Conformal transformations and the beginning of the Universe. Part II.

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Null geodesics as conformal objects

- Two **spacetimes**[1] (M, g) and $(\hat{M}, \hat{g})$ are **conformally related** iff there exists a diffeomorphism $\phi : M \to \hat{M}$ such that $g = \mathrm{e}^{2\Upsilon} \cdot \phi^*(\hat{g})$, with Υ a differentiable function on M.
- In the **index notation**:
  - the **metric** is $\hat{g}_{\mu\nu} = \mathrm{e}^{-2\Upsilon} g_{\mu\nu}$, the **inverse metric** is $\hat{g}^{\mu\nu} = \mathrm{e}^{2\Upsilon} g^{\mu\nu}$, and the **Levi-Civita connection** coefficients are related by $\boxed{\hat{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} - \delta^{\mu}{}_{\nu}\Upsilon_{\rho} - \delta^{\mu}{}_{\rho}\Upsilon_{\nu} + g_{\nu\rho}\Upsilon^{\mu}}$, where $\Upsilon_{\mu} = \Upsilon_{,\mu}$ and $\Upsilon^{\mu} = g^{\mu\nu}\Upsilon_{\nu}$.
- In this way the geodesic equation for a curve $x^{\mu} = x^{\mu}(t)$ is:

$\boxed{\frac{\mathrm{d}\dot{x}^{\mu}}{\mathrm{d}t} + \Gamma^{\mu}{}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = \lambda\dot{x}^{\mu}}$, or if we replace Γ by $\hat{\Gamma}$, is:

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- This shows that a **null**, i.e. satisfying $g(\dot{x}, \dot{x}) = 0$, **geodesic** in metric g is also a **null geodesic** in the metric $\hat{g}$.

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Conformal compactifications of spacetimes were introduced by **Roger Penrose** to correctly define boundary conditions for the theory of **gravitational radiation.**

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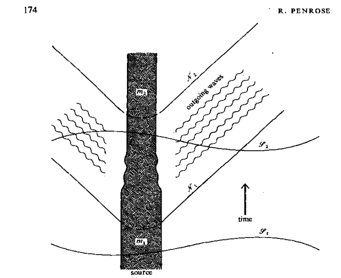

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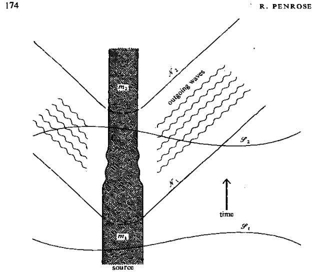

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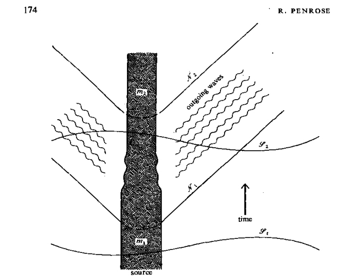

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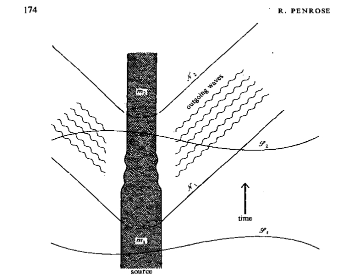

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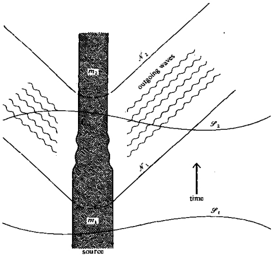

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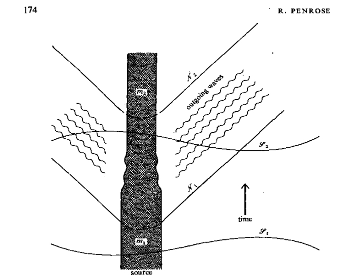

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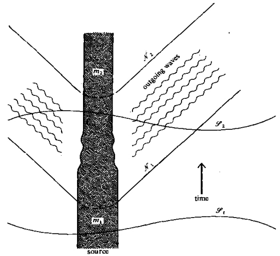

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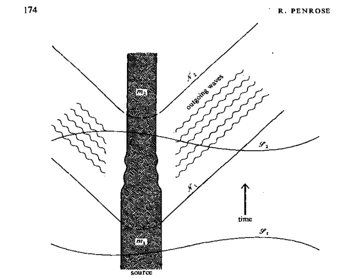

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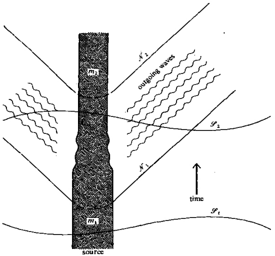

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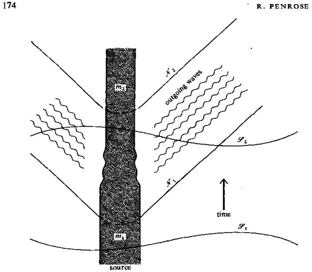

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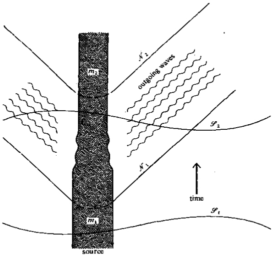

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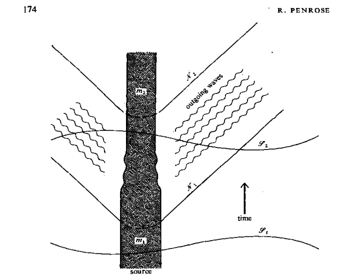

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$$\phi : M \to \mathrm{Int}\hat{M}$$

and a function Ω on $\hat{M}$, such that (i) $\hat{g} = \Omega^2 \phi_*(g)$, and (ii) $\Omega = 0$ on $\partial\hat{M}$, and (iii) $\mathrm{d}\Omega \neq 0$ at $\partial\hat{M}$.

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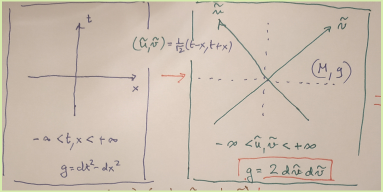
We say that a 4-dimensional Lorentzian manifold $(\hat{M}, \hat{g})$ with **boundary** $\partial\hat{M}$ is a **conformal compactification** of a spacetime (M, g) iff there exists a diffeomorphism

$$\phi : M \to \mathrm{Int}\hat{M}$$

and a function Ω on $\hat{M}$, such that (i) $\hat{g} = \Omega^2 \phi_*(g)$, and (ii) $\Omega = 0$ on $\partial\hat{M}$, and (iii) $\mathrm{d}\Omega \neq 0$ at $\partial\hat{M}$.

2-dimensional Minkowski space

- In $M = \mathbb{R}^2$ with the Minkowski metric $g = \mathrm{d}t^2 - \mathrm{d}x^2$, change coordinates to $\tilde{u} = (t - x)/\sqrt{2}$ and $\tilde{v} = (t + x)/\sqrt{2}$. This parametrizes M by $-\infty < \tilde{u}, \tilde{v} < +\infty$, and the Minkwski metric is $g = 2\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}$.
- Change coordinates in M from $(\tilde{u}, \tilde{v})$ to (u, v) such that $\tilde{u} = \operatorname{tg} u$ and $\tilde{v} = \operatorname{tg} v$. This transforms the entire $M = \mathbb{R}^2$, in a one-to-one fashion, onto the **interior** of a **diamond** $\operatorname{Int}\hat{M} = \{(u, v) \in \mathbb{R}^2 \ : \ -\pi/2 < u, v < \pi/2\}$.



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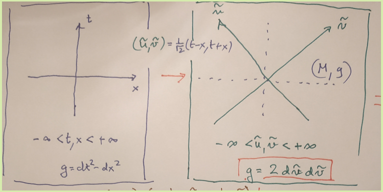

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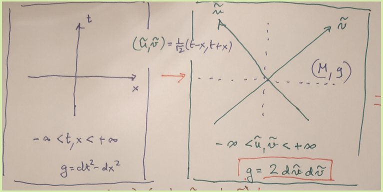

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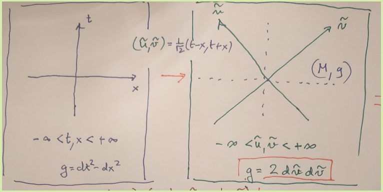

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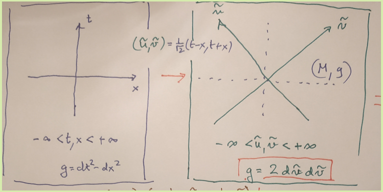

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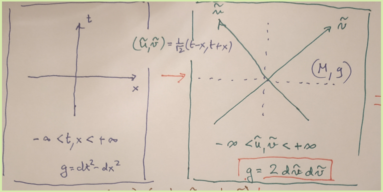

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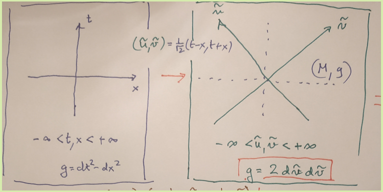

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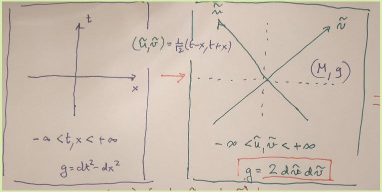

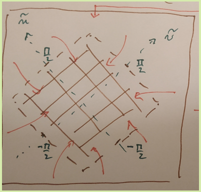

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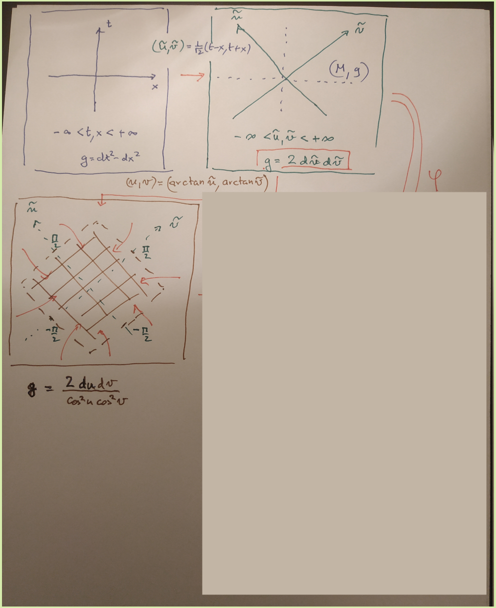

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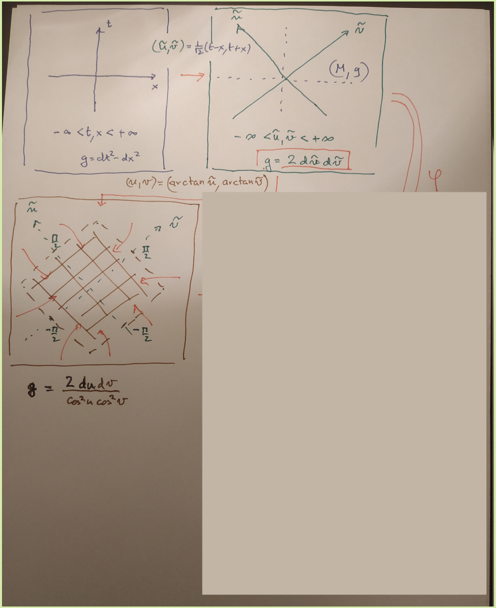

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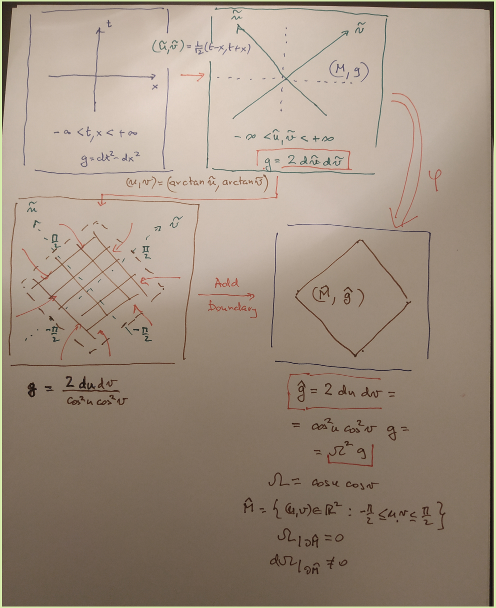

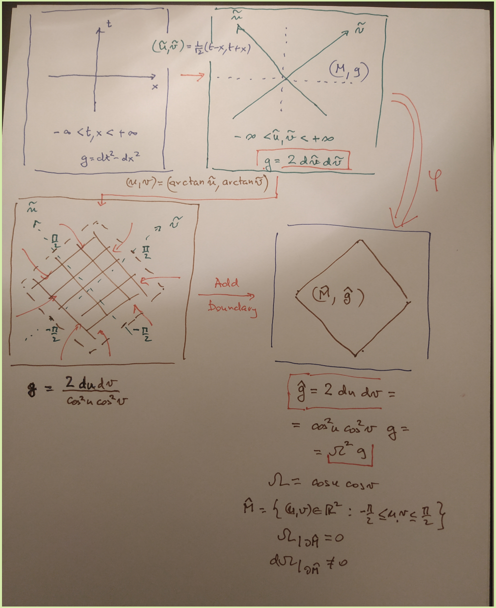
t
x
$-\infty < t, x < +\infty$
$g = dt^2 - dx^2$
$(\tilde{u}, \tilde{v}) = \frac{1}{\sqrt{2}}(t-x, t+x)$
(M, g)
$-\infty < \tilde{u}, \tilde{v} < +\infty$
$g = 2\, d\tilde{u}\, d\tilde{v}$
$(u, v) = (\arctan \tilde{u}, \arctan \tilde{v})$
φ
$\frac{\pi}{2}$
$-\frac{\pi}{2}$
Add boundary
$(\hat{M}, \hat{g})$
$g = \frac{2\, du\, dv}{\cos^2 u \cos^2 v}$
$\hat{g} = 2\, du\, dv =$
$= \cos^2 u \cos^2 v \; g =$
$= \Omega^2 g$
$\Omega = \cos u \cos v$
$\hat{M} = \{ (u,v) \in \mathbb{R}^2 : -\frac{\pi}{2} \leq u, v \leq \frac{\pi}{2} \}$
$\Omega|_{\partial \hat{M}} = 0$
$d\Omega|_{\partial \hat{M}} \neq 0$

- The compactified 2D Minkowski space $\hat{M} = \{(u,v) \; : -\frac{\pi}{2} \leq u, v \leq \frac{\pi}{2}\}$ has a boundary $\partial\hat{M}$ with the following components:
  - $\mathscr{I}^+ = \{(u,v) \; : \; u = \frac{\pi}{2}, -\frac{\pi}{2} < v < \frac{\pi}{2}\}$ or $\mathscr{I}^+ = \{(u,v) \; : \; -\frac{\pi}{2} < u < \frac{\pi}{2}, v = \frac{\pi}{2},\}$- **null infinity in the future**;
  - $\mathscr{I}^- = \{(u,v) \; : \; u = -\frac{\pi}{2}, -\frac{\pi}{2} < v < \frac{\pi}{2}\}$ or $\mathscr{I}^- = \{(u,v) \; : \; -\frac{\pi}{2} < u < \frac{\pi}{2}, v = -\frac{\pi}{2},\}$- **null infinity in the past**;
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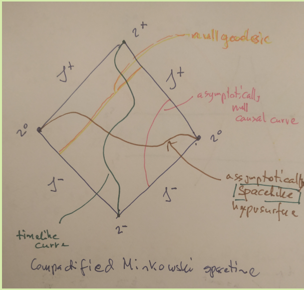

In particular i^0 is a point in which every **spacelike** hypersurfcae ends, similarly i^- is a point where every **initially timelike curve starts**, and i^+ is a point where every **finally timelike curve ends**.

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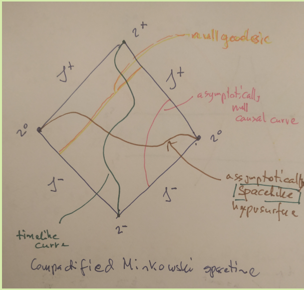

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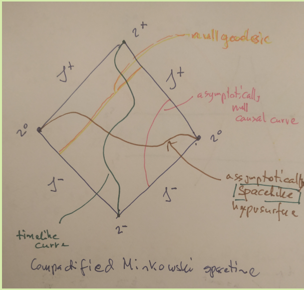

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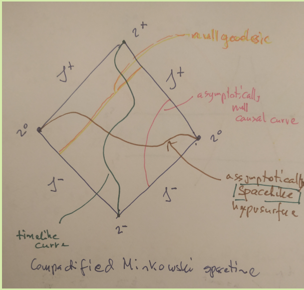

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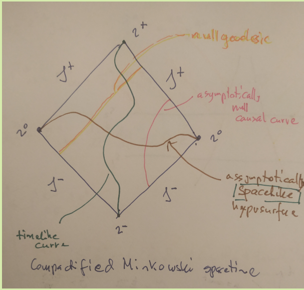

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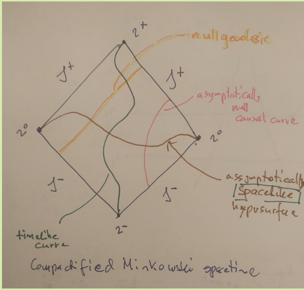

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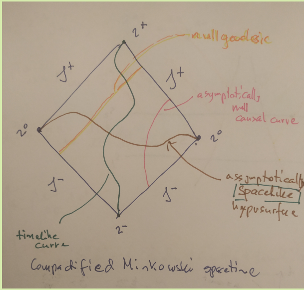

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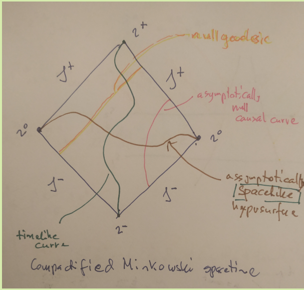

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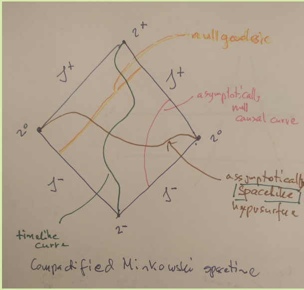

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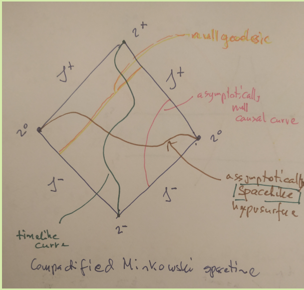

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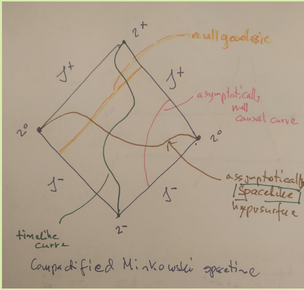

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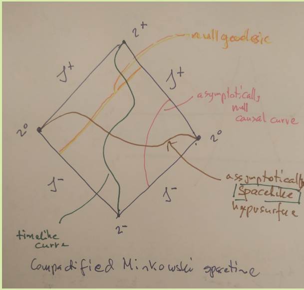

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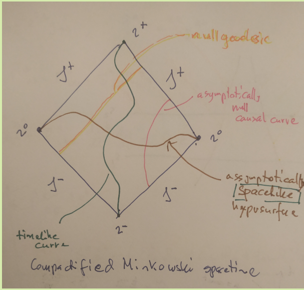

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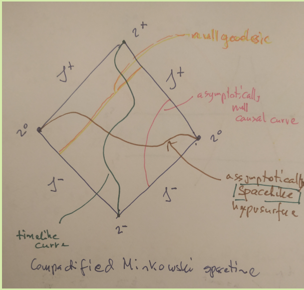

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- Now the range of coordinates (v, u) is $-\pi/2 \leq v, u \leq \pi/2$ and $v - u \geq 0$, so that the resulting picture of the conformally compactified Minkowski space with the regular metric $\hat{g} = 2\mathrm{d}u\mathrm{d}v - \frac{1}{2}\sin^2(v-u)\mathrm{d}s^2$ is as follows:

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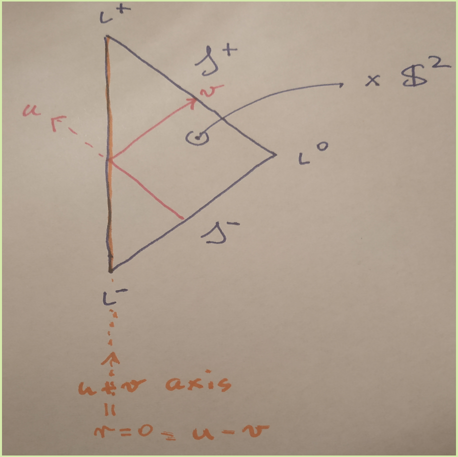
Penrose diagram for 4d Minkowski



Note that both $\mathscr{I}^\pm$ are **null** hypersurfaces.
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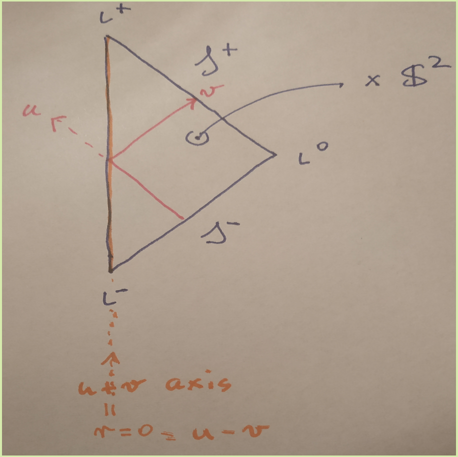

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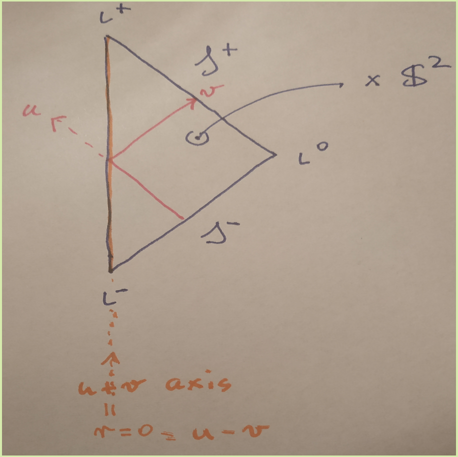

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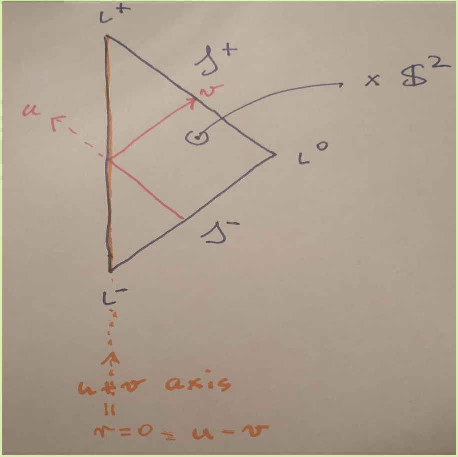

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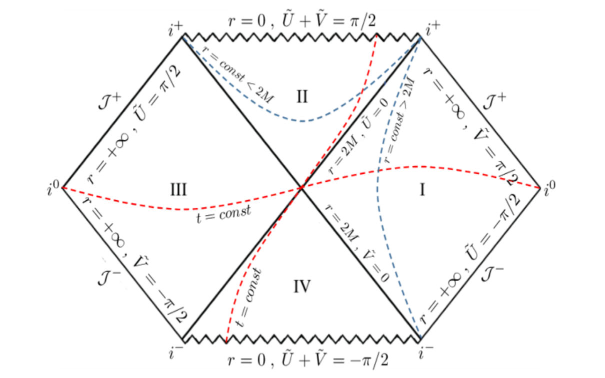

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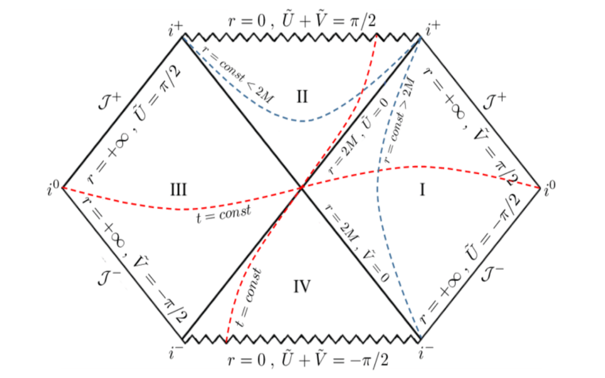

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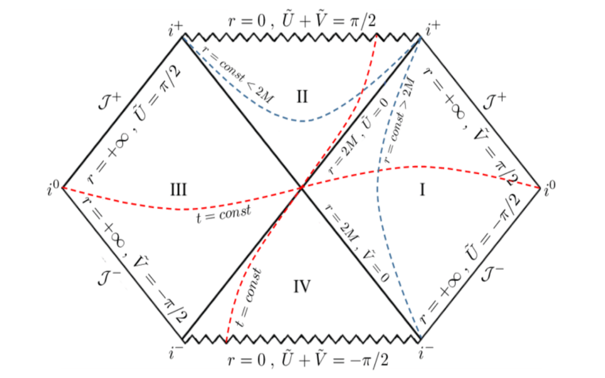

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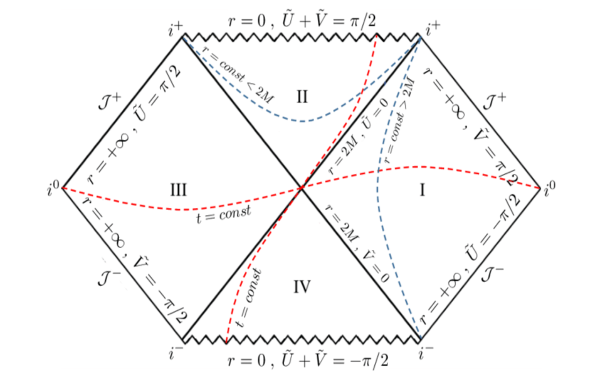

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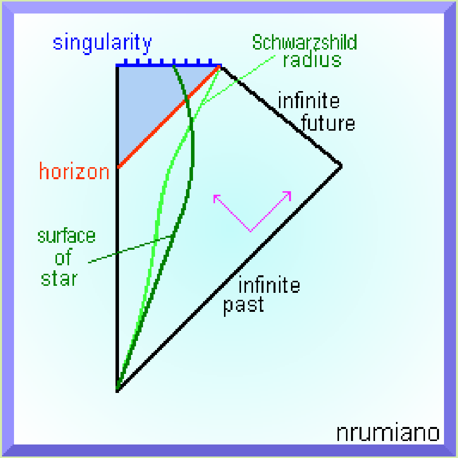

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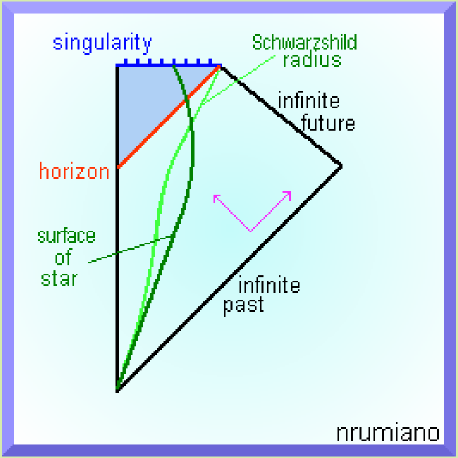

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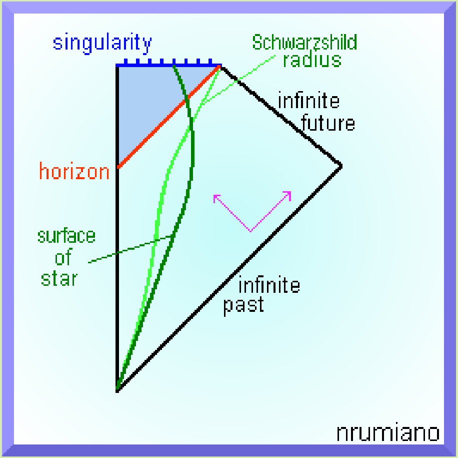

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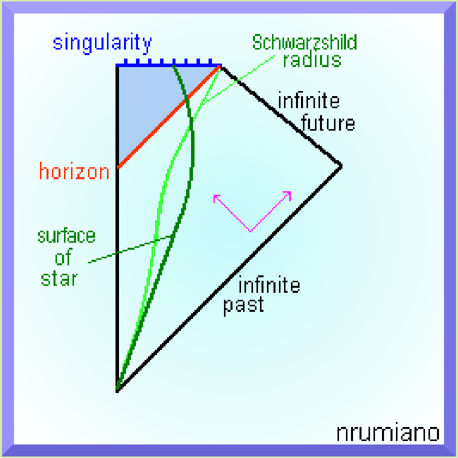

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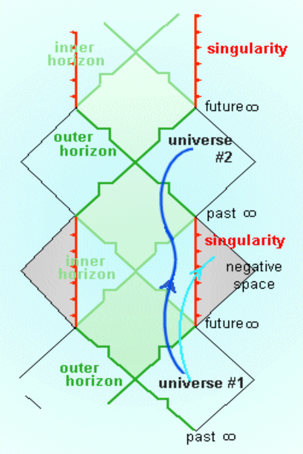

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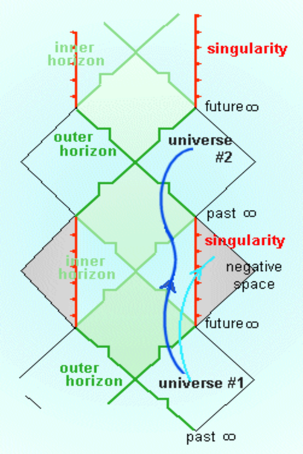

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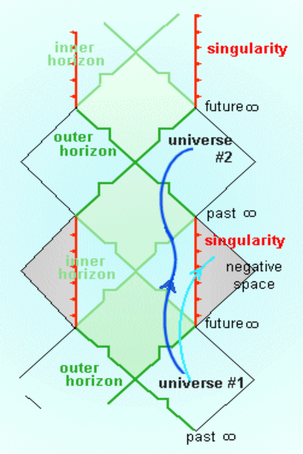

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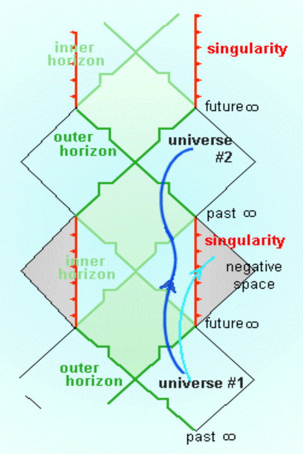

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- The deSitter space as a global manifold can be identified with a **quadric** Q in $\mathbb{R}^5$ given by the equation
$$-T^2 + X^2 + Y^2 + Z^2 + W^2 = \frac{1}{H}^2 = \mathrm{const}.$$
- It acquires a Lorentzian metric from the 5D Minkowski metric $g = -\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 + \mathrm{d}W^2$ in $\mathbb{R}^5$.
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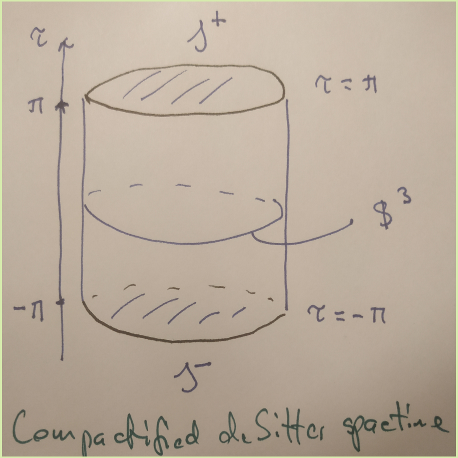
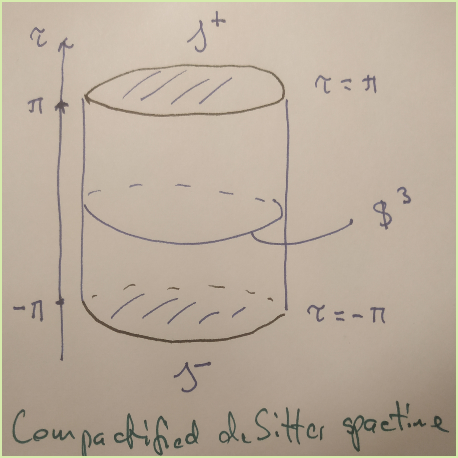
Compactified deSitter space





τ
$\mathscr{J}^+$
π
τ = π
S^3
−π
τ = −π
$\mathscr{J}^-$
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One can check that the **conformally flat** deSitter spacetime satisfies **vacuum** Einstein's equations with **positive** cosmological constant Λ. Actually the deSitter metric

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Definition

A smooth spacetime M with metric g is **asymptotically simple** if there is a smooth manifold $\hat{M}$ with boundary $\mathscr{I}$ and a metric $\hat{g}$ and a smooth scalar function Ω such that

- $M = \mathrm{Int}\hat{M}$,
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- every null geodesic in M has a future and a past endpoint on $\mathscr{I}$.

The last condition is too strong to include spacetimes with **black holes.** One releases it introducing a notion of a **weakly assymptotically simple spacetime** (WAS). We will not consider it here, but of course assymptotically simple spacetime is WAS. We have

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The last condition is too strong to include spacetimes with **black holes.** One releases it introducing a notion of a **weakly assymptotically simple spacetime** (WAS). We will not consider it here, but of course assymptotically simple spacetime is WAS. We have

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A smooth spacetime M with metric g is **asymptotically simple** if there is a smooth manifold $\hat{M}$ with boundary $\mathscr{I}$ and a metric $\hat{g}$ and a smooth scalar function Ω such that

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Theorem

The boundary $\mathscr{I}$ of a **(weakly) assymptotically simple spacetime**satisfying Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},$$

with $T^{\mu}{}_{\mu} = 0$ in the vicinity of $\mathscr{I}$, is

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Two words about the proof of the Theorem

- Using the transformation for the Levi-Civita connection coefficients for the metric $\hat{g}$ and g of a WAS spacetime, one gets the following relation between the Ricci scalars $\hat{R}$ and R:
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