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LETTER TO THE EDITOR

Covariant definition of inertial forces

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Abstract. We present a covariant definition of inertial forces in general relativity (gravitational, centrifugal, Euler and Coriolis–Lense–Thirring) which is valid in all spacetimes, including ones with no symmetry.

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The profound problem of the origin of inertia has been studied by many physicists and philosophers, including Newton, Leibnitz, Mach, Einstein and more recently Sciama, Wheeler and others. However, the obvious practical question of how to split inertia into the different kinds of inertial force (gravitational, centrifugal, Coriolis and Euler) attracted serious attention only very recently after Abramowicz, Carter, and Lasota [1] (hereafter ACL) demonstrated that, by considering separately the different kinds of inertial force, one could gain an improved understanding of relativistic dynamics. The ACL approach was later shown to be quite useful for studying astrophysical problems involving rotating matter in strong gravitational fields (see [2] for references).

ACL defined the different kinds of inertial force for spacetimes which are static, i.e. which admit a hypersurface-orthogonal timelike Killing vector field η^i,

$$\eta_{[i}\nabla_j\eta_{k]} = 0 \qquad \nabla_{(i}\eta_{k)} = 0. \tag{1}$$

One can easily check by making the direct substitution

$$n^i = \mathrm{e}^{-\Phi}\eta^i \qquad \Phi = \tfrac{1}{2}\ln(-\eta^i\eta_i) \tag{2}$$

that (1) implies that there exists a timelike and hypersurface-orthogonal unit vector n^i, for which the corresponding four-acceleration is equal to the gradient of a scalar function:

$$n^k n_k = -1 \qquad n_{[i}\nabla_j n_{k]} = 0 \qquad \dot{n}_k \equiv n^i\nabla_i n_k = \nabla_k\Phi. \tag{3}$$

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Conditions (1) are very restrictive, and it is therefore quite surprising that it turns out to be possible to fulfill conditions (3) even in the most general spacetimes. To see this, one writes the metric of an arbitrary spacetime in the form

$$g = -\mathrm{e}^{-2\Phi}\left(\frac{\partial}{\partial t}\right)^2 + 2g^{t\mu}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x^\mu}\right) + g^{\mu\nu}\left(\frac{\partial}{\partial x^\mu}\right)\left(\frac{\partial}{\partial x^\nu}\right) \tag{4}$$

where $\mu, \nu = 1, 2, 3$. It is known that one can *always* transform (4) to Gaussian (not in general normal) coordinates in which $\Phi = 0$ (see *e.g.* [3]). Here we demand a weaker condition, namely that $\nabla^t\Phi = g^{ti}\nabla_i\Phi$. If this condition is satisfied, than by a direct substitution one can check that the vector n^i defined by the corresponding 1-form

$$n_i \mathrm{d}x^i = \mathrm{e}^{\Phi}\mathrm{d}t \tag{5}$$

indeed obeys conditions (3). From this and from the demonstration by Abramowicz [4] that conditions (3) suffice to derive all of the ACL results, we deduce in this letter a covariant definition of the different kinds of inertial force which is valid for any spacetime.

The vector field n^i is not uniquely determined by conditions (3). However, locally each particular choice of n^i uniquely defines a foliation of the spacetime into slices, each of which represents space at a particular instant of time and will be denoted here by $\mathcal{S}(t)$. The geometry of space on each $\mathcal{S}(t)$ is given by the metric

$$h_{ik} = g_{ik} + n_i n_k \qquad h^i{}_k = \delta^i{}_k + n^i n_k. \tag{6}$$

It will be convenient to consider on each $\mathcal{S}(t)$ a conformally adjusted metric $\tilde{h}_{ik}$

$$\tilde{h}_{ik} = \mathrm{e}^{-2\Phi}h_{ik} \tag{7}$$

and the covariant derivative $\tilde{\nabla}_i$ corresponding to it. It will also be convenient to introduce a vector

$$\eta^i = \mathrm{e}^{\Phi}n^i. \tag{8}$$

In a static spacetime η^i is a timelike Killing vector (cf equation (2).)

Consider now the 4-velocity u^i of a test particle (with rest mass m) and write its *unique* decomposition

$$u^i = \gamma(n^i + v\tau^i). \tag{9}$$

Here τ^i is a unit vector orthogonal to n^i and v is the speed. Let $\tilde{\tau}^i$ and $\tilde{\tau}_i$ denote the contravariant and covariant components of the *unit vector parallel to* τ^i in the conformal geometry $\tilde{h}_{ik}$. One has

$$\tilde{\tau}^i = \mathrm{e}^{\Phi}\tau^i \qquad \tilde{\tau}_i = \mathrm{e}^{-\Phi}\tau_i. \tag{10}$$

After a considerable amount of quite tedious but simple algebra, the projection of the 4-acceleration can be written in the form

$$a^{\perp}_k \equiv h^j{}_k u^i \nabla_i u_j = \nabla_k\Phi + (\gamma v)^2\tilde{\tau}^k\tilde{\nabla}_k\tilde{\tau}_i + \gamma^2 v X_k + \dot{V}\tilde{\tau}_k. \tag{11}$$

We have introduced the notation

$$X_k = n^i(\nabla_i \tau_k - \nabla_k \tau_i) \qquad \dot{V} = u^i \nabla_i(-\mathcal{E} v) \qquad -\mathcal{E} = \eta^i u_i. \tag{12}$$

The quantity $\mathcal{E} = -e^{\Phi}\gamma$ corresponds to the specific energy of the particle and is conserved for geodesic motion in stationary spacetimes.

Our acceleration formula (10) is identical in form to the corresponding formula derived by ACL for the case of a static spacetime. We can therefore use the same arguments as used by ACL (and in subsequent papers) to justify physical reasons for the following, obviously covariant, definitions of the various inertial forces:

$$\text{Gravitational force}: \quad G_k = -m\nabla_k \Phi \tag{13}$$

$$\text{Coriolis–Lense–Thirring force}: \quad C_k = -m\gamma^2 v X_k \tag{14}$$

$$\text{Centrifugal force}: \quad Z_k = -m(\gamma v)^2 \tilde{\tau}^i \tilde{\nabla}_i \tilde{\tau}_k \tag{15}$$

$$\text{Euler force}: \quad E_k = -m\dot{V}\tilde{\tau}_k. \tag{16}$$

Additional arguments and features which have not been discussed before and are worth pointing out here include the following.

(a) All of the forces are defined in the *comoving frame* of the particle (with 4-velocity u^i). In contrast, the velocity of the particle in space, $v^i = h^i{}_k u^k = \gamma v \tau^i$, is measured with respect to the *reference frame* given by the vector field n^i. When $u^i \equiv n^i$, the comoving frame and the reference frame coincide. In this case $v \equiv 0$ and the only non-vanishing inertial force is the gravitational force.

(b) In a general situation, the gravitational force is due to the acceleration of the reference frame as indeed is required by Einstein's equivalence principle. The gravitational acceleration is the same for all free-falling test bodies (Galileo's result).

(c) Gravitational acceleration is equal to the gradient of the function Φ which does not depend on the intrinsic properties of test particles or on their velocities. For this reason Φ should be considered as a generalization of the Newtonian gravitational potential. Several authors have reached the same conclusion when discussing particular cases (see, for example, [5]).

(d) The Coriolis–Lense–Thirring force (14) has no Newtonian counterpart. In order to determine the vector $n^i \nabla_i \tau_k$ in (12), one must postulate how τ_k varies along n^i because the definition (7) determines τ_k only along the world line of the particle. This can be done in several different ways (different gauges). Any particular one extends τ_k to a timelike surface $\mathcal{T}$. The trajectory of the particle in space could be defined as the intersection $\mathcal{T} \cap \mathcal{S}(t)$. Independently of the choice of gauge, the Coriolis–Lense–Thirring force vanishes when the vorticity of the particle's trajectory in space is orthogonal to n^i.

The gauge used by ACL assures that the Lie derivative of τ^i with respect to η^i vanishes,

$$\mathcal{L}_\eta \tau_k \equiv \eta^i \nabla_i \tau_k - \tau^i \nabla_i \eta_k = 0. \tag{17}$$

and with this gauge the Coriolis force equals

$$C_k = -m e^{-\Phi} \tau^i (\nabla_i \eta_k + \nabla_k \eta_i) \tag{18}$$

i.e. it vanishes in static spacetimes. It may be convenient to introduce the gauge

$$n^i(\nabla_i \tau_k - \nabla_k \tau_i) = 0 \tag{19}$$

because with this, the Coriolis–Lense–Thirring force vanishes in *every* spacetime and so all of the results derived by the ACL approach for static spacetimes which follow from $C_k = 0$ may be repeated with the gauge (18) in the most general case.

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