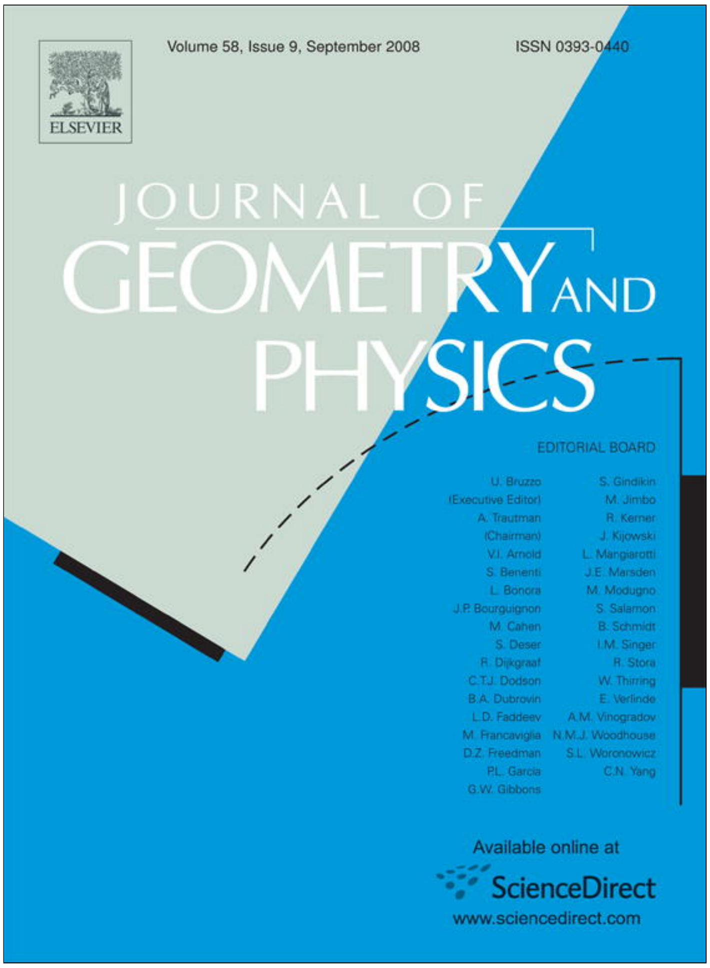
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Distinguished dimensions for special Riemannian geometries ☆

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ABSTRACT

The paper is based on relations between a ternary symmetric form defining the $\mathbf{SO}(3)$ geometry in dimension five and Cartan's works on isoparametric hypersurfaces in spheres. As observed by Bryant such a ternary form exists only in dimensions $n_k = 3k + 2$, where $k = 1, 2, 4, 8$. In these dimensions it reduces the orthogonal group to the subgroups $H_k \subset \mathbf{SO}(n_k)$, with $H_1 = \mathbf{SO}(3)$, $H_2 = \mathbf{SU}(3)$, $H_4 = \mathbf{Sp}(3)$ and $H_8 = \mathbf{F_4}$. This enables studies of special Riemannian geometries with structure groups H_k in dimensions n_k.

The necessary and sufficient conditions for the H_k geometries to admit the characteristic connection are given. As an illustration nontrivial examples of $\mathbf{SU}(3)$ geometries in dimension 8 admitting characteristic connection are provided. Among them are the examples having nonvanishing torsion and satisfying Einstein equations with respect to either the Levi-Civita or the characteristic connections.

The torsionless models for the H_k geometries have the respective symmetry groups $G_1 = \mathbf{SU}(3)$, $G_2 = \mathbf{SU}(3) \times \mathbf{SU}(3)$, $G_3 = \mathbf{SU}(6)$ and $G_4 = \mathbf{E}_6$. The groups H_k and G_k constitute a part of the 'magic square' for Lie groups. The 'magic square' Lie groups suggest studies of ten other classes of special Riemannian geometries. Apart from the two exceptional cases, they have the structure groups $\mathbf{U}(3)$, $\mathbf{S}(\mathbf{U}(3) \times \mathbf{U}(3))$, $\mathbf{U}(6)$, $\mathbf{E}_6 \times \mathbf{SO}(2)$, $\mathbf{Sp}(3) \times \mathbf{SU}(2)$, $\mathbf{SU}(6) \times \mathbf{SU}(2)$, $\mathbf{SO}(12) \times \mathbf{SU}(2)$ and $\mathbf{E}_7 \times \mathbf{SU}(2)$ and should be considered in respective dimensions 12, 18, 30, 54, 28, 40, 64 and 112. The two 'exceptional' cases are: $\mathbf{SU}(2) \times \mathbf{SU}(2)$ geometries in dimension 8 and $\mathbf{SO}(10) \times \mathbf{SO}(2)$ geometries in dimension 32.

The case of $\mathbf{SU}(2) \times \mathbf{SU}(2)$ geometry in dimension 8 is examined closer. We determine the tensor that reduces $\mathbf{SO}(8)$ to $\mathbf{SU}(2) \times \mathbf{SU}(2)$ leaving the more detailed studies of the geometries based on the magic square ideas to the forthcoming paper.

1. Introduction

In a recent paper [3] we studied 5-dimensional manifolds (M^5, g, Υ) equipped with the Riemannian metric tensor g_{ij} and a 3-tensor Υ_{ijk} such that

(i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$,
(ii) $\Upsilon_{ijj} = 0$,
(iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$.

It turns out that the quadratic condition (iii) selects from all the *symmetric totally trace free* 3-tensors in $\mathbb{R}^5$ only such a one whose stabilizer in $\mathbf{SO}(5)$ is the *irreducible* $\mathbf{SO}(3)$.

The geometry of Riemannian 5-manifolds (M^5, g, Υ) is particularly interesting if the tensor Υ satisfies the *nearly integrability* [3] condition

$$(\stackrel{LC}{\nabla}_v \Upsilon)(v, v, v) \equiv 0. \qquad (1.1)$$

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In such a case (M^5, g, Υ) is naturally equipped with a unique metric $\mathfrak{so}(3)$-valued connection ∇^T whose torsion T is *totally skew symmetric*. We call this connection the *characteristic* connection of the nearly integrable geometry (M^5, g, Υ). Existence and uniqueness of the characteristic connection enable a classification of the nearly integrable geometries (M^5, g, Υ) according to the algebraic properties of T and the curvature K of ∇^T. In Ref. [3] examples were given of the nearly integrable geometries (M^5, g, Υ) with the characteristic connection ∇^T having K and T of all the possible types from the above-mentioned classification. In particular, a 7-parameter family of such geometries admitting at each point two **SO**(3)-invariant vector spaces of ∇^T-covariantly constant spinors was given. However, in this family of examples, the characteristic ∇^T connection was flat.

Properties of Υ resemble a bit, properties of the tensor J, defining an almost hermitian structure on a Riemannian manifold. For example, condition (iii) for Υ is an algebraic condition of the same sort as the almost hermitian condition

$$J_{ij}J_{jk} = -g_{ik} \tag{1.2}$$

for J. Also, the nearly integrable condition (1.1) for Υ is similar to the nearly Kähler condition

$$(\overset{LC}{\nabla}_v J)(v) \equiv 0$$

for J. Since the almost hermitian condition (1.2) imposes a severe restriction on the dimension of the manifold to be *even*, a natural question arises if there are some restrictions on the dimensions of $\mathbb{R}^n$ in which one can have a tensor with the properties (i)–(iii). More precisely we ask the following question:

In which dimensions n the Euclidean space $(\mathbb{R}^n, g)$ *can be equipped with a tensor* Υ *satisfying conditions* (i)–(iii)?

It is rather easy to show that dimensions $n \le 4$ do not admit such a tensor. Following [3] we know that in dimension $n = 5$ the tensor Υ may be defined by

$$\Upsilon_{ijk}x^i x^j x^k = \frac{1}{2}\det\begin{pmatrix} x^5 - \sqrt{3}x^4 & \sqrt{3}x^3 & \sqrt{3}x^2 \\ \sqrt{3}x^3 & x^5 + \sqrt{3}x^4 & \sqrt{3}x^1 \\ \sqrt{3}x^2 & \sqrt{3}x^1 & -2x^5 \end{pmatrix}. \tag{1.3}$$

Thus, Υ is defined as a tensor whose components are coefficients of the homogeneous polynomial of third degree obtained as the determinant of a generic 3×3 real symmetric trace free matrix.

2. Dimensions 5, 8, 14 and 26

Robert Bryant [4] remarks that other dimensions in which Υ with properties (i)–(iii) surely exists are: $n = 8$, $n = 14$ and $n = 26$. This is essentially due to the fact that numbers 5, 8, 14 and 26 are values of the sequence $n_k = 3k + 2$ for $k = 1, 2, 4$ and 8, respectively. These four values of k correspond to the only possible dimensions of the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. To fully explain Bryant's remark we need some preparation.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ and let $A \in M_{3\times 3}(\mathbb{K})$ be a *hermitian* 3×3-matrix with entries in $\mathbb{K}$. The word 'hermitian' here means that the entries a_{ij} and a_{ji} of A are mutually conjugate in $\mathbb{K}$, i.e. $a_{ji} = \bar{a}_{ij}$ for $i, j = 1, 2, 3$. In particular, the entries $a_{11}, a_{22}, a_{33} \in \mathbb{R}$.

We may formally write

$$\det A = \sum_{\pi \in S_3} \operatorname{sgn}\pi\, a_{1\pi(1)}a_{2\pi(2)}a_{3\pi(3)},$$

which after expansion reads:

$$\det A = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}.$$

Note that despite the possible *noncommutativity*, or even *nonassociativity*, of the product, the values of the first *four* monomials in the above formal expression are well defined. This is because among the three factors in each of the four monomials, at least one is a real number a_{ii}, the other two being either both real (in the first term) or conjugate to each other (in the remaining three terms). Thus, the values of these four monomials are real numbers and do not depend on the order of their factors and the order of the multiplication. Passing to the last two terms in the formula for det A we see, that *a priori* there are a huge number of possibilities to order the factors and the brackets in these two terms. But the requirement that the sum of these terms is *real* reduces this huge number to only 12 possibilities. It turns out that out of these 12 possibilities only *two* are really different. They are all equal either to $(a_{12}a_{23})a_{31} + a_{13}(a_{32}a_{21})$ or to $(a_{21}a_{32})a_{13} + a_{31}(a_{23}a_{12})$. Note that the first expression becomes the second under the transformation $A \to \bar{A}$. Moreover, such transformation does not affect the values of the first four terms in det A. Summing up we have the following lemma.

Lemma 2.1. *Given a hermitian matrix* $A \in M_{3\times 3}(\mathbb{K})$ *with entries* $a_{ij} \in \mathbb{K}$ *such that* $a_{ji} = \bar{a}_{ij}$, $i, j = 1, 2, 3$, *where* $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, *there are only two possibilities to assign a real value to the Weierstrass formula*

$$\det A = \sum_{\pi \in S_3} \operatorname{sgn}\pi\, a_{1\pi(1)}a_{2\pi(2)}a_{3\pi(3)}$$

for the determinant of A. These two possible values are given by

$$\det_1 A = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{13}(a_{32}a_{21}) + (a_{12}a_{23})a_{31}$$

or by

$$\det_2 A = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} + a_{31}(a_{23}a_{12}) + (a_{21}a_{32})a_{13}.$$

In general $\det_1 A \neq \det_2 A$ *if* $\mathbb{K} = \mathbb{H}$ *or* $\mathbb{O}$, *but* $\det_1 A \to \det_2 A$ *when* $A \to \bar{A}$.

Let $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ be the unit octonions. We have: $e_0^2 = 1 = -e_\mu^2$, $e_\mu e_{\mu+1} = e_{\mu+3}$, $e_\mu e_\nu = -e_\nu e_\mu$, $\mu \neq \nu = 1, 2, 3, \ldots, 7$, with additional relations resulting from the cyclic permutation of each triple $(e_\mu, e_{\mu+1}, e_{\mu+3})$.

It is convenient to introduce

$$X^1 = x^1 e_0 + x^6 e_1 + x^9 e_2 + x^{10} e_4 + x^{15} e_3 + x^{16} e_5 + x^{17} e_6 + x^{18} e_7,$$
$$X^2 = x^2 e_0 + x^7 e_1 + x^{11} e_2 + x^{12} e_4 + x^{19} e_3 + x^{20} e_5 + x^{21} e_6 + x^{22} e_7,$$
$$X^3 = x^3 e_0 + x^8 e_1 + x^{13} e_2 + x^{14} e_4 + x^{23} e_3 + x^{24} e_5 + x^{25} e_6 + x^{26} e_7.$$

Then X^1, X^2, X^3 are three generic octonions. We can consider them to be the generic quaternions if $x^I = 0$ for all $I = 15, 16, 17, \ldots, 26$, and three generic complex numbers if $x^I = 0$ for all $I = 9, 10, 11, \ldots, 26$. If $x^I = 0$ for all $I = 6, 7, 8, \ldots, 26$ then X^1, X^2, X^3 are three generic real numbers. Using this we define a 3×3 hermitian matrix

$$A = \begin{pmatrix} x^5 - \sqrt{3}x^4 & \sqrt{3}X^3 & \sqrt{3}X^2 \\ \sqrt{3}\bar{X}^3 & x^5 + \sqrt{3}x^4 & \sqrt{3}X^1 \\ \sqrt{3}\bar{X}^2 & \sqrt{3}\bar{X}^1 & -2x^5 \end{pmatrix} \tag{2.1}$$

in full analogy to the matrix entering the formula (1.3). Now, we have two 'characteristic polynomials': $\det_1(A - \lambda I)$ and $\det_2(A - \lambda I)$. They can be written as:

$$\det_1(A - \lambda I) = -\lambda^3 - 3g(\vec{X}, \vec{X})\lambda + 2\Upsilon_1(\vec{X}, \vec{X}, \vec{X}),$$
$$\det_2(A - \lambda I) = -\lambda^3 - 3g(\vec{X}, \vec{X})\lambda + 2\Upsilon_2(\vec{X}, \vec{X}, \vec{X}),$$

where $(\vec{X})^I = x^I$, $I = 1, 2, 3, \ldots, n_k = 3k + 2$ and $k = 1, 2, 4, 8$. The bilinear form is:

$$g(\vec{X}, \vec{X}) = (x^1)^2 + (x^2)^2 + \cdots + (x^{n_k})^2 = g_{IJ}x^I x^J$$

and the two ternary forms are

$$\Upsilon_1(\vec{X}, \vec{X}, \vec{X}) = \frac{1}{2}\det_1 A \quad \text{and} \quad \Upsilon_2(\vec{X}, \vec{X}, \vec{X}) = \frac{1}{2}\det_2 A.$$

Now, we have the following proposition, which is our formulation of Bryant's [4] remark.

Proposition 2.2. *If* $I, J, K = 1, 2, 3, \ldots, n_k = 3k + 2$, $k = 1, 2, 4, 8$, *then the tensors* Υ^1_{IJK} *and* Υ^2_{IJK} *given by*

$$\Upsilon^a_{IJK} = \frac{1}{6}\frac{\partial^3 \Upsilon_a(\vec{X}, \vec{X}, \vec{X})}{\partial x^I \partial x^J \partial x^K} \quad a = 1, 2,$$

satisfy

(i) $\Upsilon^a_{IJK} = \Upsilon^a_{(IJK)}$,
(ii) $\Upsilon^a_{IJJ} = 0$,
(iii) $\Upsilon^a_{JKI}\Upsilon^a_{LMI} + \Upsilon^a_{LJI}\Upsilon^a_{KMI} + \Upsilon^a_{KLI}\Upsilon^a_{JMI} = g_{JK}g_{LM} + g_{LJ}g_{KM} + g_{KL}g_{JM}$.

They reduce the $\mathbf{GL}(\mathbb{R}^{n_k})$ *group, via* $\mathbf{O}(n_k)$, *to its subgroup* H_k, *where* H_k *is the irreducible* $\mathbf{SO}(3)$ *in* $\mathbf{SO}(5)$ *if* $k = 1$, *the irreducible* $\mathbf{SU}(3)$ *in* $\mathbf{SO}(8)$ *if* $k = 2$, *the irreducible* $\mathbf{Sp}(3)$ *in* $\mathbf{SO}(14)$ *if* $k = 4$ *and the irreducible* $\mathbf{F}_4$ *in* $\mathbf{SO}(26)$ *if* $k = 8$.

If $k = 1, 2$ *the tensors* Υ^1_{IJK} *and* Υ^2_{IJK} *coincide. If* $k = 3, 4$ *they belong to the same* $\mathbf{O}(n_k)$ *orbit and are related by the element* $\mathrm{diag}(1, 1, 1, 1, 1, -1, -1, \ldots, -1)$ *of* $\mathbf{O}(n_k)$. *For* $k = 3, 4$ *tensors* Υ^1_{IJK} *and* Υ^2_{IJK} *are not equivalent under the* $\mathbf{SO}(n_k)$*-action.*

The above proposition gives examples of a tensor with all the properties of tensor Υ in dimensions $n = 5, 8, 14$ and 26. It is remarkable that it can be proven that these examples exhaust all the possibilities!

To discuss this statement we need to invoke Elie Cartan's results on 'isoparametric hypersurfaces in spheres' [7].

3. Isoparametric hypersurfaces in spheres

We recall (see. e.g. [14]) that a hypersurface in a real Riemannian manifold M of constant curvature is called *isoparametric* iff it has constant principal curvatures. Tuglio Levi-Civita [13] knew that the number of such distinct curvatures was at most *two* for the Euclidean space $M=\mathbb{R}^3$. The case of $M=\mathbb{R}^n$ with arbitrary $n>3$ is similar. It was shown by Beniamino Segre [15] that irrespectively of n the number of distinct principal curvatures of an isoparametric hypersurface in $M=\mathbb{R}^n$ is at most two. Elie Cartan [6] extended this result to the isoparametric hypersurfaces in the hyperbolic spaces H^n again showing that in such a case the number of possible distinct principal curvatures is at most two. The situation is quite different for isoparametric hypersurfaces in spheres S^n. In particular Cartan in Ref. [7] found examples of isoparametric hypersurfaces in spheres with *three* different principal curvatures, each of which had the same multiplicity. He also introduced a homogeneous polynomial

$$F:\mathbb{R}^n\to\mathbb{R}$$

of degree p satisfying the differential equations

$$\text{(Cii)}\qquad \triangle F=0 \tag{3.1}$$

$$\text{(Ciii)}\qquad |\vec{\nabla}F|^2=p^2((x^1)^2+(x^2)^2+\cdots+(x^n)^2)^{(p-1)} \tag{3.2}$$

and proved that all the isoparametric hypersurfaces in S^{n-1} which have p different constant principal curvatures of the same *multiplicity* are given by

$$S_c=\{x^I\in\mathbb{R}^n\mid F=c\text{ and }(x^1)^2+(x^2)^2+\cdots+(x^n)^2=1\}, \tag{3.3}$$

i.e. that they are the level surfaces of such polynomials F restricted to the sphere.

Cartan found *all* the homogeneous harmonic polynomials of degree $p=3$ satisfying condition (Ciii). In doing that he *proved* [7] that such polynomials can exist only if $n=n_k=5,8,14,26$. In these four dimensions he found that the most general form of the polynomials is

$$F=\sum_{I,J,K=1}^{n_k}\Upsilon^a_{IJK}x^Ix^Jx^K,$$

where Υ^a_{IJK} is one of the two tensors appearing in Proposition 2.2. Writing a generic homogeneous polynomial of degree p as $F=\Upsilon_{I_1I_2\dots I_p}x^{I_1}x^{I_2}\dots x^{I_p}$ we see that it satisfies Cartan's conditions (Cii)–(Ciii) iff the *totally symmetric* tensor $\Upsilon_{I_1I_2\dots I_p}$ satisfies

$$\text{(Cii}'\text{)}\qquad \Upsilon_{JJJ_3J_4\dots J_p}=0$$

$$\text{(Ciii}'\text{)}\qquad \Upsilon_{J(J_2J_3\dots J_p}\Upsilon_{K_2K_3\dots K_p)J}=g_{(J_2K_2}g_{J_3K_3}\dots g_{J_pK_p)},$$

where $g_{IJ}=\mathrm{diag}(1,1,\dots,1)$. Note that in case $p=3$ the above tensor reduces to Υ_{IJK} and conditions (Cii′) and (Ciii′) become exactly the respective conditions (ii) and (iii) of Proposition 2.2. Since Υ_{IJK} is totally symmetric also the condition (i) is satisfied. Thus, Cartan's finding of *all* isoparametric hypersurfaces with three constant distinct principal curvatures of the same multiplicity solves our problem of dimensions in which the tensor Υ may exist. Summarizing we have the following theorem, which is a reformulation of the above-mentioned Cartan's results.

Theorem 3.1. *An $\mathbb{R}^n$ with the standard Euclidean metric g_{IJ} admits a tensor Υ_{IJK} with the properties*

(i) $\Upsilon_{IJK}=\Upsilon_{(IJK)}$,
(ii) $\Upsilon_{IJJ}=0$,
(iii) $\Upsilon_{JKI}\Upsilon_{LMI}+\Upsilon_{LJI}\Upsilon_{KMI}+\Upsilon_{KLI}\Upsilon_{JMI}=g_{JK}g_{LM}+g_{LJ}g_{KM}+g_{KL}g_{JM}$

if and only if $n=n_k=3k+2$ for $k=1,2,4,8$. Modulo the action of the $\mathbf{SO}(n_k)$ *group all such tensors are given by* Proposition 2.2.

4. Representations of SU(3), Sp(3) and F$_4$

It is known that there are *real* irreducible representations of the group $\mathbf{SU}(3)$ in dimensions:

1, 8, 20, 27, 70.

Also, there are *real* irreducible representations of the group $\mathbf{Sp}(3)$ in dimensions:

1, 14, 21, 70, 84, 90, 126, 189, 512, 525

and there are *real* irreducible representations of the group $\mathbf{F}_4$ in dimensions:

1, 26, 52, 273, 324, 1053, 1274, 4096, 8424.

To see how these representations appear we consider a vector space $\mathbb{R}^{n_k}$, $n_k=5,8,14,26$ equipped with the Riemannian metric g and the corresponding tensor Υ^1_{IJK} of Proposition 2.2. As we know the stabilizer H_k of Υ^1 is a subgroup of $\mathbf{SO}(n_k)$, which when n_k is 5, 8, 14 and 26 is, respectively, $\mathbf{SO}(3)$, $\mathbf{SU}(3)$, $\mathbf{Sp}(3)$ and $\mathbf{F}_4$. Now, the tensor Υ^1 can be used to decompose the tensor product representation $\bigotimes^2 \mathbb{R}^{n_k}$ of the group H_k onto the *real* irreducible components as follows. First, we define an endomorphism

$$\hat{\Upsilon} : \bigotimes^2 \mathbb{R}^{n_k} \longrightarrow \bigotimes^2 \mathbb{R}^{n_k}, \qquad W^{IK} \stackrel{\hat{\Upsilon}}{\longmapsto} 4\,\Upsilon^1_{IJM}\Upsilon^1_{KLM} W^{JL}, \tag{4.1}$$

which preserves the decomposition $\bigotimes^2 \mathbb{R}^{n_k} = \bigwedge^2 \mathbb{R}^{n_k} \oplus \bigodot^2 \mathbb{R}^{n_k}$. Second, we look for its eigenspaces, which surely are H_k-invariant. We have the following proposition.

Proposition 4.1. (1) *If* $n_k = 5$ *then*

$$\bigotimes^2 \mathbb{R}^{n_k} = {}^5\bigwedge^2_3 \oplus {}^5\bigwedge^2_7 \oplus {}^5\bigodot^2_1 \oplus {}^5\bigodot^2_5 \oplus {}^5\bigodot^2_9,$$

where

$$\begin{aligned}
{}^5\bigodot^2_1 &= \{S \in \bigotimes^2 \mathbb{R}^5 \mid \hat{\Upsilon}(S) = 14 \cdot S\} = \{S = \lambda \cdot g,\ \lambda \in \mathbb{R}\},\\
{}^5\bigwedge^2_3 &= \{F \in \bigotimes^2 \mathbb{R}^5 \mid \hat{\Upsilon}(F) = 7 \cdot F\} = \mathfrak{so}(3),\\
{}^5\bigodot^2_5 &= \{S \in \bigotimes^2 \mathbb{R}^5 \mid \hat{\Upsilon}(S) = -3 \cdot S\},\\
{}^5\bigwedge^2_7 &= \{F \in \bigotimes^2 \mathbb{R}^5 \mid \hat{\Upsilon}(F) = -8 \cdot F\},\\
{}^5\bigodot^2_9 &= \{S \in \bigotimes^2 \mathbb{R}^5 \mid \hat{\Upsilon}(S) = 4 \cdot S\}.
\end{aligned}$$

The real vector spaces ${}^5\bigwedge^2_i \subset \bigwedge^2 \mathbb{R}^5$ *and* ${}^5\bigodot^2_j \subset \bigodot^2 \mathbb{R}^5$ *of respective dimensions i and j are irreducible representations of the group* $\mathbf{SO}(3)$.
(2) *If* $n_k = 8$ *then*

$$\bigotimes^2 \mathbb{R}^{n_k} = {}^8\bigwedge^2_8 \oplus {}^8\bigwedge^2_{20} \oplus {}^8\bigodot^2_1 \oplus {}^8\bigodot^2_8 \oplus {}^8\bigodot^2_{27},$$

where

$$\begin{aligned}
{}^8\bigodot^2_1 &= \{S \in \bigotimes^2 \mathbb{R}^8 \mid \hat{\Upsilon}(S) = 20 \cdot S\} = \{S = \lambda \cdot g,\ \lambda \in \mathbb{R}\},\\
{}^8\bigwedge^2_8 &= \{F \in \bigotimes^2 \mathbb{R}^8 \mid \hat{\Upsilon}(F) = 10 \cdot F\} = \mathfrak{su}(3),\\
{}^8\bigodot^2_8 &= \{S \in \bigotimes^2 \mathbb{R}^8 \mid \hat{\Upsilon}(S) = -6 \cdot S\},\\
{}^8\bigwedge^2_{20} &= \{F \in \bigotimes^2 \mathbb{R}^8 \mid \hat{\Upsilon}(F) = -8 \cdot F\},\\
{}^8\bigodot^2_{27} &= \{S \in \bigotimes^2 \mathbb{R}^8 \mid \hat{\Upsilon}(S) = 4 \cdot S\}.
\end{aligned}$$

The real vector spaces ${}^8\bigwedge^2_i \subset \bigwedge^2 \mathbb{R}^8$ *and* ${}^8\bigodot^2_j \subset \bigodot^2 \mathbb{R}^8$ *of respective dimensions i and j are irreducible representations of the group* $\mathbf{SU}(3)$. *The representations* $\bigwedge^2_8$ *and* $\bigodot^2_8$ *are equivalent.*
(3) *If* $n_k = 14$ *then*

$$\bigotimes^2 \mathbb{R}^{n_k} = {}^{14}\bigwedge^2_{21} \oplus {}^{14}\bigwedge^2_{70} \oplus {}^{14}\bigodot^2_1 \oplus {}^{14}\bigodot^2_{14} \oplus {}^{14}\bigodot^2_{90},$$

where

$$\begin{aligned}
{}^{14}\bigodot^2_1 &= \{S \in \bigotimes^2 \mathbb{R}^{14} \mid \hat{\Upsilon}(S) = 32 \cdot S\} = \{S = \lambda \cdot g,\ \lambda \in \mathbb{R}\},\\
{}^{14}\bigwedge^2_{21} &= \{F \in \bigotimes^2 \mathbb{R}^{14} \mid \hat{\Upsilon}(F) = 16 \cdot F\} = \mathfrak{sp}(3),\\
{}^{14}\bigodot^2_{14} &= \{S \in \bigotimes^2 \mathbb{R}^{14} \mid \hat{\Upsilon}(S) = -12 \cdot S\},\\
{}^{14}\bigwedge^2_{70} &= \{F \in \bigotimes^2 \mathbb{R}^{14} \mid \hat{\Upsilon}(F) = -8 \cdot F\},\\
{}^{14}\bigodot^2_{90} &= \{S \in \bigotimes^2 \mathbb{R}^{14} \mid \hat{\Upsilon}(S) = 4 \cdot S\}.
\end{aligned}$$

The real vector spaces ${}^{14}\bigwedge^2_i \subset \bigwedge^2 \mathbb{R}^{14}$ *and* ${}^{14}\bigodot^2_j \subset \bigodot^2 \mathbb{R}^{14}$ *of respective dimensions i and j are irreducible representations of the group* $\mathbf{Sp}(3)$.
(4) *If* $n_k = 26$ *then*

$$\bigotimes^2 \mathbb{R}^{n_k} = {}^{26}\bigwedge^2_{52} \oplus {}^{26}\bigwedge^2_{273} \oplus {}^{26}\bigodot^2_1 \oplus {}^{26}\bigodot^2_{26} \oplus {}^{26}\bigodot^2_{324},$$

where

$$\begin{aligned}
{}^{26}\bigodot^2_1 &= \{S \in \bigotimes^2 \mathbb{R}^{26} \mid \hat{\Upsilon}(S) = 56 \cdot S\} = \{S = \lambda \cdot g,\ \lambda \in \mathbb{R}\},\\
{}^{26}\bigwedge^2_{52} &= \{F \in \bigotimes^2 \mathbb{R}^{26} \mid \hat{\Upsilon}(F) = 28 \cdot F\} = \mathfrak{f}_4,\\
{}^{26}\bigodot^2_{26} &= \{S \in \bigotimes^2 \mathbb{R}^{26} \mid \hat{\Upsilon}(S) = -24 \cdot S\},\\
{}^{26}\bigwedge^2_{273} &= \{F \in \bigotimes^2 \mathbb{R}^{26} \mid \hat{\Upsilon}(F) = -8 \cdot F\},\\
{}^{26}\bigodot^2_{324} &= \{S \in \bigotimes^2 \mathbb{R}^{26} \mid \hat{\Upsilon}(S) = 4 \cdot S\}.
\end{aligned}$$

The real vector spaces ${}^{26}\bigwedge^2_i \subset \bigwedge^2 \mathbb{R}^{26}$ *and* ${}^{26}\bigodot^2_j \subset \bigodot^2 \mathbb{R}^{26}$ *of respective dimensions i and j are irreducible representations of the group* $\mathbf{F}_4$.

Remark 4.2. According to this proposition we may identify spaces $\mathbb{R}^{n_k}$ with the representation spaces ${}^{n_k}\bigodot^2_{n_k}$ corresponding to the eigenvalues $2 - n_k$ of $\hat{\Upsilon}$. Noting that the dimension of the group H_k is

$$\dim H_k = 4k - 1 + (k-1)\log_2 k, \quad k = 1, 2, 4, 8$$

and introducing $s_k = \frac{9}{2}k(k+1)$, we see that

- the eigenvalues of $\hat{\Upsilon}$ corresponding to spaces ${}^{n_k}\bigodot^2_1$ are $4 + 2n_k$,
- the eigenvalues corresponding to spaces ${}^{n_k}\bigwedge^2_{\dim H_k}$ are $2 + n_k$,
- the eigenvalues corresponding to spaces ${}^{n_k}\bigwedge^2_{(s_k+1-\dim H_k)}$ are *always* -8,
- the eigenvalues corresponding to spaces ${}^{n_k}\bigodot^2_{s_k}$ are *always* $+4$.

We also note that we may identify the Lie algebras $\mathfrak{h}_k$ of H_k with the representations ${}^{n_k}\bigwedge^2_{\dim H_k}$.

5. H_k structures on Riemannian manifolds

Definition 5.1. An H_k structure on an n_k-dimensional Riemannian manifold (M, g) is a structure defined by means of a rank 3 tensor field Υ satisfying

(i) $\Upsilon_{IJK} = \Upsilon_{(IJK)}$,
(ii) $\Upsilon_{IJJ} = 0$,
(iii) $\Upsilon_{JKI}\Upsilon_{LMI} + \Upsilon_{LJI}\Upsilon_{KMI} + \Upsilon_{KLI}\Upsilon_{JMI} = g_{JK}g_{LM} + g_{LJ}g_{KM} + g_{KL}g_{JM}$.

Definition 5.2. Two H_k structures (M, g, Υ) and $(\bar{M}, \bar{g}, \bar{\Upsilon})$ defined on two respective n_k-manifolds M and $\bar{M}$ are (locally) *equivalent* iff there exists a (local) diffeomorphism $\phi : M \to \bar{M}$ such that

$$\phi^*(\bar{g}) = g \quad \text{and} \quad \phi^*(\bar{\Upsilon}) = \Upsilon.$$

If $\bar{M} = M, \bar{g} = g, \bar{\Upsilon} = \Upsilon$ the equivalence ϕ is called a (local) *symmetry* of (M, g, Υ). The group of (local) symmetries is called a *symmetry group of* (M, g, Υ).

As we know the tensor field Υ reduces the structure group of the bundle of orthonormal frames over M to one of the groups H_k of Proposition 2.2. We also know that the Lie algebra $\mathfrak{h}_k$ of H_k is isomorphic to $\mathfrak{h}_k \simeq {}^{n_k}\Lambda^2_{\dim H_k} \subset \otimes^2 \mathbb{R}^{n_k}$ of Proposition 4.1. Thus, at each point, every element F of the Lie algebra $\mathfrak{h}_k$ may be considered to be an endomorphism of $\mathbb{R}^{n_k}$. This defines an element

$$f = \exp(F) \in H_k \subset \mathbf{SO}(n_k) \subset \mathbf{GL}(n_k, \mathbb{R})$$

and, point by point, induces the natural action $\rho(f)$ of the group H_k on the vector-valued 1-forms

$$\theta = (\theta^1, \theta^2, \theta^3, \ldots, \theta^{n_k}) \in \mathbb{R}^{n_k} \otimes \Omega^1(M)$$

by:

$$\theta \mapsto \tilde{\theta} = \rho(f)(\theta) = f \cdot \theta. \tag{5.1}$$

This, enables for local description of an H_k structure on M by means of a coframe

$$\theta = (\theta^I) = (\theta^1, \theta^2, \theta^3, \ldots, \theta^{n_k}) \tag{5.2}$$

on M, given up to the H_k transformations (5.1).

For such a class of coframes the Riemannian metric g is

$$g = \theta_1^2 + \theta_2^2 + \theta_3^2 + \cdots + \theta_{n_k}^2,$$

and the tensor Υ, reducing the structure group from $\mathbf{SO}(n_k)$ to H_k, is

$$\Upsilon = \Upsilon^1_{IJK}\theta^I\theta^J\theta^K, \tag{5.3}$$

where Υ^1 is defined in Proposition 2.2.

Definition 5.3. An orthonormal coframe $(\theta^1, \theta^2, \theta^3, \ldots, \theta^{n_k})$ in which the tensor Υ of an H_k structure (M, g, Υ) is of the form (5.3) is called a *coframe adapted to* (M, g, Υ), an *adapted coframe*, for short.

Given an H_k structure as above, we consider an *arbitrary* $\mathfrak{h}_k$-valued connection on M. This may be locally represented by means of an $\mathfrak{h}_k$-valued 1-form Γ given by

$$\Gamma = (\Gamma^I{}_J) = \gamma^\alpha E_\alpha, \quad \alpha = 1, 2, \ldots, \dim H_k, \tag{5.4}$$

where γ^α are 1-forms on M and for each α the symbols $E_\alpha = (E_\alpha{}^I{}_J)$ denote constant $n_k \times n_k$-matrices which form a basis of the Lie algebra $\mathfrak{h}_k$. The explicit expressions for E_α are presented in Appendix A. The connection Γ, having values in $\mathfrak{h}_k \subset \mathfrak{so}(n_k)$, is necessarily metric. Via the Cartan structure equations,

$$\mathrm{d}\theta^I + \Gamma^I{}_J \wedge \theta^J = T^I \tag{5.5}$$

$$\mathrm{d}\Gamma^I{}_J + \Gamma^I{}_K \wedge \Gamma^K{}_J = R^I{}_J, \tag{5.6}$$

it defines the torsion 2-form T^I and the $\mathfrak{h}_k$-curvature 2-form $R^I{}_J$. Using these forms we define the torsion tensor $T^I{}_{JK} \in (\mathbb{R}^{n_k} \otimes \bigwedge^2 \mathbb{R}^{n_k})$ and the $\mathfrak{h}_k$-curvature tensor $r^\alpha_{JK} \in (\mathfrak{h}_k \otimes \bigwedge^2 \mathbb{R}^{n_k})$, respectively, by

$$T^I = \frac{1}{2} T^I{}_{JK} \theta^J \wedge \theta^K$$

and

$$R^I{}_J = \frac{1}{2} r^\alpha_{KL} \theta^K \wedge \theta^L E_\alpha{}^I{}_J. \tag{5.7}$$

The connection satisfies the first Bianchi identity

$$R^I{}_J \wedge \theta^J = DT^I \tag{5.8}$$

and the second Bianchi identity

$$DR^I{}_J = 0, \tag{5.9}$$

with the covariant differential defined by

$$DT^I = \mathrm{d}T^I + \Gamma^I{}_J \wedge T^J, \qquad DR^I{}_J = \mathrm{d}R^I{}_J + \Gamma^I{}_K \wedge R^K{}_J - R^I{}_K \wedge \Gamma^K{}_J.$$

Since the H_k preserves g and Υ we have the following proposition.

Proposition 5.4. *Every $\mathfrak{h}_k$-valued connection Γ of (5.4) is metric*

$$\stackrel{\Gamma}{\nabla}_v(g) \equiv 0$$

and preserves tensor Υ

$$\stackrel{\Gamma}{\nabla}_v(\Upsilon) \equiv 0 \quad \forall v \in \mathrm{T}M.$$

6. Characteristic connection

In this section we consider H_k structures (M, g, Υ) with Levi-Civita connection $\stackrel{LC}{\Gamma} \in \mathfrak{so}(n_k) \otimes \mathbb{R}^{n_k}$ *uniquely* decomposable according to

$$\stackrel{LC}{\Gamma} = \Gamma + \frac{1}{2} T, \tag{6.1}$$

where $\Gamma \in \mathfrak{h}_k \otimes \mathbb{R}^{n_k}$ and $T \in \bigwedge^3 \mathbb{R}^{n_k}$.

Such H_k structures are interesting, since for them, contrary to the generic case, the decomposition (6.1) defines a unique $\mathfrak{h}_k$-valued connection Γ. Moreover, given the unique decomposition (6.1), we may rewrite the Cartan structure equations

$$\mathrm{d}\theta^I + \stackrel{LC}{\Gamma}{}^I{}_J \wedge \theta^J = 0$$

for the Levi-Civita connection $\stackrel{LC}{\Gamma}$ into the form

$$\mathrm{d}\theta^I + \Gamma^I{}_J \wedge \theta^J = \frac{1}{2} T^I{}_{JK} \theta^J \wedge \theta^K$$

and to interpret T as the *totally skew symmetric torsion* of Γ.

Definition 6.1. An $\mathfrak{h}_k$-valued connection Γ of an H_k structure (M, g, Υ) admitting the unique decomposition

$$\overset{LC}{\Gamma} = \Gamma + \frac{1}{2}T, \quad \text{with } \Gamma \in \mathfrak{h}_k \otimes \mathbb{R}^{n_k} \text{ and } T \in \bigwedge^3 \mathbb{R}^{n_k}$$

is called the characteristic connection.

Since $\Gamma \in \mathfrak{h}_k \otimes \mathbb{R}^{n_k}$ and $T \in \bigwedge^3 \mathbb{R}^{n_K}$ it is obvious from (6.1) that the Levi-Civita connection $\overset{LC}{\Gamma}$ of H_k structures which admit characteristic connections must satisfy

$$\overset{LC}{\Gamma} \in [\mathfrak{h}_k \otimes \mathbb{R}^{n_k}] + \bigwedge^3 \mathbb{R}^{n_k}. \tag{6.2}$$

Moreover, since $\dim(\mathfrak{h}_k \otimes \mathbb{R}^{n_k}) + \dim(\bigwedge^3 \mathbb{R}^{n_k}) < \dim(\mathfrak{so}(n_k) \otimes \mathbb{R}^{n_k})$ then it is obvious that the unique decomposition (6.1) is not possible for all H_k structures. Our aim now is to characterize H_k structures admitting characteristic connection.

Following [3] we introduce the following definition.

Definition 6.2. An H_k structure (M, g, Υ) is called *nearly integrable* iff

$$(\overset{LC}{\nabla}_v \Upsilon)(v, v, v) \equiv 0 \tag{6.3}$$

for the Levi-Civita connection $\overset{LC}{\nabla}$ and for every vector field v on M.

The condition (6.3), when written in an adapted coframe (5.2), is

$$\overset{LC}{\Gamma}_{M(JI} \Upsilon_{KL)M} \equiv 0, \tag{6.4}$$

where $\overset{LC}{\Gamma}_{MJ} = \overset{LC}{\Gamma}_{MJK} \theta^K$ denotes the $\mathfrak{so}(n_k)$-valued 1-form corresponding to the Levi-Civita connection $\overset{LC}{\nabla}$. This motivates an introduction of the map

$$\Upsilon' : \bigwedge^2 \mathbb{R}^{n_k} \otimes \mathbb{R}^{n_k} \mapsto \bigodot^4 \mathbb{R}^{n_k}$$

such that

$$\begin{aligned}
\Upsilon'(\overset{LC}{\Gamma})_{IJKL} &= 12 \overset{LC}{\Gamma}_{M(JI} \Upsilon_{KL)M} \\
&= \overset{LC}{\Gamma}_{MJI} \Upsilon_{MKL} + \overset{LC}{\Gamma}_{MKI} \Upsilon_{JML} + \overset{LC}{\Gamma}_{MLI} \Upsilon_{JKM} \\
&\quad + \overset{LC}{\Gamma}_{MIJ} \Upsilon_{MKL} + \overset{LC}{\Gamma}_{MKJ} \Upsilon_{IML} + \overset{LC}{\Gamma}_{MLJ} \Upsilon_{IKM} \\
&\quad + \overset{LC}{\Gamma}_{MIK} \Upsilon_{MJL} + \overset{LC}{\Gamma}_{MJK} \Upsilon_{IML} + \overset{LC}{\Gamma}_{MLK} \Upsilon_{IJM} \\
&\quad + \overset{LC}{\Gamma}_{MIL} \Upsilon_{MJK} + \overset{LC}{\Gamma}_{MJL} \Upsilon_{IMK} + \overset{LC}{\Gamma}_{MKL} \Upsilon_{IJM}.
\end{aligned} \tag{6.5}$$

Comparing this with (6.4) we have the following proposition.

Proposition 6.3. *An H_k structure (M, g, Υ) is nearly integrable if and only if its Levi-Civita connection* $\overset{LC}{\Gamma} \in \ker \Upsilon'$.

It is worth noting that each of the last four rows of (6.5) resembles the l.h.s. of equality

$$X_{MJ} \Upsilon_{MKL} + X_{MK} \Upsilon_{JML} + X_{ML} \Upsilon_{JKM} = 0$$

satisfied by every matrix $X \in \mathfrak{h}_k = {}^{n_k}\bigwedge^2_{\dim H_k}$. Thus, $\mathfrak{h}_k \otimes \mathbb{R}^{n_k} \subset \ker \Upsilon'$. Due to the first equality in (6.5) we also have $\bigwedge^3 \mathbb{R}^{n_k} \subset \ker \Upsilon'$. This proves the following lemma.

Lemma 6.4. *Since*

$$\mathfrak{h}_k \otimes \mathbb{R}^{n_k} \subset \ker \Upsilon' \quad \textit{and} \quad \bigwedge^3 \mathbb{R}^{n_k} \subset \ker \Upsilon'$$

then

$$([\mathfrak{h}_k \otimes \mathbb{R}^{n_k}] + \bigwedge^3 \mathbb{R}^{n_k}) \subset \ker \Upsilon'.$$

Thus, comparing this with (6.2) we have the following proposition.

Proposition 6.5. *Among all H_k structures only the nearly integrable ones may admit characteristic connection.*

It is known [3] that if $n_k = 5$ then the nearly integrability condition is also sufficient for the existence of a characteristic connection. To see that it is no longer true for all n_k we need to see how the intersections $[\mathfrak{h}_k \otimes \mathbb{R}^{n_k}] \cap \bigwedge^3 \mathbb{R}^{n_k}$ and the algebraic sums $[\mathfrak{h}_k \otimes \mathbb{R}^{n_k}] + \bigwedge^3 \mathbb{R}^{n_k}$ depend on the dimension n_k. After some algebra we arrive at the following proposition.

Proposition 6.6. • *If* $n_k = 5$ *or* $n_k = 14$ *then*

$$\ker \Upsilon' = [\mathfrak{h}_k \otimes \mathbb{R}^{n_k}] \oplus \bigwedge^3 \mathbb{R}^{n_k}.$$

• *If* $n_k = 8$ *then*

$$\ker \Upsilon' = [\mathfrak{su}(3) \otimes \mathbb{R}^8] + \bigwedge^3 \mathbb{R}^8 \quad \textit{and} \quad [\mathfrak{su}(3) \otimes \mathbb{R}^8] \cap \bigwedge^3 \mathbb{R}^8 = {}^8\bigodot^2_1.$$

• *If* $n_k = 26$ *then*

$$\ker \Upsilon' = [\mathfrak{f}_4 \otimes \mathbb{R}^{26}] \oplus \bigwedge^3 \mathbb{R}^{26} \oplus {}^{26}\bigwedge^2_{52}.$$

• *In particular, for* $n_k = 5, 14$ *and* 26 *we have* $[\mathfrak{h}_k \otimes \mathbb{R}^{n_k}] \cap \bigwedge^3 \mathbb{R}^{n_k} = \{0\}$.

This implies the following theorem.

Theorem 6.7. *In dimensions* $n_k = 5$ *and* $n_k = 14$ *the necessary and sufficient condition for an* H_k *structure* (M, g, Υ) *to admit a characteristic connection is that* (M, g, Υ) *is nearly integrable*

$$(\stackrel{LC}{\nabla}_v \Upsilon)(v, v, v) \equiv 0.$$

Proposition 6.6 also implies that the nearly integrable H_k structures in dimension $n_k = 8$ admit decomposition (6.1). However, in this dimension condition (6.1) determines the connection Γ and the torsion T up to an additional freedom. Due to the 1-dimensional intersection $[\mathfrak{su}(3) \otimes \mathbb{R}^8] \cap \bigwedge^3 \mathbb{R}^8 = {}^8\bigodot^2_1$ we see that in such a case there is a 1-parameter family of connections $\Gamma(\lambda) \in \mathfrak{su}(3) \otimes \mathbb{R}^8$ and 1-parameter family of skew symmetric torsions $T(\lambda) \in \bigwedge^3 \mathbb{R}^8$ such that

$$\stackrel{LC}{\Gamma} = \Gamma(\lambda) + \frac{1}{2} T(\lambda). \tag{6.6}$$

It is clear that for $n_k = 8$, the requirement (6.1) uniquely determines $\Gamma \in \mathfrak{su}(3) \otimes \mathbb{R}^8$ and $T \in \bigwedge^3 \mathbb{R}^8$ only if we restrict ourselves to the nearly integrable **SU**(3) structures for which the Levi-Civita connection $\stackrel{LC}{\Gamma}$ is in the 118-dimensional space ${}^8\mathcal{V}$ such that ${}^8\mathcal{V} \oplus {}^8\bigodot^2_1 = \ker \Upsilon'$. It follows that this space has the following decomposition under the **SU**(3)-action ${}^8\mathcal{V} = 2\,{}^8\bigodot^2_{27} \oplus 2\,{}^8\bigwedge^2_{20} \oplus 3\,{}^8\bigodot^2_8$. It is convenient to extend this notation and to introduce vector spaces ${}^{n_k}\mathcal{V}$ to be subspaces of $\ker \Upsilon'$ such that:

$$\begin{aligned}
&{}^{n_k}\mathcal{V} = \ker \Upsilon' \quad \text{for } n_k = 5, 14\\
&{}^8\mathcal{V} = 2\,{}^8\bigodot^2_{27} \oplus 2\,{}^8\bigwedge^2_{20} \oplus 3\,{}^8\bigodot^2_8 \subsetneq \ker \Upsilon',\\
&{}^{26}\mathcal{V} = [\mathfrak{f}_4 \otimes \mathbb{R}^{26}] \oplus \bigwedge^3 \mathbb{R}^{26} \subsetneq \ker \Upsilon'.
\end{aligned}$$

Using these we have the following definition

Definition 6.8. An H_k structure (M, g, Υ) is called *restricted nearly integrable* iff its Levi-Civita connection $\stackrel{LC}{\Gamma} \in {}^{n_k}\mathcal{V}$.

Remark 6.9. Note that for $n_k = 5$ or $n_k = 14$ the term: *restricted nearly integrable* is the same as: *nearly integrable*.

Looking again at Proposition 6.6 we see that the above restriction for the nearly integrable **SU**(3) or $\mathbf{F}_4$ structures in respective dimensions $n_k = 8$ and $n_k = 26$ is precisely the one that gives the sufficient conditions for the existence and uniqueness of the characteristic connection. Summarizing we have the following theorem.

Theorem 6.10. *A necessary and sufficient condition for an* H_k *structure* (M, g, Υ) *to admit a characteristic connection is that this structure is restricted nearly integrable.*

Remark 6.11. Note, that if $n_k = 5$ then, out of the *a priori* 50 independent components of the Levi-Civita connection $\stackrel{LC}{\Gamma}$, the (restricted) nearly integrable condition (6.1) excludes 25. Thus, heuristically, the (restricted) nearly integrable **SO**(3) structures constitute 'a half' of all the possible **SO**(3) structures in dimension 5.

If $n_k = 8$ the Levi-Civita connection has 224 components. The restricted nearly integrable condition reduces it to 118. For $n_k = 14$ these numbers reduce from 1274 to 658. For $n_k = 26$ the reduction is from 8450 to 3952.

7. Classification of the restricted nearly integrable H_k structures

We classify the possible types of the restricted nearly integrable H_k structures according to the H_k irreducible decompositions of the spaces $\bigwedge^3 \mathbb{R}^{n_k}$ in which the torsion T of their characteristic connections resides. Using a computer algebra package for Lie group computations 'LiE' [12] we easily arrive at the following proposition.

Proposition 7.1. *Let (M, g, Υ) be a restricted nearly integrable H_k structure. The H_k irreducible decomposition of the skew symmetric torsion T of the characteristic connection for (M, g, Υ) is given by:*

- $T \in {}^5\bigwedge^2_7 \oplus {}^5\bigwedge^2_3$, *for* $n_k = 5$,
- $T \in {}^8\bigodot^2_{27} \oplus {}^8\bigwedge^2_{20} \oplus {}^8\bigodot^2_8 \oplus {}^8\bigodot^2_1$, *for* $n_k = 8$,
- $T \in {}^{14}V_{189} \oplus {}^{14}V_{84} \oplus {}^{14}\bigwedge^2_{70} \oplus {}^{14}\bigwedge^2_{21}$, *for* $n_k = 14$,
- $T \in {}^{26}V_{1274} \oplus {}^{26}V_{1053} \oplus {}^{26}\bigwedge^2_{273}$, *for* $n_k = 26$.

Here ${}^{n_k}V_j$ denotes irreducible j-dimensional representations of H_k which were not present in the H_k decomposition of $\bigotimes^2 \mathbb{R}^{n_k}$.

This provides an analog of the Gray–Hervella [10] classification for the restricted nearly integrable H_k structures.

We close this section with a remark on possible types of the curvature R of the characteristic connections.

Remark 7.2. In the below formulae ${}^{n_k}V_j$ denote the j-dimensional irreducible representation space for H_k which did not appear in Proposition 4.1.

- If $n_k = 5$ then $R \in {}^5\bigodot^2_9 \oplus {}^5\bigwedge^2_7 \oplus 2\,{}^5\bigodot^2_5 \oplus {}^5\bigwedge^2_3 \oplus {}^5\bigodot^2_1$
- If $n_k = 8$ then $R \in {}^8V_{70} \oplus 3\,{}^8\bigodot^2_{27} \oplus 2\,{}^8\bigwedge^2_{20} \oplus 4\,{}^8\bigodot^2_8 \oplus {}^8\bigodot^2_1$.
- If $n_k = 14$ then $R \in {}^{14}V_{525} \oplus {}^{14}V_{512} \oplus 2\,{}^{14}V_{189} \oplus {}^{14}V_{126} \oplus 2\,{}^{14}\bigodot^2_{90} \oplus 2\,{}^{14}\bigwedge^2_{70} \oplus {}^{14}\bigwedge^2_{21} \oplus 2\,{}^{14}\bigodot^2_{14} \oplus {}^{14}\bigodot^2_1$.
- If $n_k = 26$ then $R \in {}^{26}V_{8424} \oplus {}^{26}V_{4096} \oplus {}^{26}V_{1274} \oplus {}^{26}V_{1053} \oplus {}^{26}V'_{1053} \oplus 2\,{}^{26}\bigodot^2_{324} \oplus {}^{26}\bigwedge^2_{273} \oplus {}^{26}\bigwedge^2_{52} \oplus {}^{26}\bigodot^2_{26} \oplus {}^{26}\bigodot^2_1$.

Note that, due to the restricted nearly integrability condition, it is rather unlikely that R may attain values in all of the above irreducible parts.

8. Dimensions 12, 18, 28, 30, 40, 54, 64 and 112; the 'exceptional' 8 and 32

8.1. Torsionless models

It is obvious that the simplest restricted nearly integrable H_k structures have the characteristic connection Γ with vanishing torsion $T \equiv 0$. For them

$$\overset{LC}{\Gamma} = \Gamma \in \mathfrak{h}_k \otimes \mathbb{R}^{n_k},$$

hence their Riemannian holonomy group is reduced from $\mathbf{SO}(n_k)$ to the group H_k. Since $H_k \subset \mathbf{SO}(n_k)$ in the respective dimensions $n_k = 5, 8, 14$ and 26 are not present in the Berger list of the Riemannian holonomy groups [1], the only possible restricted nearly integrable H_k structures with $T \equiv 0$ must be locally isometric to the symmetric spaces $M = G_k/H_k$. The Lie group G_k appearing here must have dimension $\dim G_k = n_k + \dim H_k$. Looking at Cartan's list [5] of the irreducible symmetric spaces (see e.g. [2] pp. 312–317) we have the following theorem.

Theorem 8.1. *All H_k structures with vanishing torsion are locally isometric to one of the symmetric spaces*

$$M = G_k/H_k,$$

where the possible Lie groups G are given in the following table:

$\dim M$	Group H_k	Group G_k	Group G_k	Group G_k
$n_k = 5$	$\mathbf{SO}(3)$	$\mathbf{SU}(3)$	$\mathbf{SO}(3) \times_\rho \mathbb{R}^5$	$\mathbf{SL}(3, \mathbb{R})$
$n_k = 8$	$\mathbf{SU}(3)$	$\mathbf{SU}(3) \times \mathbf{SU}(3)$	$\mathbf{SU}(3) \times_\rho \mathbb{R}^8$	$\mathbf{SL}(3, \mathbb{C})$
$n_k = 14$	$\mathbf{Sp}(3)$	$\mathbf{SU}(6)$	$\mathbf{Sp}(3) \times_\rho \mathbb{R}^{14}$	$\mathbf{SU}^*(6) \simeq \mathbf{SL}(3, \mathbb{H})$
$n_k = 26$	$\mathbf{F}_4$	$\mathbf{E}_6$	$\mathbf{F}_4 \times_\rho \mathbb{R}^{26}$	$\mathbf{E}_6^{-26} \simeq \mathbf{SL}(3, \mathbb{O})$

Here ρ is the irreducible representation of H_k in $\mathbb{R}^{n_k}$.

Remark 8.2. Let $\mathfrak{g}_k$ be the Lie algebra of the group G_k of Theorem 8.1. We note that since the torsionless models for the H_k structures are the symmetric spaces $M = G_k/H_k$, then arbitrary restricted nearly integrable H_k structures may be analyzed in terms of a *Cartan $\mathfrak{g}_k$-valued connection* on the Cartan bundle $H_k \to P \to M$. In such a language the *torsionless* models with respect to the $\mathfrak{h}_k$ connection are simply the *flat* models for the corresponding Cartan $\mathfrak{g}_k$-valued connection on P.

Remark 8.3. According to [19] the manifold $M = \mathbf{SU}(3)/\mathbf{SO}(3)$ is a unique irreducible Riemannian symmetric space $M = G/H$ with the property that $(\operatorname{rank} G - \operatorname{rank} H) = 1$ and that M is not isometric to an odd dimensional real Grassmann manifold. It is interesting to note [9] (see [19] p. 324) that the other *compact* torsionless H_k structures correspond to manifolds $M = \mathbf{SU}(3)$, $M = \mathbf{SU}(6)/\mathbf{Sp}(3)$ and $M = \mathbf{E}_6/\mathbf{F}_4$, which are examples of a very few irreducible symmetric Riemannian manifolds $M = G/H$ with $(\operatorname{rank} G - \operatorname{rank} H) = 2$.

8.2. The 'magic square'

We now concentrate on the Lie algebras $\mathfrak{h}_k$ and $\mathfrak{g}_k$ corresponding to groups H_k and G_k appearing in the second and the third columns of the table included in Theorem 8.1. We note that these Lie algebras constitute the first two columns of the 'magic square' [17,18]:

$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathfrak{su}(3)$	$2\,\mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

In agreement with the previous notation let us denote by H_k, G_k, $\mathcal{G}_k$ and $\tilde{\mathcal{G}}_k$ the compact Lie groups corresponding to the Lie algebras of the first, the second, the third and the fourth respective columns of the magic square. Since G_k/H_k are the torsionless compact models for H_k geometries, it may seem reasonable to consider spaces $\mathcal{G}_k/G_k$ and $\tilde{\mathcal{G}}_k/\mathcal{G}_k$ as the torsionless models for new special Riemannian geometries with a characteristic connection. Unfortunately the homogeneous spaces $\mathcal{G}_k/G_k$ and $\tilde{\mathcal{G}}_k/\mathcal{G}_k$ are *reducible*. However, if we replace the second column in the magic square by

$\mathfrak{su}(3)\oplus\mathbb{R}$
$2\,\mathfrak{su}(3)\oplus\mathbb{R}$
$\mathfrak{su}(6)\oplus\mathbb{R}$
$\mathfrak{e}_6\oplus\mathbb{R}$

then the Lie groups G_k corresponding to *these* Lie algebras define the *irreducible* Riemannian symmetric spaces $\mathcal{G}_k/G_k$. Similarly if we replace the third column in the magic square by

$\mathfrak{sp}(3)\oplus\mathfrak{su}(2)$
$\mathfrak{su}(6)\oplus\mathfrak{su}(2)$
$\mathfrak{so}(12)\oplus\mathfrak{su}(2)$
$\mathfrak{e}_7\oplus\mathfrak{su}(2)$

then the Lie groups $\mathcal{G}_k$ corresponding to *these* Lie algebras define the *irreducible* Riemannian symmetric spaces $\tilde{\mathcal{G}}_k/\mathcal{G}_k$. Thus starting from the second and the third columns of the table in Theorem 8.1, via the magic square, we arrived at 12 symmetric spaces.

$\mathbf{SU}(3)/\mathbf{SO}(3)$	$\mathbf{Sp}(3)/\mathbf{U}(3)$	$\mathbf{F}_4/(\mathbf{Sp}(3)\times\mathbf{SU}(2))$
$\mathbf{SU}(3)$	$\mathbf{SU}(6)/\mathbf{S}(\mathbf{U}(3)\times\mathbf{U}(3))$	$\mathbf{E}_6/(\mathbf{SU}(6)\times\mathbf{SU}(2))$
$\mathbf{SU}(6)/\mathbf{Sp}(3)$	$\mathbf{SO}(12)/\mathbf{U}(6)$	$\mathbf{E}_7/(\mathbf{SO}(12)\times\mathbf{SU}(2))$
$\mathbf{E}_6/\mathbf{F}_4$	$\mathbf{E}_7/(\mathbf{E}_6\times\mathbf{SO}(2))$	$\mathbf{E}_8/(\mathbf{E}_7\times\mathbf{SU}(2))$

These 12 symmetric spaces can be considered as torsionless models for special geometries on Riemannian manifolds M with the following dimensions and structure groups:

$\dim M$ n_k	Structure group H_k	$\dim M$ $2(n_k+1)$	Structure group Extended G_k	$\dim M$ $4(n_k+2)$	Structure group Extended $\mathcal{G}_k$
5	$\mathbf{SO}(3)$	12	$\mathbf{U}(3)$	28	$\mathbf{Sp}(3)\times\mathbf{SU}(2)$
8	$\mathbf{SU}(3)$	18	$\mathbf{S}(\mathbf{U}(3)\times\mathbf{U}(3))$	40	$\mathbf{SU}(6)\times\mathbf{SU}(2)$
14	$\mathbf{Sp}(3)$	30	$\mathbf{U}(6)$	64	$\mathbf{SO}(12)\times\mathbf{SU}(2)$
26	$\mathbf{F}_4$	54	$\mathbf{E}_6\times\mathbf{SO}(2)$	112	$\mathbf{E}_7\times\mathbf{SU}(2)$

A quick look at Cartan's list of the irreducible symmetric spaces of compact type suggests that the special Riemannian geometries appearing in this list should be supplemented by the two 'exceptional' possibilities:

(1) $\dim M = 32$, with the structure group $\mathbf{SO}(10)\times\mathbf{SO}(2)$ and with the torsionless model of compact type $M=\mathbf{E}_6/(\mathbf{SO}(10)\times\mathbf{SO}(2))$
(2) $\dim M = 8$, with the structure group $\mathbf{SU}(2)\times\mathbf{SU}(2)$ and with the torsionless model of compact type $M=\mathbf{G}_2/(\mathbf{SU}(2)\times\mathbf{SU}(2))$.

Although these two possibilities are not implied by the magic square, we are convinced that their place is in the above table: item (1) should stay in the second column for $\dim M$ in the row between dimensions 30 and 54, and item (2) should stay in the third column for $\dim M$ in the 'zeroth' row, before dimension 28.

It is interesting if all these geometries admit characteristic connection. Also, we do not know which objects in $\mathbb{R}^{\dim M}$ reduce the orthogonal groups $\mathbf{SO}(\dim M)$ to the above-mentioned structure groups. Are these symmetric tensors, as was in the case of the groups H_k?

9. Examples in dimension 8

In the following sections we will briefly discuss the two different 8-dimensional cases, namely: the restricted nearly integrable **SU**(3) geometries and the **SU**(2) × **SU**(2) geometries. In particular, we provide nontrivial examples of restricted nearly integrable **SU**(3) structures. We also explain how to define an **SU**(2) × **SU**(2) structure by means of a symmetric tensor of the *sixth* order.

9.1. **SU**(3) *structures*

It is interesting to note that in the decomposition of $\bigwedge^3\mathbb{R}^8$ onto the **SU**(3)-invariant components (see Proposition 7.1) there exists a 1-dimensional subspace ${}^8\bigodot^2_1$. This space, in the adapted coframe of Definition 5.3, is spanned by a 3-form

$$\psi=\tau_1\wedge\theta^6+\tau_2\wedge\theta^7+\tau_3\wedge\theta^8+\theta^6\wedge\theta^7\wedge\theta^8, \tag{9.1}$$

where (τ_1,τ_2,τ_3) are 2-forms

$$\begin{aligned}\tau_1&=\theta^1\wedge\theta^4+\theta^2\wedge\theta^3+\sqrt{3}\theta^1\wedge\theta^5\\ \tau_2&=\theta^1\wedge\theta^3+\theta^4\wedge\theta^2+\sqrt{3}\theta^2\wedge\theta^5\\ \tau_3&=\theta^1\wedge\theta^2+2\theta^4\wedge\theta^3\end{aligned}$$

spanning the 3-dimensional irreducible representation ${}^5\bigwedge^2_3\simeq\mathfrak{so}(3)$ of **SO**(3).

Note that the 3-form ψ can be considered in $\mathbb{R}^8$ without any reference to tensor Υ. It is remarkable that this 3-form *alone* reduces the **GL**$(8,\mathbb{R})$ to the irreducible **SU**(3) in the same way as Υ does.[1] If one thinks that formula (9.1) is written in the orthonormal coframe θ then one gets the reduction from **GL**$(8,\mathbb{R})$ via **SO**(8) to the irreducible **SU**(3). Thus, in dimension 8, the H_k structure can be defined either in terms of the *totally symmetric* Υ or in terms of the *totally skew symmetric* ψ.[2]

Remark 9.1. In this sense the 3-form ψ and the 2-forms (τ_1,τ_2,τ_3) play the same role in the relations between **SU**(3) structures in dimension *eight* and **SO**(3) structures in dimension *five* as the 3-form

$$\phi=\sigma_1\wedge\theta^5+\sigma_2\wedge\theta^6+\sigma_3\wedge\theta^7+\theta^5\wedge\theta^6\wedge\theta^7$$

and the self-dual 2-forms

$$\begin{aligned}\sigma_1&=\theta^1\wedge\theta^3+\theta^4\wedge\theta^2\\ \sigma_2&=\theta^4\wedge\theta^1+\theta^3\wedge\theta^2\\ \sigma_3&=\theta^1\wedge\theta^2+\theta^3\wedge\theta^4\end{aligned}$$

play in the relations [16] between $\mathbf{G}_2$ structures in dimension *seven* and **SU**(2) structures in dimension *four*.

We also note that the **SU**(3)-invariant 3-form ψ can be used to find the explicit decomposition of an arbitrary 3-form ω in $\bigwedge^3\mathbb{R}^8$ onto the irreducible components mentioned in Proposition 7.1. Indeed, given an **SU**(3) structure (M,g,Υ) on an 8-manifold M we may write an arbitrary 3-form ω in the adapted coframe θ of Definition 5.3 as $\omega=\frac{1}{6}\omega_{IJK}\theta^I\wedge\theta^J\wedge\theta^K$. Using $\psi=\frac{1}{6}\psi_{IJK}\theta^I\wedge\theta^J\wedge\theta^K$, we associate with ω a tensor $\psi(\omega)_{IJ}=\psi_{IKL}\omega_{JKL}$. Since $\psi(\omega)$ is an element of $\bigotimes^2\mathbb{R}^8$, it may be analyzed by means of the endomorphism $\hat{\Upsilon}$ naturally associated with $\Upsilon=\Upsilon_{IJK}\theta^I\theta^J\theta^K$ via (4.1). It follows that the 3-form ω is

- in ${}^8\bigodot^2_1$ iff $\hat{\Upsilon}(\psi(\omega))=20\psi(\omega)$,
- in ${}^8\bigodot^2_8$ iff $\hat{\Upsilon}(\psi(\omega))=-6\psi(\omega)$,
- in ${}^8\bigwedge^2_{20}$ iff $\hat{\Upsilon}(\psi(\omega))=-8\psi(\omega)$,
- in ${}^8\bigodot^2_{27}$ iff $\hat{\Upsilon}(\psi(\omega))=4\psi(\omega)$.

Now, if we have a nearly integrable **SU**(3) structure in dimension 8, it is easy to check what is the type of its totally skew symmetric torsion T_{IJK}. For this it suffices to consider a 3-form $T=\frac{1}{6}T_{IJK}\theta^I\wedge\theta^J\wedge\theta^K$, to associate with it $\psi(T)$ and to apply the endomorphism $\hat{\Upsilon}$.

[1] We note that ψ is a stable form in dimension 8 and, as such, was considered by Nigel Hitchin in [11].

[2] Simon Chiossi [8] asks if there is another situation in which a subgroup $H\subset$ **SO**(n) of **GL**$(n,\mathbb{R})$ is a stabilizer of either a totally symmetric tensor or, independently, of a totally skew symmetric tensor. The answer is yes. The fifth exterior power of the 14-dimensional representation ${}^{14}\bigodot^2_{14}$ of **Sp**(3) has a 1-dimensional **Sp**(3)-invariant subspace. Thus, it defines a 5-form whose stabilizer under the action of **GL**$(14,\mathbb{R})$ includes **Sp**(3) ⊂ **SO**(14). It turns out that this 5-form alone, independently of tensor Υ of Section 2, reduces the **GL**$(n,\mathbb{R})$, via **SO**(14) to **Sp**(3). The explicit expression for this form in the adapted coframe of Section 5 is given in Appendix B.

9.1.1. Structures with $(8+l)$-dimensional symmetry group

In the following we will apply the following construction.

Let G be an $(8+l)$-dimensional Lie group with a Lie subgroup H of dimension $l \le 8 = \dim \mathbf{SU}(3)$. We arrange a labeling of a left invariant coframe $(\theta^I, \gamma^\alpha)$ on G in such a way that the vector fields X_α, $\alpha = 1, 2, \dots, l$, of the frame (e_I, X_α) dual to $(\theta^I, \gamma^\alpha)$ generate H. Suppose now that the groups G and H are such that the following tensors

$$g = g_{IJ}\theta^I\theta^J = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 + (\theta^4)^2 + (\theta^5)^2 + (\theta^6)^2 + (\theta^7)^2 + (\theta^8)^2$$

$$\Upsilon = \Upsilon_{IJK} = \frac{1}{2}\det\begin{pmatrix} \theta^5 - \sqrt{3}\theta^4 & \sqrt{3}(\theta^3 + \mathrm{i}\theta^8) & \sqrt{3}(\theta^2 + \mathrm{i}\theta^7) \\ \sqrt{3}(\theta^3 - \mathrm{i}\theta^8) & \theta^5 + \sqrt{3}\theta^4 & \sqrt{3}(\theta^1 + \mathrm{i}\theta^6) \\ \sqrt{3}(\theta^2 - \mathrm{i}\theta^7) & \sqrt{3}(\theta^1 - \mathrm{i}\theta^6) & -2\theta^5 \end{pmatrix} \tag{9.2}$$

are *invariant* on G when Lie transported along the flows of vector fields X_α. In such a case both g and Υ project to well defined nondegenerate tensors on the 8-dimensional homogeneous space $M = G/H$. These projected tensors, which we will also respectively denote by g and Υ, define the $\mathbf{SU}(3)$ structure (M, g, Υ) on M.

9.1.2. Restricted nearly integrable structures with maximal symmetry groups

It follows that the *restricted* nearly integrable $\mathbf{SU}(3)$ structures with *maximal* symmetry groups are locally equivalent to the torsionless models of Theorem 8.1. Thus the possible maximal symmetry groups G are:

$$G_{\lambda>0} = \mathbf{SU}(3) \times \mathbf{SU}(3), \qquad G_{\lambda=0} = \mathbf{SU}(3) \times_\rho \mathbb{R}^8 \quad \text{or} \quad G_{\lambda<0} = \mathbf{SL}(3, \mathbb{C}).$$

The three cases are distinguishable by means of the sign of a real constant λ, which is related to the Ricci scalar of the Levi-Civita/characteristic connection of the corresponding torsionless $\mathbf{SU}(3)$ structure.

We illustrate this statement by using the left invariant coframe $(\theta^I, \gamma^\alpha)$ on G discussed in the preceding section. Here it satisfies the following differential system

$$\mathrm{d}\theta^I + \Gamma^I{}_J \wedge \theta^J = 0, \qquad \mathrm{d}\Gamma^I{}_J + \Gamma^I{}_K \wedge \Gamma^K{}_J = R^I{}_J,$$

where the characteristic connection matrix Γ^I_J is related to the 1-forms γ^α, $\alpha = 1, 2, \dots, 8$, via:

$$\Gamma = \gamma^\alpha E_\alpha = \begin{pmatrix} 0 & -\gamma^1 & -\gamma^2 & -\gamma^3 & -\sqrt{3}\gamma^3 & -\gamma^4 & -\gamma^5 & -\gamma^6 \\ \gamma^1 & 0 & -\gamma^3 & \gamma^2 & -\sqrt{3}\gamma^2 & -\gamma^5 & -\gamma^4 - \gamma^7 & -\frac{\gamma^8}{\sqrt{3}} \\ \gamma^2 & \gamma^3 & 0 & 2\gamma^1 & 0 & \gamma^6 & -\frac{\gamma^8}{\sqrt{3}} & -\gamma^7 \\ \gamma^3 & -\gamma^2 & -2\gamma^1 & 0 & 0 & -\frac{\gamma^8}{\sqrt{3}} & -\gamma^6 & -2\gamma^5 \\ \sqrt{3}\gamma^3 & \sqrt{3}\gamma^2 & 0 & 0 & 0 & -\gamma^8 & \sqrt{3}\gamma^6 & 0 \\ \gamma^4 & \gamma^5 & -\gamma^6 & \frac{\gamma^8}{\sqrt{3}} & \gamma^8 & 0 & -\gamma^1 & \gamma^2 \\ \gamma^5 & \gamma^4 + \gamma^7 & \frac{\gamma^8}{\sqrt{3}} & \gamma^6 & -\sqrt{3}\gamma^6 & \gamma^1 & 0 & -\gamma^3 \\ \gamma^6 & \frac{\gamma^8}{\sqrt{3}} & \gamma^7 & 2\gamma^5 & 0 & -\gamma^2 & \gamma^3 & 0 \end{pmatrix}, \tag{9.3}$$

and the curvature R^I_J is given by

$$R = -\frac{\lambda}{12}\kappa^\alpha E_\alpha,$$

where the 2-forms κ^α are given by

$$\begin{aligned} \kappa^1 &= \theta^1 \wedge \theta^2 - 2\theta^3 \wedge \theta^4 + \theta^6 \wedge \theta^7 \\ \kappa^2 &= \theta^1 \wedge \theta^3 - \theta^2 \wedge \theta^4 + \sqrt{3}\theta^2 \wedge \theta^5 - \theta^6 \wedge \theta^8 \\ \kappa^3 &= \theta^1 \wedge \theta^4 + \sqrt{3}\theta^1 \wedge \theta^5 + \theta^2 \wedge \theta^3 + \theta^7 \wedge \theta^8 \\ \kappa^4 &= 4\theta^1 \wedge \theta^6 + 2\theta^2 \wedge \theta^7 - 2\theta^3 \wedge \theta^8 \\ \kappa^5 &= \theta^1 \wedge \theta^7 + \theta^2 \wedge \theta^6 + 2\theta^4 \wedge \theta^8 \\ \kappa^6 &= \theta^1 \wedge \theta^8 - \theta^3 \wedge \theta^6 + \theta^4 \wedge \theta^7 - \sqrt{3}\theta^5 \wedge \theta^7 \\ \kappa^7 &= -2\theta^1 \wedge \theta^6 + 2\theta^2 \wedge \theta^7 + 4\theta^3 \wedge \theta^8 \\ \kappa^8 &= \sqrt{3}(\theta^2 \wedge \theta^8 + \theta^3 \wedge \theta^7 + \theta^4 \wedge \theta^6 + \sqrt{3}\theta^5 \wedge \theta^6). \end{aligned} \tag{9.4}$$

Now, the system guarantees that the Lie derivatives of the structural tensors g and Υ of (9.2) with respect to all vector fields X_α dual on G to γ^α vanish:

$$L_{X_\alpha} g = 0, \qquad L_{X_\alpha} \Upsilon \equiv 0.$$

Thus, the quotient spaces are equipped with $\mathbf{SU}(3)$ structures locally equivalent to the natural $\mathbf{SU}(3)$ structures on

$$M_{\lambda>0} = \mathbf{SU}(3), \qquad M_{\lambda=0} = \mathbb{R}^8, \qquad M_{\lambda<0} = \mathbf{SL}(3,\mathbb{C})/\mathbf{SU}(3).$$

Since in such cases the Riemannian holonomy group is reduced to $\mathbf{SU}(3)$, the Riemannian curvature coincides with the curvature of the characteristic connection. The Ricci tensor of these curvatures is Einstein, $Ric = \lambda g$.

9.1.3. *Examples with 11-dimensional symmetry group and torsion in* $^8\bigodot^2_{27}$

Below we present a 2-parameter family of restricted nearly integrable $\mathbf{SU}(3)$ structures which are a local deformation of the torsionless model $M_{\lambda>0} = \mathbf{SU}(3)$. These structures have a 11-dimensional symmetry group G described by the Maurer–Cartan coframe $(\theta^I, \gamma^1, \gamma^2, \gamma^3)$ defined below.

$$\begin{aligned}
\mathrm{d}\theta^1 &= \gamma^1\wedge\theta^2 + \gamma^2\wedge\theta^3 + \gamma^3\wedge\theta^4 + \sqrt{3}\gamma^3\wedge\theta^5 + \frac{k}{\sqrt{3}}\kappa^8\\
\mathrm{d}\theta^2 &= -\gamma^1\wedge\theta^1 - \gamma^2\wedge\theta^4 + \sqrt{3}\gamma^2\wedge\theta^5 + \gamma^3\wedge\theta^3 - k\kappa^6\\
\mathrm{d}\theta^3 &= -2\gamma^1\wedge\theta^4 - \gamma^2\wedge\theta^1 - \gamma^3\wedge\theta^2 - k\kappa^5\\
\mathrm{d}\theta^4 &= 2\gamma^1\wedge\theta^3 + \gamma^2\wedge\theta^2 - \gamma^3\wedge\theta^1 + \frac{k}{2}\kappa^7\\
\frac{1}{\sqrt{3}}\mathrm{d}\theta^5 &= -\gamma^2\wedge\theta^2 - \gamma^3\wedge\theta^1 - \frac{k}{6}(2\kappa^4 + \kappa^7)\\
\mathrm{d}\theta^6 &= \gamma^1\wedge\theta^7 - \gamma^2\wedge\theta^8 + (2k-t)\kappa^3 - 2(k+7t)\theta^7\wedge\theta^8\\
\mathrm{d}\theta^7 &= -\gamma^1\wedge\theta^6 + \gamma^3\wedge\theta^8 + (2k-t)\kappa^2 + 2(k+7t)\theta^6\wedge\theta^8\\
\mathrm{d}\theta^8 &= \gamma^2\wedge\theta^6 - \gamma^3\wedge\theta^7 + (2k-t)\kappa^1 - 2(k+7t)\theta^6\wedge\theta^7\\
\mathrm{d}\gamma^1 &= \gamma^2\wedge\gamma^3 + (k+15t)(t-2k)\kappa^1 + (k+15t)(k-t)\theta^6\wedge\theta^7\\
\mathrm{d}\gamma^2 &= -\gamma^1\wedge\gamma^3 + (k+15t)(t-2k)\kappa^2 - (k+15t)(k-t)\theta^6\wedge\theta^8\\
\mathrm{d}\gamma^3 &= \gamma^1\wedge\gamma^2 + (k+15t)(t-2k)\kappa^3 + (k+15t)(k-t)\theta^7\wedge\theta^8.
\end{aligned}$$

The 2-forms κ^α appearing here are given in (9.4); k and t are real constants.

It is easy to check that in all directions spanned by the three vector fields X_α dual to γ^α we have

$$L_{X_\alpha} g = 0, \qquad L_{X_\alpha} \Upsilon = 0,$$

where the structural tensors g and Υ are given by (9.2). Thus the quotient 8-manifold $M = G/H$, where H is generated by X_α, is equipped with an $\mathbf{SU}(3)$ structure (M, g, Υ). As mentioned above, this structure is restricted nearly integrable. It has the characteristic connection Γ given by (9.3) with

$$\gamma^4 = (k-t)(\theta^4 + \sqrt{3}\theta^5), \qquad \gamma^5 = (k-t)\theta^3, \qquad \gamma^6 = (k-t)\theta^2,$$
$$\gamma^7 = 2(t-k)\theta^4, \qquad \gamma^8 = \sqrt{3}(t-k)\theta^1.$$

The torsion T of Γ is of a pure type. It lies in the 27-dimensional representation $^8\bigodot^2_{27}$. The torsion 3-form T reads:

$$\begin{aligned}
T = t(&\theta^1\wedge\theta^2\wedge\theta^8 + \theta^1\wedge\theta^3\wedge\theta^7 + \theta^1\wedge\theta^4\wedge\theta^6 + \sqrt{3}\theta^1\wedge\theta^5\wedge\theta^6 + \theta^2\wedge\theta^3\wedge\theta^6\\
&- \theta^2\wedge\theta^4\wedge\theta^7 + \sqrt{3}\theta^2\wedge\theta^5\wedge\theta^7 - 2\theta^3\wedge\theta^4\wedge\theta^8 - 15\theta^6\wedge\theta^7\wedge\theta^8).
\end{aligned}$$

Remarkably this form is coclosed, so the Ricci tensor Ric^Γ of the characteristic connection is *symmetric*. Moreover, it is diagonal,

$$\mathrm{Ric}^\Gamma = \mathrm{diag}(\lambda, \lambda, \lambda, \lambda, \lambda, \mu, \mu, \mu),$$

with two constant eigenvalues

$$\lambda = 12(k^2 + 15kt - 8t^2), \qquad \mu = 12\left(k^2 + \frac{5}{3}kt\right).$$

These two eigenvalues coincide when $t = 0$ and $t = \frac{5}{3}k$. In the first case the $\mathbf{SU}(3)$ structure is locally equivalent to the torsionless model $M_{\lambda>0} = \mathbf{SU}(3)$. The case

$$t = \frac{5}{3}k$$

is interesting since it provides an example of a restricted nearly integrable $\mathbf{SU}(3)$ structure with the characteristic connection Γ satisfying the Einstein equations

$$\mathrm{Ric}^{\Gamma}=\frac{136}{3}k^2 g$$

and having torsion of a nontrivial pure type ${}^8\bigodot^2_{27}$.

We further note that for all values of t and k the Ricci tensor for the Levi-Civita connection of this structure is also diagonal,

$$\mathrm{Ric}^{LC}=\mathrm{diag}(\lambda',\lambda',\lambda',\lambda',\lambda',\mu',\mu',\mu'),$$

with eigenvalues given by $\lambda'=\lambda+3t^2$, $\mu'=\mu+115t^2$. This Ricci tensor is Einstein in the torsionless case $t=0$ and when

$$t=\frac{10}{13}k.$$

In this later case the Ricci tensor reads

$$\mathrm{Ric}^{LC}=\frac{16128}{169}k^2 g.$$

9.1.4. Examples with 9-dimensional symmetry group and vectorial torsion

Let G be a 9-dimensional group with a left invariant coframe (θ^I,γ^1) on it as in Section 9.1.1. Let (e_I,X_1) be a basis of vector fields on G dual to (θ^I,γ^1). We assume that (θ^I,γ^1) satisfies the following differential system

$$\begin{aligned}
\mathrm{d}\theta^1&=\gamma^1\wedge\theta^2-\frac{1}{2}t_1\theta^1\wedge\theta^3+\frac{1}{2}t_1\theta^2\wedge\theta^4-\frac{1}{2\sqrt{3}}t_1\theta^2\wedge\theta^5+\frac{1}{2}t_2\theta^3\wedge\theta^7+\frac{1}{2}t_2\theta^4\wedge\theta^6\\
\mathrm{d}\theta^2&=-\gamma^1\wedge\theta^1+\frac{1}{2}t_1\theta^1\wedge\theta^4+\frac{1}{2\sqrt{3}}t_1\theta^1\wedge\theta^5+\frac{1}{2}t_1\theta^2\wedge\theta^3+\frac{1}{2}t_2\theta^3\wedge\theta^6-\frac{1}{2}t_2\theta^4\wedge\theta^7\\
\mathrm{d}\theta^3&=-2\gamma^1\wedge\theta^4+\frac{1}{\sqrt{3}}t_1\theta^4\wedge\theta^5\\
\mathrm{d}\theta^4&=2\gamma^1\wedge\theta^3-\frac{1}{\sqrt{3}}t_1\theta^3\wedge\theta^5\\
\mathrm{d}\theta^5&=\frac{1}{\sqrt{3}}t_1\theta^1\wedge\theta^2-\frac{1}{\sqrt{3}}t_2\theta^1\wedge\theta^6-\frac{1}{\sqrt{3}}t_2\theta^2\wedge\theta^7+\frac{1}{\sqrt{3}}t_1\theta^3\wedge\theta^4+\frac{1}{\sqrt{3}}t_1\theta^6\wedge\theta^7\\
\mathrm{d}\theta^6&=\gamma^1\wedge\theta^7+\frac{1}{2}t_2\theta^1\wedge\theta^4+\frac{1}{2}t_2\theta^2\wedge\theta^3+\frac{1}{2}t_1\theta^3\wedge\theta^6-\frac{1}{2}t_1\theta^4\wedge\theta^7+\frac{1}{2\sqrt{3}}t_1\theta^5\wedge\theta^7\\
\mathrm{d}\theta^7&=-\gamma^1\wedge\theta^6+\frac{1}{2}t_2\theta^1\wedge\theta^3-\frac{1}{2}t_2\theta^2\wedge\theta^4-\frac{1}{2}t_1\theta^3\wedge\theta^7-\frac{1}{2}t_1\theta^4\wedge\theta^6-\frac{1}{2\sqrt{3}}t_1\theta^5\wedge\theta^6\\
\mathrm{d}\theta^8&=-t_2\theta^3\wedge\theta^4\\
\mathrm{d}\gamma^1&=-\frac{1}{6}t_1^2\theta^1\wedge\theta^2+\frac{1}{6}t_1t_2\theta^1\wedge\theta^6+\frac{1}{6}t_1t_2\theta^2\wedge\theta^7+\frac{1}{6}(3t_2^2-4t_1^2)\theta^3\wedge\theta^4-\frac{1}{6}t_1^2\theta^6\wedge\theta^7.
\end{aligned}$$

Here the real parameters t_1,t_2 are constants.

Let H be a 1-parameter subgroup of G generated by the vector field X_1. Now we consider g and Υ of (9.2). It is easy to check that the above differential equations for the system (θ^I,γ^1) guarantee that on G the Lie derivatives with respect to X_1 of g and Υ identically vanish:

$$L_{X_1}g\equiv 0,\qquad L_{X_1}\Upsilon\equiv 0.$$

Thus on the homogeneous space $M=G/H$ we have an $\mathbf{SU}(3)$ structure (M,g,Υ). This $\mathbf{SU}(3)$ structure has the following properties.

- It is a restricted nearly integrable structure.
- It has a 9-dimensional symmetry group G.
- Its characteristic connection is given by (9.3), where

$$\gamma^2=-\frac{1}{2}t_1\theta^1,\qquad \gamma^3=\frac{1}{2}t_1\theta^2,\qquad \gamma^4=\frac{1}{2}t_2\theta^4-\frac{1}{\sqrt{3}}t_2\theta^5+\frac{1}{2}t_1\theta^8,$$

$$\gamma^5=\frac{1}{2}t_2\theta^3,\qquad \gamma^6=-\frac{1}{2}t_1\theta^6,\qquad \gamma^7=-t_2\theta^4,\qquad \gamma^8=-\frac{\sqrt{3}}{2}t_1\theta^7.$$

- The skew symmetric torsion T of Γ is of purely 'vectorial' type: $T \in {}^8\bigodot^2_8$. Explicitly the torsion 3-form is

$$T = t_1\left(-\frac{2}{\sqrt{3}}\theta^1\wedge\theta^2\wedge\theta^5 + \theta^1\wedge\theta^6\wedge\theta^8 + \theta^2\wedge\theta^7\wedge\theta^8 + \frac{1}{\sqrt{3}}\theta^3\wedge\theta^4\wedge\theta^5 - \frac{2}{\sqrt{3}}\theta^5\wedge\theta^6\wedge\theta^7\right) + t_2\left(\frac{1}{\sqrt{3}}\theta^1\wedge\theta^5\wedge\theta^6 + \frac{1}{\sqrt{3}}\theta^2\wedge\theta^5\wedge\theta^7 + \theta^3\wedge\theta^4\wedge\theta^8\right).$$

- The Ricci tensor Ric^Γ of the characteristic connection is symmetric:

$$\mathrm{Ric}^\Gamma = \begin{pmatrix} -\frac{1}{3}(4t_1^2+t_2^2) & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3}t_1t_2 & 0 \\ 0 & -\frac{1}{3}(4t_1^2+t_2^2) & 0 & 0 & 0 & \frac{2}{3}t_1t_2 & 0 & 0 \\ 0 & 0 & -\frac{7}{3}t_1^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{7}{3}t_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -t_1^2 & 0 & 0 & 0 \\ 0 & \frac{2}{3}t_1t_2 & 0 & 0 & 0 & -\frac{1}{3}(4t_1^2+t_2^2) & 0 & 0 \\ -\frac{2}{3}t_1t_2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}(4t_1^2+t_2^2) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t_1^2 \end{pmatrix};$$

hence the torsion 3-form T is coclosed.

- The Ricci tensor Ric^{LC} of the Levi-Civita connection is

$$\mathrm{Ric}^{LC} = \begin{pmatrix} -\frac{1}{6}(t_1^2+t_2^2) & 0 & 0 & 0 & 0 & 0 & -\frac{4}{3}t_1t_2 & 0 \\ 0 & -\frac{1}{6}(t_1^2+t_2^2) & 0 & 0 & 0 & \frac{4}{3}t_1t_2 & 0 & 0 \\ 0 & 0 & \frac{1}{6}(-13t_1^2+3t_2^2) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6}(-13t_1^2+3t_2^2) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6}(3t_1^2+2t_2^2) & 0 & 0 & -\frac{1}{2\sqrt{3}}t_1t_2 \\ 0 & \frac{4}{3}t_1t_2 & 0 & 0 & 0 & -\frac{1}{6}(t_1^2+t_2^2) & 0 & 0 \\ -\frac{4}{3}t_1t_2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6}(t_1^2+t_2^2) & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}}t_1t_2 & 0 & 0 & \frac{1}{2}t_2^2 \end{pmatrix}.$$

We note that if $t_1 = 0$ or $t_2 = 0$ both the Ricci tensors Ric^Γ and Ric^{LC} in the example above are diagonal. Below we present another 2-parameter family of examples with this property.

Now the 9-dimensional group G has the basis of the left invariant forms (θ^I, γ^1) such that:

$$\begin{aligned}
\mathrm{d}\theta^1 &= \gamma^1\wedge\theta^2 - c\theta^2\wedge\theta^8 + (t-4c)\left(\frac{1}{2}\theta^3\wedge\theta^7 + \frac{1}{2}\theta^4\wedge\theta^6 + \frac{3c\sqrt{3}}{6c-t}\theta^5\wedge\theta^6\right)\\
\mathrm{d}\theta^2 &= -\gamma^1\wedge\theta^1 + c\theta^1\wedge\theta^8 + (t-4c)\left(\frac{1}{2}\theta^3\wedge\theta^6 - \frac{1}{2}\theta^4\wedge\theta^7 + \frac{3c\sqrt{3}}{6c-t}\theta^5\wedge\theta^7\right)\\
\mathrm{d}\theta^3 &= -2\gamma^1\wedge\theta^4 + 2c(\theta^1\wedge\theta^7 + \theta^2\wedge\theta^6 + \theta^4\wedge\theta^8)\\
\mathrm{d}\theta^4 &= 2\gamma^1\wedge\theta^3 + 2c(\theta^1\wedge\theta^6 - \theta^2\wedge\theta^7 - \theta^3\wedge\theta^8)\\
\mathrm{d}\theta^5 &= \frac{6c-t}{\sqrt{3}}(\theta^1\wedge\theta^6 + \theta^2\wedge\theta^7)\\
\mathrm{d}\theta^6 &= \gamma^1\wedge\theta^7 + (t-4c)\left(\frac{1}{2}\theta^1\wedge\theta^4 + \frac{3c\sqrt{3}}{6c-t}\theta^1\wedge\theta^5 + \frac{1}{2}\theta^2\wedge\theta^3\right) - c\theta^7\wedge\theta^8\\
\mathrm{d}\theta^7 &= -\gamma^1\wedge\theta^6 + (t-4c)\left(\frac{1}{2}\theta^1\wedge\theta^3 - \frac{1}{2}\theta^2\wedge\theta^4 + \frac{3c\sqrt{3}}{6c-t}\theta^2\wedge\theta^5\right) + c\theta^6\wedge\theta^8\\
\mathrm{d}\theta^8 &= -2c\theta^1\wedge\theta^2 + (4c-t)\theta^3\wedge\theta^4 - 2c\theta^6\wedge\theta^7\\
\mathrm{d}\gamma^1 &= (2c-t)\left(-c\theta^1\wedge\theta^2 + \frac{1}{2}(4c-t)\theta^3\wedge\theta^4 - c\theta^6\wedge\theta^7\right),
\end{aligned}$$

where c and t are constants. The homogeneous space $M = G/H$, where H is a 1-dimensional subgroup of G generated by X_1 dual to γ^1, is equipped with a restricted nearly integrable $\mathbf{SU}(3)$ structure (M, g, Υ) via (9.2). The characteristic connection is given by (9.3) with

$$\gamma^2 = c\theta^7, \quad \gamma^3 = c\theta^6, \quad \gamma^4 = \frac{1}{2}(t-2c)\theta^4 + \frac{1}{\sqrt{3}}\frac{t^2-18c^2}{6c-t}\theta^5,$$
$$\gamma^5 = \frac{1}{2}(t-2c)\theta^3, \quad \gamma^6 = -c\theta^2, \quad \gamma^7 = (2c-t)\theta^4, \quad \gamma^8 = \sqrt{3}c\theta^1.$$

In this 2-parameter family of examples the skew symmetric torsion is again of purely vectorial type $T \in {}^8\bigodot^2_8$; its corresponding 3-form is given by

$$T = t\left(\frac{1}{\sqrt{3}}\theta^1\wedge\theta^5\wedge\theta^6 + \frac{1}{\sqrt{3}}\theta^2\wedge\theta^5\wedge\theta^7 + \theta^3\wedge\theta^4\wedge\theta^8\right).$$

As announced above both the Ricci tensors are now diagonal for all values of c and t. Introducing the eigenvalues

$$\lambda = \frac{1}{3}[(6c-t)^2 - 2t^2], \qquad \mu = 4c(3c-t),$$
$$\lambda' = \frac{1}{3}\left[(6c-t)^2 - \frac{3}{2}t^2\right], \qquad \mu' = \frac{1}{3}(6c-t)^2, \qquad \nu' = \frac{1}{3}\left[(6c-t)^2 + \frac{1}{2}t^2\right]$$

we have

$$\mathrm{Ric}^{\Gamma} = \mathrm{diag}(\lambda,\lambda,\mu,\mu,\mu,\lambda,\lambda,\mu), \qquad \mathrm{Ric}^{LC} = \mathrm{diag}(\lambda',\lambda',\nu',\nu',\mu',\lambda',\lambda',\nu').$$

Of course the group G is a symmetry group of this restricted nearly integrable **SU**(3) structure.

We close this section with the following theorem, whose proof based on the Bianchi identities, is purely computational.

Theorem 9.2. *Let (M, g, Υ) be an arbitrary restricted nearly integrable* **SU**(3) *structure in dimension eight. Assume that the torsion T of the characteristic connection of this structure is of purely vectorial type, $T \in {}^8\bigodot^2_8$. Then the 3-form T corresponding to the torsion is coclosed*

$$\mathrm{d}(*T) \equiv 0.$$

This theorem, in particular, implies that the Ricci tensor of the characteristic connection for such structures is symmetric.

9.2. **SU**(2) × **SU**(2) *structures*

In this section we consider **SU**(2) × **SU**(2) structures in dimension eight modeled on the torsionless structure $\mathbf{G}_2/(\mathbf{SU}(2)\times\mathbf{SU}(2))$. The approach presented here should be useful in studies of the other exceptional case concerning the **SO**(10) × **SO**(2) structures in dimension 32.

In full analogy with H_k structures we start with the identification of $\mathbb{R}^8$ with a space of the antisymmetric block matrices $M_{7\times7}(\mathbb{R}) \ni \iota(\vec{X}) = \begin{pmatrix}\mathbf{0}_{3\times3} & \alpha\\ -\alpha^t & \mathbf{0}_{4\times4},\end{pmatrix}$, in which the matrices $\alpha \in M_{3\times4}(\mathbb{R})$ have 3 rows and 4 columns. The entries of α satisfy the following four relations

$$\alpha_{16} - \alpha_{34} - \alpha_{25} = 0, \qquad \alpha_{26} - \alpha_{37} + \alpha_{15} = 0,$$
$$\alpha_{36} + \alpha_{27} + \alpha_{14} = 0, \qquad \alpha_{35} + \alpha_{17} - \alpha_{24} = 0. \tag{9.5}$$

These four relations reduce the 12 free parameters present in an arbitrary 3 × 4 matrix to 8 parameters. Now defining

$$\mathbb{M}^8 = \left\{\iota(\vec{X}) \in M_{7\times7}(\mathbb{R}) :\ \iota(\vec{X}) = \begin{pmatrix}\mathbf{0}_{3\times3} & \alpha\\ -\alpha^t & \mathbf{0}_{4\times4},\end{pmatrix} \text{ with } \alpha \text{ satisfying (9.5)}\right\}, \tag{9.6}$$

we have an isomorphism $\iota : \mathbb{R}^8 \to \mathbb{M}^8$ between the vector spaces $\mathbb{R}^8$ and $\mathbb{M}^8$.

Now we define a representation ρ of the group **SU**(2) × **SU**(2) in $\mathbb{R}^8$, which will enable us to define an **SU**(2) × **SU**(2) structure in dimension eight.

We use two different representations of **SU**(2) in dimension seven: The representation ρ_1 generated by 7 × 7-matrices

$$h_i = \exp(t_i s_i), \quad i = 1,2,3,\ t_i \in \mathbb{R} \text{ (no summation!)},$$

such that

$$s_1 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&\frac{1}{2}&0&0\\ 0&0&0&-\frac{1}{2}&0&0&0\\ 0&0&0&0&0&0&-\frac{1}{2}\\ 0&0&0&0&0&\frac{1}{2}&0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&\frac{1}{2}&0\\ 0&0&0&0&0&0&\frac{1}{2}\\ 0&0&0&-\frac{1}{2}&0&0&0\\ 0&0&0&0&-\frac{1}{2}&0&0 \end{pmatrix},$$

$$s_3=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&\frac12\\0&0&0&0&0&-\frac12&0\\0&0&0&0&\frac12&0&0\\0&0&0&-\frac12&0&0&0\end{pmatrix},$$

and the representation ρ_2 generated by 7×7 matrices

$$\chi_i=\exp(\tau_i\sigma_i),\quad i=1,2,3,\ \tau_i\in\mathbb{R}\,(\text{no summation!}),$$

such that

$$\sigma_1=\begin{pmatrix}0&-1&0&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&-\frac12\\0&0&0&0&0&-\frac12&0\\0&0&0&0&\frac12&0&0\\0&0&0&\frac12&0&0&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&0&1&0&0&0&0\\0&0&0&0&0&0&0\\-1&0&0&0&0&0&0\\0&0&0&0&0&\frac12&0\\0&0&0&0&0&0&-\frac12\\0&0&0&-\frac12&0&0&0\\0&0&0&0&\frac12&0&0\end{pmatrix},$$

$$\sigma_3=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&1&0&0&0&0\\0&-1&0&0&0&0&0\\0&0&0&0&-\frac12&0&0\\0&0&0&\frac12&0&0&0\\0&0&0&0&0&0&-\frac12\\0&0&0&0&0&\frac12&0\end{pmatrix}.$$

We note that

$$[s_j,s_k]=\epsilon_{ijk}s_i,\qquad [\sigma_j,\sigma_k]=\epsilon_{ijk}\sigma_i,\qquad [s_i,\sigma_j]=0,\quad i,j,k=1,2,3,$$

where ϵ_{ijk} is the Levi-Civita symbol in 3-dimensions.

Now, we consider all the 7×7-matrices of the form

$$h=h(t_1,t_2,t_3,\tau_1,\tau_2\tau_3)=h_1h_2h_3\chi_1\chi_2\chi_3.$$

They constitute a 7-dimensional representation ρ_7 of the full group $\mathbf{SU}(2)\times\mathbf{SU}(2)$.

Remarkably, $h\iota(\vec{X})h^t$ is an element of $\mathbb{M}^8$ for all the elements $\iota(\vec{X})$ of $\mathbb{M}^8$. Moreover, due to the fact that $[s_i,\sigma_j]=0$ for all i,j, the map

$$(\mathbf{SU}(2)\times\mathbf{SU}(2))\times\mathbb{M}^8\ni(h,\iota(\vec{X}))\mapsto h\iota(\vec{X})h^t\in\mathbb{M}^8$$

is a good action of $\mathbf{SU}(2)\times\mathbf{SU}(2)$ on $\mathbb{M}^8$. Thus, using the isomorphism ι we get the 8-dimensional representation ρ of $\mathbf{SU}(2)\times\mathbf{SU}(2)$ given by

$$\mathbb{R}^8\ni\vec{X}\mapsto\rho(h)\vec{X}=\iota^{-1}[h\iota(\vec{X})h^t]\in\mathbb{R}^8.\tag{9.7}$$

Given an element $\vec{X}\in\mathbb{R}^8$ we consider its characteristic polynomial

$$\begin{aligned}P_{\vec{X}}(\lambda)&=\det(\iota(\vec{X})-\lambda I)\\&=-\lambda^7-6g(\vec{X},\vec{X})\lambda^5-9g(\vec{X},\vec{X})^2\lambda^3+2\gamma(\vec{X},\vec{X},\vec{X},\vec{X},\vec{X},\vec{X})\lambda.\end{aligned}\tag{9.8}$$

This polynomial is invariant under the $\mathbf{SU}(2)\times\mathbf{SU}(2)$-action given by the representation ρ of (9.7),

$$P_{\rho(h)\vec{X}}(\lambda)=P_{\vec{X}}(\lambda).$$

Thus, all the coefficients of $P_{\vec{X}}(\lambda)$, which are multilinear in $\vec{X}$, are $\mathbf{SU}(2)\times\mathbf{SU}(2)$-invariant.

It is convenient to use a basis $\mathbf{e}_I$ in $\mathbb{R}^8$ such that the isomorphism $\iota : \mathbb{R}^8 \to \mathbb{M}^8$ takes the form:

$$\vec{X} = x^I\mathbf{e}_I \mapsto \iota(\vec{X}) = \begin{pmatrix} 0 & 0 & 0 & -x^3+\sqrt{3}x^4 & -x^5+\sqrt{3}x^6 & -x^7+\sqrt{3}x^8 & -2x^1 \\ 0 & 0 & 0 & -x^1-\sqrt{3}x^2 & x^7+\sqrt{3}x^8 & -x^5-\sqrt{3}x^6 & 2x^3 \\ 0 & 0 & 0 & -2x^7 & x^1-\sqrt{3}x^2 & -x^3-\sqrt{3}x^4 & -2x^5 \\ x^3-\sqrt{3}x^4 & x^1+\sqrt{3}x^2 & 2x^7 & 0 & 0 & 0 & 0 \\ x^5-\sqrt{3}x^6 & -x^7-\sqrt{3}x^8 & -x^1+\sqrt{3}x^2 & 0 & 0 & 0 & 0 \\ x^7-\sqrt{3}x^8 & x^5+\sqrt{3}x^6 & x^3+\sqrt{3}x^4 & 0 & 0 & 0 & 0 \\ 2x^1 & -2x^3 & 2x^5 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

With this choice the bilinear form g of (9.8) reads:

$$g(\vec{X},\vec{X}) = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 + (x^7)^2 + (x^8)^2.$$

The 6-linear form γ of (9.8) defines a tensor γ_{IJKLMN} via

$$\gamma(\vec{X},\vec{X},\vec{X},\vec{X},\vec{X},\vec{X}) = \gamma_{IJKLMN}x^Ix^Jx^Kx^Lx^Mx^N.$$

This tensor has the following properties.

- It reduces $\mathbf{GL}(8,\mathbb{R})$, via $\mathbf{SO}(8)$, to $\mathbf{SU}(2)\times\mathbf{SU}(2)$.
- The 6th order polynomial

  $$\Phi = \gamma(\vec{X},\vec{X},\vec{X},\vec{X},\vec{X},\vec{X})$$

  of variables $x^I, I = 1,2,\ldots,8$, satisfies

  (a) $\triangle\Phi = -72g(\vec{X},\vec{X})^2$

  (b) $|\vec{\nabla}\Phi|^2 = -72\Phi g(\vec{X},\vec{X})^2$

  (c) $\vec{X}\vec{\nabla}\Phi = 6\Phi$.

The properties (a)–(c) show that Φ cannot be interpreted as the Cartan polynomial (3.1) and (3.2) defining an isoparametric hypersurface in S^7. But we can modify it so that the redefined polynomial satisfies (3.1) and (3.2). Indeed, using properties (a)–(c) it is easy to see that the 6th order homogeneous polynomial $F = \Phi + g(\vec{X},\vec{X})^3$ is a solution of

(Cii) $\triangle F = 0$

(Ciii) $|\vec{\nabla}F|^2 = 6^2g(\vec{X},\vec{X})^5$.

Thus, via (3.3), the polynomial F defines an isoparametric hypersurface in S^7 which has $p = 6$ distinct constant principal eigenvalues. Note that since both Φ and $g(\vec{X},\vec{X})$ are $\mathbf{SU}(2)\times\mathbf{SU}(2)$-invariant, the polynomial F is also so. Hence a stabilizer, under the action of $\mathbf{GL}(8,\mathbb{R})$, of a 6th order symmetric tensor Υ_{IJKLMN} defined by

$$F = \Phi + g(\vec{X},\vec{X})^3 = \Upsilon(\vec{X},\vec{X},\vec{X},\vec{X},\vec{X},\vec{X}) = \Upsilon_{IJKLMN}x^Ix^Jx^Kx^Lx^Mx^N \tag{9.9}$$

contains the group $\mathbf{SU}(2)\times\mathbf{SU}(2)$. Actually we have the following proposition.

Proposition 9.3. *The 6th order symmetric tensor Υ_{IJKLMN} defined above reduces the $\mathbf{GL}(8,\mathbb{R})$ group, via $\mathbf{SO}(8)$, to the irreducible $\mathbf{SU}(2)\times\mathbf{SU}(2)$ associated with the representation ρ of (9.7).*

Following the case of H_k structures we use the tensor Υ_{IJKLMN} of (9.9) to define an endomorphism

$$\hat{\Upsilon} : \bigotimes^2\mathbb{R}^8 \longrightarrow \bigotimes^2\mathbb{R}^8, \qquad W^{IK} \stackrel{\hat{\Upsilon}}{\longmapsto} \frac{5^2}{2^5}\Upsilon_{IJMNPQ}\Upsilon_{KLMNPQ}W^{JL}, \tag{9.10}$$

which preserves the decomposition $\bigotimes^2\mathbb{R}^8 = \bigwedge^2\mathbb{R}^8 \oplus \bigodot^2\mathbb{R}^8$. Its eigenspaces, are $\mathbf{SU}(2)\times\mathbf{SU}(2)$-invariant and define representations of dimension 1, 5, 6, 7, 9, 15, 21. Explicitly we have the following proposition.

Proposition 9.4. *The $\mathbf{SU}(2)\times\mathbf{SU}(2)$ irreducible decomposition of $\bigotimes^2\mathbb{R}^8$ is given by*

$$\bigotimes^2\mathbb{R}^8 = \bigodot^2_1 \oplus \bigodot^2_5 \oplus \bigodot^2_9 \oplus \bigodot^2_{21} \oplus \bigwedge^2_6 \oplus \bigwedge^2_7 \oplus \bigwedge^2_{15},$$

where

$$\bigodot^2_1 = \{S \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(S) = 175\cdot S\} = \{S = \lambda\cdot g,\ \lambda\in\mathbb{R}\},$$
$$\bigodot^2_5 = \{S \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(S) = -21\cdot S\},$$
$$\bigwedge^2_6 = \{F \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(F) = 35\cdot F\} = \mathfrak{su}(2)\oplus\mathfrak{su}(2),$$
$$\bigwedge^2_7 = \{F \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(F) = -25\cdot F\},$$
$$\bigodot^2_9 = \{S \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(S) = 7\cdot S\},$$
$$\bigwedge^2_{15} = \{F \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(F) = -49\cdot F\},$$
$$\bigodot^2_{21} = \{S \in \bigotimes^2\mathbb{R}^8 \mid \hat{\Upsilon}(S) = 27\cdot S\}.$$

The real vector spaces $\bigwedge^2_i \subset \bigwedge^2 \mathbb{R}^8$ *and* $\bigodot^2_j \subset \bigodot^2 \mathbb{R}^8$ *of respective dimensions i and j are irreducible representations of the group* $\mathbf{SU}(2) \times \mathbf{SU}(2)$.

Remark 9.5. Note that $\bigwedge^2_6$, is isomorphic to the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ represented as the Lie subalgebra of 8×8-matrices. In this 8-dimensional representation ρ' the bases of the two $\mathfrak{su}(2)$ algebras are, respectively, Σ^L_i and Σ^R_i, $i = 1, 2, 3$, where

$$\Sigma^L_1 = \frac{1}{4}\begin{pmatrix} 0 & 0 & -5 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 3 & 0 & 0 & 0 & 0 \\ 5 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 \end{pmatrix},$$

$$\Sigma^L_2 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & -5 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\sqrt{3} & -3 \\ 5 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 3 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma^L_3 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -3\sqrt{3} & -3 \\ 0 & 0 & 0 & 0 & 5 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 3 & 0 & 0 \\ 0 & 0 & -5 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & -3 & 0 & 0 & 0 & 0 \\ 1 & 3\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma^R_1 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & 0 & 0 & 1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & -1 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 1 & 0 & 0 & 0 & 0 \\ 1 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma^R_2 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & -1 \\ -1 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma^R_3 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & -\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 1 & 0 & 0 & 0 & 0 \\ -1 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \end{pmatrix}$$

and we have

$$[\Sigma^L_j, \Sigma^L_k] = -\epsilon_{ijk}\Sigma^L_i, \qquad [\Sigma^R_j, \Sigma^R_k] = -\epsilon_{ijk}\Sigma^R_i, \qquad [\Sigma^L_i, \Sigma^R_j] = 0, \quad i, j, k = 1, 2, 3.$$

Of course the Lie algebra representation ρ' is the derivative of the 8-dimensional representation ρ of $\mathbf{SU}(2) \times \mathbf{SU}(2)$ considered in (9.7).

Now, we define the $\mathbf{SU}(2) \times \mathbf{SU}(2)$ structure on an 8-manifold as a structure equipped with the Riemannian metric g and the 6-tensor Υ, which in an orthonormal coframe θ^I is given by $\Upsilon = \Upsilon_{IJKLMN}\theta^I\theta^J\theta^K\theta^L\theta^M\theta^N$ with Υ_{IJKLMN} of (9.9). Having this, we may use the above proposition (and the basis of $\bigwedge^2_6 = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ given above) as the starting point for addressing the question about the description of such structures in terms of $(\mathfrak{su}(2) \oplus \mathfrak{su}(2))$-valued connections. Regardless of the open question if and when the characteristic connection for such structures exists, the torsionless models here will locally be isometric to the symmetric spaces $\mathbf{G}_2/(\mathbf{SU}(2) \times \mathbf{SU}(2))$, $\mathbb{R}^8 = [(\mathbf{SU}(2) \times \mathbf{SU}(2)) \times_\rho \mathbb{R}^8]/(\mathbf{SU}(2) \times \mathbf{SU}(2))$ and $\mathbf{G}^2_2/(\mathbf{SU}(2) \times \mathbf{SU}(2))$ with the standard $\mathbf{SU}(2) \times \mathbf{SU}(2)$ structure on them.

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Appendix A. Basis for the n_k-dimensional representations of the Lie algebras $\mathfrak{h}_k$

Here we give the explicit formulae for the generic elements X_{n_k} of the Lie algebras $\mathfrak{h}_k$ in terms of the $n_k \times n_k$ antisymmetric matrices. We denote the basis of the Lie algebra $\mathfrak{h}_k$ by E_α, $\alpha = 1, 2, \ldots, \dim H_k$ and write

$$X_{n_k} = x^\alpha E_\alpha.$$

The explicit form of the matrices E_α for each value of $n_k = 5, 8, 14$ and 26 can be read off from the formulae below.

For $n_k = 5$ we have:

$$X_5 = \begin{pmatrix} 0 & -x^1 & -x^2 & -x^3 & -\sqrt{3}x^3 \\ x^1 & 0 & -x^3 & x^2 & -\sqrt{3}x^2 \\ x^2 & x^3 & 0 & 2x^1 & 0 \\ x^3 & -x^2 & -2x^1 & 0 & 0 \\ \sqrt{3}x^3 & \sqrt{3}x^2 & 0 & 0 & 0 \end{pmatrix}. \tag{A.1}$$

For $n_k = 8$ we have:

$$X_8 = \begin{pmatrix} 0 & -x^1 & -x^2 & -x^3 & -\sqrt{3}x^3 & -x^4 & -x^5 & -x^6 \\ x^1 & 0 & -x^3 & x^2 & -\sqrt{3}x^2 & -x^5 & -x^4 - x^7 & -\frac{x^8}{\sqrt{3}} \\ x^2 & x^3 & 0 & 2x^1 & 0 & x^6 & -\frac{x^8}{\sqrt{3}} & -x^7 \\ x^3 & -x^2 & -2x^1 & 0 & 0 & -\frac{x^8}{\sqrt{3}} & -x^6 & -2x^5 \\ \sqrt{3}x^3 & \sqrt{3}x^2 & 0 & 0 & 0 & -x^8 & \sqrt{3}x^6 & 0 \\ x^4 & x^5 & -x^6 & \frac{x^8}{\sqrt{3}} & x^8 & 0 & -x^1 & x^2 \\ x^5 & x^4 + x^7 & \frac{x^8}{\sqrt{3}} & x^6 & -\sqrt{3}x^6 & x^1 & 0 & -x^3 \\ x^6 & \frac{x^8}{\sqrt{3}} & x^7 & 2x^5 & 0 & -x^2 & x^3 & 0 \end{pmatrix}. \tag{A.2}$$

For $n_k = 14$ we have:

$$X_{14}=\begin{pmatrix}
0 & -x^1 & -x^2 & -x^3 & -\sqrt{3}x^3 & -x^4 & -x^5 & -x^6 & -x^9 & -x^{10} & -x^{11} & -x^{12} & -x^{13} & -x^{14}\\
x^1 & 0 & -x^3 & x^2 & -\sqrt{3}x^2 & -x^5 & -x^4-x^7 & -\frac{x^8}{\sqrt{3}} & -x^{11} & -x^{12} & -x^9-x^{15} & -x^{10}-x^{16} & -\frac{x^{17}}{\sqrt{3}} & -\frac{x^{18}}{\sqrt{3}}\\
x^2 & x^3 & 0 & 2x^1 & 0 & x^6 & -\frac{x^8}{\sqrt{3}} & -x^7 & x^{13} & x^{14} & -\frac{x^{17}}{\sqrt{3}} & -\frac{x^{18}}{\sqrt{3}} & -x^{15} & -x^{16}\\
x^3 & -x^2 & -2x^1 & 0 & 0 & -\frac{x^8}{\sqrt{3}} & -x^6 & -2x^5 & -\frac{x^{17}}{\sqrt{3}} & -\frac{x^{18}}{\sqrt{3}} & -x^{13} & -x^{14} & -2x^{11} & -2x^{12}\\
\sqrt{3}x^3 & \sqrt{3}x^2 & 0 & 0 & 0 & -x^8 & \sqrt{3}x^6 & 0 & -x^{17} & -x^{18} & \sqrt{3}x^{13} & \sqrt{3}x^{14} & 0 & 0\\
x^4 & x^5 & -x^6 & \frac{x^8}{\sqrt{3}} & x^8 & 0 & -x^1 & x^2 & -x^{19} & -x^{20} & x^{12} & -x^{11} & -x^{14} & x^{13}\\
x^5 & x^4+x^7 & \frac{x^8}{\sqrt{3}} & x^6 & -\sqrt{3}x^6 & x^1 & 0 & -x^3 & x^{12} & -x^{11} & x^{16}-x^{19} & -x^{15}-x^{20} & -\frac{x^{18}}{\sqrt{3}} & \frac{x^{17}}{\sqrt{3}}\\
x^6 & \frac{x^8}{\sqrt{3}} & x^7 & 2x^5 & 0 & -x^2 & x^3 & 0 & -x^{14} & x^{13} & -\frac{x^{18}}{\sqrt{3}} & \frac{x^{17}}{\sqrt{3}} & z^{10} & -z^9\\
x^9 & x^{11} & -x^{13} & \frac{x^{17}}{\sqrt{3}} & x^{17} & x^{19} & -x^{12} & x^{14} & 0 & x^{21} & -x^1 & x^5 & x^2 & -x^6\\
x^{10} & x^{12} & -x^{14} & \frac{x^{18}}{\sqrt{3}} & x^{18} & x^{20} & x^{11} & -x^{13} & -x^{21} & 0 & -x^5 & -x^1 & x^6 & x^2\\
x^{11} & x^9+x^{15} & \frac{x^{17}}{\sqrt{3}} & x^{13} & -\sqrt{3}x^{13} & -x^{12} & -x^{16}+x^{19} & \frac{x^{18}}{\sqrt{3}} & x^1 & x^5 & 0 & x^7+x^{21} & -x^3 & -\frac{x^8}{\sqrt{3}}\\
x^{12} & x^{10}+x^{16} & \frac{x^{18}}{\sqrt{3}} & x^{14} & -\sqrt{3}x^{14} & x^{11} & x^{15}+x^{20} & -\frac{x^{17}}{\sqrt{3}} & -x^5 & x^1 & -x^7-x^{21} & 0 & \frac{x^8}{\sqrt{3}} & -x^3\\
x^{13} & \frac{x^{17}}{\sqrt{3}} & x^{15} & 2x^{11} & 0 & x^{14} & \frac{x^{18}}{\sqrt{3}} & -z^{10} & -x^2 & -x^6 & x^3 & -\frac{x^8}{\sqrt{3}} & 0 & z^4\\
x^{14} & \frac{x^{18}}{\sqrt{3}} & x^{16} & 2x^{12} & 0 & -x^{13} & -\frac{x^{17}}{\sqrt{3}} & z^9 & x^6 & -x^2 & \frac{x^8}{\sqrt{3}} & x^3 & -z^4 & 0
\end{pmatrix}, \tag{A.3}$$

where $z^4 = x^4 + x^7 + x^{21}$, $z^9 = x^9 + x^{15} + x^{20}$, $z^{10} = x^{10} + x^{16} - x^{19}$.

The size of the formula for X_{14} forces us to skip the 26-dimensional representation of $\mathfrak{f}_4$. It can be easily obtained by looking for the Lie algebra element stabilizing Υ^1 of Proposition 2.2.

Appendix B. A 5-form reducing GL(14, $\mathbb{R}$) to Sp(3) ⊂ SO(14)

An explicit expression for a nonzero element ϕ of the only 1-dimensional $\mathbf{Sp}(3)$-invariant subspace in $\bigwedge^{5}{}^{14}\odot^2_{14}$ is written below. This 5-form reduces $\mathbf{GL}(14,\mathbb{R})$ to $\mathbf{Sp}(3) \subset \mathbf{SO}(3)$ and, in the adapted coframe of Section 5, contains 129 terms as follows:

$$\begin{aligned}
\phi = {} & 120\sqrt{3}\theta^1\wedge\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^5 - 240\sqrt{3}\theta^1\wedge\theta^2\wedge\theta^5\wedge\theta^6\wedge\theta^7 - 192\sqrt{3}\theta^1\wedge\theta^2\wedge\theta^5\wedge\theta^9\wedge\theta^{11}\\
& - 144\sqrt{3}\theta^1\wedge\theta^2\wedge\theta^5\wedge\theta^{10}\wedge\theta^{12} - 72\theta^1\wedge\theta^2\wedge\theta^6\wedge\theta^9\wedge\theta^{14} + 144\theta^1\wedge\theta^2\wedge\theta^6\wedge\theta^{10}\wedge\theta^{13}\\
& - 54\theta^1\wedge\theta^2\wedge\theta^7\wedge\theta^{11}\wedge\theta^{14} + 72\theta^1\wedge\theta^2\wedge\theta^7\wedge\theta^{12}\wedge\theta^{13} - 360\theta^1\wedge\theta^2\wedge\theta^8\wedge\theta^9\wedge\theta^{10}\\
& - 162\theta^1\wedge\theta^2\wedge\theta^8\wedge\theta^{11}\wedge\theta^{12} - 18\theta^1\wedge\theta^2\wedge\theta^8\wedge\theta^{13}\wedge\theta^{14} - 360\theta^1\wedge\theta^3\wedge\theta^4\wedge\theta^6\wedge\theta^8\\
& - 144\theta^1\wedge\theta^3\wedge\theta^4\wedge\theta^9\wedge\theta^{13} - 72\theta^1\wedge\theta^3\wedge\theta^4\wedge\theta^{10}\wedge\theta^{14} - 120\sqrt{3}\theta^1\wedge\theta^3\wedge\theta^5\wedge\theta^6\wedge\theta^8\\
& - 48\sqrt{3}\theta^1\wedge\theta^3\wedge\theta^5\wedge\theta^9\wedge\theta^{13} - 24\sqrt{3}\theta^1\wedge\theta^3\wedge\theta^5\wedge\theta^{10}\wedge\theta^{14} - 216\theta^1\wedge\theta^3\wedge\theta^6\wedge\theta^9\wedge\theta^{12}\\
& + 288\theta^1\wedge\theta^3\wedge\theta^6\wedge\theta^{10}\wedge\theta^{11} - 360\theta^1\wedge\theta^3\wedge\theta^7\wedge\theta^9\wedge\theta^{10} - 162\theta^1\wedge\theta^3\wedge\theta^7\wedge\theta^{11}\wedge\theta^{12}\\
& - 18\theta^1\wedge\theta^3\wedge\theta^7\wedge\theta^{13}\wedge\theta^{14} - 54\theta^1\wedge\theta^3\wedge\theta^8\wedge\theta^{11}\wedge\theta^{14} + 72\theta^1\wedge\theta^3\wedge\theta^8\wedge\theta^{12}\wedge\theta^{13}\\
& + 120\sqrt{3}\theta^1\wedge\theta^4\wedge\theta^5\wedge\theta^7\wedge\theta^8 + 36\sqrt{3}\theta^1\wedge\theta^4\wedge\theta^5\wedge\theta^{11}\wedge\theta^{13} + 12\sqrt{3}\theta^1\wedge\theta^4\wedge\theta^5\wedge\theta^{12}\wedge\theta^{14}\\
& - 720\theta^1\wedge\theta^4\wedge\theta^6\wedge\theta^9\wedge\theta^{10} - 18\theta^1\wedge\theta^4\wedge\theta^6\wedge\theta^{13}\wedge\theta^{14} - 72\theta^1\wedge\theta^4\wedge\theta^8\wedge\theta^9\wedge\theta^{14}\\
& + 144\theta^1\wedge\theta^4\wedge\theta^8\wedge\theta^{10}\wedge\theta^{13} - 720\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^6\wedge\theta^9\wedge\theta^{10} - 108\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^6\wedge\theta^{11}\wedge\theta^{12}\\
& - 6\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^6\wedge\theta^{13}\wedge\theta^{14} - 144\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^7\wedge\theta^9\wedge\theta^{12} + 192\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^7\wedge\theta^{10}\wedge\theta^{11}\\
& - 24\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^8\wedge\theta^9\wedge\theta^{14} + 48\sqrt{3}\theta^1\wedge\theta^5\wedge\theta^8\wedge\theta^{10}\wedge\theta^{13} + 144\theta^1\wedge\theta^6\wedge\theta^7\wedge\theta^9\wedge\theta^{13}\\
& + 72\theta^1\wedge\theta^6\wedge\theta^7\wedge\theta^{10}\wedge\theta^{14} + 288\theta^1\wedge\theta^6\wedge\theta^8\wedge\theta^9\wedge\theta^{11} + 216\theta^1\wedge\theta^6\wedge\theta^8\wedge\theta^{10}\wedge\theta^{12}\\
& - 54\theta^1\wedge\theta^9\wedge\theta^{10}\wedge\theta^{11}\wedge\theta^{14} + 72\theta^1\wedge\theta^9\wedge\theta^{10}\wedge\theta^{12}\wedge\theta^{13} - 360\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^7\wedge\theta^8\\
& - 108\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^{11}\wedge\theta^{13} - 36\theta^2\wedge\theta^3\wedge\theta^4\wedge\theta^{12}\wedge\theta^{14} + 120\sqrt{3}\theta^2\wedge\theta^3\wedge\theta^5\wedge\theta^7\wedge\theta^8\\
& + 36\sqrt{3}\theta^2\wedge\theta^3\wedge\theta^5\wedge\theta^{11}\wedge\theta^{13} + 12\sqrt{3}\theta^2\wedge\theta^3\wedge\theta^5\wedge\theta^{12}\wedge\theta^{14} - 360\theta^2\wedge\theta^3\wedge\theta^6\wedge\theta^9\wedge\theta^{10}\\
& - 162\theta^2\wedge\theta^3\wedge\theta^6\wedge\theta^{11}\wedge\theta^{12} - 18\theta^2\wedge\theta^3\wedge\theta^6\wedge\theta^{13}\wedge\theta^{14} - 216\theta^2\wedge\theta^3\wedge\theta^7\wedge\theta^9\wedge\theta^{12}\\
& + 288\theta^2\wedge\theta^3\wedge\theta^7\wedge\theta^{10}\wedge\theta^{11} - 72\theta^2\wedge\theta^3\wedge\theta^8\wedge\theta^9\wedge\theta^{14} + 144\theta^2\wedge\theta^3\wedge\theta^8\wedge\theta^{10}\wedge\theta^{13}\\
& + 120\sqrt{3}\theta^2\wedge\theta^4\wedge\theta^5\wedge\theta^6\wedge\theta^8 + 48\sqrt{3}\theta^2\wedge\theta^4\wedge\theta^5\wedge\theta^9\wedge\theta^{13} + 24\sqrt{3}\theta^2\wedge\theta^4\wedge\theta^5\wedge\theta^{10}\wedge\theta^{14}\\
& + 324\theta^2\wedge\theta^4\wedge\theta^7\wedge\theta^{11}\wedge\theta^{12} + 18\theta^2\wedge\theta^4\wedge\theta^7\wedge\theta^{13}\wedge\theta^{14} + 54\theta^2\wedge\theta^4\wedge\theta^8\wedge\theta^{11}\wedge\theta^{14}
\end{aligned}$$

$$
\begin{aligned}
&-72\theta^2\wedge\theta^4\wedge\theta^8\wedge\theta^{12}\wedge\theta^{13}-144\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^6\wedge\theta^9\wedge\theta^{12}+192\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^6\wedge\theta^{10}\wedge\theta^{11}\\
&-240\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^7\wedge\theta^9\wedge\theta^{10}-324\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^7\wedge\theta^{11}\wedge\theta^{12}-6\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^7\wedge\theta^{13}\wedge\theta^{14}\\
&-18\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^8\wedge\theta^{11}\wedge\theta^{14}+24\sqrt{3}\theta^2\wedge\theta^5\wedge\theta^8\wedge\theta^{12}\wedge\theta^{13}+108\theta^2\wedge\theta^6\wedge\theta^7\wedge\theta^{11}\wedge\theta^{13}\\
&+36\theta^2\wedge\theta^6\wedge\theta^7\wedge\theta^{12}\wedge\theta^{14}+288\theta^2\wedge\theta^7\wedge\theta^8\wedge\theta^9\wedge\theta^{11}+216\theta^2\wedge\theta^7\wedge\theta^8\wedge\theta^{10}\wedge\theta^{12}\\
&+24\theta^2\wedge\theta^9\wedge\theta^{11}\wedge\theta^{12}\wedge\theta^{14}-48\theta^2\wedge\theta^{10}\wedge\theta^{11}\wedge\theta^{12}\wedge\theta^{13}+120\sqrt{3}\theta^3\wedge\theta^4\wedge\theta^5\wedge\theta^6\wedge\theta^7\\
&+96\sqrt{3}\theta^3\wedge\theta^4\wedge\theta^5\wedge\theta^9\wedge\theta^{11}+72\sqrt{3}\theta^3\wedge\theta^4\wedge\theta^5\wedge\theta^{10}\wedge\theta^{12}+72\theta^3\wedge\theta^4\wedge\theta^6\wedge\theta^9\wedge\theta^{14}\\
&-144\theta^3\wedge\theta^4\wedge\theta^6\wedge\theta^{10}\wedge\theta^{13}+54\theta^3\wedge\theta^4\wedge\theta^7\wedge\theta^{11}\wedge\theta^{14}-72\theta^3\wedge\theta^4\wedge\theta^7\wedge\theta^{12}\wedge\theta^{13}\\
&+360\theta^3\wedge\theta^4\wedge\theta^8\wedge\theta^9\wedge\theta^{10}+162\theta^3\wedge\theta^4\wedge\theta^8\wedge\theta^{11}\wedge\theta^{12}+72\theta^3\wedge\theta^4\wedge\theta^8\wedge\theta^{13}\wedge\theta^{14}\\
&+24\sqrt{3}\theta^3\wedge\theta^5\wedge\theta^6\wedge\theta^9\wedge\theta^{14}-48\sqrt{3}\theta^3\wedge\theta^5\wedge\theta^6\wedge\theta^{10}\wedge\theta^{13}-18\sqrt{3}\theta^3\wedge\theta^5\wedge\theta^7\wedge\theta^{11}\wedge\theta^{14}\\
&+24\sqrt{3}\theta^3\wedge\theta^5\wedge\theta^7\wedge\theta^{12}\wedge\theta^{13}+120\sqrt{3}\theta^3\wedge\theta^5\wedge\theta^8\wedge\theta^9\wedge\theta^{10}-54\sqrt{3}\theta^3\wedge\theta^5\wedge\theta^8\wedge\theta^{11}\wedge\theta^{12}\\
&+108\theta^3\wedge\theta^6\wedge\theta^8\wedge\theta^{11}\wedge\theta^{13}+36\theta^3\wedge\theta^6\wedge\theta^8\wedge\theta^{12}\wedge\theta^{14}+144\theta^3\wedge\theta^7\wedge\theta^8\wedge\theta^9\wedge\theta^{13}\\
&+72\theta^3\wedge\theta^7\wedge\theta^8\wedge\theta^{10}\wedge\theta^{14}-6\theta^3\wedge\theta^9\wedge\theta^{12}\wedge\theta^{13}\wedge\theta^{14}+12\theta^3\wedge\theta^{10}\wedge\theta^{11}\wedge\theta^{13}\wedge\theta^{14}\\
&-18\sqrt{3}\theta^4\wedge\theta^5\wedge\theta^6\wedge\theta^{11}\wedge\theta^{14}+24\sqrt{3}\theta^4\wedge\theta^5\wedge\theta^6\wedge\theta^{12}\wedge\theta^{13}-24\sqrt{3}\theta^4\wedge\theta^5\wedge\theta^7\wedge\theta^9\wedge\theta^{14}\\
&+48\sqrt{3}\theta^4\wedge\theta^5\wedge\theta^7\wedge\theta^{10}\wedge\theta^{13}-72\sqrt{3}\theta^4\wedge\theta^5\wedge\theta^8\wedge\theta^9\wedge\theta^{12}+96\sqrt{3}\theta^4\wedge\theta^5\wedge\theta^8\wedge\theta^{10}\wedge\theta^{11}\\
&+144\theta^4\wedge\theta^6\wedge\theta^8\wedge\theta^9\wedge\theta^{13}+72\theta^4\wedge\theta^6\wedge\theta^8\wedge\theta^{10}\wedge\theta^{14}-108\theta^4\wedge\theta^7\wedge\theta^8\wedge\theta^{11}\wedge\theta^{13}\\
&-36\theta^4\wedge\theta^7\wedge\theta^8\wedge\theta^{12}\wedge\theta^{14}-18\theta^4\wedge\theta^9\wedge\theta^{10}\wedge\theta^{13}\wedge\theta^{14}+3\theta^4\wedge\theta^{11}\wedge\theta^{12}\wedge\theta^{13}\wedge\theta^{14}\\
&-192\sqrt{3}\theta^5\wedge\theta^6\wedge\theta^7\wedge\theta^9\wedge\theta^{11}-144\sqrt{3}\theta^5\wedge\theta^6\wedge\theta^7\wedge\theta^{10}\wedge\theta^{12}+48\sqrt{3}\theta^5\wedge\theta^6\wedge\theta^8\wedge\theta^9\wedge\theta^{13}\\
&+24\sqrt{3}\theta^5\wedge\theta^6\wedge\theta^8\wedge\theta^{10}\wedge\theta^{14}+36\sqrt{3}\theta^5\wedge\theta^7\wedge\theta^8\wedge\theta^{11}\wedge\theta^{13}+12\sqrt{3}\theta^5\wedge\theta^7\wedge\theta^8\wedge\theta^{12}\wedge\theta^{14}\\
&+108\sqrt{3}\theta^5\wedge\theta^9\wedge\theta^{10}\wedge\theta^{11}\wedge\theta^{12}-6\sqrt{3}\theta^5\wedge\theta^9\wedge\theta^{10}\wedge\theta^{13}\wedge\theta^{1}4-\sqrt{3}\theta^5\wedge\theta^{11}\wedge\theta^{12}\wedge\theta^{13}\wedge\theta^{14}\\
&-360\theta^6\wedge\theta^7\wedge\theta^8\wedge\theta^9\wedge\theta^{10}-162\theta^6\wedge\theta^7\wedge\theta^8\wedge\theta^{11}\wedge\theta^{12}-18\theta^6\wedge\theta^7\wedge\theta^8\wedge\theta^{13}\wedge\theta^{14}\\
&-108\theta^6\wedge\theta^9\wedge\theta^{10}\wedge\theta^{11}\wedge\theta^{13}-36\theta^6\wedge\theta^9\wedge\theta^{10}\wedge\theta^{12}\wedge\theta^{14}+48\theta^7\wedge\theta^9\wedge\theta^{11}\wedge\theta^{12}\wedge\theta^{13}\\
&+24\theta^7\wedge\theta^{10}\wedge\theta^{11}\wedge\theta^{12}\wedge\theta^{14}-12\theta^8\wedge\theta^9\wedge\theta^{11}\wedge\theta^{13}\wedge\theta^{14}-6\theta^8\wedge\theta^{10}\wedge\theta^{12}\wedge\theta^{13}\wedge\theta^{14}.
\end{aligned}
$$

References

[1] M. Berger, Sur les groupes d'holonomie des varietes a connexion affine aet des varites riemanniennes, Bull. Soc. Math. France 83 (1955) 279–330.
[2] A.L. Besse, Einstein Manifolds, Springer, 1987.
[3] M. Bobieński, P. Nurowski, Irreducible $SO(3)$ geometries in dimension five, J. Reine Angew. Math. 605 (2007) 51–93. doi:10.1515/CRELLE.2007.027, math.DG/0507152.
[4] R.L. Bryant, (2005) private communication.
[5] E. Cartan, Sur une classe remarquable d'espaces de Riemann, Bull. Soc. Math. France 54 (1926) 214–246; Bull. Soc. Math. France 55 (1927) 114–134.
[6] E. Cartan, Familles de surfaces isoparametriques dans les espaces a courbure constante, Ann. Mat. 17 (1938) 177–191.
[7] E. Cartan, Sur des familes remarquables d'hypersurfaces isoparametriques dans les espaces spheriques, Math. Z. 45 (1938) 335–367.
[8] S. Chiossi, (2006) private communications.
[9] Th. Friedrich, (2006) private communication.
[10] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura. Appl. 123 (4) (1980) 35–58.
[11] N. Hitchin, Stable forms and special metrics, math.DG/0107101, 2001.
[12] M. van Leeuwen, A computer package for Lie group computations, http://www-math.univ-poitiers.fr/~maavl/LiE, 1996.
[13] T. Levi-Civita, Famiglie di superficie isoparametriche nell'ordinario spacio euclideo, Atti Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur 26 (1937) 355–362.
[14] H.-F. Münzner, Isoparametric Hyperfläschen in Sphären, I and II, Math. Ann. 251 (1980) 57–71; Math. Ann. 256 (1981) 215–232.
[15] B. Segre, Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni, Atti Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur 27 (1938) 203–207.
[16] S. Salamon, Riemannian geometry and holonomy groups, in: Pitman Research Notes in Math., 201, Longman, 1989.
[17] J. Tits, Algebres alternatives, algebres de Jordan et algebres de Lie exceptionnelles, Indag. Math. 28 (1966) 223–237.
[18] E.B. Vinberg, A construction of exceptional simple Lie groups (Russian), Tr. Semin. Vektorn. Tensorn. Anal. 13 (1966) 7–9.
[19] J.A. Wolf, Spaces of Constant Curvature, McGraw Hill, 1967.