & 0 & 0 & 0 \\ b_3 & f_{31} & f_{32} & f_{33} & 0 & 0 \\ c_1 & u_{11} & u_{12} & 0 & h_{11} & h_{12} \\ c_2 & u_{21} & u_{22} & 0 & h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ n \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \theta^4 \\ \theta^1 \\ \theta^2 \\ \theta^3 \\ \Omega_3 \\ \Omega_2 \end{pmatrix},$$

of a $(3, 2, 1)$-para-CR structure $[l_1, l_2, l_3, m_1, m_2, n]$ defined on $\mathcal{J}$ by $l_1 = \lambda = \mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y$, $l_2 = \nu_1 = \mathrm{d}p - R\mathrm{d}x - s\mathrm{d}y$, $l_3 = \nu_2 = \mathrm{d}q - s\mathrm{d}x - T\mathrm{d}y$, $n = \nu_3 = \mathrm{d}s - D_yR\mathrm{d}x - D_xT\mathrm{d}y$, $m_1 = \mu_1 = \mathrm{d}x$, $m_2 = \mu_2 = \mathrm{d}y$. However, when looking for the local invariants for such structures, we can easily normalize the unwanted a_{11} and a_{12} parameters in these transformations to $a_{11} = 0$ and $a_{22} = 0$ by the requirement that the invariant forms (θ^i) satisfy a consequence of (7.23), i.e.:

$$(8.4)\qquad \mathrm{d}\theta^4 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = 0.$$

It is easy to see that (8.4) necessarily implies $a_{11} = 0$ and $a_{22} = 0$. Since condition (7.23) is needed to have a conformal metric on the solution space, from now on we will assume (8.4), and as a consequence

$$a_{11} = a_{12} = 0.$$

In such a case the para-CR transformations (8.3) become the contact transformations [1] for the associated system of PDEs $z_{xx} = R(x, y, z, z_x, z_y, z_{xy})$, $z_{yy} = T(x, y, z, z_x, z_y, z_{xy})$. As in the previous sections we assume in addition that $D_x^2 T = D_y^2 R$, but release the $1 - R_s T_s > 0$ condition to $1 - R_s T_s \neq 0$. We have the following theorem.

THEOREM 8.2. – *Given a pair of PDEs on the plane $z_{xx} = R(x, y, z, z_x, z_y, z_{xy})$ & $z_{yy} = T(x, y, z, z_x, z_y, z_{xy})$ satisfying $D_x^2 T = D_y^2 R$ and $1 - R_s T_s \neq 0$, the condition*

$$J_1 \equiv 0, \qquad \& \qquad J_2 \equiv 0,$$

where

$$J_1 = (R_s T_s - 4) D_x R_s + R_s (2 D_y R_s - R_s D_x T_s) + 8 R_q - 6 R_q R_s T_s + 4 R_p R_s + 2 R_s^2 T_q - 2 R_p R_s^2 T_s + 2 R_s^3 T_p$$

$$J_2 = (R_s T_s - 4) D_y T_s + T_s (2 D_x T_s - T_s D_y R_s) + 8 T_p - 6 R_s T_p T_s + 4 T_q T_s + 2 R_p T_s^2 - 2 R_s T_q T_s^2 + 2 R_q T_s^3,$$

is preserved under the contact transformations of the variables. If this condition is satisfied the 4-dimensional solution space of the PDEs is naturally equipped with a conformal class $[g]$ of metrics. If

$$1 - R_s T_s > 0$$

these conformal metrics have split signature. If

$$1 - R_s T_s < 0$$

the metrics have Lorentzian signature. The conformal class $[g]$ is invariant under the contact transformations of the variables of the PDEs.

We also have a useful Proposition, which gives local representatives of the conformal class $[g]$ from the above Theorem:

PROPOSITION 8.3. – *If $R_s T_s \neq 4$ a representative g of the conformal class $[g]$ can be chosen so that it is given by*

$$g = 2\lambda\omega \; + \; 2(R_s T_s - 4)(T_s v_1^2 - 2 v_1 v_2 + R_s v_2^2), \tag{8.5}$$

[1] Note that the situation here is similar to the situation in the *point* invariant case. There the para-CR transformations (7.22) of a $(1, 2, 3)$-type para-CR structure associated with the system of PDEs (7.15) differed from the point transformations (8.1), by the appearence of the nonzero parameters f_{13} and f_{23} in (7.22. But one of the consequences of equations (7.23)-(7.27) was that $f_{13} = f_{23} = 0$, (see (7.28)), which proved that the para-CR transformations (7.22) and the point transformations (8.1) were equivalent.

where

$$\omega = (4D_xT_s - 2T_sD_yR_s + 4R_pT_s - 2R_s^2T_pT_s - 2R_sT_qT_s + 4R_qT_s^2)\nu_1$$
$$+ (4D_yR_s - 2R_sD_xT_s + 4R_sT_q - 2R_qR_sT_s^2 - 2R_pR_sT_s + 4R_s^2T_p)\nu_2$$
$$+ 2(4 - R_sT_s)(R_sT_s - 1)\nu_3 + v\lambda,$$

$$\lambda = \mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y,$$

$$\nu_1 = \mathrm{d}p - R\mathrm{d}x - s\mathrm{d}y, \quad \nu_2 = \mathrm{d}q - s\mathrm{d}x - T\mathrm{d}y, \quad \nu_3 = \mathrm{d}s - D_yR\mathrm{d}x - D_xT\mathrm{d}y,$$

and

$$2v = 8D_xT_q - 4D_y^2R_s + 4(D_xT_s)D_yR_s + 4R_sD_xT_p - 4R_sD_yT_q - 4R_s^2D_yT_p$$
$$+ 8R_qT_p - 14R_sT_pD_yR_s + 4R_pR_sT_p + 3R_s^2T_pD_xT_s - 6R_s^3T_p^2 - 4T_qD_yR_s$$
$$+ 4R_sT_qD_xT_s - 6R_s^2T_pT_q + 8T_sD_yR_q - 2(D_yR_s)^2T_s + 4R_pT_sD_yR_s - 2R_sT_sD_xT_q$$
$$+ R_sT_sD_y^2R_s + 4R_sT_sD_yR_p - R_s^2T_sD_xT_p + R_s^2T_sD_yT_q + R_s^3T_sD_yT_p + 8R_zT_s$$
$$+ 2R_qR_sT_pT_s + 2R_pR_s^2T_pT_s + 8R_qT_qT_s - 3R_sT_qT_sD_yR_s + 4R_pR_sT_qT_s$$
$$- 2R_s^3T_pT_qT_s - 2R_s^2T_q^2T_s + 4R_qT_s^2D_yR_s - 2R_sT_s^2D_yR_q - R_s^2T_s^2D_yR_p - 2R_sR_zT_s^2$$
$$+ 2R_qR_s^2T_pT_s^2 + 2R_qR_sT_qT_s^2 + 8R_sT_z - 2R_s^2T_sT_z.$$

If $R_sT_s \neq 0$ *another representative* g *of* $[g]$ *may be chosen so that:*

$$g = 2\lambda\omega' + T_s\nu_1^2 - 2\nu_1\nu_2 + R_s\nu_2^2, \tag{8.6}$$

where

$$\omega' = \frac{-D_yT_s + 2T_p - R_sT_pT_s + T_qT_s}{T_s}\nu_1 + \frac{-D_xR_s + 2R_q - R_qR_sT_s + R_pR_s}{R_s}\nu_2$$
$$+ (1 - R_sT_s)\nu_3 - \frac{v'}{2R_s^3T_s}\lambda,$$

with λ, ν_1, ν_2 *and* ν_3 *as before, and*

$$v' = 2R_s^2(D_xR_s)D_yT_s - 4R_qR_s^2D_yT_s - R_pR_s^3D_yT_s - 4R_s^2T_pD_xR_s$$
$$+ 8R_qR_s^2T_p + 2R_pR_s^3T_p + 2(D_xR_s)^2T_s - 8R_qT_sD_xR_s + 8R_q^2T_s$$
$$- 2R_pR_sT_sD_xR_s + 4R_pR_qR_sT_s - R_s^2T_sD_xD_yR_s + 2R_s^2T_sD_yR_q + R_s^3T_sD_xT_q$$
$$+ R_s^3T_sD_yR_p - R_qR_s^3T_pT_s - 3R_s^2T_qT_sD_xR_s + 6R_qR_s^2T_qT_s + R_pR_s^3T_qT_s$$
$$+ 2R_qR_sT_s^2D_xR_s - 4R_q^2R_sT_s^2 + R_s^3R_zT_s^2 + R_s^4T_sT_z.$$

PROOF. – (of the Proposition and the Theorem). We start by forcing the *contact* invariant forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$ given in (8.3) to satisfy the *first four* equations (7.13). We do it in several steps. The first step consists in the requirement that $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$ satisfy consequences of equations (7.13), namely equations (7.23)-(7.27). The first of these conditions implies $d\theta^4 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = 0$, and this, as noted before, implies $a_{11} = a_{12} = 0$.

Let us now, unless otherwise stated, assume that $1 - R_s T_s > 0$. Then the conditions (7.23)-(7.27) can be easily fulfilled by taking $u_{11} = u_{12} = u_{21} = u_{22} = 0$ in (8.3), since this enables us to identify forms (8.3) with (7.22). After this identification the imposition of the rest of conditions (7.23)-(7.27) may be obtained by making the same normalizations of parameters h_{11}, h_{21}, h_{12}, h_{21}, f_{21} and f_{11} as in the proof of Theorem 7.5. It follows however, that one can achieve (7.23)-(7.27) *without* the restriction $u_{11} = u_{12} = u_{21} = u_{22} = 0$ on the parameters u_{11}, u_{12}, u_{21} and u_{22}. We checked that the most general normalizations to achieve (7.23)-(7.25) is to take h_{11}, h_{21}, h_{12}, h_{21}, f_{21} and f_{11} as in (7.29)-(7.30) and to restrict u_{11}, u_{12}, u_{21} and u_{22} by only *one* constraint

$$u_{22} f_{11} - u_{21} f_{12} + u_{12} f_{21} - u_{11} f_{22} = 0. \tag{8.7}$$

If this is not zero, equation (7.23) has an unwanted term proportional to $\theta^1 \wedge \theta^2 \wedge \theta^4$ on the right hand side. Even without the restriction (8.7), but assuming (7.29)-(7.30), we get that $d\theta^1 \wedge \theta^1 \wedge \theta^3 \wedge \theta^4 \wedge \Omega_3$ and $d\theta^2 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \Omega_2$ are still given by (7.31). This proves that the conditions $J_1 \equiv J_2 \equiv 0$ are neccessary for a conformal metric g to be defined on the solution space. It also proves that these conditions are *contact invariant*. This surely holds when our assumption $1 - R_s Ts > 0$ is satisfied. (That this assumption is only a technical one will be clear soon). So from now on we assume the normalizations (7.29)-(7.30), (8.7) and that the invariants J_1 and J_2 are both zero, $J_1 \equiv J_2 \equiv 0$.

Now it follows that the conditions $d\theta^1 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = -\Omega_3 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$, $d\theta^2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^4 = -\Omega_2 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$, $d\theta^2 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0$ and $d\theta^1 \wedge \theta^1 \wedge \theta^3 \wedge \theta^4 = 0$ are equivalent to precisely *the same* normalizations (7.35), (7.36) and (7.37) of f_{33}, b_2 and b_1 as in the proof of Theorem 7.6. Further repetition, step by step, of the absorbtion/normalization procedure described in the proof of Theorem 7.6 leads to the last relevant normalization, which determines the coefficient b_3. Here, again this coeffcient turns out to be precisely as in the proof of Theorem 7.6. That the present expressions for the determined parameters h_{11}, h_{21}, h_{12}, $h_{21}, f_{21}, f_{11}, f_{33}$, b_1, b_2 and b_3 do *not depend* on the parameters u_{11}, u_{12}, u_{21} and u_{22} is remarkable. They are invisible because they turn out to parametrize only that part of the contact transformations, which is related to the orthogonal group preserving the metric g we are going to construct.

Indeed, assuming $J_1 \equiv J_2 \equiv 0$ and the above discussed normalizations for h_{11}, h_{21}, h_{12}, $h_{21}, f_{21}, f_{11}, f_{33}$, b_1, b_2, b_3, we calculate $G = 2(\theta^1\theta^2 + \theta^3\theta^4)$. A direct calculation shows then, that the resulting expression for G has *no* $u_{11}, u_{12}, u_{21}, u_{22}$ dependence! Moreover, the so obtained G is also *independent* of still undetermined parameters a, b_3, f_{31}, f_{32}, c_1 and c_2. Its dependence on the parameters f_{12} and f_{22} is only *conformal*. By this we mean that the parameters f_{12} and f_{22} only appear as a common factor $f_{12} f_{22}$ in front of the entire expression for G. This means that *all* the remaining free parameters a, b_3, f_{31}, f_{32}, c_1, c_2, u_{11}, u_{12}, u_{21} and u_{22} are *group parameters* of the dilation group $\mathbf{CO}(G)$ preserving conformally the bilinear form G.

If one wants the explicit expressions for G, with the above normalizations for h_{11}, h_{21}, h_{12}, $h_{21}, f_{21}, f_{11}, f_{33}$, b_1, b_2 and b_3, in terms of the functions R and T defining the system $z_{xx} = R$ & $z_{yy} = T$, one has to decide how to mod the resulting formula by the constraints $J_1 \equiv J_2 \equiv 0$.

It follows that if we write $J_1 \equiv J_2 \equiv 0$ in the form $D_x R_s = \dots$ and $D_y T_s = \dots$, and eliminate these derivatives from G, then we obtain G, which up to a factor, coincides with g from formula (8.5). Similarly, if we write these condtions as $D_x T_s = \dots$ and $D_y R_s = \dots$, we get the result that G differs from formula (8.6) only by a factor. This proves that the bilinear forms g as in (8.5) and (8.6) are conformally invariant on $\mathcal{J}$, and that they change conformally when the system $z_{xx} = R$ & $z_{yy} = T$ undergoes *contact* transformation of the variables.

The last thing is to prove that $[g]$ is actually defined on the solution space of the PDEs, and that it is nondegenerate there with signature depending on the sign of $1 - R_s T_s$.

Let us start to comment on these last issues with a remark about the technicality of our assumption $1 - R_s T_s > 0$. We needed the assumption $1 - R_s T_s > 0$ starting with the normalization (7.30). It was needed there to maintain the invariant forms θ^i to be *real*. But this was only made for simplicity, since we did not want to deal with the complex numbers in the proof. Moreover, from the point of view of the conformal metric we wanted to construct, this was a good simplification since in the resulting formulae (8.5), (8.6) for g the square root $\sqrt{1 - R_s T_s}$ does not appear at all! Concluding this issue, we say that if we were in the situation when $1 - R_s T_s < 0$, our normalizing procedure for the forms θ^i would make them complex, but the resulting G would nevertheless be real and given by (8.5) or (8.6). Thus all the conformal properties of g established so far are also valid in the $1 - R_s T_s < 0$ case.

There is one more technical issue here. The reason for having two different expressions for g, as in (8.5) and (8.6), is to have local expressions valid everywhere off the set $1 - R_s T_s = 0$. Since solving for $D_x R_s$ and $D_y T_s$ in $J_1 \equiv J_2 \equiv 0$ we devide by $(4 - R_s T_s)$, the metric (8.5) is only defined if $R_s T_s \neq 4$; similarly, because of the division by $R_s T_s$, the metric (8.6) is defined only if $R_s T_s \neq 0$. Off the set $R_s T_s = 0 = 4 - R_s T_s$ the conformal metrics (8.5) and (8.6) coincide, since they are local manifestations of the same formula $G = 2(\theta^1 \theta^2 + \theta^3 \theta^4)$ on $\mathcal{J}$.

Finally we comment on how G descends to the solution space of the PDEs.

We start with an observation that the bilinear form (8.5) satisfies $g(D_x, \cdot) = g(D_y, \cdot) \equiv 0$, i.e. it is *degenerate* along the vector fields D_x and D_y on $\mathcal{J}$. The first product $2\lambda\omega$ in (8.5) has obviously signature $(+, -)$. Thus to determine the signature of (8.5) we need to determine the signature of the product

$$2(R_s T_s - 4)(T_s v_1^2 - 2 v_1 v_2 + R_s v_2^2).$$

Since the quadratic form $T_s v_1^2 - 2v_1v_2 + R_s v_2^2$ has $\Delta = 4(1 - R_sT_s)$ as its discriminant, then the signature of the product $2(R_sT_s - 4)(T_s v_1^2 - 2v_1v_2 + R_s v_2^2)$ is: $\pm(+,-)$ iff $1 - R_sT_s > 0$ and $\pm(+,+)$ iff $1 - R_sT_s < 0$. Thus, assuming that $R_sT_s \neq 4$, we conclude that, modulo the degenerate directions D_x and D_y along which g is vanishing, the bilinear form (8.5) has either *split* (iff $1 - R_sT_s > 0$), or *Lorentzian* signature (iff $1 - R_sT_s < 0$) on $\mathcal{J}$.

A straightforward, but lengthy (!), calculation shows that the Lie derivatives of g, from formula (8.5), with respect to the degenerate directions D_x and D_y are:

$$\mathcal{L}_{D_x} g = a(D_x) g, \quad \& \quad \mathcal{L}_{D_y} g = a(D_y) g,$$

where

$$\begin{aligned} a(D_x) &= (4 - R_sT_s)^{-2} \times \\ &\Big(8D_yR_s + 16R_p - 8R_sD_xT_s + 8R_s^2T_p + 8R_sT_q - 24R_qT_s - 4R_sT_sD_yR_s - \\ &16R_pR_sT_s + 3R_s^2T_sD_xT_s - 4R_s^3T_pT_s - 4R_s^2T_qT_s + 10R_qR_sT_s^2 + 4R_pR_s^2T_s^2\Big) \end{aligned}$$

and

$$\begin{aligned} a(D_y) &= (4 - R_sT_s)^{-2} \\ &\times \Big(8D_xT_s + 16T_q - 8T_sD_yR_s + 8R_qT_s^2 + 8R_pT_s - 24R_sT_p - 4R_sT_sD_xT_s \\ &- 16R_sT_qT_s + 3R_sT_s^2D_yR_s - 4R_qR_sT_s^3 - 4R_pR_sT_s^2 + 10R_s^2T_pT_s + 4R_s^2T_qT_s^2\Big) \end{aligned}$$

Recalling the fact that the distribution $H^+ = \mathrm{Span}(D_x, D_y)$ is integrable on $\mathcal{J}$, we see that the bilinear form g descends to a *conformal metric* g on the *4-dimensional leaf space* $\mathcal{J}/H^+$, and that the descended metric has *split* signature iff $1 - R_sT_s > 0$ and *Lorentzian* signature iff $1 - R_sT_s < 0$ and $R_sT_s \neq 4$. Obviously the leaf space $\mathcal{J}/H^+$ may be identified with the 4-dimensional solution space of the PDEs.

Analogous considerations can be performed for the metric (8.6) if $R_sT_s \neq 0$. This is also degenerate along D_x and D_y in $\mathcal{J}$. It also, apart from the degenerate directions D_x and D_y, has signature Lorentzian/split. For this metric we have

$$\mathcal{L}_{D_x} g = \frac{D_xR_s - 2R_q}{R_s} g, \quad \& \quad \mathcal{L}_{D_y} g = \frac{D_yT_s - 2T_p}{T_s} g,$$

so again (8.6) descends to a conformal metric of *split* (iff $1 - R_sT_s > 0$ and $R_sT_s \neq 0$) or *Lorentzian* signature (if $1 - R_sT_s < 0$) on $\mathcal{J}/H^+$. If $R_sT_s \neq 0$ and $R_sT_s \neq 4$, these two conformal classes coincide on $\mathcal{J}/H^+$ as we explained before.

This finishes the proofs of the Theorem and the Proposition. □

REMARK 8.4. – In the proof we did not show that, contrary to the $(1,2,3)$ para-CR forms (7.22) which satisfy (7.32), we can force the $(3,2,1)$ para-CR forms (8.3) to satisfy their *torsionless* counterpart, i.e. the first four of equations (7.13). But this is very easy: one first makes the normalizations $u_{11} = u_{12} = u_{21} = u_{22} = 0$ and all the other ones from Theorems 7.5 and 7.6, and after achieving (7.32) uses a transformation, which is an identity on the obtained $(\theta^1, \theta^2, \theta^3, \theta^4)$ and changes the obtained Ω_2 and Ω_3 according to:

$$\Omega_3 \to \Omega_3' = \Omega_3 - t^1{}_{23}\theta^2, \qquad \Omega_2 \to \Omega_2' = \Omega_2 - t^2{}_{13}\theta^1, \tag{8.8}$$

where $t^1{}_{23}$ and $t^2{}_{13}$ are torsions given by (7.33). Since the obtained θ^1 and θ^2 are linear combinations of l_1, l_2 and l_3 *only* (because $f_{13} = f_{23} = 0$ is the chosen normalization (7.29)!), then transformation (8.8) is an allowed $(3,2,1)$-para-CR transformation ([2]) for the type $(3,2,1)$ para-CR forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2, \Omega_3)$. But this transformation *absorbs* the torsion terms in (7.32) and makes the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_2', \Omega_3')$ to satisfy the torsionless part of equations (7.13). This means that the type $(3,2,1)$ para-CR structures originating from the system $z_{xx} = R$ & $z_{yy} = T$ with $D_x^2 T = D_y^2 R$, $J_1 \equiv J_2 \equiv 0$, $R_s T_s \neq 1$, contrary to the corresponding $(1,2,3)$ para-CR structures, define quite a general conformal geometry on the solution space, and that their *invariants* can be described in terms of the *curvature of the Cartan normal conformal connection* associated with this conformal geometry. This observation, and an equivalent statement of Theorem 8.2 and Proposition 8.3, in a slightly different language, was first made by E.T. Newman and his collaborators [6]. According to Newman [6], using all the type $(3,2,1)$ para-CR structures coming from the system $z_{xx} = R$ & $z_{yy} = T$ satisfying $J_1 \equiv J_2 \equiv 0$, one can obtain *all* the conformal classes of the Lorentzian 4-metrics. This statement is not clear to us, and requires further justification. For example, similarly to the attempts in [18], we were unable to calculate the Weyl tensor of the metrics (8.5) and (8.6). This was mainly because of the huge length of the intermediate expressions encountered during the calculations of the the Cartan normal conformal connection. Thus we were unable to see if it is general enough to cover all the conformal Lorentzian/split signature 4-metrics. Finding the conformally Einstein or Bach conditions for these metrics in terms of the defining functions R and T would be very interesting, and would complete the Newman programme.

Although, we were unable to calculate the *full* Weyl tensor of the metric (8.6), we succeded in calculating *two* of its components. These components *must* vanish if we want the metric (8.5) to be conformally flat. Thus vanishing of these components is a conformal property, and in turn, is a *contact invariant* property of the equations $z_{xx} = R$ & $z_{yy} = T$ satisfying $D_x^2 T = D_y^2 R$ & $J_1 \equiv J_2 \equiv 0$. It is also a

([2]) Note however that this is *not* a type $(1,2,3)$ para-CR transformation, and that if only such transformations are considered one can not absorb the torsion terms in (7.32).

para-CR invariant property of the corresponding type $(3,2,1)$ para-CR structure. Defining the forms $(\omega_1,\omega_2,\omega_3,\omega_4)$ by $(\omega_1,\omega_2,\omega_3,\omega_4)=(\nu_1,\nu_2,\lambda,\omega')$, so that the metric (8.6) can be written as:

$$g = 2\omega_2\omega_4 + T_s\omega_1^2 - 2\omega_1\omega_2 + R_s\omega_2^2,$$

we calculated the components ${C^1}_{424}$ and ${C^2}_{414}$ of the Weyl tensor of this metric to be:

$${C^1}_{424} = \frac{2R_{sss}(1-R_sT_s)+3R_{ss}(R_sT_s)_s}{4(1-R_sT_s)^4}, \qquad {C^2}_{414} = \frac{2T_{sss}(1-R_sT_s)+3T_{ss}(R_sT_s)_s}{4(1-R_sT_s)^4}.$$

This proves the following theorem.

THEOREM 8.5. – *For the system of PDEs $z_{xx}=R(x,y,z,z_x,z_y,z_{xy})$ & $z_{yy}=T(x,y,z,z_x,z_y,z_{xy})$ satisfying $D_x^2T=D_y^2R$ and the metricity conditions*

$$J_1\equiv 0, \qquad \& \qquad J_2\equiv 0,$$

each of the conditions

$$K_1=2R_{sss}(1-R_sT_s)+3R_{ss}(R_sT_s)_s=0, \qquad K_2=2T_{sss}(1-R_sT_s)+3T_{ss}(R_sT_s)_s=0,$$

is invariant with respect to contact transformations of the variables.

The new invariants K_1 and K_2 from the above Theorem justify the title of this section: although we were unable to define the invariants of the type $(3,2,1)$ para-CR structures in full generality, we discussed a class of such structures whose invariants are just the conformal invariants of certain 4-metrics. In the next section we provide an example of the system $z_{xx}=R$ & $z_{yy}=T$ satisfying $J_1\equiv J_2\equiv 0$, whose corresponding conformal 4-metrics are quite interesting.

8.3 – *An example of* $(3,2,1)$ *para-CR structures with nontrivial conformally Einstein metrics.*

Given a pair of PDEs $z_{xx}=R$ & $z_{yy}=T$ it is not easy to find the most general solution of the *integrability conditions* $D_x^2T=D_y^2R$ and the *metricity conditions* $J_1\equiv J_2\equiv 0$. But particular examples of functions R and T satisfying both sets of conditions can be given. The simplest of them, but as we will see, still nontrivial, is given in the following proposition.

PROPOSITION 8.6. – *Let the functions $R=R(x,y,z,p,q,s)$ and $T=T(x,y,z,p,q,s)$ be functions of variable s alone,*

$$R=r(s) \qquad \& \qquad T=t(s),$$

and assume that their derivatives r' and t' satisfy

$$1 - r't' \neq 0.$$

Then such R and T satisfy simultaneously equations $D_x^2 T = D_y^2 R$ and $J_1 \equiv J_2 \equiv 0$.

PROOF. – Applying the operators D_x and D_y from definitions (7.17) on functions $R = r(s)$ and $T = t(s)$, we obtain

$$D_x R = r' D_y R, \quad D_y R = r' D_x T, \quad D_x T = t' D_y R \quad \& \quad D_y T = t' D_x T.$$

These are *linear* equations for functions $D_x R$, $D_y R$, $D_x T$ and $D_y T$. Hence, by an elementary argument, they have a unique solution

$$D_x R = 0, \quad D_y R = 0, \quad D_x T = 0, \quad D_y T = 0,$$

when $1 - r't' \neq 0$. Thus, with our assumptions, the operators D_x and D_y, when acting on differentiable functions $f = f(s)$ of only variable s, are identically vanishing. This, in particular, means that $D_x^2 T = 0 = D_y^2 R$. Looking at the definitions of J_1 and J_2, in which each term involves at least one derivative of R or T with respect to p, q and D_x or D_y, we see that J_1 and J_2 are identically zero as well. □

Now, having a solution $R = r(s)$, $T = t(s)$ to the integrability and the metricity conditions, we apply the theory from Section 8.2, and calculate the conformal metric on the solution space of the system

$$z_{xx} = r(z_{xy}) \qquad \& \qquad z_{yy} = t(z_{xy}).$$

Modulo a conformal factor the explicit formula for the metric g as in (8.5) reads:

$$\begin{aligned} g_0 = \ & 2\,(1 - r't')\,\big(\mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y\big)\mathrm{d}s \ + \ t'\,(\mathrm{d}p - r\,\mathrm{d}x - s\mathrm{d}y)^2 \\ & - 2\,(\mathrm{d}p - r\,\mathrm{d}x - s\mathrm{d}y)(\mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y) \ + \ r'\,(\mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y)^2, \end{aligned} \tag{8.9}$$

where x, y, z, p, q, s are coordinates on $\mathcal{J}$, $r = r(s)$, and $t = t(s)$, $r' = \mathrm{d}r/\mathrm{d}s$, $t' = \mathrm{d}t/\mathrm{d}s$. We know from the previous section that although this bilinear form is manifestly defined on $\mathcal{J}$, it transforms conformally when Lie dragged along $D_x = \partial_x + p\partial_z + r\partial_p + s\partial_q$ and $D_y = \partial_y + q\partial_z + s\partial_p + t\partial_q$, and descends to a conformal metric on the 4-dimensional solution space $\mathcal{J}/H^+$. It has split signature iff $1 - r't' > 0$ and Lorentzian signature iff $1 - r't' < 0$.

The conformal invariants of this metric are para-CR invariants of the $(3, 2, 1)$ para-CR structure $[l_1, l_2, l_3, m_1, m_2, n]$ with $l_1 = \mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y$, $l_2 = \mathrm{d}p - r\,\mathrm{d}x - s\mathrm{d}y$, $l_3 = \mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y$, $n = \mathrm{d}s$, $m_1 = \mathrm{d}x$, $m_2 = \mathrm{d}y$. These conformal invariants are given in terms of the Cartan normal conformal connection for the class $[g_0]$. It is described by the following theorem

THEOREM 8.7. – *Consider a metric* $g = e^{2h} g_0$, *where* $h = h(s)$ *is an arbitrary smooth function and* g_0 *is as in* (8.9). *Let* $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)$ *be a coframe on* $\mathcal{J}$ *defined by* $\omega_1 = \mathrm{d}q - s\mathrm{d}x - t\mathrm{d}y$, $\omega_2 = \mathrm{d}s$, $\omega_3 = \mathrm{d}z - p\,\mathrm{d}x - q\mathrm{d}y$, $\omega_4 = \mathrm{d}p - r\,\mathrm{d}x - s\mathrm{d}y$, $\omega_5 = \mathrm{d}x$, $\omega_6 = \mathrm{d}y$, *so that the metric is*

$$g = e^{2h}\big(2(1 - r't')\omega_2\omega_3 + r'\omega_1^2 - 2\omega_1\omega_4 + t'\omega_4^2\big). \tag{8.10}$$

Then the curvature of the Cartan normal conformal connection for g, when written on $\mathcal{J}$, *reads:*

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{-Z_2 r' - Z_1 t'}{2(1-r't')} & 0 & 0 & \dfrac{2Z_2 + Z_1 t'^2 - Z_2 r' t'}{2(1-r't')} & 0 \\ 0 & 0 & \frac{1}{2}(Z_2 r' - Z_1 t') & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \omega_2 \wedge \omega_4 \tag{8.11}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(Z_2 r' - Z_1 t') & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{Z_1 r' t' - 2Z_1 - Z_2 r'^2}{2(1-r't')} & 0 & 0 & \dfrac{Z_2 r' + Z_1 t'}{2(1-r't')} & 0 \\ 0 & 0 & -Z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \omega_1 \wedge \omega_2,$$

where

$$Z_1 = \frac{2(r't' - 1)r^{(3)} - 3r''(t'r')'}{4(1 - r't')^2}, \qquad Z_2 = \frac{2(r't' - 1)t^{(3)} - 3t''(t'r')'}{4(1 - r't')^2}.$$

In particular the metric g is conformally flat iff $Z_1 \equiv Z_2 \equiv 0$, *i.e. iff the functions r and t satisfy the system of third order ODEs:*

$$r^{(3)} = \frac{-3r''(r't')'}{2(1 - r't')} \qquad \& \qquad t^{(3)} = \frac{-3t''(r't')'}{2(1 - r't')}.$$

In general the metric g is of (conformal) Petrov type $N \oplus N'$ *in the split signature case, and of Petrov type* $N \oplus \bar{N}$ *in the Lorentzian case.*

The proof of this theorem consists in a straightforward, but lengthy calculation, which we made using Mathematica. We omit it here. With the use of Mathematica we also were able to check that the following theorem is true:

THEOREM 8.8. – *For every choice of sufficiently smooth functions* $r = r(s)$ *and* $t = t(s)$ *there exists a function* $h = h(s)$ *such that the metric* (8.10) *is Ricci flat. The function* h *in which the metric* $g = e^{2h} g_0$ *is Ricci flat is a solution to the 2nd order ODE:*

$$h'' = h'^2 - \frac{(t'r')'}{1 - r't'} h' + \frac{2\big(r^{(3)}t' + t^{(3)}r'\big)(1 - r't') + 2r''t'' + 4r't'r''t'' + 3t'^2 r''^2 + 3r'^2 t''^2}{8(1 - r't')^2}.$$

Thus, among the type $(3, 2, 1)$ para-CR structures originating from PDEs $z_{xx} = R$ & $z_{yy} = T$ we found conformally Ricci flat but conformally non-flat metrics. It further follows that these metrics, in addition to being conformally Ricci flat and of type $N \oplus N'$, have *reduced holonomy*. This is because they have a *covariantly constant null direction*, which is alligned with the vector field ∂_z. In the Lorentzian case, i.e. when $1 - r't' < 0$, they are known in General Relativity theory as *pp*-waves (see e.g. [13] for a definition and [12] for a discussion of their conformal properties).

It would be very interesting to find type $(3, 2, 1)$ para-CR structures defined by $z_{xx} = R$ & $z_{yy} = T$, which define conformally Einstein metrics (8.5)-(8.6) other than *pp*-waves or their split signature counterparts discussed here.

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