Old and new on 'Rolling without slipping or twisting'

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Plan

1. $(2,3,5)$ distributions
2. Rolling without slipping or twisting
3. Ice dancing: rolling $\mathbb{R}P^2$ on its dual

$(2,3,5)$ distributions

Definition

Let X_1, X_2 be two vector fields in the neighbourhood $\mathcal{U}$ of the origin in $\mathbb{R}^5$.
Then the *distribution* $\mathcal{D} = \mathrm{Span}(X_1, X_2)$ is $(2,3,5)$ iff

$$[X_1, X_2] = X_3 \quad \& \quad [X_1, X_3] = X_4 \quad \& \quad [X_2, X_3] = X_5,$$

and $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$ at all points in $\mathcal{U}$.

Example

Take $\mathcal{U}$ parameterized by (x, y, p, q, z), and

$$X_1 = \partial_x + p\partial_y + q\partial_p + \tfrac{1}{2}q^2\partial_z \quad \& \quad X_2 = \partial_q.$$

Then

$$X_3 = -\partial_p - q\partial_z, \quad X_4 = \partial_y, \quad X_5 = -\partial_z.$$

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Equivalence of distributions

- Two distributions: $\mathcal{D}$ on a manifold M and $\bar{\mathcal{D}}$ on a manifold $\bar{M}$ are (locally) *equivalent* iff there exists a (local) diffeomorphism $\phi : M \to \bar{M}$ such that $\phi_*\mathcal{D} = \bar{\mathcal{D}}$.
- Self-equivalences: $\phi : M \to M$, $\phi_*\mathcal{D} = \mathcal{D}$, of a distribution $\mathcal{D}$ are called symmetries of $\mathcal{D}$.
- Locally self-equivalences are described in terms of vector fields X on M such that $[X, \mathcal{D}] \subset \mathcal{D}$. They are called *ininitesimal symmetries* of $\mathcal{D}$. Infinitesimal symmetries form a Lie algebra $\mathfrak{g}_{\mathcal{D}}$ of symmetries of $\mathcal{D}$.

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Symmetries of $(2,3,5)$ distributions

- Every $(2,3,5)$ distribution is locally equivalent to a distribution $\mathcal{D}_F = \mathrm{Span}(X_1, X_2)$ on $\mathcal{U}$ parameterized by (x,y,p,q,z), with

$$X_1 = \partial_x + p\partial_y + q\partial_p + F\partial_z \quad \& \quad X_2 = \partial_q,$$

where X_1 is given in terms of a smooth function $F = F(x,y,p,q,z)$ such that $F_{qq} \neq 0$. A $(2,3,5)$ distribution represented locally as a distribution $\mathcal{D}_F$ is called to be in a *Goursat form*.
- The Lie algebra of infinitesimal symmetries of a $(2,3,5)$ distribution is at most 14-dimensional.
- It is 14-dimensional if and only if the distribution is locally equivalent to $\mathcal{D}_F$ with $F = \frac{1}{2}q^2$, and in such case $\mathfrak{g}_{\mathcal{D}_F}$ is isomorphic to the split real form of the simple exceptional Lie algebra $\mathfrak{g}_2$.

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$(2,3,5)$ distributions have local invariants

- There exists $(2,3,5)$ distributions which are NOT locally equivalent.
- For example, a distribution $\mathcal{D}_F$ with $F = q^k$ is locally equivalent to a distribution $\mathcal{D}_F$ with $F = \frac{1}{2}q^2$ if and only if $k = -1, \frac{1}{3}, \frac{2}{3}, 2$.
- A full set of local differential invariants for $(2,3,5)$ distributions, was determined by E. Cartan in 1910. In his celebrated '5variables paper' he associates a $\mathfrak{g}_2$-valued *Cartan connection* ω to any $(2,3,5)$ distribution. The *curvature* $\Omega = \mathrm{d}\omega + \omega \wedge \omega$ of this connection is the basic object used to detect if two $(2,3,5)$ distributions are locally nonequivalent.

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Harmonic curvature for $(2,3,5)$ distributions

- For example, for a $(2,3,5)$ distribution to be locally equivalent to the maximally symmetric distribution $\mathcal{D}_F$ with $F = \frac{1}{2}q^2$, it is neccessary and sufficient that a part of this curvature, called the *harmonic curvature*, identically vanishes.
- The harmonic curvature of a $(2,3,5)$ distribution $\mathcal{D}$ on M defines a certain quartic

$$\mathcal{C}(\xi) = \Phi_0 + 4\xi\Phi_1 + 6\xi^2\Phi_2 + 4\xi^3\Phi_3 + \xi^4\Phi_4,$$

with functions Φ_μ, $\mu = 0, 1, 2, 3, 4$ on M depending in a specific way on (quite high!) derivatives of the data defining the distribution. For example, if we take a $(2,3,5)$ distribution in the Goursat form $\mathcal{D}_F$, then the functions Φ_μ depend on the 6th derivative of the defining function F.

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Harmonic curvature of a $(2,3,5)$ distribution

- More specifically: a $(2,3,5)$ distribution in a Goursat form $\mathcal{D}_F$, with $F = f(q)$, has Cartan quartic $\mathcal{C}(\xi) = \xi^4\Phi_4$, with

$$\Phi_4 = 10f^{(6)}f''^3 - 80f''^2f^{(3)}f^{(5)} - 51f''^2{f^{(4)}}^2 + 336f''{f^{(3)}}^2f^{(4)} - 224{f^{(3)}}^4.$$

- As such it is equivalent to $\mathcal{D}_{\frac{1}{2}q^2}$ if and only if $\Phi_4 \equiv 0$, which is a rather nasty 6th order ODE for $f(q)$.

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A problem and a nice tool

- **Problem** : Find physical systems whose configuration space is naturally equipped with $(2,3,5)$ distribution;
  Select those whose distribution is maximally symmetric, to have a 'physical' realization of the simple exceptional Lie group/algebra $\mathbf{G}_2/\mathfrak{g}_2$.
- **Cartan's submaximality result**: If a $(2,3,5)$ distribution $\mathcal{D}$ has the Lie algebra of infinitesimal symmetries $\mathfrak{g}_{\mathcal{D}}$ of dimension $\dim \mathfrak{g}_{\mathcal{D}} > 7$ then $\mathcal{D}$ is maximally symmetric and locally equivalent to $\mathcal{D}_{\frac{1}{2}q^2}$.

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Configuration space

The configuration space $\mathcal{C}(\Sigma_1, \Sigma_2)$ of a system of two rigid bodies, which roll on each other, is a 5-dimensional circle bundle

$$\mathbb{S}^1 \to \mathcal{C}(\Sigma_1, \Sigma_2) \xrightarrow{\pi} \Sigma_1 \times \Sigma_2.$$

Here we idealized the two bodies, assuming that they are bounded by two surfaces Σ_1, Σ_2, equipped with the respective Riemannian metrics g_1 and g_2.

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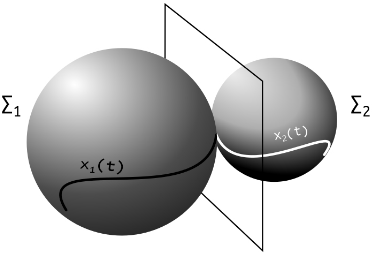
Explicitly:

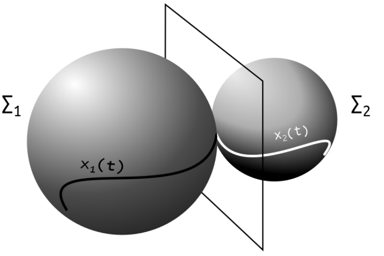
$$\mathcal{C}(\Sigma_1, \Sigma_2) = \{(x_1, x_2, A_\phi) \mid A_\phi : \mathrm{T}_{x_1}\Sigma_1 \to \mathrm{T}_{x_2}\Sigma_2\},$$



where A_ϕ is an $\mathbf{SO}(2)$ matrix identifying the tangent spaces $\mathrm{T}_{x_1}\Sigma_1$ and $\mathrm{T}_{x_2}\Sigma_2$ at the respective points $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$ at which the two bodies contact each other.

Rolling

- When the two bodies roll on each other, they draw a curve $\gamma(t) = (x_1(t), x_2(t), A_{\phi(t)})$ in $\mathcal{C}(\Sigma_1, \Sigma_2)$.
- They also draw two curves: $x_1 = x_1(t)$ in Σ_1, and $x_2 = x_2(t)$ in Σ_2. These are the traces of the point of contact between the bodies, left on each body during the process of rolling.



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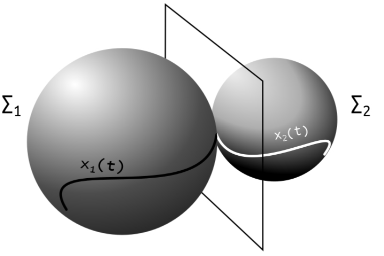

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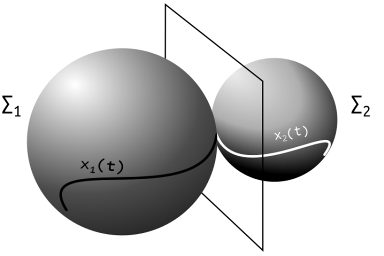

Rolling without slipping

- The two bodies roll *without slipping* iff at every moment t the tangent vector $\dot{x}_1(t)$ to $x_1(t)$ coincides with the tangent vector $\dot{x}_2(t)$ to $x_2(t)$,

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Rolling without slipping or twisting

- To define rolling *without slipping or twisting* we use the Levi-Civita connections ${}^1\nabla$ and ${}^2\nabla$ associated with the respective metrics g_1 on Σ_1 and g_2 on Σ_2.
- Two bodies roll without slipping or twisting iff they roll without slipping, $\dot{x}_2(t) = A_{\phi(t)}\dot{x}_1(t)$, and if the following implication holds:

$$ {}^1\nabla_{\dot{x}_1(t)} Y_{x_1(t)} = 0 \quad \text{then} \quad {}^2\nabla_{\dot{x}_2(t)}(A_{\phi(t)} Y_{x_2(t)}) = 0, $$

- This means: if a vector field Y is parallel along $x_1(t)$ then the A_ϕ-transformed vector field $A_\phi Y$ is parallel along $x_2(t)$.

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Rolling distribution

- At each point (x_1, x_2, A_ϕ) of the configuration space, the non-slip, $\dot{x}_2(t) = A_{\phi(t)}\dot{x}_1(t)$, and non-twist, ${}^1\nabla_{\dot{x}_1(t)} Y_{x_1(t)} = 0 \quad \text{then} \quad {}^2\nabla_{\dot{x}_2(t)}(A_{\phi(t)} Y_{x_2(t)}) = 0$, conditions define THREE independent LINEAR constraints on velocities $(\dot{x}_1, \dot{x}_2, \dot{A}_\phi)$. Thus, if we consider rolling without slipping or twisting, the vector space $\mathcal{D}_{(x_1, x_2, A_\phi)}$ of possible velocities of the system is TWO-dimensional.
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- We've just produced a lot of examples of physcial systems, which are naturally 5-dimensional, and which are naturally equipped with a structure of a 2-distribution.
- Find pairs of surfaces $((\Sigma_1, g_1), (\Sigma_2, g_2))$ for which this distribution is $(2,3,5)$. Among them find such for which the symmetry of the rolling distribution is $\mathbf{G}_2$.

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A bit of thinking

- For the rolling distribution to have **G_2** symmetry it is neccessary (and sufficient) that the Cartan quartic $\mathcal{C}(\xi)$ vanishes identically.
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- Clever (e.g.: Robert Bryant):
- Take as Σ_1 a ball of radius r with a standard round sphere Riemannina metric g_1 on it, and as Σ_2 a ball of radius R with a standard round sphere Riemannian metric g_2 on it.
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Surprising solution for $\kappa_2 = 0$

- It turns out that taking $g_2 = g_0$, i.e. the flat metric, there are solutions for g_1 with a Killing symmetry, such that (Σ_1, g_1) when rolling without slipping or twisting on the PLANE, has rolling distribution with $\mathbf{G}_2$ symmetry.
- More precisely:

Surfaces of revolution on the plane with G_2 symmetry

Together with Daniel An we have the following:

Theorem

Modulo homotheties all metrics corresponding to surfaces with a Killing vector, which when rolling 'without slipping or twisting' on the **plane** $\mathbb{R}^2$, *have the velocity distribution* $\mathcal{D}$ *with local symmetry* G_2 *are given by:*

$$\begin{aligned} g_{1o} &= \rho^4 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \\ g_{1+} &= (\rho^2+1)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \\ g_{1-} &= (\rho^2-1)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \end{aligned}$$

Theorem (continued)

Theorem

or, collectively as:

$$\boxed{g_1 = (\rho^2 + \epsilon)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \quad \text{where} \quad \epsilon = 0, \pm 1.}$$

Their curvature is given by

$$\kappa_1 = \frac{2}{(\rho^2 + \epsilon)^3}.$$

Surfaces of revolution on the plane with G_2 symmetry

Theorem

Let $\mathcal{U}$ be a region of one of the Riemann surfaces (Σ_1, g_1) of the previous Theorem, in which the curvature κ_1 is nonnegative. In the case $\epsilon = +1$, such a region can be isometrically embedded in flat $\mathbb{R}^3$ as a surface of revolution. The embedded surface, when written in the Cartesian coordinates (X, Y, Z) in $\mathbb{R}^3$, is algebraic, with the embedding given by

$$(X^2 + Y^2 + 2)^3 - 9Z^2 = 0, \qquad \epsilon = +1.$$

Theorem (continued)

Theorem

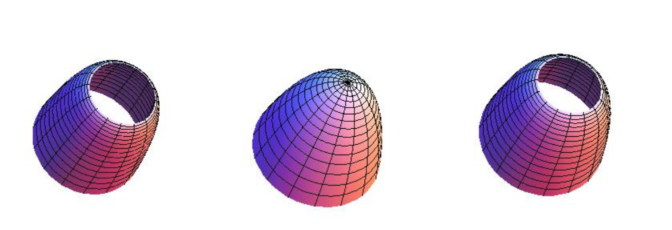
In the case $\epsilon = -1$, one can find an isometric embedding in $\mathbb{R}^3$ of a portion of $\mathcal{U}$ given by $\varphi \in [0, 2\pi[$, $\rho \geq \sqrt{2}$. This embedding gives another surface of revolution which is also algebraic, and in the Cartesian coordinates (X, Y, Z), given by

$$\boxed{(X^2 + Y^2 - 2)^3 - 9Z^2 = 0, \qquad \epsilon = -1.}$$

In the case $\epsilon = 0$, one can embed a portion of $\mathcal{U}$ with $\rho \geq 1$ in $\mathbb{R}^3$ as a surface of revolution

$$\boxed{Z = f(\sqrt{X^2 + Y^2}), \quad \text{with} \quad f(t) = \int_{\rho=1}^{t} \sqrt{\rho^4 - 1}\, \mathrm{d}\rho.}$$

How do they look?



Rysunek: The Mathematica print of the three surfaces of revolution, whose induced metric from $\mathbb{R}^3$ is given, from left to right, by respective metrics g_{1-}, g_{1+} and g_{1o}. The middle figure embeds all (Σ_1, g_{1+}). In the left figure only the portion of (Σ_1, g_{1-}) with *positive* curvature is embedded, and in the right figure only points of (Σ_1, g_{1o}) with $\rho > 1$ are embedded. It is why the left and right figures have holes on the top. All three surface, when rolling on a plane 'without twisting or slipping' have the rolling distribution with symmetry G_2.

Twistor interpretation of rolling

- Given the surfaces (Σ_1, g_2) and (Σ_2, g_2) which we want roll on each other, we now consider a 4-manifold $M = \Sigma_1 \times \Sigma_2$ and equipe it with the split signature metric $g = g_1 \ominus g_2$.
- This defines a *circle twistor bundle*

$$\mathbb{S}^1 \to \mathbb{T}(\Sigma_1 \times \Sigma_2) \stackrel{\pi}{\to} \Sigma_1 \times \Sigma_2,$$

  of *real selfdual 2-planes* over M.
- Chosing an orthonormal frame (e_1, f_1) for g_1 and an orthonormal frame (e_2, f_2) for g_2 the fibers of this bundle over a point $(x_1, x_2) \in M$ are planes

$$N_\phi = \mathrm{Span}(e_1 + e_2 \cos\phi + f_2 \sin\phi,\ f_1 - e_2 \sin\phi + f_2 \cos\phi).$$

  Here ϕ is a fiber coordiante $\phi \in [0, 2\pi]$.
- Since $g_i(e_i, f_i) = 0$, $g_i(e_i, e_i) = g_i(f_i, f_i) = 1$, $i = 1, 2$, the planes N_ϕ are *real totally null*. Hence self-dual with a proper choice of orientation in M.

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Here ϕ is a fiber coordiante $\phi \in [0, 2\pi]$.
- Since $g_i(e_i, f_i) = 0$, $g_i(e_i, e_i) = g_i(f_i, f_i) = 1$, $i = 1, 2$, the planes N_ϕ are *real totally null*. Hence self-dual with a proper choice of orientation in M.

Twistor interpretation of rolling

- Given the surfaces (Σ_1, g_2) and (Σ_2, g_2) which we want roll on each other, we now consider a 4-manifold $M = \Sigma_1 \times \Sigma_2$ and equipe it with the split signature metric $g = g_1 \ominus g_2$.
- This defines a *circle twistor bundle*

$$\mathbb{S}^1 \to \mathbb{T}(\Sigma_1 \times \Sigma_2) \xrightarrow{\pi} \Sigma_1 \times \Sigma_2,$$

of *real selfdual 2-planes* over M.

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Twistor interpretation of rolling

- There is a bundle isomorphism

$$\Phi \; : \; \mathbb{T}(\Sigma_1 \times \Sigma_2) \to C(\Sigma_1, \Sigma_2)$$

given by $\Phi(x_1, x_2, N_\phi) = (x_1, x_2, A_\phi)$, with

$$A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

- This defines a rank 2-distribution $\mathcal{D}_{\mathbb{T}} = \Phi_*^{-1}\mathcal{D}$ on the circle twistor bundle, which we call *twistor distribution*.
- In this way we have a *twistorial interpretation* of rolling without slipping or twisting: curves in the circle twistor bundle tangent to the twistor distribution are trajectories of such rolling.

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Twistor circle bundle may have non-product base

- Twistor circle bundle can be defined over ANY 4-manifold M equipped with a split signature metric g. We do NOT need either $M = \Sigma_1 \times \Sigma_2$ or $g = g_1 \ominus g_2$.
- Given any split signature metric g in dimension 4, we can always find an orthonormal frame (e_1, f_1, e_2, f_2) in which $g(e_1, e_1) = g(f_1, f_1) = 1 = -g(e_2, e_2) = -g(f_2, f_2)$ with all other products zero.
- In such frame

$$N_\phi = \mathrm{Span}(e_1 + e_2 \cos\phi + f_2 \sin\phi,\ f_1 - e_2 \sin\phi + f_2 \cos\phi),$$

is totally null as before, and the circles of N_ϕ, $\phi \in [0, 2\pi]$ constitute fibers of the *circle twistor bundle* $\mathbb{T}(M)$ over the neighbourhood of M where the frame (e_1, f_1, e_2, f_2) is defined.

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Twistor circle bundle have twistor distribution

- Let $\mathbb{S}^1 \to \mathbb{T}(M) \xrightarrow{\pi} M$ be a twistor circle bundle of totally null selfdual planes over a 4-dimensional manifold M equipped with the split signature metric g.
- There is a natural rank three distribution $\mathcal{D}^2$ defined an $\mathbb{T}(M)$.
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Again with Daniel An we have:

Theorem

If the Weyl tensor of metric g on M has nonvanishing self-dual part, then the distribution $\mathcal{D}^2$ on $\mathbb{T}(M)$ satisfies

$$\mathcal{D}^2 = [\mathcal{D}, \mathcal{D}],$$

almost everywhere, with uniquely defined distribution $\mathcal{D}$, which is $(2,3,5)$.

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Twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$

- The rank 2 distribution $\mathcal{D}$ on $\mathbb{T}(M)$ as defined on the previous slide is called **twistor distribution** on $\mathbb{T}(M)$.
- **Fact**: In case when $M = \Sigma_1 \times \Sigma_2$ and $g = g_1 \ominus g_2$ the so defined twistor distribution on $\mathbb{T}(M) \simeq \Phi^{-1}(\mathcal{C}(\Sigma_1, \Sigma_2))$ coincides with the rolling distribution $\Phi_*^{-1}\mathcal{D}$.

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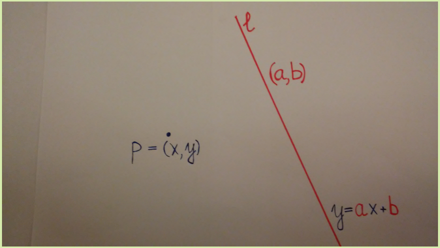
Returning question

Can we find all 4-manifolds M with split signature metric g for which the twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ has $\mathbf{G}_2$ symmetry?

Ice dancing

- On an ice ring consider a one leg skater ℓ and a spectator p. The pair (p, ℓ) is going to perform a certain movement on the ring. The rules of the movement (dance) are as follows.
- Idealization: We have a pair (p, ℓ) of a point $p \in \mathbb{R}^2$ and a line $\ell \in \mathbb{R}^2$.



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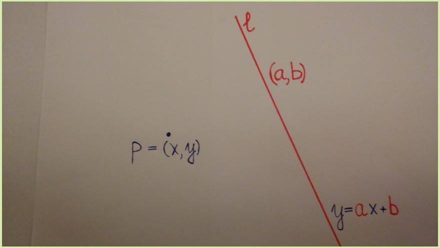

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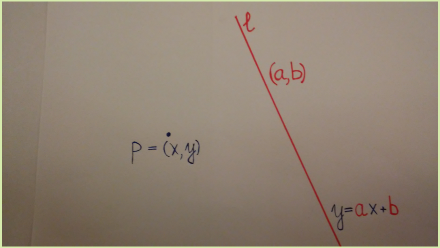

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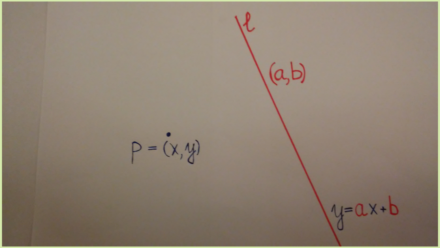

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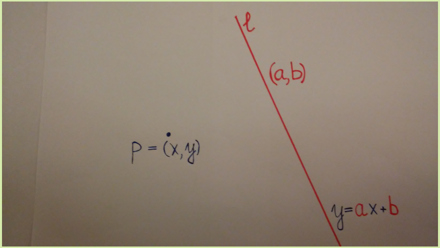

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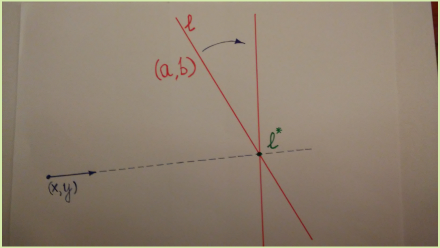
Configuration space

- Our configuration space is then 4-dimensional manifold $M = (\mathbb{R}P^2 \times (\mathbb{R}P^2)^*) \setminus I$, where I is the singular locus consisting of pairs (p, ℓ) such that $p \in \ell$.
- A movement of a pair (p, ℓ) draws a curve $\gamma(t) = (p(t), \ell(t))$ in M.
- What does it mean that a line $\ell(t)$ is moving at the moment t? It means that it is *rotating* around some point $\ell^*(t)$ on it.



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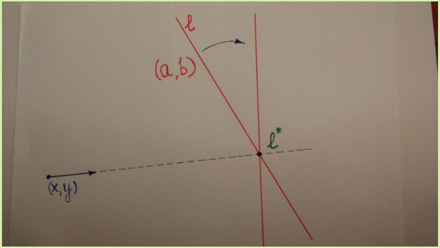

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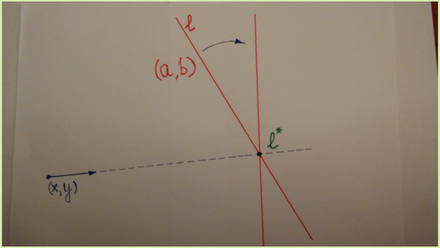

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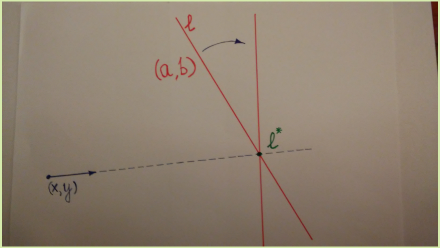
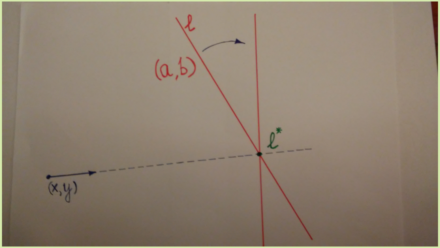


Dancing condition

The rule for the coordinated movement of a pair (p, ℓ) - the 'dancing condition' - is as follows:
at every moment t *the point* $p(t)$ *goes in the direction of the rotation point* $\ell^*(t)$ *of the line* $\ell(t)$.



Conformal class

- The dancing condition singles out a unique conformal class $[g]$ of split signature metrics on M. Indeed in the parametrization (x, y, a, b) of a point (x, y) and a line $y = ax + b$ we have :
- The rotation point $\ell^*(t) = (x^*(t), y^*(t))$ satisfies $y^* = ax^* + b$ and $y^* = (a + \dot{a}dt)x^* + b + \dot{b}dt$. This gives: $(x^*, y^*) = (-\frac{\dot{b}}{\dot{a}}, -a\frac{\dot{b}}{\dot{a}} + b)$.
- Dancing condition: $\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix} \parallel \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix}$ gives: $(x - x^*)\dot{y} - (y - y^*)\dot{x} = 0$.
- Inserting and grouping the terms we get:

$$(x\dot{a} + \dot{b})\dot{y} - (a\dot{b} + (y - b)\dot{a})\dot{x} = 0.$$

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Dancing metric

- Thus, every curve $(x(t), y(t), a(t), b(t)) \subset M$ satisfying dancing condition is a *null* curve in the $(2,2)$ signature metric

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- Since the conformal class $[g]$ with the metric representative g as above was defined only in terms of points, lines and their incidences in $\mathbb{R}P^2$, by construction it is $\mathbf{SL}(3,\mathbb{R})$ invariant. Thus the conformal class $[g]$ has at least 8-dimensional Lie algebra of conformal symmetries.
- It follows that $[g]$ contains precisely one metric $g_E \in [g]$ which has $\mathbf{SL}(3,\mathbb{R})$ as the group of *isometries*. We call g_E the *dancing metric* on M.

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- Since the conformal class $[g]$ with the metric representative g as above was defined only in terms of points, lines and their incidences in $\mathbb{R}P^2$, by construction it is $\mathbf{SL}(3,\mathbb{R})$ invariant.Thus the conformal class $[g]$ has at least 8-dimensional Lie algebra of conformal symmetries.
- It follows that $[g]$ contains precisely one metric $g_E \in [g]$ which has $\mathbf{SL}(3,\mathbb{R})$ as the group of *isometries*. We call g_E the *dancing metric* on M.

Properties of the dancing metric

- The dancing metric is Einstein, and has has nonvanishing selfdual Weyl tensor.
- In the local coordinates (x, y, a, b) on M it reads:

$$g_E = \frac{(x\mathrm{d}a + \mathrm{d}b)\mathrm{d}y - (a\mathrm{d}b + (y-b)\mathrm{d}a)\mathrm{d}x}{(y - ax - b)^2}.$$

- Considering the circle twistor bundle $\mathbb{T}(M)$ over (M, g_E), we have the twistor distribution $\mathcal{D}$ in $\mathbb{T}(M)$ which is $(2, 3, 5)$.
- Moreover, since the eight infinitesimal isometries, by the functoriality of the construction, give rise to eight infinitesimal symetries of $\mathcal{D}$, we conclude that the twistor distribution associated with (M, g_E) has at least 8-dimensional Lie algebra of infinitesimal symetries, hence it has 14 symetries by Cartan's submaximality result.

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Dancing metric and $\mathbf{G}_2$

With Gil Bor we have the following:

Theorem

The twistor distribution $\mathcal{D}$ on the circle twistor bundle $\mathbb{T}(M)$ over the manifold $M = (\mathbb{R}P^2 \times (\mathbb{R}P^2)^*) \setminus I$ equipped with the dancing metric g_E has the split real form of the simple exceptional Lie group $\mathbf{G}_2$ as a group of its symmetries.

Remark Note that we've found a geometric realization of the group $\mathbf{G}_2$ using only projective notions in $\mathbb{R}^2$.

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Rolling $\mathbb{R}P^2$ on $(\mathbb{R}P^2)^*$ without slipping or twisting

- Note that for (M, g_E) we have well defined projections $\alpha : \mathbb{T}(M) \to \mathbb{R}P^2$ and $\beta : \mathbb{T}(M) \to (\mathbb{R}P^2)^*$. In coordinates (x, y, a, b, N_ϕ) on $\mathbb{T}(M)$ we have:

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- Thus, a curve $\tilde{\gamma}(t) = (x(t), y(t), a(t), b(t), N_{\phi(t)})$ in $\mathbb{T}(M)$ tangent to the twistor distribution $\mathcal{D}$, $\dot{\tilde{\gamma}}(t) \in \mathcal{D}$, defines curves $p(t) = (x(t), y(t))$ and $\ell(t) = (a(t), b(t))$, respectively in $\mathbb{R}P^2$ and $(\mathbb{R}P^2)^*$.
- By analogy with the usual rolling of surfaces, we may think of these $p(t)$ and $\ell(t)$ as traces of a process of rolling without slipping or twisting of $\mathbb{R}P^2$ on $(\mathbb{R}P^2)^*$.

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Rolling $\mathbb{R}P^2$ on $(\mathbb{R}P^2)^*$ without slipping or twisting

- In particular, one may think of conditions on curves $p(t)$ and $\ell(t)$, respectively in $\mathbb{R}P^2$ and $(\mathbb{R}P^2)^*$, that are analogs of **a)** rolling without slipping, and **b)** rolling without slipping or twisting, expressed only in terms of projective terms in $\mathbb{R}^2$.
- Obviously, the condition which is an analog of rolling without slipping is that the curves $p(t) \subset \mathbb{R}P^2$ and $\ell(t) \subset (\mathbb{R}P^2)^*$ are such that at every moment the pair $(p(t), \ell(t))$ satisfies the *dancing condition*.

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- Rolling without slipping or twisting is more tricky to express in purely projective terms.
- I only say that the condition to make the rolling also 'without twisting' is a rather demanding one: the dancers - the point p and a line ℓ - should be aware of the 5th order derivative of their motions to comply with it.

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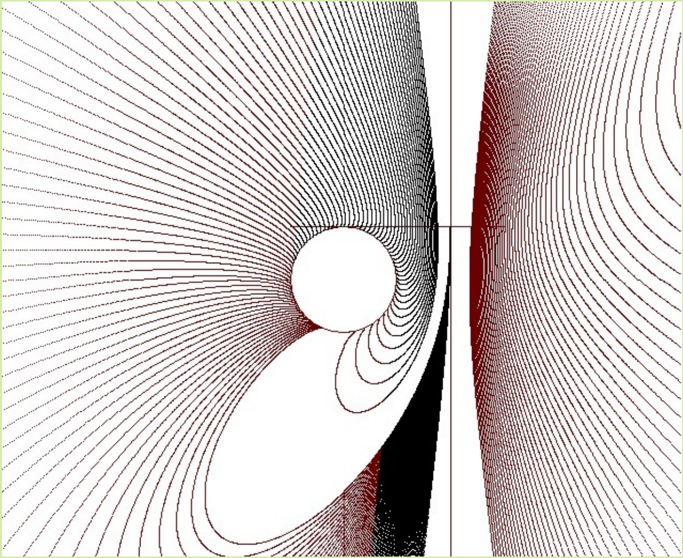
Dancing curves $\ell(t)$ when $p(t)$ is on a circle



Relevant references

- Introduction and more: "Differential equations and conformal structures" *J. Geom. Phys.*, **55**, 19-49 (2005), avaliable at: `http://www.fuw.edu.pl/~nurowski/prace/confode.pdf`
- Rolling of surfaces with a surprising example: "Twistor space for rolling bodies", *Commun. Math. Phys.*, **326**, 393-414 (2014), coauthor D. An, avaliable at: `http://www.fuw.edu.pl/~nurowski/prace/daniel_pawel.pdf`
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