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Ambient Metrics for n-Dimensional pp-Waves*

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Abstract: We provide an explicit formula for the FEFFERMAN- GRAHAM ambient metric of an n-dimensional conformal pp-wave in those cases where it exists. In even dimensions we calculate the obstruction explicitly. Furthermore, we describe all 4-dimensional pp-waves that are Bach-flat, and give a large class of Bach-flat examples which are conformally Cotton-flat, but not conformally Einstein. Finally, as an application, we use the obtained ambient metric to show that even-dimensional pp-waves have vanishing critical Q-curvature.

1. Introduction

Plane fronted gravitational waves, called *pp-waves*, are Lorentzian 4-manifolds (M, g) admitting a *covariantly constant null* vector field K. In addition, their Ricci tensor Ric satisfies

$$Ric = \Phi\, \kappa \otimes \kappa, \tag{1}$$

where κ is the 1-form on M defined by $\kappa := K \lrcorner g$. Physicists require also that the function Φ is nonnegative for a pp-wave. This is because Φ, via the *Einstein field equations*, is directly related to the energy momentum tensor of its gravitational field.

pp-waves are important in general relativity theory since they generalize the concept of a *plane wave of classical electrodynamics* [41], as well as because of the fact that every 4-dimensional spacetime has a *special* pp-wave as a well defined limit [40], the Penrose limit, as it is called.

Higher dimensional generalizations of the 4-dimensional pp-waves were studied in [42], appeared in Kaluza-Klein theory [28,25,29,9], and later in string theory [5,6,4,35,11,36,12,18,3,37]. Their property of possessing a covariantly constant null vector

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field K, implies that they have *reduced Lorentzian holonomy* from the full orthogonal group $\mathrm{SO}(1, n-1)$ to the subgroup preserving the null vector K. In fact, they can be characterised by having *Abelian* holonomy $\mathbb{R}^{n-2}$ [30,32]. As such they admit many parallel spinors: The dimension of the space of parallel spinors on an n-dimensional pp-wave is at least half of the dimension of the spinor module, [30].

In local coordinates $(x^i, u, r)_{i=1,\ldots,n-2}$ in $\mathbb{R}^n$, the n-dimensional pp-wave metric can be written as

$$g = \sum_{i=1}^{n-2} (\mathrm{d}x^i)^2 \; + \; 2\mathrm{d}u \; (\mathrm{d}r + h\mathrm{d}u)\,.$$

Here h is an arbitrary smooth real function of the first $(n-1)$ coordinates, $h = h(x^i, u)$. The covariantly constant null vector field is $K = \partial_r$. Another property of this metric is that it has vanishing scalar curvature. Hence, if it is *Einstein* then it is *Ricci flat*. This happens if and only if $\Delta h = \sum_{i=1}^{n-2} \frac{\partial^2 h}{\partial (x^i)^2} = 0$.

Conformal classes of pp-wave metrics have remarkable properties. One of them has been described by their discoverer H. W. Brinkmann already in 1925. In his seminal paper [8] Brinkmann not only studied spaces that were later called *Brinkmann waves*, namely Lorentzian manifolds with parallel null vector field, but he also showed the following [8, Theorems IV and VIII]: *A 4-dimensional, not locally conformally flat Einstein manifold* (M, g) *locally admits a function* Υ *such that the conformally rescaled metric* $\mathrm{e}^{2\Upsilon} g$ *is again Einstein, but* not *homothetic to* g*, if and only if* (M, g) *is a Ricci-flat* pp*-wave (or its counterpart in neutral signature*[1]*)*. In this case, the rescaled metric is also Ricci-flat and the gradient of Υ is a null vector. This occurs because the Weyl tensor W of a pp-wave is *null* and *aligned* with K, i.e. $K \lrcorner W = 0$, which makes these metrics not *weakly generic* in the terminology of [20].

In this paper we discuss another remarkable conformal property of n-dimensional pp-wave metrics, which is related to the *ambient metric construction* of Fefferman and Graham [15,16], a construction that provides the geometric framework of AdS/CFT correspondence[2]. The ambient metric construction mimics the situation in the flat model of conformal geometry: Here the n-dimensional sphere equipped with the flat conformal structure can be viewed as the projectivisation of the light-cone in $(n+2)$-dimensional Minkowski space. Letting the spheres wander along the light cone recovers the metrics in the conformal class. For a conformal class $[g]$ in signature (p, q) on an $n = (p+q)$-dimensional manifold M the *ambient metric* is a metric $\widetilde{g}$ of signature $(p+1, q+1)$ on the product of M with two intervals, $\widetilde{M} := (-\varepsilon, \varepsilon) \times M \times (1-\delta, 1+\delta)$, $\varepsilon > 0$, $\delta > 0$, that is *compatible with the conformal structure* (for details see Definition 1) and, moreover, is *Ricci flat*. The Ricci-flat condition ensures that the the ambient metric depends uniquely on the conformal structure and encodes all properties of the conformal class $[g]$ but has the downside that the ambient metric does not always exist. Starting with a formal power series

[1] Be aware that the coordinates in the relevant Sect. 4.2 of Brinkmann's paper [8] have to be understood as complex and complex conjugate in order to obtain Lorentzian metrics. If they are considered as real coordinates the resulting metric has neutral signature.

[2] Note that in some papers from the physics literature the term Fefferman-Graham metric has a different meaning than ours. What physicists call Fefferman-Graham metric, e.g. in [2 or 13], is a related concept that Fefferman and Graham call the Poincaré-Einstein metric. How to obtain one from another is well known and we shall explain it in Sect. 7.

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$$\widetilde{g} = 2\,(t\mathrm{d}\rho + \rho\mathrm{d}t)\,\mathrm{d}t + t^2\left(g + \sum_{k=1}^{\infty}\rho^k\mu_k\right) \tag{2}$$

with $\rho \in (-\varepsilon, \varepsilon)$, $t \in (1-\delta, 1+\delta)$ Fefferman and Graham showed that if n is *odd*, the Ricci-flatness of the ambient metric gives equations for $\mu_1, \mu_2, \ldots$ that can be solved *in principle*, but the calculations have been carried out only for very special conformal classes, mainly those that are related to Einstein spaces [34,31,19]. If $n = 2s$ is *even*, there is a conformally invariant *obstruction* to the existence of a Ricci-flat ambient metric, called the *Fefferman-Graham obstruction*. This obstruction is the nonvanishing of the obstruction tensor $\mathcal{O}$, given by the term μ_s. In $n = 4$ this obstruction tensor is the *Bach tensor* for g. In higher dimensions the *leading term* of $\mathcal{O}$ is $\Delta_g^s(g)$, but there are a lot of lower order terms, which, again, are determined *in principle*, but whose calculation is very cumbersome.

One important feature of the ambient metric is that if the metric g is *real analytic* then its corresponding ambient metric $\widetilde{g}$ (if it exists) is *also real analytic* [15,16,27]. Another feature of the ambient metric is that if the conformal class of g includes an Einstein metric g_E, then the power series in the ambient metric $\widetilde{g}_E$ *truncates* at $k = 2$; in particular, for $n > 3$, even the obstruction tensor vanishes. In such case the metric is given as a *second order polynomial* in each of the variables t and ρ. However, if the metric g is *not conformally Einstein*, then, except for a few examples [19,39], no explicit formulae for μ_k, $k > 3$ are known.

In this context our main result is the following remarkable conformal property of n-dimensional pp-waves: for them *all* the coefficients μ_k in the ambient metric, the obstruction tensor in even dimensions, and hence, the condition under which the ambient metric truncates at a given order can be calculated *explicitly*. In Sect. 4 we prove

Theorem 1. *Let $g = \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2 + 2\mathrm{d}u\ (\mathrm{d}r + h\mathrm{d}u)$ be an n-dimensional pp-wave metric with a real analytic function $h = h(x^1, \ldots, x^{n-2}, u)$. Then the Fefferman-Graham ambient metric for the conformal class $[g]$ exists if and only if n is* odd *and h is arbitrary, or if $n = 2s$ is* even *and $\Delta^s h = 0$. In both cases the ambient metric is given by a formal power series*

$$\widetilde{g} = 2\mathrm{d}\,(t\rho)\,\mathrm{d}t + t^2\left(g + \left(\sum_{k=1}^{\infty}\frac{\Delta^k h}{k!\,p_k}\rho^k\right)\mathrm{d}u^2\right),$$

with $p_k := \prod_{j=1}^{k}(2j - n)$ and $\Delta := \sum_{i=1}^{n-2}\partial_i^2$. In particular, if $n = 2s$ is even, the obstruction tensor $\mathcal{O}$ is given by $\mathcal{O} = \Delta^s h\ \mathrm{d}u^2$.

Thus if $n = 2s$ is even, the ambient metric $\widetilde{g}$ is a *polynomial* of order $s - 1$ in the variable ρ. If n is *odd*, since the metric g is real analytic, the Fefferman-Graham result guarantees that the *above metric* $\widetilde{g}$ is also *real analytic*. This in particular means that the power series $\sum_{k=1}^{\infty}\frac{\Delta^k h}{k!p_k}\rho^k$ converges to a real analytic function in variable ρ.

Theorem 1 provides us with a variety of examples of conformal structures with *explicit* ambient metrics and which, in general, are *not* conformally Einstein. For example, every polynomial h in the x^i's of order lower than k, with coefficients being functions of u, represents a pp-wave with ambient metric truncated at order lower than $k/2$. In Sect. 6 we construct more general examples than those defined by h being polynomials in the x^is. In particular, in dimension *four* we find *all* Bach-flat 4-dimensional pp-waves and

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we prove that most of them are *not conformally Einstein*. They are defined by quite general functions h and have ambient metrics which are linear in variable ρ. It is interesting to note that these pp-waves, although Bach-flat and conformal to Cotton-flat, are not conformally Einstein.

Theorem 1 implies also another interesting feature of the pp-waves: their obstruction tensor $\mathcal{O}$ (in *even* dimensions) involves only the terms of the highest possible order in the derivatives of their metric; since *all* the lower order terms that are usually present in the obstruction tensor are *vanishing*, the pp-waves are, in a sense, the closest cousins of the conformally Einstein metrics.

Using the explicit form of the ambient metric and the main result of [24], in Sect. 7 we show that for even-dimensional pp-waves the critical Q-curvature vanishes. This result is in correspondence with the fact that for a pp-wave all scalar invariants constructed from the curvature tensor vanish (for the proof in arbitrary dimension see [10]). In the final Sect. 8 we study the holonomy of the ambient metric of a pp-wave in relation to results in [31]. We show that it is contained in the stabiliser of a totally null plane.

2. The Fefferman-Graham Ambient Metric

An important tool in order to construct invariants in conformal geometry is the so-called *Fefferman-Graham ambient metric* or *ambient space* (see [15 and 16]). Let $(M, [g])$ be a a smooth n-dimensional manifold M with conformal structure $[g]$ of signature (p, q) with the conformal frame bundle $\mathcal{P}^0$. It can also be characterised by a principle $\mathbb{R}^+$-fibre bundle $\pi : \mathcal{Q} \to M$ defined as the ray sub-bundle in the bundle of metrics of signature (p, q) given by metrics in the conformal class c. The action of $\mathbb{R}^+$ on $\mathcal{Q}$ shall be denoted by φ:

$$\varphi(t, g_x) = t^2 g_x.$$

From [16] we adopt the following notation.

Definition 1. *Let $(M, [g])$ be a conformal structure of signature (p, q) over an n-dimensional manifold M, and $\pi : \mathcal{Q} \to M$ the corresponding ray bundle. A semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$ of signature $(p+1, q+1)$ is called* pre-ambient space *if*

(1) *there is a free $\mathbb{R}^+$-action $\widetilde{\varphi}$ on $\widetilde{M}$, and*
(2) *an embedding $\iota : \mathcal{Q} \to \widetilde{M}$ is $\mathbb{R}^+$-equivariant.*
(3) *If F is the fundamental vector field of $\widetilde{\varphi}$, and $\mathcal{L}$ denotes the Lie derivative, then $\mathcal{L}_F \widetilde{g} = 2\widetilde{g}$, i.e. the metric $\widetilde{g}$ is homogeneous of degree 2 with respect to the $\mathbb{R}^+$-action.*
(4) *Any $g_x \in \mathcal{Q}$ satisifies the equality $(\iota^* \widetilde{g})_{g_x} = g_x\,(d\pi(.), d\pi(.))$ in $\odot^2 T^*_{g_x}\mathcal{Q}$.*

A pre-ambient space is called ambient space *if its Ricci curvature vanishes.*

Under the assumption that the conformal structure is given by a real analytic metric, in odd dimensions a Ricci-flat ambient metric always exists and is also real analytic.

In even dimensions $n \geq 4$, the existence of a Ricci-flat ambient metric is obstructed by the nonvanishing of the obstruction tensor $\mathcal{O}$, [16, pp. 22]. This is a symmetric trace-free and divergence-free $(2, 0)$-tensor, which is conformally invariant of weight $(2-n)$, i.e. if $\hat{g} = \mathrm{e}^{2\varphi} g \in [g]$, then $\hat{\mathcal{O}} = \mathrm{e}^{(2-n)\varphi}\mathcal{O}$. It is given by

$$\mathcal{O} = \Delta_g^{n/2-2}\left(\Delta_g \mathsf{P} - \nabla^2 J\right) + \text{lower order terms},$$

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where $\mathsf{P} = \frac{1}{n-2}\left(Ric - \frac{scal}{2(n-1)}g\right)$ is the Schouten tensor, J its trace, and Δ_g denotes the Laplacian of $g \in [g]$. For a conformal class in even dimension that is given by a real analytic metric with vanishing obstruction tensor, the ambient metric exists and is also real analytic.

Fixing a metric g in the conformal class, in [15,16] it is shown that an ambient space near M can be written as

$$\widetilde{M} = (-\epsilon, \epsilon) \times M \times (1-\delta, 1+\delta)$$

with the ambient metric

$$\widetilde{g} = 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2 g(\rho),$$

in which $g(\rho)$ is a one-paramemter family of metrics on M with $g(0) = g$. This is referred to as $\widetilde{g}$ being in *normal form*. As the ambient metric is analytic, one can write the family $g(\rho)$ as a power series in ρ,

$$\widetilde{g} = 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2\left(g + \rho g' + \frac{1}{2}\rho^2 g'' + \frac{1}{6}\rho^3 g''' + \ldots\right),$$

with $g' = \partial_\rho g(0)$. We summarise the results for the ambient metric in

Theorem 2 ([15,16 and 27]). *Let $(M, [g])$ be a real analytic manifold M of dimension $n \geq 2$ equipped with a conformal structure defined by a real analytic semi-Riemannian metric g.*

(1) *If n is odd, or if n is even with $\mathcal{O} = 0$, then there exists an ambient space $(\widetilde{M}, \widetilde{g})$ with real analytic Ricci-flat metric $\widetilde{g}$.*
(2) *If n is odd the ambient space is unique modulo diffeomorphisms that restrict to the identity along $\mathcal{Q} \subset \widetilde{M}$ and commute with $\widetilde{\varphi}$. If n is even with $\mathcal{O} = 0$, the ambient space is unique, modulo the same set of diffeomorphisms and modulo terms of order $\geq n/2$ in ρ, where ρ is the coordinate in the normal form of the ambient metric.*

The Ricci-flat condition then determines symmetric $(2, 0)$-tensors μ_k such that

$$\widetilde{g} = 2t\mathrm{d}\rho\mathrm{d}t + 2\rho\mathrm{d}t^2 + t^2\left(g + \sum_{k=1}^{\infty}\rho^k\mu_k\right).$$

In [16] the first μ_k are determined explicitly:

$$\begin{aligned}
(\mu_1)_{ab} &= 2\mathsf{P}_{ab},\\
(n-4)(\mu_2)_{ab} &= -B_{ab} + (n-4)\mathsf{P}_a{}^c\mathsf{P}_{bc},\\
3(n-4)(n-6)(\mu_3)_{ab} &= \Delta_g B_{ab} - 2W_{cabd}B^{cd} - 4(n-6)\mathsf{P}_{c(a}B_{b)}{}^c - 4\mathsf{P}_c{}^c B_{ab}\\
&\quad + 4(n-4)\mathsf{P}^{cd}\nabla_d C_{(ab)c} - 2(n-4)C^c{}_a{}^d C_{dbc}\\
&\quad + (n-4)C_a{}^{cd}C_{bcd} + 2(n-4)\nabla_d\mathsf{P}^c{}_c C_{(ab)}{}^d\\
&\quad - 2(n-4)W_{cabd}\mathsf{P}^c{}_e\mathsf{P}^{ed},
\end{aligned} \tag{3}$$

where W_{abcd} is the Weyl tensor, P_{ab} is the Schouten tensor, $C_{abc} := \nabla_c\mathsf{P}_{ab} - \nabla_b\mathsf{P}_{ac}$ is the Cotton tensor, and $B_{ab} = \nabla_c C_{ab}{}^c - \mathsf{P}_{cd}W^c{}_{ab}{}^d$ is the Bach tensor.

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3. *pp*-Waves and Their Curvature

A *pp*-wave is a Lorentzian manifold with a parallel null vector field K, i.e. $\nabla K = 0$, $K \neq 0$, and $g(K, K) = 0$, whose curvature tensor satisfies the trace condition

$$R_{ab}^{\ \ ef} R_{efcd} = 0. \tag{4}$$

If we denote by κ the one-form given by $\kappa := K \lrcorner g$ the curvature condition (4) is equivalent to each of the following, in which $[ab]$ denotes the skew symmetrisation with respect to a and b, [42]:

(1) $\kappa_{[a} R_{bc]de} = 0$;
(2) there is a symmetric (2, 0)-tensor ϱ with $K \lrcorner \varrho = 0$, such that $R_{abcd} = \kappa_{[a} \varrho_{b][c} \kappa_{d]}$;
(3) there is a function φ, such that $R^{e\ \ f}_{\ ab} R_{ecdf} = \varphi \kappa_a \kappa_b \kappa_c \kappa_d$.

The Ricci tensor of a *pp*-wave is given by $Ric = \Phi\, \kappa \otimes \kappa$, for a smooth function Φ. In dimension $n = 4$ this is even equivalent to the curvature condition (4).

In [31] we gave another equivalent definition, without using coordinates or traces, but identifying a *pp*-wave as a Lorentzian manifold with parallel null vector field K, whose curvature satisfies

$$\operatorname{Im}\left(\mathcal{R}(U, V)_{|K^\perp}\right) \subset \mathbb{R} \cdot K \text{ for all } U, V \in TM. \tag{5}$$

This equivalence allows for several generalisations [32] and for an easy proof of another equivalence that is related to holonomy: An n-dimensional Lorentzian manifold is a *pp*-wave if and only if its holonomy group is contained in the Abelian subgroup $\mathbb{R}^{n-2}$ of the stabiliser in $SO(1, n-1)$ of a null vector [30].

Locally, an n-dimensional *pp*-wave admits coordinates $(x^1, \ldots, x^{n-2}, u, r)$ such that the metric is given by

$$g = \sum_{i=1}^{n-2} (\mathrm{d}x^i)^2 \ + \ 2\mathrm{d}u \ (\mathrm{d}r + h\mathrm{d}u)\,, \tag{6}$$

with h being a smooth real function of the first $(n-1)$ coordinates, $h = h(x^i, u)$, [42]. In these coordinates the parallel null vector field K is given by ∂_r and, up to symmetries, the only non-vanishing curvature terms of a *pp*-wave are

$$R(\partial_i, \partial_u, \partial_j, \partial_u) \ = \ \partial_i \partial_j h.$$

Here we use the obvious notation $\partial_r := \frac{\partial}{\partial r}$, $\partial_u := \frac{\partial}{\partial u}$ and $\partial_i := \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n-2$. Hence, the function determining the Ricci-tensor is given by $\Phi = -\Delta h$ with $\Delta h = \sum_{i=1}^{n-2} \partial_i^2 h$, i.e.

$$Ric = -\Delta h \, \mathrm{d}u^2. \tag{7}$$

Hence, the image of the Ricci-tensor is totally null, and the scalar curvature vanishes. With this at hand, one can easily calculate the tensors related to the conformal geometry of a *pp*-wave. First, there is the Schouten-tensor

$$\mathsf{P} \ = \ \frac{1}{n-2} Ric \ = \ -\frac{\Delta h}{n-2} \, \mathrm{d}u^2. \tag{8}$$

Secondly, the Weyl tensor is given by

$$W(\partial_i, \partial_u, \partial_j, \partial_u) \;=\; \partial_i \partial_j h - \delta_{ij} \frac{\Delta h}{n-2}, \tag{9}$$

and for $n > 3$ we obtain that $\partial_i \partial_j h = \delta_{ij} \frac{\Delta h}{n-2}$ as an equivalent condition on h for g being conformally flat.

Next, we calculate the Cotton tensor C. As $\nabla \mathsf{P} = -\frac{1}{n-2} \mathrm{d}(\Delta h) \otimes \mathrm{d}u^2$ one obtains that

$$C(\partial_u, \partial_i, \partial_u) = -C(\partial_u, \partial_u, \partial_i) = \frac{\partial_i \Delta h}{n-2} \tag{10}$$

are the only non-vanishing components of the Cotton tensor. Hence, $\partial_i \Delta h = 0$ is the condition on h for 3-dimensional conformally flat pp-waves.

Furthermore, we obtain the Bach tensor B,

$$B = -\frac{\Delta^2 h}{n-2}\, \mathrm{d}u^2. \tag{11}$$

This enables us to calculate the next terms in the ambient metric expansion in Eqs. (3) beyond $\mu_1 = 2\mathsf{P} = \frac{\Delta h}{n-2} \mathrm{d}u^2$, namely

$$\begin{aligned} \mu_2 &= \quad -\tfrac{1}{n-4} B &&= \tfrac{\Delta^2 h}{(n-2)(n-4)} \mathrm{d}u^2, \\ \mu_3 &= \tfrac{1}{2(n-4)(n-6)} \Delta B &&= \tfrac{\Delta^3 h}{3(n-2)(n-2)(n-4)} \mathrm{d}u^2. \end{aligned}$$

The very simple structure of μ_1, μ_2, and μ_3 above, and in particular the appearance of the consecutive powers of the Laplacian, suggests that this pattern may be also present in the next terms in the ambient metric expansion. That this is really the case will be proven in the next section.

4. The *pp*-Wave Ambient Metric

Looking at the very simple form of the pp-wave metric (6) and the general formula for the ambient metrics (2), our ansatz for the ambient metric for this g is

$$\bar{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 \left(2\mathrm{d}u \;\; (\mathrm{d}r + (h + H)\mathrm{d}u) \;\; + \;\; \sum_{i=1}^{n-2} (\mathrm{d}x^i)^2 \right), \tag{12}$$

where $H = H(\rho, x^i, u)$, and

$$H(\rho, x^i, u)_{|\rho=0} = 0. \tag{13}$$

If we were able to find an analytic function H satisfying (13) and for which the metric (12) was *Ricci flat* then, by the *uniqueness* of the Fefferman-Graham Theorem 2, we would conclude that $\bar{g}$ with this H is the ambient metric for (6). Thus to check our guess it is enough to calculate the *Ricci tensor* for (12) and to check if its *vanishing* is possible for the function H in the postulated form (13).

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Lemma 1. *The Ricci tensor of the metric* (12) *is*

$$Ric(\bar{g}) = \big((2-n)H_\rho + 2\rho H_{\rho\rho} - \triangle H - \triangle h\big)\,\mathrm{d}u^2.$$

Here $\triangle H = \sum_{i=1}^{n-2} \frac{\partial^2 H}{\partial (x^i)^2}$, $H_\rho = \frac{\partial H}{\partial \rho}$, *etc.*

Proof. We start with a coframe

$$\theta^0 = \mathrm{d}(\rho t),\ \theta^i = t\mathrm{d}x^i,\ \theta^{n-1} = t^2(\mathrm{d}r + (h+H)\mathrm{d}u),\ \theta^n = \mathrm{d}u,\ \theta^{n+1} = \mathrm{d}t, \tag{14}$$

in which the metric $\bar{g}$ reads:

$$\bar{g} = \bar{g}_{\mu\nu}\theta^\mu\theta^\nu = 2\theta^0\theta^{n+1} + 2\theta^{n-1}\theta^n + \sum_{i=1}^{n-2}(\theta^i)^2, \qquad \mu,\nu = 0,1,\ldots,n+1.$$

It has the following differentials:

$$\begin{aligned}
\mathrm{d}\theta^0 &= 0,\\
\mathrm{d}\theta^i &= -t^{-1}\theta^i \wedge \theta^{n+1}, \qquad \forall i = 1,\ldots,n-2,\\
\mathrm{d}\theta^{n-1} &= tH_\rho\theta^0 \wedge \theta^n + t\sum_{i=1}^{n-2}(h_i + H_i)\theta^i \wedge \theta^n - 2t^{-1}\theta^{n-1} \wedge \theta^{n+1} + \rho t H_\rho \theta^n \wedge \theta^{n+1},\\
\mathrm{d}\theta^n &= 0,\\
\mathrm{d}\theta^{n+1} &= 0.
\end{aligned}$$

In this coframe the Levi-Civita connection 1-forms, i.e. matrix-valued 1-forms satisfying $\mathrm{d}\theta^\mu + \Gamma^\mu_{\ \nu} \wedge \theta^\nu = 0$, $\Gamma_{\mu\nu} + \Gamma_{\nu\mu} = 0$, $\Gamma_{\mu\nu} = \bar{g}_{\mu\sigma}\Gamma^\sigma_{\ \nu}$, are:

$$\begin{aligned}
\Gamma_{0n} &= -tH_\rho\theta^n,\\
\Gamma_{in} &= -t(h_i + H_i)\theta^n,\\
\Gamma_{n-1\ n} &= t^{-1}\theta^{n+1}\\
\Gamma_{i\ n+1} &= t^{-1}\theta^i,\\
\Gamma_{n-1\ n+1} &= t^{-1}\theta^n\\
\Gamma_{n\ n+1} &= t^{-1}\theta^{n-1} - \rho t H_\rho \theta^n.
\end{aligned} \tag{15}$$

Modulo the symmetry $\Gamma_{\mu\nu} = -\Gamma_{\nu\mu}$ all other connection 1-forms are zero.

The curvature 2-forms $\Omega_{\mu\nu} = \mathrm{d}\Gamma_{\mu\nu} + \Gamma_{\mu\rho} \wedge \Gamma^\rho_{\ \nu}$, have the following nonvanishing components:

$$\begin{aligned}
\Omega_{0n} &= -H_{\rho\rho}\theta^0 \wedge \theta^n - \sum_{i=1}^{n-2} H_{i\rho}\theta^i \wedge \theta^n - \rho H_{\rho\rho}\theta^n \wedge \theta^{n+1},\\
\Omega_{in} &= -H_{i\rho}\theta^0 \wedge \theta^n - \sum_{k=1}^{n-2}(\delta_{ik}H_\rho + H_{ik} + h_{ik})\theta^k \wedge \theta^n - \rho H_{i\rho}\theta^n \wedge \theta^{n+1},\\
\Omega_{nn+1} &= -\rho H_{\rho\rho}\theta^0 \wedge \theta^n - \sum_{i=1}^{n-2}\rho H_{i\rho}\theta^i \wedge \theta^n - \rho^2 H_{\rho\rho}\theta^n \wedge \theta^{n+1},
\end{aligned} \tag{16}$$

together with the components that are implied by the symmetry $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$.

The Riemann tensor $R_{\mu\nu\rho\sigma}$, defined by $\Omega_{\mu\nu} = \frac{1}{2}R_{\mu\nu\rho\sigma}\theta^\rho \wedge \theta^\sigma$, can be read off from Eqs. (16). Using this and the inverse of the metric $g^{\mu\nu}$, $g_{\mu\rho}g^{\rho\nu} = \delta_\mu^{\ \nu}$, we calculate the

Ricci tensor $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$. It turns out that it has $R_{nn} = -2R_{0nnn+1} + \sum_{i=1}^{n-2} R_{inin}$ as its only nonvanishing component. Explicitly:

$$R_{nn} = 2\rho H_{\rho\rho} - (n-2)H_\rho - \triangle H - \triangle h.$$

This finishes the proof of the lemma.

The lemma shows that the metric $\bar{g}$ is Ricci flat if and only if the function H satisfies the following PDE:

$$(2-n)H_\rho + 2\rho H_{\rho\rho} - \triangle H = \triangle h. \tag{17}$$

For $\bar{g}$ to be the ambient metric for (6) we in addition require the initial condition (13). By looking for the solution of the initial value problem (17), (13) in the form of a power series

$$H = \sum_{k=0}^{\infty} a_k \rho^k, \tag{18}$$

we immediately get $a_0 = 0$ from the initial condition (13). Then inserting (18) in (17), we easily arrive at

Proposition 1. *If $n = 2s + 1$, $s \geq 1$, then the initial value problem (17), (13) has a* unique *power series solution. It is given by:*

$$H = \sum_{k=1}^{\infty} \frac{\triangle^k h}{k! \prod_{i=1}^{k} (2i - n)} \rho^k. \tag{19}$$

If $n = 2s$ the power series solution exists only if $\triangle^s h = 0$. If this is the case, the solution is also unique and given by the power series (19), which truncates to a polynomial *of order $(s-1)$ in the variable ρ.*

This proposition proves our Theorem 1 of the Introduction. Note that the solution we found is a solution to Eq. 3.17 in [16] that was derived for the Taylor expansion of the ambient metric, here specified for a pp-wave. In particular, for $n = 2s$ the obstruction tensor of an n-dimensional pp-wave is given by

$$\mathcal{O} = \Delta^s h \, \mathrm{d}u^2.$$

With this result at hand, every polynomial h in the x^i's of order lower than $2k$, with coefficients being functions of u, gives an example of a pp-wave for which the ambient metric truncates to a polynomial of order lower than k. This gives plenty of examples of explicit ambient metrics, also in even dimensions. Moreover, choosing h properly, one gets examples for which the conformal class does not contain an Einstein metric. This will be the aim of Sect. 6. But first we address the issue of convergence of H in odd dimensions.

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5. Convergence in Three Dimensions

In odd dimensions the solution to the Ricci-flat equation, H in (19), may be given by an infinite series. Since H contains only natural powers of ρ, general arguments as in [16] ensure that H converges for an analytic function h and is analytic as well, [21]. Here we give a simple argument that proves convergence for $n = 3$:

Proposition 2. *Let h be a function on $\mathbb{C} \times \mathbb{R}$ of variables (z, u) which is an entire holomorphic function in $z = x + iy \in \mathbb{C}$, is continuous in $u \in \mathbb{R}$, and is real for $z = x \in \mathbb{R}$. Then the series*

$$H(x, u, \rho) = \sum_{k=1}^{\infty} \frac{(\triangle^k h)(x, u)}{k! \prod_{i=1}^{k}(2i-3)} \rho^k \tag{20}$$

converges uniformly on compact subsets of $\mathbb{R}^3$.

Proof. Let $R > 1$ be a real number and let $C = \sup\{|h(z, u)|\}$ over all values of (z, u) such that $|z - x| \leq (R + 2\epsilon)$, $|u| \leq \nu > 0$, and $|x| \leq \epsilon > 0$. Then by the Cauchy-Schwarz inequality, the k^{th} derivative of h at every real point $(x, u) \in [-\epsilon, \epsilon] \times [-\nu, \nu]$ satisfies $|h^{(k)}(x, u)| \leq \frac{Ck!}{R^k}$. This provides the following estimate for the values of the powers of the Laplacian $\triangle^k h = \frac{d^{2k}h}{dz^{2k}}$:

$$\forall (x, u) \in [-\epsilon, \epsilon] \times [-\nu, \nu] \quad \textit{we have} \quad |(\triangle^k h)(x, u)| \leq \frac{C(2k)!}{R^{2k}}. \tag{21}$$

Now we rewrite (20) to the equivalent form

$$H = \rho \triangle h - \sum_{k=1}^{\infty} \frac{\triangle^{k+1} h}{(k+1)! \cdot 1 \cdot 3 \cdot \dots \cdot (2k-1)} \rho^{k+1}.$$

To show that H converges it is enough to show the convergence of the power series above. This can be done by using the estimate (21):

$$|\sum_{k=1}^{\infty} \frac{\triangle^{k+1} h}{(k+1)! \cdot 1 \cdot 3 \cdot \dots \cdot (2k-1)} \rho^{k+1}| \leq C \sum_{k=1}^{\infty} \frac{(2k+2)!}{(k+1)! \cdot 1 \cdot 3 \cdot \dots \cdot (2k-1)} \left(\frac{|\rho|}{R^2}\right)^{k+1}$$

$$= C \sum_{k=1}^{\infty} \frac{(2 \cdot 4 \cdot \dots \cdot 2k) \cdot (2k+1)(2k+2)}{(k+1)!} \left(\frac{|\rho|}{R^2}\right)^{k+1} = C \sum_{k=1}^{\infty} b_k \left(\frac{|\rho|}{R^2}\right)^{k+1}.$$

Since

$$\frac{|b_{k+1}|}{|b_k|} = \frac{2(k+1)(2k+3)(2k+4)}{(k+2)(2k+1)(2k+2)} \longrightarrow 2 \quad \text{as} \quad k \to \infty,$$

then this series converges for $|\rho| \leq \frac{R^2}{2}$. This finishes the proof.

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6. Bach Flat Metrics that are not Conformally Einstein

With Eq. (11) it is obvious how to obtain Bach-flat pp-waves. It is more difficult to find those that are not conformally Einstein. In this section we want to give examples of 4-dimensional pp-waves that are both Bach flat and not conformal to Einstein. But first we have to review some necessary conditions of being conformal to Einstein given in [20] for any dimension. In this section, when we write 'conformal to' we mean 'locally conformal to'.

From the formulae for the transformation of the Schouten tensor under conformal changes of the metric one obtains that a metric is conformal to an Einstein metric if and only if there exists a scaling function Υ such that

$$\mathsf{P} - \nabla \mathrm{d}\Upsilon + (\mathrm{d}\Upsilon)^2 \text{ is pure trace.} \tag{22}$$

In the following we write Y for the gradient of Υ. In [20, Prop. 2.1] the following necessary conditions for the metric to be conformal to Einstein were derived from Eq. (22):

$$C + W(Y, ., ., .) = 0, \tag{23}$$
$$B + (n-4)W(Y, ., ., Y) = 0. \tag{24}$$

Note that the first condition is satisfied for a gradient Y if and only if the metric is conformally equivalent to a metric with vanishing Cotton tensor, i.e. if it is *conformally Cotton-flat*. We further mention that the property of being conformally Cotton-flat is also neccessary for the metric to be conformally Einstein [20].

For a pp-wave conditions (23) and (24) are equivalent to the following:

Proposition 1. *If the pp-wave (6) is conformally Einstein but not conformally flat and $n > 3$, then there is a vector field Y on M, whose components $Y^i := \mathrm{d}x^i(Y)$, $i = 1, \ldots, n-2$, and $Y^{n-1} := \mathrm{d}u(Y)$ satisfy the equations*

$$\partial_i \Delta h - Y^i \Delta h + (n-2) \sum_{k=1}^{n-2} Y^k \partial_k \partial_i h = 0, \tag{25}$$

$$\Delta^2 h - (n-4)\Delta h \sum_{k=1}^{n-2} \left(Y^k\right)^2 + (n-2)(n-4) \sum_{k,l=1}^{n-2} Y^k Y^l \partial_k \partial_l h = 0, \tag{26}$$

for $i = 1, \ldots, n-2$, and

$$Y^{n-1} = 0. \tag{27}$$

Proof. Writing $Y = Y^k \partial_k + Y^{n-1}\partial_u + \mathrm{d}r(Y)\partial_r$, Eq. (23) and the formulae in Sect. 3 give

$$0 = Y^{n-1} W(\partial_u, \partial_i, \partial_u, \partial_j),$$
$$0 = \frac{\partial_i \Delta h}{n-2} + Y^k \left(\partial_k \partial_i h - \delta_{ki} \frac{\Delta h}{n-2}\right).$$

These, when $n > 3$, imply both $Y^{n-1} = 0$ and Eq. (25). Equation (24) gives that

$$0 = -\frac{\Delta^2 h}{n-2} - (n-4) Y^k Y^l \left(\partial_k \partial_l h - \delta_{kl} \frac{\Delta h}{n-2}\right),$$

which implies Eq. (26). □

Writing Y as the gradient of Υ,

$$Y = \sum_{k=1}^{n-2} \partial_k \Upsilon \partial_k + \partial_r \Upsilon \partial_u + (\partial_u \Upsilon - h \partial_r \Upsilon)\, \partial_r,$$

the proposition implies that $\mathrm{d}u(Y) = \partial_r \Upsilon = 0$. Hence,

$$\partial_r\, (\mathrm{d}r(Y)) = \partial_r\, (\partial_u \Upsilon - h \partial_r \Upsilon) = 0,$$

and we obtain

Corollary 1. *Let g be a pp-wave that is conformally Einstein but not conformally flat in dimension $n > 3$, and let Y be the gradient of the scaling function Υ satisfying Eq.* (22). *Then the function $Y^n = \mathrm{d}r(Y)$ does not depend on the r-variable.*

Example 1. For $n = 3$ a third order polynomial h in x with coefficients being functions of u defines a pp-wave with non-vanishing Cotton tensor. Hence, it is not conformally flat and therefore not conformally Einstein.

Example 2. Set $M = \mathbb{R}^n$ and $h = (x^1)^4 + \cdots + (x^{n-2})^4$. Then, $\partial_i \partial_j h \neq \delta_{ij} \frac{\Delta h}{n-2}$ on open sets in M and hence, g is not conformally flat. On the other hand, Eq. (26) can never be satisfied in $0 \in M$, because here all second order derivatives of h vanish, but $\Delta^2 h = 24(n-2)$. Thus, the pp-wave defined by $h = (x^1)^4 + \cdots + (x^{n-2})^4$ is not conformally Einstein.

Now we turn to dimension $n = 2s = 4$. Here the formula (19) makes sense only if $\triangle^2 h = 0$. In such case the formula truncates to $H = \frac{1}{2}\rho \Delta h$. Thus it is clear that for the 4-dimensional pp-waves the Fefferman-Graham obstruction is *precisely* $\triangle^2 h$, which is a multiple of the Bach tensor, and does not involve any lower order terms in the derivatives of the metric functions. In order to write down all such metrics, it is convenient to pass to the *complex notation* by introducing coordinates $z = \frac{x^1 + i x^2}{\sqrt{2}}$, $\bar{z} = \frac{x^1 - i x^2}{\sqrt{2}}$. In this notation the *most general* 4-dimensional pp-wave metric *satisfying* $\triangle^2 h = 0$ is given by

$$g_4 = 2\mathrm{d}u \left(\mathrm{d}r + \left(\bar{z}\alpha + z\bar{\alpha} + \beta + \bar{\beta}\right) \mathrm{d}u\right) + 2\mathrm{d}z\mathrm{d}\bar{z}.$$

Here $\alpha = \alpha(z, u)$, $\beta = \beta(z, u)$ are *holomorphic* functions of z. This metric is *Bach-flat*, and in *some* cases, such as when $a_z + \bar{\alpha}_{\bar{z}} = const$, is conformal to an Einstein metric. Its ambient metric is given by

$$\tilde{g}_4 = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 \left(2\mathrm{d}u[\mathrm{d}r + \left(\bar{z}\alpha + z\bar{\alpha} + \beta + \bar{\beta} - \rho(a_z + \bar{\alpha}_{\bar{z}})\right) \mathrm{d}u] + 2\mathrm{d}z\mathrm{d}\bar{z}\right),$$

and by construction is *Ricci flat*. We get

Proposition 2. *A* 4*-dimensional pp-wave g_4 is Bach flat if and only if*

$$g_4 = 2\mathrm{d}u \left(\mathrm{d}r + \left(\bar{z}\alpha + z\bar{\alpha} + \beta + \bar{\beta}\right) \mathrm{d}u\right) + 2\mathrm{d}z\mathrm{d}\bar{z},$$

with $\alpha = \alpha(z, u)$, $\beta = \beta(z, u)$ functions of a complex variable z and a real variable u which are holomorphic in z.

In general, this Bach-flat metric is *not* conformally Einstein:

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Theorem 3. *A* 4*-dimensional Bach-flat pp-wave*

$$g_4 = 2\mathrm{d}u\,(\mathrm{d}r + (\bar{z}\alpha + z\bar{\alpha})\,\mathrm{d}u) + 2\mathrm{d}z\mathrm{d}\bar{z} \tag{28}$$

with $\beta \equiv 0$ is conformally equivalent to a metric with vanishing Cotton tensor. Moreover, the following three properties are equivalent:

(1) $\partial_z^2\alpha \equiv 0$,
(2) *g_4 is conformally flat,*
(3) *g_4 is conformally Einstein.*

In particular, any such metric with $\partial_z^2\alpha \not\equiv 0$ is not conformally Einstein.

Proof. First, in the complex coordinates $(z, \bar{z})$ we have: $\Delta h = 2\,(\partial_z\alpha + \partial_{\bar{z}}\bar{\alpha})$. Next, using

$$\partial_1 = \frac{1}{\sqrt{2}}\,(\partial_z + \partial_{\bar{z}})\,, \quad \partial_2 = \frac{\mathrm{i}}{\sqrt{2}}\,(\partial_z - \partial_{\bar{z}})\,,$$

in the formula (9) we see that the Weyl tensor vanishes if and only if $\partial_z^2\alpha = 0$. This proves the equivalence of (1) and (2).

For the remaining statements we try to find a vector field Y that solves the necessary condition (23) for g to be conformally Einstein. We use this equation in the form (25), as in Proposition 1. Recall that in this proposition we proved that such a vector does not have a ∂_u-component. Thus we look for Y of the form

$$Y = F\partial_z + \overline{F}\partial_{\bar{z}} + f\partial_r,$$

where $F = F(z, \bar{z}, r, u)$ is a complex and $f = f(z, \bar{z}, r, u)$ is a real function. Equation (25) gives

$$0 = \partial_z^2\alpha\,(1 + \bar{z}F) + \partial_{\bar{z}}^2\bar{\alpha}\left(1 + z\overline{F}\right), \tag{29}$$

$$0 = \partial_z^2\alpha\,(1 + \bar{z}F) - \partial_{\bar{z}}^2\bar{\alpha}\left(1 + z\overline{F}\right), \tag{30}$$

which immediately implies

$$\partial_z^2\alpha\,(1 + \bar{z}F) = 0.$$

Assuming that g_4 is not conformally flat, i.e. $\partial_z^2\alpha \not\equiv 0$ we get

$$F(z) = -1/\bar{z}.$$

Thus we found that the vector Y solves (23) if and only if $Y = -\frac{1}{\bar{z}}\partial_z - \frac{1}{z}\partial_{\bar{z}} + f\partial_r$. Now, g_4 is conformally Cotton-flat if we find f such that this Y is a gradient. Setting

$$Y^\flat = g_4(Y, .) = -\frac{1}{z}\mathrm{d}z - \frac{1}{\bar{z}}\mathrm{d}\bar{z} + f\mathrm{d}u,$$

we see that Y is locally a gradient, i.e. $\mathrm{d}Y^\flat = 0$, if and only if f is a function of variable u alone. Every $f = f(u)$ gives a solution to the conformally Cotton equation.

To prove that (3) implies (2), assume that g_4 is not conformally flat but conformally Einstein. Then we plug in the vector $Y^\flat$ we have obtained as a solution of Eq. (25), and its corresponding

$$\nabla Y^\flat = \mathrm{d}f \otimes \mathrm{d}u - \left(\frac{\alpha + z\partial_{\bar{z}}\bar{\alpha}}{\bar{z}} + \frac{\bar{\alpha} + \bar{z}\partial_z\alpha}{z}\right)\mathrm{d}u^2 + \frac{1}{z^2}\mathrm{d}z^2 + \frac{1}{\bar{z}^2}\mathrm{d}\bar{z}^2$$

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into

$$\mathsf{P} - \nabla Y^\flat + (Y^\flat)^2.$$

According to Eq. (22) this must be a pure trace, if the metric g_4 is conformally Einstein. But this can not happen since $\mathsf{P} - \nabla Y^\flat + (Y^\flat)^2$ has a nowhere vanishing $\mathrm{d}z\mathrm{d}\bar{z}$-term given by $\frac{2}{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$, and an identically vanishing $\mathrm{d}r\mathrm{d}u$-term. Thus $\mathsf{P} - \nabla Y^\flat + (Y^\flat)^2$ is never proportional to g_4, which in turn, can not be conformally Einstein.

In the light of discussions in [20], the metrics (28) provide interesting examples because, apart from being Bach-flat, they are conformally Cotton-flat, but *not* conformally Einstein even though the necessary conditions (23) and (24) are both satisfied for a gradient. This phenomenon is special to Lorentzian and probably to other indefinite signature metrics.

We strongly believe that a similar argument works in any dimension, even though one might not be able to describe the functions with $\Delta^s h = 0$. But under certain assumptions it might be possible to deduce a contradiction between Eq.'s (25) – (26) and the fact that the function $\mathrm{d}r(Y)$ is independent of the r-coordinate as it occurs for $n = 4$.

We want to conclude this section by returning to the result of Brinkmann in [8] mentioned in the Introduction. If a 4-dimensional pp-wave is Einstein, and hence Ricci-flat, the function h is given by $\alpha + \overline{\alpha}$ for a holomorphic function α. Again, this metric is conformally flat if and only if $\partial_z^2 \alpha = 0$. If it is not conformally flat but conformally Einstein, then the vector field Y is null and a multiple of ∂_r, namely $Y = f\partial_r$ with a function $f = f(u)$ that depends on the variable u only. As $\mathsf{P} = 0$, Eq. (22) then is equivalent to $f' = f^2$. Hence, any such function yields a conformal rescaling of a Ricci-flat pp-wave to another Einstein metric that is in fact Ricci-flat. The new metric may be isometric to the original one but in general this is not the case (see also [14]). Finally, note that a non-trivial solution of $f' = f^2$ is not defined on all of $\mathbb{R}$, and thus, in general, f does *not yield a global* rescaling to another Einstein metric.

7. The Critical Q-Curvature of a pp-Wave

For a semi-Riemannian manifold of (M, g) even dimension $n = 2s$, in [7] T. Branson introduced a series $\{Q_{2k}\}_{k=1\ldots s}$ of scalar invariants constructed from the curvature tensor involving $2k$ derivatives of the metric[3]. As such, for a pp-wave all Q_{2k} are zero. This follows from the general fact that all scalar invariants constructed from the Riemannian curvature tensor of a pp-wave vanish (for a proof in arbitrary dimension see [10]). However, as an application of Theorem 1, in this section we will use the pp-wave ambient metric in order to show that the *critical Q-curvature* Q_n of a pp-wave vanishes. The so-called *subcritical* Q-curvatures $Q_2, \ldots, Q_{n-2}$ are defined by the inhomogeneous part of the GJMS-operators P_{2k}, namely

$$P^g_{2k}(1) = (s-k)Q_{2k}.$$

The GJMS-operators P_{2k} introduced in [23] are conformally covariant operators. We will not give a definition of the *critical* Q-curvature Q_n here (please refer to [17], for example). Instead we will explain a formula for the critical Q-curvature given in [24] that expresses it in terms of the volume of the Poincaré metric.

[3] Regarding this section, we would like to thank Andreas Juhl for explaining to us some facts about Q-curvature.

Let $(M, [g])$ be a smooth manifold of even dimension $n = 2s$ with conformal class $[g]$. To this manifold one can assign a Poincaré metric g_+. g_+ is a metric on $M_+ = M \times (0, a)$ given by

$$g_+ = \frac{1}{x^2}\left(\mathrm{d}x^2 + g_x\right),$$

where g_x is a 1-parameter family of metrics with the same signature as g and with initial condition $g_0 = g$ such that g_+ is asymptotically Einstein, which means that $Ric(g_+) + ng_+$ vanishes up to terms of order $(n-2)$ in x. The Poincaré-metric is unique up to addition of terms of the form $x^n S_x$, where S_x is a 1-parameter family of symmetric $(2, 0)$-tensors such that S_0 is trace-free (for details see [15,16]). For a Poincaré metric one can show, see [22] for details, that $\sqrt{\det(g_x)/\det(g)}$ has the Taylor expansion

$$\sqrt{\frac{\det(g_x)}{\det(g)}} = 1 + v^{(2)}x^2 + v^{(4)}x^4 + \cdots + v^{(n-2)}x^{n-2} + v^{(n)}x^n + \cdots, \tag{31}$$

defining smooth functions $v^{(2k)}$. Then in [24] it is shown that the critical Q-curvature Q_n of $(M, [g])$ is given as

$$2nc_{\frac{n}{2}} Q_n = nv^{(n)} + \sum_{k=1}^{s-1}(n-2k)\mathcal{A}_{2k}^{*}v^{(n-2k)}. \tag{32}$$

Here $\mathcal{A}_{2k}$ are the linear differential operators that appear in the expansion of a harmonic function for a Poincaré-metric, the star denotes the formal adjoint, and $c_{\frac{n}{2}}$ is a constant.

Furthermore, one has to recall how the Poincaré-metric can be obtained by the ambient metric. Assume that

$$\widetilde{g} = 2\mathrm{d}(\rho t)\mathrm{d}t + t^2 g(\rho)$$

is a pre-ambient metric for $[g]$ that is Ricci-flat up to terms of order s and higher. Such a metric always exists and is unique up to terms of order $n/2$ in ρ. Now, on

$$M_+ = \{(\rho, p, t) \in \widetilde{M} \mid p \in M, t^2\rho = -1\},$$

the Poincaré-metric is given by

$$g_+ = \frac{1}{x^2}\left(\mathrm{d}x^2 + \frac{1}{2}g(x^2)\right).$$

Note that if the pre-ambient metric is Ricci-flat, then the Poincaré-metric obtained in this way is Einstein. We can use the ambient metric of a pp-wave to prove

Theorem 4. *The critical Q-curvature of an even-dimensional pp-wave vanishes.*

Proof. Let (M, g) be a pp-wave of even dimension $n = 2s$. In Sect. 4 we have also shown that its pre-ambient metric that is Ricci-flat up to terms of order $n/2$ is given by formula (12) with H as in (19). Using the coframe in (14) we can write down the volume form $\omega(\rho)$ of the ρ-dependent family of pp-waves,

$$g(\rho) = 2\mathrm{d}u\ \ (\mathrm{d}r + (h + H)\mathrm{d}u)\ \ + \ \sum_{i=1}^{n-2}(\mathrm{d}x^i)^2,$$

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namely

$$\omega(\rho) = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^{n-2} \wedge (\mathrm{d}r + (h + H)\mathrm{d}u) \wedge \mathrm{d}u = \omega(0).$$

For the family $g_x = \frac{1}{2}g(x^2)$ defining the Poincaré metric this implies that $\det(g_x) = \det(g_0)$. Hence, all the $v^{(2k)}$ in (31) are zero and so is the critical Q-curvature by the result of [24] given in formulae (32). □

Recall that for a pp-wave (M, g) the vanishing of the scalar curvature implies that the Laplacian Δ_g is conformally covariant. Calculations using formulae in [26] show that the first GJMS-operators P_2, P_4 and P_6 are equal to the corresponding powers of the Laplacian Δ_g, Δ_g^2 and Δ_g^3. We conjecture that for pp-waves this is also the case for the higher P_{2k}.

8. Conformal and Ambient Holonomy

We conclude with a brief remark about the holonomy of the ambient metric and the holonomy of the normal conformal Cartan connection, also called the *conformal holonomy*, of a pp-wave. Holonomy groups describe the reduction of generic structures down to more special structures, in the semi-Riemannian, the conformal, and in other geometric settings. For a conformal manifold of signature (r, s) the conformal holonomy is contained in $\mathrm{SO(r + 1, s + 1)}$. If it is a proper subgroup, then the conformal structure is reduced to a more special structure. Examples are Lorentzian Fefferman spaces, for an overview see [1], where the conformal holonomy reduces to the special unitary group, or conformal structures in signature $(2, 3)$ with non-compact G_2 as structure group, [38,39].

In [31] it is proven that the conformal holonomy of an n-dimensional Lorentzian conformal class that is given by a metric with parallel null line and totally null Ricci tensor is contained in the stabiliser in $\mathrm{SO(2, n)}$ of a totally null plane $\mathcal{N}$. Of course, pp-waves are special examples of such metrics and hence, their conformal holonomy reduces to this stabiliser. But we get the same result also for the holonomy of the ambient metric of a pp-wave.

Proposition 3. *The metric $\overline{g}$ defined in Eq. (12) admits a holonomy invariant distribution of totally null planes $\mathcal{N}$ spanned by ∂_r and ∂_ρ. In particular, all curvature operators $\bar{R}(V, W)$, $V, W \in T\bar{M}$, leave invariant the fibres of $\mathcal{N}$ and of $\mathcal{N}^\perp$, which is spanned by ∂_r, ∂_ρ, and ∂_i.*

Proof. The easiest way to see this is to consider the dual frame to the co-frame in (14) given by

$$E_0 = \frac{1}{t}\partial_\rho,\ E_i = \frac{1}{t}\partial_i,\ E_{n-1} = \frac{1}{t^2}\partial_r,\ E_n = \partial_u - (h + H)\partial_r,\ E_{n+1} = \partial_t - \frac{\rho}{t}\partial_\rho.$$

Using the relation $\bar{g}(\bar{\nabla} E_\mu, E_\nu) = \Gamma_{\mu\nu}$ one can read off from the formulae for the connection 1-forms in (15) that

$$\mathcal{N} = \mathrm{span}(E_0, E_{n-1}) = (\mathrm{span}(E_0, E_i, E_{n-1}))^\perp$$

is invariant under the Levi-Civita connection. □

Corollary 2. *Let G be the holonomy group of the ambient metric of a pp-wave in odd dimension or in dimension $2s$ with $\Delta^s h = 0$. Then G is contained in the stabiliser in* $\mathrm{SO}(2, \mathrm{n})$ *of a totally null plane in* $\mathbb{R}^{2,n}$.

In general, it is possible to show that the conformal holonomy is always contained in the ambient holonomy [33]. For a conformal class with an Einstein-metric or a Ricci-flat metric both holonomy groups are the same [31,34]. For a pp-wave, not necessarily conformal Einstein, we have just seen that both are contained in the isotropy group of a totally null plane. Hence, it is very likely that the conformal holonomy is actually *equal* to the ambient holonomy. But to give a proof of this is beyond the scope of this paper.

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