# Conformal geometry of differential equations

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Transformations mixing independent and dependent variables, as above are called *point transformations*.

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$$y \to \bar{y} = \bar{y}(x, y, y')$$
$$y' \to \bar{y}' = \bar{y}'(x, y, y')$$

with

$$\bar{y}_{y'} - \bar{y}'\bar{x}_{y'} = 0$$
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- ⋆ Indeed the tangency of the two graphs at $x$ means that

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- ⋆ Thus the solution space $\mathbb{R}^3$ of the equation $y''' = 0$, with the solutions parametrized by $(a_0, a_1, a_2)$, is naturally equipped with a *conformal Lorentzian* metric

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- ⋆ In this metric two neighbouring solutions are *null separated* iff they are *tangent* at some point.
- ⋆ What shall one assume about a third order ODE to have a natural conformal Lorentzian metric on its (3-dimensional) solution space?

- $\star$ Writing a general 3rd order ODE as

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$$F_y + (\mathcal{D} - \tfrac{2}{3}F_q)\underbrace{(\tfrac{1}{6}\mathcal{D}F_q - \tfrac{1}{9}F_q^2 - \tfrac{1}{2}F_p)}_{K} \equiv 0. \qquad (W)$$

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- The metric reads:

$$g = [\mathrm{d}y - p\mathrm{d}x][\mathrm{d}q - \tfrac{1}{3}F_q\mathrm{d}p + K\mathrm{d}y + (\tfrac{1}{3}qF_q - F - pK)\mathrm{d}x] - [\mathrm{d}p - q\mathrm{d}x]^2.$$

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- ⋆ Wünschman: There is a *one-to-one correspondence* between *equivalence classes of 3rd order ODEs satisfying* $(W)$ *considered modulo contact transformations of variables* and *3-dimensional Lorentzian conformal geometries.*
- ⋆ In particular: all contact invariants of such classes of equations are expressible in terms of the conformal invariants of the associated conformal Lorentzian metrics.

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  - ⋆ In case when the ODE $y''' = F(x, y, y', y'')$ satisfies Wünschmann condition, he constructed a natural principal fiber bundle $P \to S$ over its solution space $S$, with a certain $\mathfrak{so}(2,3)$-valued Cartan connection $\omega$.

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  - ⋆ He showed that the curvature $R = \mathrm{d}\omega + \omega \wedge \omega$ of $\omega$ encodes all the contact invariants of the ODE.
  - ⋆ Since $\mathbf{SO}(2,3)$ is a conformal group for the 3-dimensional Lorentzian metrics, $\omega$ may be identified with the *Cartan normal conformal connection* associated with the conformal class $[g]$.

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$$w_1 = D^2 Q_{pp} - 4DQ_{py} - DQ_{pp}Q_p + 4Q_pQ_{py} - 3Q_{pp}Q_y + 6Q_{yy}$$

or

$$w_2 = Q_{pppp},$$

where $p = y'$ and $D = \partial_x + p\partial_y + Q\partial_p$, is a *point invariant property* of the ODE.

- Cartan E (1924) "Varietes a connexion projective" *Bull. Soc. Math.* **LII** 205-41:

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  - ⋆ Since $\mathfrak{sl}(3, \mathbb{R})$ is naturally included in $\mathfrak{sl}(4, \mathbb{R})$, and this in turn is isomorphic to $\mathfrak{so}(3, 3)$, $\mathfrak{sl}(4, \mathbb{R}) = \mathfrak{so}(3, 3)$, i.e. a *conformal algebra* for metrics of signature $(2, 2)$ in *four* dimensions, we ask the following question:

- Is it possible to describe the Lie/Cartan *point invariants* $w_1$, $w_2$, *of a second order ODE* $y'' = Q(x, y, y')$ in terms of the *conformal invariants* of a *split signature conformal metric in four dimensions*?

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  - ⋆ Given 2nd order ODE: $y'' = Q(x, y, y')$ consider a parametrization of the first jet space $J^1$ by $(x, y, p = y')$.
  - ⋆ on $J^1 \times \mathbb{R}$ consider a metric

$$g = 2[(\mathrm{d}p - Q\mathrm{d}x)\mathrm{d}x - (\mathrm{d}y - p\mathrm{d}x)(\mathrm{d}r + \tfrac{2}{3}Q_p\mathrm{d}x + \tfrac{1}{6}Q_{pp}(\mathrm{d}y - p\mathrm{d}x))], \qquad (F)$$

where $r$ is a coordinate along $\mathbb{R}$ in $J^1 \times \mathbb{R}$.

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- ⋆ If ODE $y'' = Q(x, y, y')$ undergoes a point transformation of variables then the metric $(F)$ transforms conformally.
- ⋆ All the point invariants of a point equivalence class of ODEs $y'' = Q(x, y, y')$ are expressible in terms of the conformal invariants of the associated conformal class of metrics $(F)$.
- ⋆ The metrics $(F)$ are very special among all the split signature metrics on 4-manifolds. Their Weyl tensor $C$ has algebraic type $(N, N)$ in the Cartan-Petrov-Penrose classification. Both, the selfdual $C^+$ and the antiselfdual $C^-$, parts of $C$ are expressible in terms of only $\mathrm{one}$ component.

$\star$ $C^+$ is proportional to

$$w_1 = D^2 Q_{pp} - 4DQ_{py} - DQ_{pp}Q_p + 4Q_pQ_{py} - 3Q_{pp}Q_y + 6Q_{yy},$$

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⋆ Cartan normal conformal connection associated with any conformal class $[g]$ of metrics $(F)$ is reduced to to the Cartan $\mathfrak{sl}(3, \mathbb{R})$ connection naturally defined on the Cartan bundle $P \to J^1$.

## What's interesting in $z' = (y'')^2$?

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  - ⋆ considered equations of the form $z' = F(x, y, y', y'', z)$ for two real functions $y = y(x)$ and $z = z(x)$.
  - ⋆ He observed that, the general solution to the equation $z' = y''^2$ can *not* be written in an *integral free* form

$$x = x(t, w(t), w'(t), ....w^{(k)}(t)),$$

$$y = y(t, w(t), w'(t), ....w^{(k)}(t)),$$

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Check, that its general solution may be written in the *integral-free form*:

$$x = \frac{1}{2}w''(t)$$

$$y = \frac{1}{2}tw''(t) - \frac{1}{2}w'(t)$$

$$z = \frac{1}{2}t^2w''(t) - tw'(t) + w(t),$$

where $w = w(t)$ is an *arbitray* sufficiently smooth real function.

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The situation is quite *different* for $z' = F(x, y, y', y'', z)$, as it was shown by Hilbert on the example of $z' = (y'')^2$.

- Cartan E (1910) “Les systemes de Pfaff a cinq variables et les equations aux derivees partielles du second ordre” *Ann. Sc. Norm. Sup.* **27** 109-192:

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## The third example

# Cartan's construction

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- Each equation $(H)$ may be represented by forms

$$\omega^1 = \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x$$
$$\omega^2 = \mathrm{d}y - p\mathrm{d}x$$
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on a $5$-dimensional manifold $J$ parametrized by $(x, y, p = y', q = y'', z)$.

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- every solution to the equation is a curve $\gamma(t) = (x(t), y(t), p(t), q(t), z(t))$ in $J$ on which the forms $(\omega^1, \omega^2, \omega^3)$ simultaneously vanish.

- Transformation that transforms solutions to solution may mix the forms $(\omega^1, \omega^2, \omega^3)$ among themselves, thus:

Definition

Two equations $z' = F(x, y, y', y'', z)$ and $\bar{z}' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'', \bar{z})$ represented by the respective forms

$$\omega^1 = \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x, \quad \omega^2 = \mathrm{d}y - p\mathrm{d}x, \quad \omega^3 = \mathrm{d}p - q\mathrm{d}x;$$

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\bar{\omega}^1 &= \mathrm{d}\bar{z} - \bar{F}(\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{z})\mathrm{d}\bar{x}, \quad \bar{\omega}^2 = \mathrm{d}\bar{y} - \bar{p}\mathrm{d}\bar{x}, \quad \bar{\omega}^3 = \mathrm{d}\bar{p} - \bar{q}\mathrm{d}\bar{x},
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are (locally) *equivalent* iff there exists a (local) diffeomorphism $\phi : (x, y, p, q, z) \to (\bar{x}, \bar{y}, \bar{p}, \bar{q}, \bar{z})$ such that

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$$\phi^* \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \lambda \\ \kappa & \mu & \nu \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}$$

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- If $F_{qq} \equiv 0$ then such equations have integral-free solutions.
- There are nonequivalent equations among the equations having $F_{qq} \neq 0$. All these equations are beyond the class of equations with integral-free solutions.

# Equations $z' = F(x, y, y', y'', z)$ with $F_{y''y''} \neq 0$

Given $z' = F(x, y, y', y'', z)$ take its corresponding forms

$$\omega^1 = \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x, \quad \omega^2 = \mathrm{d}y - p\mathrm{d}x, \quad \omega^3 = \mathrm{d}p - q\mathrm{d}x;$$

and suplement them with $\omega^4 = \mathrm{d}q$ and $\omega^5 = \mathrm{d}x$.

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and suplement them with $\omega^4 = \mathrm{d}q$ and $\omega^5 = \mathrm{d}x$. Define

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \end{pmatrix} = \begin{pmatrix} s_1 & s_2 & s_3 & 0 & 0 \\ s_4 & s_5 & s_6 & 0 & 0 \\ s_7 & s_8 & s_9 & 0 & 0 \\ s_{10} & s_{11} & s_{12} & s_{13} & s_{14} \\ s_{15} & s_{16} & s_{17} & s_{18} & s_{19} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \end{pmatrix}.$$

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We also have formulae for the differentials of the forms $\Omega_\mu$, $\mu = 1, 2, ..., 9$.

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Together with these expressions the system provides all the local invariants for the equivalence class of equations satisfying $F_{qq} \neq 0$.

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An equivalence class of equations $z' = F(x, y, y', y'', z)$ with $F_{y''y''} \neq 0$ *uniquely* defines a 14-dimensional manifold $P \to J$ and a preferred coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_9)$ on it such that

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\mathrm{d}\theta^4 &= \theta^1 \wedge \Omega_7 + \tfrac{4}{3}\theta^3 \wedge \Omega_6 + \theta^4 \wedge \Omega_1 + \theta^5 \wedge \Omega_2 \\
\mathrm{d}\theta^5 &= \theta^2 \wedge \Omega_7 - \tfrac{4}{3}\theta^3 \wedge \Omega_5 + \theta^4 \wedge \Omega_3 + \theta^5 \wedge \Omega_4.
\end{aligned}$$

We also have formulae for the differentials of the forms $\Omega_\mu$, $\mu = 1, 2, ..., 9$.

Together with these expressions the system provides all the local invariants for the equivalence class of equations satisfying $F_{qq} \neq 0$.
We pass to the interpretetion in terms of Cartan connection:

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On this fibre bundle the following matrix of $1$-forms:

$$\omega=\begin{pmatrix}-\Omega_1-\Omega_4 & -\Omega_8 & -\Omega_9 & -\frac{1}{\sqrt{3}}\Omega_7 & \frac{1}{3}\Omega_5 & \frac{1}{3}\Omega_6 & 0\\ \theta^1 & \Omega_1 & \Omega_2 & \frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{3}\theta^3 & 0 & \frac{1}{3}\Omega_6\\ \theta^2 & \Omega_3 & \Omega_4 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{3}\theta^3 & -\frac{1}{3}\Omega_5\\ \frac{2}{\sqrt{3}}\theta^3 & \frac{2}{\sqrt{3}}\Omega_5 & \frac{2}{\sqrt{3}}\Omega_6 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{\sqrt{3}}\Omega_7\\ \theta^4 & \Omega_7 & 0 & \frac{2}{\sqrt{3}}\Omega_6 & -\Omega_4 & \Omega_2 & \Omega_9\\ \theta^5 & 0 & \Omega_7 & -\frac{2}{\sqrt{3}}\Omega_5 & \Omega_3 & -\Omega_1 & -\Omega_8\\ 0 & \theta^5 & -\theta^4 & \frac{2}{\sqrt{3}}\theta^3 & -\theta^2 & \theta^1 & \Omega_1+\Omega_4\end{pmatrix},$$

is a Cartan connection with values in the Lie algebra of $G_2$.

The curvature of this connection $R = \mathrm{d}\omega + \omega \wedge \omega$ 'measures' how much a given equivalence class of equations is 'distorted' from the flat Hilbert case corresponding to $F = q^2$.

## $(3,2)$-signature conformal metric

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Given an equivalence class of equation $z' = F(x, y, y', y'', z)$ consider its corresponding bundle $P$ with the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_9)$.

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$$\tilde{g} = 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}\theta^3\theta^3$$

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This form is *degenerate* on $P$ and has signature $(3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

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The $9$ degenerate directions generate the vertical space of $P$.

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- This $\mathfrak{so}(4,3)$-valued connection *is reduced* to a subalgebra $\mathfrak{g}_2 \subset \mathfrak{so}(4,3)$ and may be identified with the Cartan $\mathfrak{g}_2$ connection $\omega$ on $P$.

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All metrics with conformal $G_2$ holonomy are given by the described construction.

If $[g_F]$ includes and Einstein metric then this holonomy is a proper subgroup of $G_{2(2)}$.

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# Fefferman-Graham ambient metrics

Given a conformal class of metrics $[g]$ on $M$ and given a representative $g \in [g]$, Fefferman and Graham define a metric $\hat{g}$ on $R_+ \times I \times M$, which encodes the conformal properties of $[g]$, and which is *Ricci flat*.

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$$\hat{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2\Big(g + 2\rho P + \rho^2\mu_2 + \rho^3\mu_3 + \rho^4\mu_4 + ...\Big)$$

where $t \in \mathbb{R}_+$, $\rho \in I = ]-\epsilon, \epsilon[$, $P$ is the Schouten tensor for $g$, and $\mu_i$ are symmetric 2-tensors on $M$, with leading terms of order $2i$, $i = 2, 3, \dots$, in the derivatives of $g$.

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If the dimension of $M$ is odd and $g$ is real analytic, $\hat{g}$ is *real analytic* in $\rho$ and is *uniquely* determined by the condition $Ric(\hat{g}) \equiv 0$. It is then called *Feferman-Graham ambient metric* fpr $[g]$.

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PN (2008) *Conformal structures with explicit ambient metrics and conformal G2 holonomy*, IMA Volumes in Mathematics and its Applications, **144** 515-526 (2008):

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An example of such equation is given by
$F = (y'')^2 + s_1 y' + s_2(y')^2 + s_3(y')^3 + s_4(y')^4 + s_5(y')^5 + s_6(y')^6$,
where $s_4 + 5s_5 y' + 15 s_6 (y')^2 \neq 0$.

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with at least one of $s_4$, $s_5$, or $s_6$ non zero, and let $[g_F]$ be the conformal class defined by the metric $g_F$ as on the previous slide. Then the holonomy of the ambient metric for $[g_F]$ is *equal* to $G_{2(2)} \subset SO(4,3)$.

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In particular this metric is Ricci flat and admits a covariantly constant spinor.

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  is called an *irreducible* $\mathbf{SO}(3)$ *structure* in dimension five.

- An irreducible $\mathbf{SO}(3)$ structure $(M^5, g, \Upsilon)$ is called *nearly integrable* if $\Upsilon$ is a *Killing tensor* for $g$:

$$\overset{LC}{\nabla}_X \Upsilon(X,X,X) = 0, \qquad \forall X \in TM^5.$$

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- Thus, nearly integrable $\mathbf{SO}(3)$ structures provide *low-dimensional examples* of *Riemannian* geometries which can be described in terms of a *unique metric* connection $(\Gamma)$ with *totally skew symmetric* torsion $(T)$.

- This sort of geometries are studied extensively by the string theorists.

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- We do not know if *nonhomogeneous* examples exist.

- Perhaps these structures are so rigid that they must be homogeneous.

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- A polynomial $I$, in variables $a_i$, is called an *algebraic invariant* of $w_4(x,y)$ if it changes according to

$$I \to I' = (\det b)^p \; I, \qquad b \in \mathbf{GL}(2,\mathbb{R})$$

under the action of this $5$-dimensional representation on $a_i$s.

- The lowest order invariants of $w_4(x, y)$ are:

$$I_2 = 3a_2^2 - 4a_1a_3 + a_0a_4$$

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- Defining $\Upsilon_{ijk}$ and $g_{ij}$ via

$$\Upsilon_{ijk}a_ia_ja_k = 3\sqrt{3}I_3$$

$$g_{ij}a_ia_j = I_2,$$

one can check that the so defined $g_{ij}$ and $\Upsilon_{ijk}$ satisfy the desidered relations i)-iii).

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- The stabilizer of the conformal class $[(g,\Upsilon)]$ is the irreducible $\mathbf{GL}(2,\mathbb{R})$ in dimension five.

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$$\overset{W}{\nabla}_X g + A(X)g = 0.$$

- An irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $(M^5, [(g, \Upsilon, A)])$ is called *nearly integrable* iff tensor $\Upsilon$ is a *conformal* Killing tensor for $\overset{W}{\nabla}$:

$$\overset{W}{\nabla}_X \Upsilon(X,X,X) + \tfrac{1}{2}A(X)\Upsilon(X,X,X) = 0, \qquad \forall X \in \mathrm{T}M^5.$$

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- To achieve the uniqueness one requires the that torsion $T$ of $\nabla$, considered as an element of $\bigotimes^3 \mathrm{T}^* M^5$, seats in a 10-dimensional subspace $\bigwedge^3 \mathrm{T}^* M^5$.

- In terms of the connection $1$-forms of the Weyl connection $\overset{W}{\Gamma}$, and the characteristic connection $\Gamma$, we have

$$\overset{W}{\Gamma} = \Gamma + \tfrac{1}{2}T,$$

where $\overset{W}{\Gamma} \in \mathfrak{co}(3,2) \otimes \mathrm{T}^*M^5$, $\Gamma \in \mathfrak{gl}(2,\mathbb{R}) \otimes \mathrm{T}^*M^5$ and $T \in \bigwedge^3 \mathrm{T}^*M^5$.

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- The converse is also true: if an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure in dimension five admits a connection $\nabla$ satisfying

$$\nabla_X g + A(X)g = 0, \qquad \nabla_X \Upsilon + \frac{3}{2}A(X)\Upsilon = 0,$$

and having totally skew symmetric torsion $T \in \bigwedge^3 \mathrm{T}^*M^5$ then it is nearly integrable.

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- Can we produce examples of the nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometries in dimension five? Can we produce examples with 'pure' torsion in $\bigwedge_3$ or $\bigwedge_7$? Can we produce nonhomogeneous examples?

# A well known fact

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- Ordinary differential equation $y^{(5)} = 0$ has $\mathbf{GL}(2, \mathbb{R}) \times_\rho \mathbb{R}^5$ as its group of contact symmetries. Here $\rho : \mathbf{GL}(2, \mathbb{R}) \to \mathbf{GL}(5, \mathbb{R})$ is the 5-dimensional irreducible representation of $\mathbf{GL}(2, \mathbb{R})$.

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- This, in particular, means that $y^{(5)} = 0$ may be described in terms of a *flat* $\mathfrak{gl}(2,\mathbb{R})$-valued connection on the principal fibre bundle $\mathbf{GL}(2,\mathbb{R}) \to P \to M^5$ over the solution space $M^5$ of the ODE.

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- What about more complicated $5$th order ODEs?

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$$150F_4DF_3 + 200F_2F_4 - 140F_4^2DF_4 + 130F_3F_4^2 + 14F_4^4 = 0$$

$$1250D^2F_2 - 6250DF_1 + 1750DF_3DF_4 - 2750F_2DF_4 -$$

$$875F_3DF_3 + 1250F_2F_3 - 500F_4DF_2 + 700(DF_4)^2F_4 +$$

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- Every nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structure obtained in this way has torsion of its characteristic connection of the 'pure' type $T \in \bigwedge_3$.

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- We call the three conditions on $F$ the Wünschmann-like conditions.

## Examples of $F$ satisfying the Wünschmann-like conditions

The three differential equations

$$y^{(5)} = c\Big(\frac{5y^{(3)3}(5-27cy''^2)}{9(1+cy''^2)^2} + 10\frac{y''y^{(3)}y^{(4)}}{1+cy''^2}\Big),$$

with $c = +1, 0, -1,$ represent the only three contact nonequivalent classes of Wünschmann-like ODEs having the corresponding nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures $(M^5, [g, \Upsilon, A])$ with the characteristic connection with vanishing torsion.

In all three cases the holonomy of the Weyl connection $\overset{W}{\Gamma}$ of structures $(M^5, [g, \Upsilon, A])$ is reduced to the $\mathbf{GL}(2,\mathbb{R})$. For all the three cases the Maxwell 2-form $\mathrm{d}A \equiv 0$. The corresponding Weyl structure is flat for $c = 0$. If $c = \pm 1$, then in the conformal class $[g]$ there is an Einstein metric of positive $(c = +1)$ or negative $(c = -1)$ Ricci scalar. In case $c = 1$ the manifold $M^5$ can be identified with the homogeneous space $\mathbf{SU}(1,2)/\mathbf{SL}(2,\mathbb{R})$ with an Einstein $g$ descending from the Killing form on $\mathbf{SU}(1,2)$. Similarly in $c = -1$ case the manifold $M^5$ can be identified with the homogeneous space $\mathbf{SL}(3,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})$ with an Einstein $g$ descending from the Killing form on $\mathbf{SL}(3,\mathbb{R})$. In both cases with $c \neq 0$ the metric $g$ is not conformally flat.

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$$F = \frac{5(8y_3^3 - 12y_2y_3y_4 + 3y_1y_4^2)}{6(2y_1y_3 - 3y_2^2)},$$

$$F = \frac{5y_4^2}{3y_3} \pm y_3^{5/3},$$

represent four nonequivalent nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures corresponding to the different signs in the second expression and to the different signs of the denominator in the first expression. These structures have $6$-dimensional symmety group.

$$F = \frac{1}{9(y_1^2 + y_2)^2} \times$$

$$\Big(5w\big(y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_4(y_1^2 + y_2)\big) +$$

$$45y_4(y_1^2 + y_2)(2y_1 y_2 + y_3) - 4y_1^9 - 18y_1^7 y_2 - 54y_1^5 y_2^2 - 90y_1^3 y_2^3 + 270y_1 y_2^4 +$$

$$15y_1^6 y_3 + 45y_1^4 y_2 y_3 - 405y_1^2 y_2^2 y_3 + 45y_2^3 y_3 + 60y_1^3 y_3^2 - 180y_1 y_2 y_3^2 - 40y_3^3\Big),$$

where

$$w^2 = y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12y_1 y_2 y_3 + 4y_3^2 - 3y_1^2 y_4 - 3y_2 y_4.$$

This again has $6$-dimensional symmetry group.

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$$90z^{4/3}(3q-4z^{2/3})\frac{\mathrm{d}^2q}{\mathrm{d}z^2}-54z^{4/3}(\frac{\mathrm{d}q}{\mathrm{d}z})^2+30z^{1/3}(6q-5z^{2/3})\frac{\mathrm{d}q}{\mathrm{d}z}-25q=0,$$

in which $z=\frac{y_4^3}{y_3^4}$.

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This equation may be solved explicitely giving example of ODEs having its nearly integrable structure being nonhomogeneous.

# What about other orders of ODEs?

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- If a $3$rd order ODE $y''' = F(x, y, y', y'')$ satisfies the Wünschmann condition

$$9D^2F_2 - 18F_2DF_2 - 27DF_1 + 4F_2^3 - 18F_1F_2 + 54F_y = 0,$$

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- This conformal structure in dimension *three* is related to the quadratic $\mathbf{GL}(2, \mathbb{R})$ invariant $\Delta = a_0a_2 - a_1^2$ of $w_2(x, y) = a_0x^2 + 2a_1xy + a_2y^2$.

- If a $4$th order ODE $y^{(4)} = F(x, y, y', y'', y''')$ satisfies the Wünschmann-like conditions

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$$80DF_2F_3 + 160F_1F_3 - 72DF_3F_3^2 + 88F_2F_3^2 + 9F_3^4 + 16000F_y = 0,$$

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$$4D^2F_3 - 8DF_2 + 8F_1 - 6DF_3F_3 + 4F_2F_3 + F_3^3 = 0,$$

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$$80DF_2F_3 + 160F_1F_3 - 72DF_3F_3^2 + 88F_2F_3^2 + 9F_3^4 + 16000F_y = 0,$$

$$D = \partial_x + y_1\partial_y + y_2\partial_{y_1} + y_3\partial_{y_2} + F\partial_{y_3},$$

- If a $4$th order ODE $y^{(4)} = F(x, y, y', y'', y''')$ satisfies the Wünschmann-like conditions

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then it defines an irreducible $\mathbf{GL}(2, \mathbb{R})$ structure on the $4$-dimensional space $M^4$ of its solutions.

- This $\mathbf{GL}(2,\mathbb{R})$ structure in dimension *four* may be understood in terms of a *conformal* Weyl-like structure associated with the *quartic* $\mathbf{GL}(2,\mathbb{R})$ invariant

$$I_4 = -3a_1^2a_2^2 + 4_0a_2^3 + 4a_1^3a_3 - 6a_0a_1a_2a_3 + a_0^2a_3^2,$$

of

$$w_3(x,y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$$

and a certain $1$-form $A$ on $M^4$.

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- It seems that rich $\mathbf{GL}(2,\mathbb{R})$ geometries, with lots of examples, are possible in orders $3\leq n\leq 5$ *only*!