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Symmetric (2,3,5) distributions, an interesting ODE of 7th order and Plebański metric[☆]

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ABSTRACT

We show that the unique 7th order ODE having 10 contact symmetries appears naturally in the theory of generic 2-distributions in dimension five.

1. Introduction

Recently Dunajski and Sokolov have found [1] a general solution to an interesting 7th order ODE:

$$10{y^{(3)}}^3 y^{(7)} - 70{y^{(3)}}^2 y^{(4)} y^{(6)} - 49{y^{(3)}}^2 {y^{(5)}}^2 + 280 y^{(3)} {y^{(4)}}^2 y^{(5)} - 175{y^{(4)}}^4 = 0.$$

This equation can be characterized as a unique (modulo contact transformations of variables) 7th order ODE which has a 10-dimensional group of local contact symmetries [2–4]. As mentioned by Dunajski and Sokolov, this equation was known already in 1904 by Noth. Since we cannot find any earlier reference to this equation, we will call this *Noth's equation* in the following.

In this short note, we show that Noth's equation also turns out to be a natural geometric condition for a certain class of generic 2-distributions in dimension five.

We say that a 2-distribution $\mathcal{D} = \mathrm{Span}(X_4, X_5)$, where X_4 and X_5 are two vector fields on a 5-dimensional manifold M, is *generic*, or (2,3,5) on M, if the system of five vector fields

$$(X_1, X_2, X_3, X_4, X_5) = ([X_5, [X_4, X_5]],\ [X_4, [X_4, X_5]],\ [X_4, X_5],\ X_4,\ X_5)$$

forms a frame on M. Locally, a generic 2-distribution $\mathcal{D}$ on M can be defined as the annihilator of three 1-forms $(\omega_1, \omega_2, \omega_3)$ on M, defined in terms of a single real function $f = f(x, y, p, q, z)$, such that $f_{qq} \neq 0$, via

$$\omega_1 = \mathrm{d}y - p\mathrm{d}x, \qquad \omega_2 = \mathrm{d}p - q\mathrm{d}x, \qquad \omega_3 = \mathrm{d}z - f(x, y, p, q, z)\mathrm{d}x. \tag{1.1}$$

Here (x, y, p, q, z) is a local coordinate system in M.

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The local geometry of (2,3,5) distributions is nontrivial: there exist generic distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M which do *not* admit a local diffeomorphism $\varphi : M \to M$ such that $\varphi_* \mathcal{D}_1 = \mathcal{D}_2$. For example, distributions corresponding to a function $f = q^2$ and $f = q^3$ in (1.1) do not admit such a diffeomorphism. In such case, we say that they are *locally nonequivalent*. The full set of differential invariants of (2,3,5) distributions considered modulo local diffeomorphism was given by Cartan in [5]. For each (2,3,5) distribution he associated a Cartan connection, with values in the split real form of the exceptional Lie algebra $\mathfrak{g}_2$, whose curvature provided the invariants. These invariants can be also understood in terms of a certain conformal class of metrics [6], defined on M by $\mathcal{D}$. This conformal class is defined as follows:

Let $\mathcal{D}$ be defined as the annihilator of 1-forms $(\omega_1, \omega_2, \omega_3)$, as e.g. in (1.1). We supplement them by the 1-forms ω_4 and ω_5, in such a way that $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5 \neq 0$. In case of (1.1), we take $\omega_4 = \mathrm{d}q$ and $\omega_5 = \mathrm{d}x$. Consider the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ defined on M via

$$\begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 & 0\\ b_{21} & b_{22} & b_{23} & 0 & 0\\ b_{31} & b_{32} & b_{33} & 0 & 0\\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45}\\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix}, \tag{1.2}$$

with some functions b_{ij}, $i, j = 1, 2, \ldots, 5$, on M such that $\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 \neq 0$. It follows that for a (2,3,5) distribution $\mathcal{D}$ one can always find functions b_{ij} and 1-forms Ω_μ, $\mu = 1, 2, \ldots, 7$, on M such that

$$\begin{aligned}
\mathrm{d}\theta^1 &= \theta^1 \wedge (2\Omega_1 + \Omega_4) + \theta^2 \wedge \Omega_2 + \theta^3 \wedge \theta^4\\
\mathrm{d}\theta^2 &= \theta^1 \wedge \Omega_3 + \theta^2 \wedge (\Omega_1 + 2\Omega_4) + \theta^3 \wedge \theta^5\\
\mathrm{d}\theta^3 &= \theta^1 \wedge \Omega_5 + \theta^2 \wedge \Omega_6 + \theta^3 \wedge (\Omega_1 + \Omega_4) + \theta^4 \wedge \theta^5\\
\mathrm{d}\theta^4 &= \theta^1 \wedge \Omega_7 + \tfrac{4}{3}\theta^3 \wedge \Omega_6 + \theta^4 \wedge \Omega_1 + \theta^5 \wedge \Omega_2\\
\mathrm{d}\theta^5 &= \theta^2 \wedge \Omega_7 - \tfrac{4}{3}\theta^3 \wedge \Omega_5 + \theta^4 \wedge \Omega_3 + \theta^5 \wedge \Omega_4.
\end{aligned} \tag{1.3}$$

And now, it turns out that the (3, 2)-signature conformal class $[g_\mathcal{D}]$ represented on M by the metric

$$g_\mathcal{D} = g_{ij}\theta^i \otimes \theta^j = \theta^1 \otimes \theta^5 + \theta^5 \otimes \theta^1 - \theta^2 \otimes \theta^4 - \theta^4 \otimes \theta^2 + \tfrac{4}{3}\theta^3 \otimes \theta^3 \tag{1.4}$$

is well defined, and that its Weyl tensor can be used to get all the basic differential invariants of the distribution $\mathcal{D}$. The simplest of these invariants, the so called *Cartan's quartic* $C(\zeta)$ of $\mathcal{D}$, can be obtained in terms of the Weyl tensor of the conformal class $[g_\mathcal{D}]$ as follows [7,8]:

Calculate the Weyl tensor $W = W_{ijkl}\theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^l$ for the metric $g_\mathcal{D}$ in the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$. (Use the metric $g_\mathcal{D}$ to lower the index i from the natural placement $W^i{}_{jkl}$ to W_{ijkl}, $W_{ijkl} = g_{ip}W^p{}_{jkl}$). Then Cartan's quartic for $\mathcal{D}$ is

$$C(\zeta) := A_1 + 4A_2\zeta + 6A_3\zeta^2 + 4A_4\zeta^3 + A_5\zeta^4$$

with the functions A_I, $I = 1, 2, \ldots, 5$, given by

$$A_1 = W_{4114}, \quad A_2 = W_{4124}, \quad A_3 = W_{4125}, \quad A_4 = W_{4225}, \quad A_5 = W_{5225}.$$

The simplest equivalence class of (2,3,5) distributions corresponds to the vanishing of Cartan's quartic, $C(\zeta) \equiv 0$, or equivalently, $A_I \equiv 0$ for all $I = 1, 2, \ldots, 5$. Modulo local diffeomorphisms there is only one such distribution $\mathcal{D}$. It may be represented by (1.1) with $f = q^2$. This distribution has maximal group of local symmetries. This group is isomorphic to the split real form of the exceptional group G_2 [5].

This provokes a problem: find all functions $f = f(x, y, p, q, z)$, which via (1.1), define a generic distribution $\mathcal{D}$, which is locally diffeomorphically equivalent to the most symmetric one, the one with $f = q^2$.

The general solution to this problem requires rather elaborate calculations, and it follows that the PDEs required for f to correspond to vanishing A_Is are quite ugly. However in the restricted case when the function f depends only on a single variable q the solution is quite nice (see [6], Eq. (57)). For completeness, we recall this solution in the next section.

2. Cartan quartic for the distribution with $f = f(q)$

If the distribution is given as the annihilator of

$$\begin{aligned}
\omega_1 &= \mathrm{d}y - p\mathrm{d}x\\
\omega_2 &= \mathrm{d}p - q\mathrm{d}x\\
\omega_3 &= \mathrm{d}z - f(q)\mathrm{d}x,
\end{aligned} \tag{2.1}$$

the conformal class $[g_{\mathcal{D}}]$ can be represented by (1.4), with the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ given by

$$\theta^1 = \omega_1 - \frac{1}{f''}(f'\omega_2 - \omega_3)$$
$$\theta^2 = \frac{1}{f''}(f'\omega_2 - \omega_3)$$
$$\theta^3 = \frac{4f''^2 - f'f^{(3)}}{4f''^2}\omega_2 + \frac{f^{(3)}}{4f''^2}\omega_3$$
$$\theta^4 = \frac{(7f^{(3)^2} - 4f''f^{(4)})}{40f''^3}(f'\omega_2 - \omega_3) + \omega_4 - \omega_5$$
$$\theta^5 = -\omega_4,$$

where

$$\omega_4 = \mathrm{d}q, \qquad \omega_5 = \mathrm{d}x.$$

With this choice of θ^is the Cartan quartic is

$$C(\zeta) = \frac{a_5}{100f''^4}\zeta^4$$

with

$$a_5 = 10f^{(6)}f''^3 - 80f''^2f^{(3)}f^{(5)} - 51f''^2f^{(4)^2} + 336f''f^{(3)^2}f^{(4)} - 224f^{(3)^4}.$$

We see, in particular, that the only nonvanishing component of Cartan's quartic is A_5, and that the quartic has a *quadruple* root, which makes it of type IV, in the terminology of [9].

We have the following corollary.

Corollary 2.1. *Necessary and sufficient conditions for the distribution*

$$\mathcal{D} = \mathrm{Span}(\partial_q, \partial_x + p\partial_y + q\partial_p + f(q)\partial_z)$$

to have split real form of the exceptional Lie group G_2 as a group of its local symmetries are

$$f'' \neq 0 \tag{2.2}$$

and

$$10f^{(6)}f''^3 - 80f''^2f^{(3)}f^{(5)} - 51f''^2f^{(4)^2} + 336f''f^{(3)^2}f^{(4)} - 224f^{(3)^4} = 0. \tag{2.3}$$

Thus apart from the genericity condition (2.2) the function f must satisfy quite a complicated 6th order ODE (2.3).

Strangely enough, this ODE is closely related to the Noth equation, studied by Dunajski and Sokolov, and mentioned in the Section 1. We have the following proposition.

Proposition 2.2. *Suppose that a real, sufficiently many times differentiable, function $f = f(q)$ satisfies (2.3). Let $\Theta = \Theta(x_5)$ be another real, sufficiently many times differentiable, function of a real variable x_5, whose second and third derivatives with respect to x_5 are related to f via an equation:*

$$f(-\Theta^{(3)}) + x_5\Theta^{(3)} - \Theta'' = 0. \tag{2.4}$$

Assume in addition that $\Theta^{(4)} \neq 0$. Then the function $\Theta = \Theta(x_5)$ satisfies the following 8th order ODE:

$$10\Theta^{(4)^3}\Theta^{(8)} - 70\Theta^{(4)^2}\Theta^{(5)}\Theta^{(7)} - 49\Theta^{(4)^2}\Theta^{(6)^2} + 280\Theta^{(4)}\Theta^{(5)^2}\Theta^{(6)} - 175\Theta^{(5)^4} = 0.$$

Proof. The proof consists in a successive differentiation of the equation $f(-\Theta^{(3)}) = -x_5\Theta^{(3)} + \Theta''$ using the chain rule. We have $-\Theta^{(4)}f' = -\Theta^{(3)} - x_5\Theta^{(4)} + \Theta^{(3)}$, i.e. $f' = x_5$. Then, in the same way:

$$f^{(p)} = -\frac{1}{\Theta^{(4)}}\frac{\mathrm{d}}{\mathrm{d}x_5}f^{(p-1)} \qquad \text{for} \quad p = 2, 3, \ldots,$$

i.e. $f'' = -\frac{1}{\Theta^{(4)}}$, $f^{(3)} = -\frac{\Theta^{(5)}}{\Theta^{(4)^3}}$, $f^{(4)} = \frac{\Theta^{(6)}}{\Theta^{(4)^4}} - 3\frac{\Theta^{(5)^2}}{\Theta^{(4)^5}}$, etc. Inserting these derivatives of f into the definition of a_5 we get

$$a_5 = -\frac{10\Theta^{(4)^3}\Theta^{(8)} - 70\Theta^{(4)^2}\Theta^{(5)}\Theta^{(7)} - 49\Theta^{(4)^2}\Theta^{(6)^2} + 280\Theta^{(4)}\Theta^{(5)^2}\Theta^{(6)} - 175\Theta^{(5)^4}}{\Theta^{(4)^{12}}}.$$

Thus if the equation $a_5 = 0$ for f is satisfied, i.e. if (2.3) holds, then the function $\Theta = \Theta(x_5)$ satisfies the 8th order ODE from the proposition, as claimed. □

Magically, Eq. (2.3), when transformed via (2.4) into the 8th order ODE from Proposition 2.2 and then reduced by one order via $y = \Theta'$, becomes the 7th order ODE of Noth. The magic is in a peculiar form of the transformation (2.4) relating f and Θ. The geometric reason for this transformation is explained in the next section.

3. Distribution with $f = f(q)$ as a twistor distribution

A particular class of (2,3,5) distributions is associated with 4-dimensional split signature metrics. This is carefully explained in [7], see Section 2, for every split signature metric. Here we concentrate on a special case, when the metric is given in terms of a one real function of four variables, called *Plebański second heavenly function*.

Let $\Theta = \Theta(x, y, z, w)$ be a real, sufficiently smooth, function of four real variables (x, y, z, w) which satisfies the differential equation called *second heavenly equation of Plebański*:

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0. \tag{3.1}$$

Such a function enables us to define a 4-metric g, on a manifold $\mathcal{U}$ parametrized by (x, y, z, w), via

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - \Theta_{xx}\mathrm{d}z^2 - \Theta_{yy}\mathrm{d}w^2 + 2\Theta_{xy}\mathrm{d}w\mathrm{d}z. \tag{3.2}$$

This is Plebański's second heavenly metric. It can be written in the form

$$g = \tau^1 \otimes \tau^2 + \tau^2 \otimes \tau^1 + \tau^3 \otimes \tau^4 + \tau^4 \otimes \tau^3,$$

where

$$\begin{aligned}\tau^1 &= \mathrm{d}x - \Theta_{yy}\mathrm{d}w + \Theta_{xy}\mathrm{d}z\\ \tau^2 &= \mathrm{d}w\\ \tau^3 &= \mathrm{d}y - \Theta_{xx}\mathrm{d}z + \Theta_{xy}\mathrm{d}w\\ \tau^4 &= \mathrm{d}z.\end{aligned}$$

Since g has split signature on $\mathcal{U}$, there is a natural circle bundle $\mathbb{S}^1 \to \mathbb{T}(\mathcal{U}) \to \mathcal{U}$ over $\mathcal{U}$ [7]. In this bundle, which we call as the *circle twistor bundle* for $(\mathcal{U}, g)$, every point in the fiber over $x \in \mathcal{U}$ is a certain *real* totally null selfdual 2-plane at x. There is an entire circle of such planes at x. The bundle $\mathbb{T}(\mathcal{U}) \xrightarrow{\pi} \mathcal{U}$ is naturally equipped with a 2-dimensional distribution $\mathcal{D}$. Its plane $\mathcal{D}_p$ at a point $p \in \mathbb{T}(\mathcal{U})$, which as we know can be identified with a certain real totally null 2-plane $N(p)$ at $\pi(p)$, is the tautological *horizontal* lift of $N(p)$ from $\pi(p)$ to p. *Horizontality* in $\mathbb{T}(\mathcal{U})$ is induced by the Levi-Civita connection of g from $\mathcal{U}$. (See [7], Section 2, for details.) In case of the Plebański metric (3.2), given in terms of the heavenly function $\Theta = \Theta(x, y, z, w)$, the circle twistor bundle can be locally parametrized by (x, y, z, w, ξ) and the twistor distribution can be defined as the annihilator of the 1-forms

$$\begin{aligned}\tilde{\omega}_1 &= \mathrm{d}\xi - \Big((\partial_x + \xi\partial_y)^3\Theta\Big)\mathrm{d}z\\ \tilde{\omega}_2 &= \mathrm{d}w + \xi\mathrm{d}z\\ \tilde{\omega}_3 &= \mathrm{d}y - \xi\mathrm{d}x - \Big((\partial_x + \xi\partial_y)^2\Theta\Big)\mathrm{d}z.\end{aligned}$$

A little tweak (see [9], Thm 3.3.5), which we have learned from Ian Anderson [10], and which he attributes to Goursat [11] (see also [12], p. 7), consists in introducing new coordinates $(x_1, x_2, x_3, x_4, x_5)$ on $\mathbb{T}(\mathcal{U})$:

$$x_1 = z, \qquad x_2 = w, \qquad x_3 = -\xi, \qquad x_4 = y - \xi x, \qquad x_5 = x, \tag{3.3}$$

and enables us to conclude that the twistor distribution for the Plebański metric (3.2) can equivalently be defined by the annihilator of the forms

$$\begin{aligned}\omega_1 &= \mathrm{d}x_2 - x_3\mathrm{d}x_1\\ \omega_2 &= \mathrm{d}x_3 + \Theta_{555}\mathrm{d}x_1\\ \omega_3 &= \mathrm{d}x_4 - (\Theta_{55} - x_5\Theta_{555})\mathrm{d}x_1.\end{aligned} \tag{3.4}$$

Here $\Theta_{55} = \frac{\partial^2\Theta}{\partial x_5^2}$, $\Theta_{555} = \frac{\partial^3\Theta}{\partial x_5^3}$, and because Θ is originally function of only *four* variables (x, y, z, w), we have $\Theta_3 + x_5\Theta_4 = 0$.

Now we can demystify transformation (2.4): Consider the case when the function Θ is a function of x only, $\Theta = \Theta(x_5)$. Note that in this case the second heavenly equation (3.1) is automatically satisfied. Then, comparing the formulae (2.1) and (3.4), we see that the relation between the function f in (2.1) and the function Θ in (3.4) is

$$q = -\Theta_{555}, \qquad f(q) = \Theta_{55} - x_5\Theta_{555}.$$

This inevitably leads to

$$f(-\Theta_{555}) = \Theta_{55} - x_5\Theta_{555},$$

which is the relation (2.4).

For an explicit derivation of Noth's ODE in terms of the Plebański second heavenly metric, we find explicit formulae for the conformal class $[g_{\mathcal{D}}]$ associated with the distribution $\mathcal{D}$ defined by (3.4) with $\Theta = \Theta(x_5)$. Since for this the function Θ is a function of *one* variable only, we will denote the derivatives w.r.t. x_5 by primes, double primes, etc. First we extend the forms (3.4) by

$$\omega_4 = \mathrm{d}x_1, \qquad \omega_5 = \mathrm{d}x_5 \tag{3.5}$$

to a coframe on $\mathbb{T}(\mathcal{U})$, and then find a suitable representatives of the forms $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5)$ defining, via (1.4), the conformal class $[g_{\mathcal{D}}]$. These forms can be taken to be

$$\begin{aligned}
\theta^1 &= \omega_1 - \Theta^{(4)}(x_5\omega_2 - \omega_3)\\
\theta^2 &= \Theta^{(4)}(x_5\omega_2 - \omega_3)\\
\theta^3 &= -\frac{4\Theta^{(4)} + x_5\Theta^{(5)}}{4\Theta^{(4)}}\omega_2 + \frac{\Theta^{(5)}}{4\Theta^{(4)}}\omega_3\\
\theta^4 &= -\frac{5{\Theta^{(5)}}^2 - 4\Theta^{(4)}\Theta^{(6)}}{40{\Theta^{(4)}}^3}(x_5\omega_2 - \omega_3) + \omega_4 - \Theta^{(4)}\omega_5\\
\theta^5 &= \Theta^{(4)}\omega_5,
\end{aligned}$$

with

$$\begin{aligned}
\omega_1 &= \mathrm{d}x_2 - x_3\mathrm{d}x_1\\
\omega_2 &= \mathrm{d}x_3 + \Theta^{(3)}\mathrm{d}x_1\\
\omega_3 &= \mathrm{d}x_4 - (\Theta'' - x_5\Theta^{(3)})\mathrm{d}x_1\\
\omega_4 &= \mathrm{d}x_1\\
\omega_5 &= \mathrm{d}x_5.
\end{aligned}$$

It is straightforward now to calculate Cartan's quartic for $g_{\mathcal{D}}$ with these forms θ^i. It reads

$$C(\zeta) = -\frac{\alpha_5\zeta^4}{100\Theta^{(4)}},$$

where α_5 is given by

$$\alpha_5 = 10{\Theta^{(4)}}^3\Theta^{(8)} - 70{\Theta^{(4)}}^2\Theta^{(5)}\Theta^{(7)} - 49{\Theta^{(4)}}^2{\Theta^{(6)}}^2 + 280\Theta^{(4)}{\Theta^{(5)}}^2\Theta^{(6)} - 175{\Theta^{(5)}}^4.$$

Thus under the condition $\Theta^{(4)} \neq 0$, Cartan's quartic identically vanishes if and only if $y = \Theta'$ satisfies Noth's ODE.

We have just proved the following theorem.

Theorem 3.1. *The twistor distribution $\mathcal{D}$ on the circle twistor bundle $\mathbb{S}^1 \to \mathbb{T}(M) \to M$ of the Plebański second heavenly manifold (M, g) with the metric*

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - \Theta''\mathrm{d}z^2$$

and the second heavenly function $\Theta = \Theta(x)$ such that $\Theta^{(4)} \neq 0$, has the split real form of the exceptional group G_2 as a group of its local symmetries if and only if the heavenly function Θ satisfies the ODE:

$$10{\Theta^{(4)}}^3\Theta^{(8)} - 70{\Theta^{(4)}}^2\Theta^{(5)}\Theta^{(7)} - 49{\Theta^{(4)}}^2{\Theta^{(6)}}^2 + 280\Theta^{(4)}{\Theta^{(5)}}^2\Theta^{(6)} - 175{\Theta^{(5)}}^4 = 0.$$

4. G_2 flatness for $\mathcal{D}$ of Plebański metrics implies Noth's equation

The aim of this section[1] is to argue that G_2 flatness of the twistor distribution associated with Plebański second heavenly manifolds (M, g) implies Noth's equation for the heavenly function Θ. Since the Cartan quartic for the general case looks horrible and is impossible to be displayed here, we present the details of the calculation only in the restricted case when the heavenly function Θ is independent of w and z. We will comment on the general case at the end of this section.

[1] The first version of this paper appeared on arXiv.org in 2013 without this section. Since then we have expanded our result to general Plebanski metrics. Also, the results of our arXiv.org version of the paper were used by M. Randal in [13,14].

If we assume that Θ is independent of w and z, we have $\Theta_w = 0$ and $\Theta_z = 0$, which degenerates the second heavenly equation into the form:

$$\Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = 0, \tag{4.1}$$

which is the homogeneous Monge–Ampere equation. This is a special case which is convenient to study using the same change of coordinates as before in (3.3), because in them the differential equation retains its form:

$$\Theta_{44}\Theta_{55} - \Theta_{45}^2 = 0. \tag{4.2}$$

Also, the conditions $\Theta_w = \Theta_z = \Theta_\xi = 0$ in the original coordinates imply, and are equivalent to, that the function Θ in the new coordinates satisfies

$$\Theta_1 = \Theta_2 = \Theta_3 + x_5\Theta_4 = 0.$$

It is convenient to rewrite the Monge–Ampere equation (4.2) into equivalent form,

$$\begin{aligned}\Theta_{45} &= H\Theta_{55}\\ \Theta_{44} &= H^2\Theta_{55},\end{aligned} \tag{4.3}$$

which is a system of differential equations for Θ and a certain differentiable function H. Then applying these relations, we can compute the exterior derivatives of Θ, H and their partials as follows:

$$\begin{aligned}
d\Theta &= -\Theta_4 x_5 dx_3 + \Theta_4 dx_4 + \Theta_5 dx_5\\
d\Theta_5 &= (-\Theta_4 - H\Theta_{55}x_5)dx_3 + H\Theta_{55}dx_4 + \Theta_{55}dx_5\\
d\Theta_4 &= -H^2\Theta_{55}x_5 dx_3 + H^2\Theta_{55}dx_4 + H\Theta_{55}dx_5\\
d\Theta_{55} &= -(2H\Theta_{55} + H_5\Theta_{55}x_5 + H\Theta_{555}x_5)dx_3 + (H_5\Theta_{55} + H\Theta_{555})dx_4 + \Theta_{555}dx_5\\
d\Theta_{555} &= (-3H_5\Theta_{55} - 3H\Theta_{555} - H_{55}\Theta_{55}x_5 - 2H_5\Theta_{555}x_5 - H\Theta_{5555}x_5)dx_3\\
&\quad + (H_{55}\Theta_{55} + 2H_5\Theta_{555} + H\Theta_{5555})dx_4 + \Theta_{5555}dx_5\\
d\Theta_{5555} &= (-4H_{55}\Theta_{55} - 8H_5\Theta_{555} - 4H\Theta_{5555} - H_{555}\Theta_{55}x_5\\
&\quad - 3H_{55}\Theta_{555}x_5 - 3H_5\Theta_{5555}x_5 - H\Theta_{55555}x_5)dx_3\\
&\quad + (H_{555}\Theta_{55} + 3H_{55}\Theta_{555} + 3H_5\Theta_{5555} + H\Theta_{55555})dx_4 + \Theta_{55555}dx_5
\end{aligned}$$

$$\begin{aligned}
d\Theta_{55555} &= (-5H_{555}\Theta_{55} - 15H_{55}\Theta_{555} - 15H_5\Theta_{5555} - 5H\Theta_{55555} - H_{5555}\Theta_{55}x_5\\
&\quad - 4H_{555}\Theta_{555}x_5 - 6H_{55}\Theta_{5555}x_5 - 4H_5\Theta_{55555}x_5 - H\Theta_{555555}x_5)dx_3\\
&\quad + (H_{5555}\Theta_{55} + 4H_{555}\Theta_{555} + 6H_{55}\Theta_{5555} + 4H_5\Theta_{55555} + H\Theta_{555555})dx_4\\
&\quad + \Theta_{555555}dx_5\\
d\Theta_{555555} &= (-6H_{5555}\Theta_{55} - 24H_{555}\Theta_{555} - 36H_{55}\Theta_{5555} - 24H_5\Theta_{55555} - 6H\Theta_{555555}\\
&\quad - H_{55555}\Theta_{55}x_5 - 5H_{5555}\Theta_{555}x_5 - 10H_{555}\Theta_{5555}x_5 - 10H_{55}\Theta_{55555}x_5\\
&\quad - 5H_5\Theta_{555555}x_5 - H\Theta_{5555555}x_5)dx_3\\
&\quad + (H_{55555}\Theta_{55} + 5H_{5555}\Theta_{555} + 10H_{555}\Theta_{5555} + 10H_{55}\Theta_{55555}\\
&\quad + 5H_5\Theta_{555555} + H\Theta_{5555555})dx_4 + \Theta_{5555555}dx_5
\end{aligned}$$

$$\begin{aligned}
dH &= H(H - H_5x_5)dx_3 + HH_5dx_4 + H_5dx_5\\
dH_5 &= (HH_5 - H_5^2x_5 - HH_{55}x_5)dx_3 + (H_5^2 + HH_{55})dx_4 + H_{55}dx_5\\
dH_{55} &= -(3H_5H_{55} + HH_{555})x_5dx_3 + (3H_5H_{55} + HH_{555})dx_4 + H_{555}dx_5\\
dH_{555} &= (-3H_5H_{55} - HH_{555} - 3H_{55}^2x_5 - 4H_5H_{555}x_5 - HH_{5555}x_5)dx_3\\
&\quad + (3H_{55}^2 + 4H_5H_{555} + HH_{5555})dx_4 + H_{5555}dx_5\\
dH_{5555} &= (-6H_{55}^2 - 8H_5H_{555} - 2HH_{5555} - 10H_{55}H_{555}x_5 - 5H_5H_{5555}x_5\\
&\quad - HH_{55555}x_5)dx_3\\
&\quad + (10H_{55}H_{555} + 5H_5H_{5555} + HH_{55555})dx_4 + H_{55555}dx_5.
\end{aligned}$$

Now, we can calculate the components A_i, $i = 1, 2, \ldots, 5$ of the Cartan quartic for the twistor distribution associated with this Plebański function Θ. We take the coframe given by (3.4) and (3.5), and find a new coframe (1.2) such that (1.3) is satisfied. It requires long steps of calculations to find such a coframe. The coframe we eventually found is shown below:

$$\begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta^5\end{pmatrix}=\begin{pmatrix}\Theta_{5555} & 0 & 0 & 0 & 0\\ 0 & -x_5 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ b_{41} & b_{42} & \dfrac{-5\Theta_{55555}^2+3\Theta_{5555}\Theta_{555555}}{30\Theta_{5555}^2} & 0 & -\Theta_{5555}\\ b_{51} & b_{52} & 0 & 1 & 2H\Theta_{55}\end{pmatrix}\begin{pmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega^5\end{pmatrix},$$

where

$$\begin{aligned}b_{41}=&-\frac{1}{15\Theta_{5555}^2}(3H_{5555}\Theta_{55}^2\Theta_{5555}+27H_{555}\Theta_{55}\Theta_{555}\Theta_{5555}+45H_{55}\Theta_{555}^2\Theta_{5555}\\&+18H_{55}\Theta_{55}\Theta_{5555}^2+45H_5\Theta_{555}\Theta_{5555}^2-5H_{555}\Theta_{55}^2\Theta_{55555}\\&-35H_{55}\Theta_{55}\Theta_{555}\Theta_{55555}-40H_5\Theta_{555}^2\Theta_{55555}+7H_5\Theta_{55}\Theta_{5555}\Theta_{55555}\\&+5H\Theta_{555}\Theta_{5555}\Theta_{55555}+5H\Theta_{55}\Theta_{55555}^2-3H\Theta_{55}\Theta_{5555}\Theta_{555555})\\ b_{42}=&\frac{10\Theta_{5555}\Theta_{55555}+5\Theta_{55555}^2x_5-3\Theta_{5555}\Theta_{555555}x_5}{30\Theta_{5555}^2}\end{aligned}$$

$$\begin{aligned}b_{51}=&-\frac{1}{30\Theta_{5555}^3}(-5H_{555}^2\Theta_{55}^4-70H_{55}H_{555}\Theta_{55}^3\Theta_{555}-245H_{55}^2\Theta_{55}^2\Theta_{555}^2\\&-80H_5H_{555}\Theta_{55}^2\Theta_{555}^2-560H_5H_{55}\Theta_{55}\Theta_{555}^3-320H_5^2\Theta_{555}^4+30H_{55}^2\Theta_{55}^3\Theta_{5555}\\&+20H_5H_{555}\Theta_{55}^3\Theta_{5555}-6HH_{5555}\Theta_{55}^3\Theta_{5555}+260H_5H_{55}\Theta_{55}^2\Theta_{555}\Theta_{5555}\\&-44HH_{555}\Theta_{55}^2\Theta_{555}\Theta_{5555}+280H_5^2\Theta_{55}\Theta_{555}^2\Theta_{5555}-20HH_{55}\Theta_{55}\Theta_{555}^2\Theta_{5555}\\&+80HH_5\Theta_{555}^3\Theta_{5555}-35H_5^2\Theta_{55}^2\Theta_{5555}^2-36HH_{55}\Theta_{55}^2\Theta_{5555}^2\\&-140HH_5\Theta_{55}\Theta_{555}\Theta_{5555}^2-20H^2\Theta_{555}^2\Theta_{5555}^2+30H^2\Theta_{55}\Theta_{5555}^3\\&+10HH_{555}\Theta_{55}^3\Theta_{55555}+70HH_{55}\Theta_{55}^2\Theta_{555}\Theta_{55555}+80HH_5\Theta_{55}\Theta_{555}^2\Theta_{55555}\\&-14HH_5\Theta_{55}^2\Theta_{5555}\Theta_{55555}-10H^2\Theta_{55}\Theta_{555}\Theta_{5555}\Theta_{55555}-5H^2\Theta_{55}^2\Theta_{55555}^2\\&+3H^2\Theta_{55}^2\Theta_{5555}\Theta_{555555})\\ b_{52}=&-\frac{1}{3\Theta_{5555}^2}(-H_{555}\Theta_{55}^2-7H_{55}\Theta_{55}\Theta_{555}-8H_5\Theta_{555}^2+5H_5\Theta_{55}\Theta_{5555}\\&+4H\Theta_{555}\Theta_{5555}+H\Theta_{55}\Theta_{55555}).\end{aligned}$$

Even in this restricted situation the computed Cartan's matrix coefficients A_i, $i=1,2,3,4,5$, are too complicated to list them all here. However, surprisingly we found that

$$\begin{aligned}A_5=&\frac{1}{100\Theta 5555^4}(175\Theta_{55555}^4-280\Theta_{5555}\Theta_{55555}^2\Theta_{555555}+49\Theta_{5555}^2\Theta_{555555}^2\\&+70\Theta_{5555}^2\Theta_{55555}\Theta_{5555555}-10\Theta_{5555}^3\Theta_{55555555}).\end{aligned}$$

For G_2 flatness of the twistor distribution we need that all the A_i, $i=1,2,3,4,5$ must vanish. The form of the computed A_5 shows that Noth's equation

$$\begin{aligned}175\Theta_{55555}^4-280\Theta_{5555}\Theta_{55555}^2\Theta_{555555}+49\Theta_{5555}^2\Theta_{555555}^2+\\ 70\Theta_{5555}^2\Theta_{55555}\Theta_{5555555}-10\Theta_{5555}^3\Theta_{55555555}=0,\end{aligned}\tag{4.4}$$

for Θ_5 is a necessary condition. Whether this condition is sufficient is an open question.

Interestingly, there are three well-known solutions of the Monge–Ampere equations (4.1):

(1) $\Theta=\phi(C_1x+C_2y)+C_3x+C_4y+C_5$
(2) $\Theta=(C_1x+C_2y)\phi\left(\frac{y}{x}\right)+C_3x+C_4y+C_5$
(3) $\Theta=(C_1x+C_2y+C_3)\phi\left(\frac{C_4x+C_5y+C_6}{C_1x+C_2y+C_3}\right)+C_7x+C_8y+C_9$

Rewriting these solutions in terms of the variables x_3, x_4 and x_5 using $x=x_5$, $y=x_4-x_3x_5$, we impose Noth's equation (4.4) on such Θs, which in turn impose conditions on the free function ϕ appearing in (1)–(3). The results are as follows:

ad (1) In this case all the A_is, $i=1,2,3,4,5$, vanish provided that the function ϕ_t satisfies Noth's ODE for the variable $t=C_1x_5+C_2(x_4-x_3x_5)$. Thus in this case Noth's ODE is *sufficient* to guarantee the G_2 flatness of the twistor distribution. However, by a simple change of coordinates, one can convince himself that the heavenly metric corresponding to $\Theta = \phi(C_1 x + C_2 y) + C_3 x + C_4 y + C_5$ is the same as in Theorem 3.1.

ad (2)–(3) In these cases the equations $A_i = 0, i = 1, 2, 3, 4, 5$, cannot be satisfied by function ϕ, which obey the genericity condition $\Theta_{5555} \neq 0$.

We actually have computed A_i, $i = 1, 2, \ldots, 5$ for the Plebański metric satisfying the heavenly equations with heavenly function Θ depending on all the variables (x, y, z, w). This was done with the help of Mathematica symbolic calculation program. It follows that in that, fully general case, the Noth equation (4.4) is *still necessary* for the heavenly function θ to correspond to G_2 flat twistor distribution. While this could be checked by brute force calculation, we conjecture that there should be a more geometrical argument for this, which we leave as an open question. Also it would be interesting to know if there are heavenly functions satisfying both (3.1) and (4.4) and for which the corresponding heavenly metric (3.2) is not equivalent to the one from Theorem 3.1.

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References

[1] M. Dunajski, V. Sokolov, On the 7th order ODE with submaximal symmetry, J. Geom. Phys. 61 (2011) 1258–1262.
[2] G. Noth, Differentialinvarianten und invariante Differentialgleichungen zweier zehngliedriger, Gruppen Leipz. Ber. 56 (1904) 19–48 Diss. Leipzig.
[3] P.J. Olver, Equivalence, Invariants, and Symmetry, CUP, Cambridge, 1995.
[4] V.V. Sokolov, Symmetries of evolution equations, Russian Math. Surveys 43 (1988) 165–204.
[5] E. Cartan, Les systemes de Pfaff a cinq variables et les equations aux derivees partielles du seconde ordre, Ann. Sc. Norm. Sup. 27 (1910) 109–192.
[6] P. Nurowski, Differential equations and conformal structures, J. Geom. Phys. 55 (2005) 19–49.
[7] D. An, P. Nurowski, Twistor space for rolling bodies, Comm. Math. Phys. 326 (2014) 393–414.
[8] C.R. Graham, T. Willse, Parallel tractor extension and ambient metrics of holonomy split G_2, J. Differential Geom. 92 (2012) 463–506.
[9] F. Strazzullo, Symmetry analysis of general rank–3 pfaffian systems in five variables, (Ph.D. thesis), Utah State University, Logan, Utah, 2009.
[10] I. Anderson, private communication, 2012.
[11] E. Goursat, Lecons sur le problem de Pfaff, Hermann, Paris, 1922.
[12] R.L. Bryant, Nine Lectures on Exterior Differential Systems, unpublished, 1999.
[13] M. Randall, Flat (2,3,5) distributions and Chazy's equations, SIGMA 12 (2016) 029 28 pages.
[14] M. Randall, Schwarz functions and duality, 2016, arXiv:1607.04961.