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Extensions of bundles of null directions*

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Abstract. The geometry of $\mathcal{P}$, the bundle of null directions over an Einstein spacetime, is studied. The full set of invariants of the natural G-structure on $\mathcal{P}$ is constructed using the Cartan method of equivalence. This leads to an extension of $\mathcal{P}$ which is an elliptic fibration over the spacetime. Examples are given which show that such an extension, although natural, is not unique. A reinterpretation of the Petrov classification in terms of the fibres of an extension of $\mathcal{P}$ is presented.

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1. Introduction

In 1922 Elie Cartan made the following observation [1].

> From a geometric viewpoint, it is worthwhile to note an interesting property. At each point A [of a conformally non-flat space-time] there exist four privileged null directions [...]. They can be characterized as follows: Any one of these directions, say AA', is invariant under transport around an infinitesimal parallelogram one of whose sides is AA' and the other of whose sides is along an arbitrary null direction at A. In the case of ds^2 corresponding to a single attractive mass (ds^2 of Schwarzschild) the four privileged directions reduce to two (degenerate) directions which correspond to null rays pointing to or from the centre of attraction#.

This remark implicitly anticipates elements of the so-called Petrov classification, the elegant contemporary formulation of which owes much to the work of Roger Penrose. In this formulation the anti-selfdual part of the Weyl tensor at a spacetime point corresponds to a totally symmetric spinor C_{ABCD} and a null direction at a point is defined uniquely in terms of a spinor $\xi^A(z) = \binom{1}{z}$. Then the Cartan (principal) null directions correspond to the solutions of the following equation:

$$C_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0, \qquad (1)$$

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for $z \in \mathbf{C} \cup \{\infty\}$ defining the spinor ξ^A. This equation, being fourth order in z, always has four roots, z_i say, but some of them may be repeated. Multiple roots correspond to coincidences between the corresponding Cartan null directions. The Petrov classification (or the Cartan–Petrov–Penrose classification [1, 10, 11] as it perhaps should be properly called) of metrics at a given spacetime point is based on these results. One says that the metric is algebraically general at a point if its Weyl tensor defines four distinct Cartan directions there. Otherwise the metric is algebraically special. The following five possibilities may occur:

- z_1, z_2, z_3, z_4 all different $\longleftrightarrow$ four distinct Cartan directions $\longleftrightarrow$ Petrov type I.
- $z_1 = z_2, z_3, z_4$ different $\longleftrightarrow$ three distinct Cartan directions $\longleftrightarrow$ Petrov type II.
- $z_1 = z_2 \neq z_3 = z_4$ $\longleftrightarrow$ two pairs of distinct Cartan directions $\longleftrightarrow$ Petrov type D.
- $z_1 = z_2 = z_3, z_4$ different $\longleftrightarrow$ two distinct Cartan directions $\longleftrightarrow$ Petrov type III.
- $z_1 = z_2 = z_3 = z_4$ $\longleftrightarrow$ one Cartan direction $\longleftrightarrow$ Petrov type N.

This classification can be reinterpreted as follows. Consider a curve

$$w^2 = C_{ABCD}\xi^A\xi^B\xi^C\xi^D \tag{2}$$

in $\mathbf{C}^2$ with coordinates (w, z) or, better, a compact Riemann surface $\mathcal{T}$ associated with a double-valued function $w(z) = \sqrt{C_{ABCD}\xi^A\xi^B\xi^C\xi^D}$ on $\mathbf{C} \cup \{\infty\} \cong \mathbf{S}^2$. It is well known that the topology of $\mathcal{T}$ depends on the roots of (1) and corresponds to a two-dimensional torus $\mathbf{T}^2$ if all z_i are distinct. If some coincidences between the roots occur then we have the following possibilities. $\mathcal{T}$ has the topology of a torus with one vanishing cycle in Petrov type II, it has the topology of two spheres touching each other in two different points in Petrov type D, it has the topology of a sphere with a distinguished point in Petrov type III, and it has the topology of two spheres touching each other in a single point in Petrov type N.

It turns out that equation (2), which seems to be artificially added, appears naturally in the Einstein theory [8, 9]. A fibration $\tilde{\mathcal{P}}$ can be defined over the spacetime, each fibre having the topology of the associated surface $\mathcal{T}$, with the Einstein equations taking an interesting form on the total space [8, 9]. In this paper we extend the results of [8, 9] by showing how the fibration $\tilde{\mathcal{P}}$ can be defined by using natural objects on the Penrose bundle of null directions $\mathcal{P}$ over the spacetime.

We recall that given a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$ and its bundle of null directions $\mathcal{P}$ one naturally defines a class of six 1-forms $[(F, \bar{F}, T, \Lambda, E, \bar{E})]$ on it having the following properties [6, 7] (see also section 5 of the present paper):

(a) Λ, T are real- and F, E are complex-valued 1-forms on $\mathcal{P}$.
(b) $F \wedge \bar{F} \wedge T \wedge \Lambda \wedge E \wedge \bar{E} \neq 0$ at each point p of $\mathcal{P}$.
(c) Two sets of forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ and $(F', \bar{F}', T', \Lambda', E', \bar{E}')$ are in the same class iff

$$\Lambda = \frac{1}{A}\Lambda', \tag{3}$$

$$F = e^{i\varphi}(F' + \bar{y}\Lambda'), \tag{4}$$

$$T = A(T' + \bar{y}\bar{F}' + yF' + y\bar{y}\Lambda'), \tag{5}$$

$$E = \frac{1}{w}E', \tag{6}$$

where $A > 0$, φ (real) $y, w \neq 0$ (complex) are arbitrary functions on $\mathcal{P}$. This defines a certain G-structure on $\mathcal{P}$. This structure can be studied using the Cartan method of equivalence. In this paper we solve the Cartan equivalence problem for this G-structure. We show that this naturally leads to an elliptic fibration $\tilde{\mathcal{P}}$ associated with the Einstein spacetime i.e. to the association of an elliptic curve (2) with each point of the spacetime. The extension of $\mathcal{P}$ to

$\tilde{\mathcal{P}}$ was obtained previously in [8, 9] by the continuation of solutions of a certain differential system $\mathcal{I}$, defined initially only on an open set of $\mathbf{R}^6$. Such an extension is natural but, as is discussed below, not unique.

The paper is organized as follows. Section 2 contains notation and definitions. Section 3 defines the differential system $\mathcal{I}$. Section 4 gives examples of solutions of the differential system $\mathcal{I}$ corresponding to all vacuums of type N. Section 5 uses the Cartan method of equivalence to obtain the differential system of section 4 from the natural objects defined on the bundle of null directions over the spacetime. The results of section 5 are applied in section 6 to give an effective algorithm for checking whether two metrics are isometrically equivalent. In section 7 a way of associating an elliptic curve with any point of a conformally non-flat Einstein spacetime is presented. The elliptic fibration associated in this way with any conformally non-flat Einstein spacetime constitutes a double branch cover of the bundle of null directions. This natural extension of $\mathcal{P}$ is not unique and in section 8 some examples of Einstein spacetimes, with different extension of $\mathcal{P}$, are exhibited.

2. Basic definitions

We briefly recall the definitions of the geometrical objects we need in the following. Let $\mathcal{M}$ be a four-dimensional oriented and time-oriented manifold equipped with a Lorentzian metric g of signature $(+, +, +, -)$. It is convenient to introduce a null frame $(m, \bar{m}, k, l)$ on $\mathcal{M}$ with a dual coframe $\theta^i = \left(\theta^1, \theta^2, \theta^3, \theta^4\right) = (M, \bar{M}, K, L)$ so that†

$$g = g_{ij}\theta^i\theta^j = 2(M\bar{M} - KL). \tag{7}$$

Given g and θ^i the connection 1-forms $\Gamma_{ij} = g_{ik}\Gamma^k{}_j$ are uniquely defined by

$$\mathrm{d}\theta^i = -\Gamma^i{}_j \wedge \theta^j, \qquad \Gamma_{ij} + \Gamma_{ji} = 0. \tag{8}$$

The connection coefficients Γ_{ijk} are determined by the relation $\Gamma_{ij} = \Gamma_{ijk}\theta^k$‡. Using them we define the curvature 2-forms $\mathcal{R}^k{}_i$, the Riemann tensor $R^i{}_{jkl}$, the Ricci tensor R_{ij} and the Ricci scalar R by

$$\mathcal{R}^k{}_i = \tfrac{1}{2}R^k{}_{imj}\theta^m \wedge \theta^j = \mathrm{d}\Gamma^k{}_i + \Gamma^k{}_j \wedge \Gamma^j{}_i, \qquad R_{ij} = R^k{}_{ikj}, \qquad R = g^{ij}R_{ij}.$$

We also introduce the traceless Ricci tensor by

$$S_{ij} = R_{ij} - \tfrac{1}{4}g_{ij}R.$$

Note that the vanishing of S_{ij} is equivalent to the Einstein equations $R_{ij} = \lambda g_{ij}$ for the metric g. We define the Weyl tensor $C^i{}_{jkl}$ by

$$C_{ijkl} = R_{ijkl} + \tfrac{1}{3}Rg_{i[k}g_{l]j} + R_{j[k}g_{l]i} + R_{i[l}g_{k]j},$$

and its spinorial coefficients Ψ_μ by

$$\begin{aligned}
\mathcal{R}_{23} &= \Psi_4\bar{M}\wedge K + \Psi_3(L\wedge K - M\wedge\bar{M}) + \left(\Psi_2 + \tfrac{1}{12}R\right)L\wedge M \\
&\quad + \tfrac{1}{2}S_{33}M\wedge K + \tfrac{1}{2}S_{32}(L\wedge K + M\wedge\bar{M}) + \tfrac{1}{2}S_{22}L\wedge\bar{M}, \\
\mathcal{R}_{14} &= \left(-\Psi_2 - \tfrac{1}{12}R\right)\bar{M}\wedge K - \Psi_1(L\wedge K - M\wedge\bar{M}) - \Psi_0 L\wedge M \\
&\quad - \tfrac{1}{2}S_{11}M\wedge K - \tfrac{1}{2}S_{41}(L\wedge K + M\wedge\bar{M}) - \tfrac{1}{2}S_{44}L\wedge\bar{M}, \\
\tfrac{1}{2}(\mathcal{R}_{43} - \mathcal{R}_{12}) &= \Psi_3\bar{M}\wedge K + \left(\Psi_2 - \tfrac{1}{24}R\right)(L\wedge K - M\wedge\bar{M}) + \Psi_1 L\wedge M \\
&\quad + \tfrac{1}{2}S_{31}M\wedge K + \tfrac{1}{4}(S_{12} + S_{34})(L\wedge K + M\wedge\bar{M}) + \tfrac{1}{2}S_{42}L\wedge\bar{M}.
\end{aligned} \tag{9}$$

† Expressions such as $\theta^i\theta^j$ mean the symmetrized tensor product, e.g. $\theta^i\theta^j = \frac{1}{2}(\theta^i \otimes \theta^j + \theta^j \otimes \theta^i)$. Also, we will denote by round (respectively square) brackets the symmetrization (respectively antisymmetrization) of indices, e.g. $a_{(ik)} = \frac{1}{2}(a_{ik} + a_{ki})$, $a_{[ik]} = \frac{1}{2}(a_{ik} - a_{ki})$, etc.

‡ We lower and raise indices by means of the metric and its inverse.

3. The differential system

Here we quote the major results from [8] which will be used in this paper. They describe the properties of a system of six 1-forms on $\mathbf{R}^6$ which determine a conformally non-flat Lorentzian 4-metric satisfying Einstein equations.

Theorem 1. *Let $\mathcal{P}_0$ be an open subset of $\mathbf{R}^6$. Suppose that on $\mathcal{P}_0$ we have six 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ which satisfy the following conditions:*

(a) T, Λ are real- and F, E are complex-valued 1-forms.
(b) $F \wedge \bar{F} \wedge T \wedge \Lambda \wedge E \wedge \bar{E} \neq 0$ at each point p of $\mathcal{P}_0$.
(c) There exist complex-valued 1-forms Ω and Γ on $\mathcal{P}_0$, and a certain complex function α on $\mathcal{P}_0$ such that

$$\begin{aligned} \mathrm{d}F &= (\Omega - \bar{\Omega}) \wedge F + E \wedge T + \bar{\Gamma} \wedge \Lambda \\ \mathrm{d}T &= \Gamma \wedge F + \bar{\Gamma} \wedge \bar{F} - (\Omega + \bar{\Omega}) \wedge T \\ \mathrm{d}\Lambda &= \bar{E} \wedge F + E \wedge \bar{F} + (\Omega + \bar{\Omega}) \wedge \Lambda \\ \mathrm{d}E &= 2\Omega \wedge E + \bar{F} \wedge T + \alpha \Lambda \wedge F. \end{aligned} \tag{10}$$

Then

(a) $\mathcal{P}_0$ is locally foliated by two-dimensional manifolds $\mathcal{S}_x$, which are tangent to the real distribution $\mathcal{V}$ defined by

$$F(\mathcal{V}) = T(\mathcal{V}) = \Lambda(\mathcal{V}) = 0.$$

(b) The degenerate metric

$$G = 2(F\bar{F} - T\Lambda) \tag{11}$$

on $\mathcal{P}_0$ has the signature $(+, +, +, -, 0, 0)$ and is preserved when Lie-transported along any leaf $\mathcal{S}_x$ of the foliation $\{\mathcal{S}_x\}$.
(c) The four-dimensional space $\mathcal{M}$ of all leaves of the foliation $\{\mathcal{S}_x\}$ is naturally equipped with a Lorentzian conformally non-flat metric g which is Einstein ($S_{ij} = 0$) and is defined by projecting G from $\mathcal{P}_0$ to $\mathcal{M}$.

The Einstein property of the metric g was proven in [8] by using the integrability conditions for the system (10). We summarize them in the following proposition.

Proposition 1. *If $(F, \bar{F}, T, \Lambda, E, \bar{E})$ satisfy (10) then there exist complex functions a, h on $\mathcal{P}_0$, and a real constant λ, such that*

$$\begin{aligned} \mathrm{d}\Gamma &= 2\Gamma \wedge \Omega + \alpha T \wedge \bar{F} + a(T \wedge \Lambda + F \wedge \bar{F}) + h\Lambda \wedge F \\ \mathrm{d}\Omega &= E \wedge \Gamma - \left(\alpha + \tfrac{1}{2}\lambda\right)(T \wedge \Lambda + F \wedge \bar{F}) + a\Lambda \wedge F. \end{aligned} \tag{12}$$

Moreover,

$$\begin{aligned} \mathrm{d}\alpha &= \alpha_1 F + \gamma_4 \bar{F} + \gamma_1 T + \alpha_4 \Lambda - 2aE, \\ \mathrm{d}a &= a_1 F + \alpha_4 \bar{F} + \alpha_1 T + a_4 \Lambda + hE - (3\alpha + \lambda)\Gamma - 2a\Omega, \\ \mathrm{d}h &= h_1 F - a_4 \bar{F} - a_1 T + h_4 \Lambda + 4a\Gamma - 4h\Omega, \end{aligned} \tag{13}$$

where the possible forms of Ω and Γ are

$$\begin{aligned} \Omega &= \omega_1 F + \omega_2 \bar{F} + \omega_3 T + \omega_4 \Lambda, \\ \Gamma &= \gamma_1 F - 4\omega_4 \bar{F} - 4\omega_1 T + \gamma_4 \Lambda - (3\alpha + \lambda)E \end{aligned} \tag{14}$$

and $\gamma_1, \gamma_4, \alpha_1, \alpha_4, a_1, a_4, h_1, h_4, \omega_1, \omega_2, \omega_3, \omega_4$ are certain complex functions on $\mathcal{P}_0$.

The next result gives a geometric interpretation of the functions α, h and the constant λ.

Proposition 2. *The spinorial coefficients for the Weyl tensor of the metric g on $\mathcal{M}$ read*

$$\Psi_0 = h, \qquad \Psi_1 = -a, \qquad \Psi_2 = -\alpha - \tfrac{1}{3}\lambda, \qquad \Psi_3 = 0, \qquad \Psi_4 = -1.$$

The metric is of Petrov type D iff

$$a = 0, \qquad \textit{and} \qquad h = -(3\alpha + \lambda)^2 \tag{15}$$

and of type N iff

$$h = a = 0, \qquad \textit{and} \qquad \lambda = -3\alpha. \tag{16}$$

The metric is algebraically special iff $I^3 = 27J^2$, where

$$I = -h + \tfrac{1}{3}(3\alpha + \lambda)^2, \qquad \textit{and} \qquad J = a^2 + \tfrac{1}{27}(3\alpha + \lambda)^3 + \tfrac{1}{3}h(3\alpha + \lambda).$$

From now on the differential system $\mathcal{I}$ of the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ satisfying equations (10) on $\mathcal{P}_0$ will be denoted by $(\mathcal{I}, \mathcal{P}_0)$.

4. Type-N vacuum solutions

In this section we present some specific examples of the differential systems $(\mathcal{I}, \mathcal{P}_0)$. We exhibit the forms $(F, \bar{F}, T, \Lambda, E\bar{E})$ which, via theorem 1, correspond to type-N vacuum $(\alpha = \lambda = 0)$ solutions of the Einstein equations. These examples illustrate the way in which well known solutions appear in this formalism.

Example 1. Let $\mathcal{P}_0$ be an open set of $\mathbf{R}^6$ with coordinates $(Z, \bar{Z}, U, V, z, \bar{z})$, where U, V are real and Z, z are complex. Consider the following 1-forms on $\mathcal{P}_0$:

$$\begin{aligned}
F &= \mathrm{d}\bar{Z} + z\,\mathrm{d}U \\
T &= \mathrm{d}U \\
\Lambda &= \mathrm{d}V + z\,\mathrm{d}Z + \bar{z}\,\mathrm{d}\bar{Z} + \left[z\bar{z} - \tfrac{1}{2}\left(Z^2 + \bar{Z}^2\right)\right]\mathrm{d}U \\
E &= \mathrm{d}z + Z\,\mathrm{d}U.
\end{aligned} \tag{17}$$

It is a matter of straightforward calculation to check that these forms constitute a solution to the system (10) with $\alpha = \lambda = \Omega = \Gamma = 0$.

One also easily checks that although F, T, Λ, E depend on six real coordinates, the metric

$$G = 2(F\bar{F} - T\Lambda) = 2\,\mathrm{d}Z\,\mathrm{d}\bar{Z} - 2\,\mathrm{d}U\left[\mathrm{d}V - \tfrac{1}{2}\left(Z^2 + \bar{Z}^2\right)\mathrm{d}U\right]$$

depends on $(Z, \bar{Z}, U, V)$ only. Thus, it projects to

$$g = 2\,\mathrm{d}Z\,\mathrm{d}\bar{Z} - 2\,\mathrm{d}U\left[\mathrm{d}V - \tfrac{1}{2}\left(Z^2 + \bar{Z}^2\right)\mathrm{d}U\right]$$

on a 4-manifold $\mathcal{M}$ coordinatized by $(Z, \bar{Z}, U, V)$. The spacetime $\mathcal{M}$ with this metric is a plane-fronted gravitational wave possessing six symmetries.

Example 2. Let $\mathcal{P}_0$ be again coordinatized by $(Z, \bar{Z}, U, V, z, \bar{z})$ and let $\omega_3 =$ constant be a complex parameter. Consider the forms

$$\begin{aligned}
F &= \mathrm{d}\bar{Z} + [z + (\bar{\omega}_3 - \omega_3)\bar{Z}]\,\mathrm{d}U \\
T &= \mathrm{d}U \\
\Lambda &= \mathrm{d}V + z\,\mathrm{d}Z + \bar{z}\,\mathrm{d}\bar{Z} + \left[z\bar{z} - \tfrac{1}{2}\left(Z^2 + \bar{Z}^2\right) + (\omega_3 - \bar{\omega}_3)(zZ - \bar{z}\bar{Z}) - (\omega_3 + \bar{\omega}_3)v\right]\mathrm{d}U \\
E &= \mathrm{d}z + (Z - 2\omega_3 z)\,\mathrm{d}U.
\end{aligned} \tag{18}$$

They again constitute a solution to the system (10). This solution generalizes the previous example since the corresponding $\alpha = \lambda = \Gamma = 0$, but $\Omega = \omega_3 T$. For any value of the complex parameter ω_3 the metric

$$g = 2\,\mathrm{d}Z\,\mathrm{d}\bar{Z} - 2\,\mathrm{d}U\big[\mathrm{d}V + (\omega_3 - \bar{\omega}_3)(\bar{Z}\,\mathrm{d}Z - Z\,\mathrm{d}\bar{Z}) \\ +\big((\omega_3 - \bar{\omega}_3)^2 Z\bar{Z} - (\omega_3 + \bar{\omega}_3)V - \tfrac{1}{2}(Z^2 + \bar{Z}^2)\big)\,\mathrm{d}U\big]$$

on the quotient manifold parametrized by $(Z, \bar{Z}, U, V)$ is a plane-fronted gravitational wave with six symmetries. The solution of example 1 corresponds to $\omega_3 = 0$ and is the simplest in the class. It follows that example 2 exhausts the list of all vacuum ($\lambda = 0$) solutions to the Einstein equations with six symmetries.

Example 3. A generalization of the preceding examples can be obtained by taking $\mathcal{P}_0$ with coordinates $(Z, \bar{Z}, U, V, z, \bar{z})$ and the forms

$$\begin{aligned} F &= \mathrm{e}^{-\mathrm{i}\phi}[\mathrm{d}\bar{Z} + z\,\mathrm{d}U] \\ T &= \mathrm{e}^{r}\,\mathrm{d}U \\ \Lambda &= \mathrm{e}^{-r}[\mathrm{d}V + z\,\mathrm{d}Z + \bar{z}\,\mathrm{d}\bar{Z} + (z\bar{z} - H - \bar{H})\,\mathrm{d}U] \\ E &= \mathrm{e}^{-r-\mathrm{i}\phi}[\mathrm{d}z + H_Z\,\mathrm{d}U]. \end{aligned} \tag{19}$$

Here $H = H(Z, U)$ is any holomorphic function of the variable Z, $H_Z = \partial H/\partial Z$ and the real functions r and ϕ are determined by the condition $\mathrm{e}^{2(r+\mathrm{i}\phi)} = H_{ZZ}$. One easily sees that equations (19) constitute a solution to the system (10) with $\alpha = \lambda = \Gamma = 0$ and $\Omega = -\frac{1}{2}\mathrm{d}(r + \mathrm{i}\phi)$. The corresponding type-N vacuum spacetime is a general plane wave with five (or more) symmetries.

Example 4. Another example of solutions with $\alpha = \lambda = 0$ is given by

$$\begin{aligned} F &= \mathrm{e}^{-\mathrm{i}\phi}[\mathrm{d}\bar{Z} + (V + z(Z + \bar{Z}))\,\mathrm{d}U] \\ T &= \mathrm{e}^{r}(Z + \bar{Z})\,\mathrm{d}U \\ \Lambda &= \mathrm{e}^{-r}[\mathrm{d}V + z\,\mathrm{d}Z + \bar{z}\,\mathrm{d}\bar{Z} + (z\bar{z}(Z + \bar{Z}) + (z + \bar{z})V - H - \bar{H})\,\mathrm{d}U] \\ E &= \mathrm{e}^{-r-\mathrm{i}\phi}\big[\mathrm{d}z + \big(z^2 + H_Z\big)\,\mathrm{d}U\big]. \end{aligned} \tag{20}$$

Here $\mathcal{P}_0$ is parametrized by $(Z, \bar{Z}, U, V, z, \bar{z})$, $H = H(Z, U)$ is holomorphic in Z and the real functions r and ϕ are given by $\mathrm{e}^{2(r+\mathrm{i}\phi)} = H_{ZZ}/(Z + \bar{Z})$. The solutions are type-N vacuums ($\alpha = \lambda = 0$) and have $\Omega = -\frac{1}{2}\mathrm{d}(r + \mathrm{i}\phi) - z\,\mathrm{d}U$ and $\Gamma = -\mathrm{e}^{r+\mathrm{i}\phi}\,\mathrm{d}U$. They belong to the Kundt class.

All the solutions presented so far corresponded to type-N vacuums with non-diverging rays. Generic type-N vacuums are given below.

Example 5. Consider $\mathcal{P}_0 \subset \mathbf{R}^6$ with coordinates $(Z, \bar{Z}, U, V, z, \bar{z})$, where Z, z are complex and U, V are real. Define

$$\Lambda_0 = \mathrm{d}U + \xi\,\mathrm{d}Z + \bar{\xi}\,\mathrm{d}\bar{Z},$$

where $\xi = \xi(U, Z, \bar{Z})$ is a function of variables $U, Z, \bar{Z}$ only. Let

$$\begin{aligned} F &= \mathrm{e}^{-\mathrm{i}\phi}[\mathrm{d}\bar{\xi} + V\,\mathrm{d}Z - \bar{\partial}\bar{\xi}\,\mathrm{d}\bar{Z} + z\Lambda_0] \\ T &= \mathrm{e}^{r}\Lambda_0 \\ \Lambda &= \mathrm{e}^{-r}[\mathrm{d}V + z\,\mathrm{d}\xi + \bar{z}\,\mathrm{d}\bar{\xi} + z\bar{z}\Lambda_0 + (\partial\bar{\partial}\bar{\xi} + zV - \bar{z}\bar{\partial}\bar{\xi})\,\mathrm{d}\bar{Z} + (\bar{\partial}\partial\xi + \bar{z}V - z\partial\xi)\,\mathrm{d}Z] \\ E &= \mathrm{e}^{-r-\mathrm{i}\phi}\big[\mathrm{d}z + \big(z^2 + \partial_U\bar{\partial}\bar{\xi}\big)\,\mathrm{d}\bar{Z}\big], \end{aligned} \tag{21}$$

where $\partial_U = \partial/\partial U$, $\partial = \partial_Z - \xi\partial_U$ and real functions r and ϕ are determined by $\mathrm{e}^{2(r+\mathrm{i}\phi)} = -\partial_U{}^2\bar\partial\bar\xi/(V + \bar\partial\xi)$. It follows that if the function ξ satisfies the equations

$$\partial\partial_U\bar\partial\bar\xi = 0 \qquad \mathrm{Im}\,\partial\partial\bar\partial\bar\xi = 0$$

then the above forms satisfy the system (10) with $\alpha = \lambda = 0$ and $\Omega = -\frac{1}{2}\mathrm{d}(r + \mathrm{i}\phi) - z\,\mathrm{d}\bar Z$ and $\Gamma = -\mathrm{e}^{r+\mathrm{i}\phi}\,\mathrm{d}Z$. The corresponding spacetimes are of type N.

It follows from the results of Plebański [12] that examples 3–5 constitute all the solutions to the vacuum type-N Einstein equations.

5. From the Einstein spacetime to the differential system

We first briefly summarize the Cartan method of analysing G-structures (see [4] for more details).

Let $\mathcal{X}$ be an n-dimensional manifold and $[\{\theta^i\}]$, $i = 1, \ldots, n$, be a class of n linearly independent 1-forms on $\mathcal{X}$ such that two representatives $\{\theta^i\}$ and $\{\theta'^i\}$ are in the same class iff there exists an element $(a^i_j) \in G$ of a certain group G such that $\theta'^i = a^i_j\theta^j$. Now, suppose that we have two sets $\{\theta'^i\}$ and $\{\theta^i\}$ of n linearly independent 1-forms on $\mathcal{X}$. The G-structure equivalence question is: does there exists a (local) diffeomorphism ϕ of $\mathcal{X}$ such that

$$\phi^*(\theta'^i) = a^i_j\theta^j \tag{22}$$

for some G-valued function a^i_j on $\mathcal{X}$. In other words, does the system of differential equations (22), for ϕ, have a solution? This question is not easy to answer, since the right-hand side of (22) is undetermined. Elie Cartan associates with the forms $\{\theta^i\}$ and $\{\theta'^i\}$ two systems of 1-forms Ω_μ and Ω'_μ on a manifold $\hat{\mathcal{X}}$ of dimension $\hat n \geqslant n$. Then he shows that equations like (22) for ϕ have a solution iff a simpler system

$$\hat\phi^*\Omega'_\mu = \Omega_\mu \tag{23}$$

of differential equations for a diffeomorphism $\hat\phi$ of $\hat{\mathcal{X}}$ has a solution. Examples are known (e.g. CR structures [2]) where the Cartan procedure produces $k > \hat n$ 1-forms Ω_μ of which $\hat n$ are linearly independent. Then decomposing $k - \hat n$ of the dependent 1-forms Ω_μ onto the basis of $\hat n$ independent ones we obtain functions f_I (coefficients of the decompositions) which if (23) has a solution have to satisfy $\hat\phi^*(f'_I) = f_I$. The advantage of these equations for $\hat\phi$ is that they are not differential equations. If the procedure gives enough independent functions f_I then by the implicit function theorem the whole problem reduces to evaluating whether a certain Jacobian is non-degenerate.

In this section we show that any conformally non-flat Einstein spacetime defines a differential system as in theorem 1. We consider the bundle $\mathcal{P}$ of null directions of an Einstein spacetime and study its natural G-structure using the Cartan method described above. This enables us to define a differential system on $\mathcal{P}$ that has all the properties of $(\mathcal{I}, \mathcal{P}_0)$. Here the arguments presented in [9] are approached from a different point of view.

Let $(\mathcal{M}, g)$ be a four-dimensional Lorentzian (not necessarily Einstein) manifold. Consider the set Σ_x of all null directions outgoing from a given point $x \in \mathcal{M}$. This set is topologically a sphere $\mathbf{S}^2$—the celestial sphere of an observer situated at x†. The points of this sphere can be parametrized by a complex number z belonging to the Argand plane $\mathbf{C} \cup \{\infty\}$. A direction associated with $z \neq \infty$ is generated by a null vector

$$k(z) = k + z\bar z l - zm - \bar z\bar m. \tag{24}$$

† We consider outgoing directions from x. In this sense directions generated by, for example, k and $-k$ are considered to be different and two vector fields generate the same direction if an only if they differ by a multiple of a positive real function on $\mathcal{M}$.

With $z = \infty$ we associate a direction generated by the vector l. Conversely, given a null direction outgoing from x we find that it is either parallel to the vector l or can be represented by only one null vector $k(z)$ such that $g(k(z), l) = -1$. It follows that such a vector $k(z)$ has necessarily the form (24), and that it defines a certain $z \in \mathbf{C}$. If a direction is parallel to l we associate with it $z = \infty$.

We define a fibre bundle $\mathcal{P} = \bigcup_{x\in\mathcal{M}} \Sigma_x$ over $\mathcal{M}$, so that the two-dimensional spheres Σ_x are its fibres. The canonical projection $\pi\colon \mathcal{P} \to \mathcal{M}$ is defined by $\pi(\Sigma_x) = x$. The following geometrical objects existing on $\mathcal{P}$ are relevant in the present paper (see [6] for details).

- The Levi-Civita connection associated with the metric g on $\mathcal{M}$ distinguishes a horizontal space in $\mathrm{T}\mathcal{P}$. In this way for any point $p \in \mathcal{P}$ we have a natural splitting of its tangent space onto a direct sum $\mathrm{T}_p\mathcal{P} = V_p \bigoplus H_p$, where H_p is a four-dimensional horizontal space and V_p is a two-dimensional vertical space. The vertical space V_p is tangent to the fibre $\Sigma_{\pi(p)}$ at the point p. Thus V_p has a natural complex structure related to the complex structure on the sphere $\mathbf{S}^2$. The complexification of V_p splits into eigenspaces V_p^+ and V_p^- of this complex structure. We have a horizontal lift $\tilde{v}$ of any vector v from $\pi(p) \in \mathcal{M}$ to $\mathcal{P}$. This is such a vector $\tilde{v}$ at p that $\tilde{v} \in H_p$ and $\pi_*(\tilde{v}) = v$.
- A Lorentzian metric $\tilde{g}$ can be defined on $\mathcal{P}$ by the requirements that
  - (a) the scalar product of any two horizontal vectors determined by $\tilde{g}$ is the same as the scalar product with respect to g of their push forwards to $\mathcal{M}$,
  - (b) the scalar product of any two vertical vectors with respect to $\tilde{g}$ is equal to their scalar product in the natural metric on the two-dimensional sphere (this is consistent since vertical vectors can be considered tangent vectors to $\mathbf{S}^2$),
  - (c) any two vectors such that one is horizontal and the other is vertical are orthogonal with respect to $\tilde{g}$.
- There is a natural congruence of oriented lines on $\mathcal{P}$ which is tangent to the horizontal lifts of null directions from $\mathcal{M}$. It is defined by the following recipe. Take any null vector k at $x \in \mathcal{M}$. This represents a certain null direction $p(k)$ outgoing from x. Correspondingly, this defines a point $p = p(k)$ in the fibre $\pi^{-1}(x)$. Lift k horizontally to p. This defines $\tilde{k}$ which generates a certain direction outgoing from $p \in \mathcal{P}$. Repeating this procedure for all directions outgoing from $x \in \mathcal{M}$ we attach to any point of $\pi^{-1}(x)$ a unique direction. If we do this for all points of $\mathcal{M}$, we define a field of directions on $\mathcal{P}$ which, according to its construction and the properties of $\tilde{g}$, is null. Since we considered outgoing null directions the integral curves of this field are oriented. They form the desired null congruence. This congruence is called the null spray on $\mathcal{P}$ [13].

Let X be any non-vanishing vector field tangent to the null spray on $\mathcal{P}$. Let Λ' be a real 1-form on $\mathcal{P}$ defined by $\Lambda' = -\tilde{g}(X)$. Since X is defined up to a multiplication by a strictly positive real function on $\mathcal{P}$ then Λ' is also specified up to a multiplication by a real strictly positive function, say $1/A > 0$, on $\mathcal{P}$

$$\Lambda' \to \Lambda = \frac{1}{A}\Lambda', \qquad A > 0.$$

With the horizontal space in $\mathcal{P}$ one associates another 1-form. This is a complex 1-form E' on $\mathcal{P}$ such that for any $p \in \mathcal{P}$ we have $E'(H_p) = E'(V_p^-) = 0$ and $E' \wedge \bar{E}' \neq 0$. This is defined up to a multiplication by a non-vanishing complex function, say $1/w$, on $\mathcal{P}$

$$E' \to E = \frac{1}{w}E'.$$

It is now easy to see that the metric $\tilde{g}$ on $\mathcal{P}$ can be expressed as

$$\tilde{g} = 2\left(\frac{1}{|w|^2}E'\bar{E}' + \Lambda'T' + F'\bar{F}'\right)$$

for some choice of the 1-forms T' (real) and F' (complex) on $\mathcal{P}$. The above expression can be considered a definition of the forms F' and T'. These are given up to the following transformations:

$$F' \to F = \mathrm{e}^{\mathrm{i}\phi}(F' + \bar{y}\Lambda'),$$
$$T' \to T = A(T' + \bar{y}\bar{F}' + yF' + y\bar{y}\Lambda'),$$

where ϕ (real) and y (complex) are some functions on $\mathcal{P}$.

It follows that the forms (F', T', Λ', E') may be expressed in terms of the ordered null cotetrad (7) and the corresponding connection by

$$\begin{aligned} \Lambda' &= L + z\bar{z}K + z\bar{M} + \bar{z}M, \\ F' &= M + zK, \\ E' &= \mathrm{d}z + \Gamma_{32} + z(\Gamma_{21} + \Gamma_{43}) + z^2\Gamma_{14}, \\ T' &= K, \end{aligned} \tag{25}$$

where z is given by (24) and we omit the pull-back symbol π^* in expressions such as $\pi^*(\Gamma_{32})$, etc.

In this way we see that the bundle $\mathcal{P}$ of null directions of any spacetime $\mathcal{M}$ is equipped with the class of six 1-forms $[(F', \bar{F}', T', \Lambda', E', \bar{E}')]$ defined above. Following Elie Cartan this class of forms can be used to study all the invariant properties of the underlying Lorentzian geometry. Thus, we consider a class of 1-forms $[(F', \bar{F}', T', \Lambda', E', \bar{E}')]$ on $\mathcal{P}$ with the following properties.

(a) Λ', T' are real- and F', E' are complex-valued 1-forms on $\mathcal{P}$.
(b) $F' \wedge \bar{F}' \wedge T' \wedge \Lambda' \wedge E' \wedge \bar{E}' \neq 0$ at each point p of $\mathcal{P}$.
(c) Two sets of forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ and $(F', \bar{F}', T', \Lambda', E', \bar{E}')$ are in the same class iff

$$\Lambda = \frac{1}{A}\Lambda', \tag{26}$$
$$F = \mathrm{e}^{\mathrm{i}\varphi}(F' + \bar{y}\Lambda') \tag{27}$$
$$T = A(T' + \bar{y}\bar{F}' + yF' + y\bar{y}\Lambda') \tag{28}$$
$$E = \frac{1}{w}E'. \tag{29}$$

Here $A > 0$, φ (real) y, $w \neq 0$ (complex) are arbitrary functions on $\mathcal{P}$.
(d) A particular set of forms that belong to the considered class is given explicitly by (25).

Given the representation (25) of the forms we calculate their differentials. These read as follows:

$$\mathrm{d}\Lambda' = \bar{E}' \wedge F' + E' \wedge \bar{F}' + [\gamma - \bar{\omega}_z]\Lambda' \wedge F' + [\bar{\gamma} - \omega_{\bar{z}}]\Lambda' \wedge \bar{F}' + [\omega + \bar{\omega}]\Lambda' \wedge T', \tag{30}$$

$$\begin{aligned} \mathrm{d}F' = {} & E' \wedge T' + [-\gamma_{\bar{z}} + 2\bar{\gamma}_z + \bar{\omega}_{z\bar{z}}]F' \wedge \Lambda' - \bar{\gamma}_{\bar{z}}\bar{F}' \wedge \Lambda' + [\bar{\omega}_{\bar{z}} + \bar{\gamma}]\Lambda' \wedge T' \\ & + [\bar{\gamma} + \omega_{\bar{z}}]\bar{F}' \wedge F' + [\omega - \bar{\omega}]F' \wedge T', \end{aligned} \tag{31}$$

$$\begin{aligned} \mathrm{d}T' = {} & -\gamma_{z\bar{z}}F' \wedge \Lambda' - \bar{\gamma}_{z\bar{z}}\bar{F}' \wedge \Lambda' - [\gamma_{\bar{z}} + \bar{\gamma}_z]\Lambda' \wedge T' + [\bar{\omega}_z - \omega_z - 2\gamma]T' \wedge F' \\ & + [\bar{\gamma}_z - \gamma_{\bar{z}} + \bar{\omega}_{z\bar{z}} - \omega_{z\bar{z}}]F' \wedge \bar{F}' + [\omega_{\bar{z}} - \bar{\omega}_{\bar{z}} - 2\bar{\gamma}]T' \wedge \bar{F}', \end{aligned} \tag{32}$$

$$\begin{aligned} \mathrm{d}E' = {} & 2\gamma E' \wedge F' + 2\gamma_{\bar{z}}\Lambda' \wedge E' - 2\omega_{\bar{z}}E' \wedge \bar{F}' + 2\omega E' \wedge T' \\ & + \Phi T' \wedge F' + \Psi T' \wedge \bar{F}' - \tfrac{1}{2}\Phi_{\bar{z}\bar{z}}\Lambda' \wedge \bar{F}' + \left[\tfrac{1}{12}\Psi_{zz} + \tfrac{1}{12}R\right]F' \wedge \Lambda' \\ & - \tfrac{1}{4}\Psi_z[T' \wedge \Lambda' + F' \wedge \bar{F}'] - \tfrac{1}{2}\Phi_{\bar{z}}[T' \wedge \Lambda' - F' \wedge \bar{F}'], \end{aligned} \tag{33}$$

where we have used the following abbreviations:

$$2\gamma = \Gamma_{211} + \Gamma_{431} + 2z\Gamma_{141} - \bar{z}(\Gamma_{214} + \Gamma_{434}) - 2z\bar{z}\Gamma_{144}, \tag{34}$$

$$2\omega = \Gamma_{213} + \Gamma_{433} + z(2\Gamma_{143} - \Gamma_{211} - \Gamma_{431}) - \bar{z}(\Gamma_{212} + \Gamma_{432}) + z\bar{z}(\Gamma_{214} + \Gamma_{434} - 2\Gamma_{142}) - 2z^2\Gamma_{141} + 2z^2\bar{z}\Gamma_{144}, \tag{35}$$

$$\Phi = \tfrac{1}{2}S_{33} - \bar{z}S_{23} - zS_{13} + z\bar{z}(S_{12} + S_{34}) + \tfrac{1}{2}\bar{z}^2S_{22} + \tfrac{1}{2}z^2S_{11} - \bar{z}^2zS_{24} - z^2\bar{z}S_{14} + \tfrac{1}{2}z^2\bar{z}^2S_{44}, \tag{36}$$

$$\Psi = \Psi_4 - 4\Psi_3 z + 6\Psi_2 z^2 - 4\Psi_1 z^3 + \Psi_0 z^4, \tag{37}$$

and the subscript z (or $\bar{z}$) denotes the derivative with respect to z (or $\bar{z}$).

It follows that the differential of E' carries all the information about the Ricci tensor and the Weyl coefficients Ψ_μ. In particular, the equation

$$\mathrm{d}E' \wedge \Lambda' \wedge \bar{F}' \wedge E' \equiv 0, \tag{38}$$

which is the same as

$$\Phi \equiv 0, \tag{39}$$

is equivalent to the Einstein equations for the 4-metric. Similarly,

$$\mathrm{d}E' \wedge \Lambda' \wedge F' \wedge E' \equiv 0, \tag{40}$$

which is the same as

$$\Psi \equiv 0, \tag{41}$$

is equivalent to the conformal flatness of the metric. The equation

$$\Psi = 0 \tag{42}$$

is also important. It has at most four solutions at each fibre in $\mathcal{P}$ over a given point $x \in \mathcal{M}$. These four points, via (24), correspond to four principal null directions at x.

One easily discovers that both of the equations (38) and (40) are invariant under the transformations (26)–(29) of the forms. On the other hand, in the differentials of the six considered 1-forms there are terms which may be transformed to zero by an appropriate choice of the gauge (26)–(29). Our aim now will be to use this gauge to obtain the simplest possible form of the differentials (30)–(33).

We start our analysis with the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ of (26)–(29), in which $(F', \bar{F}', T', \Lambda', E', \bar{E}')$ are given by (25).

From the geometrical point of view the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ live on a manifold $\mathcal{C}'$ that has higher dimension than $\mathcal{P}$. Actually, if p denotes a generic point of $\mathcal{P}$, then $\mathcal{C}'$ may be parametrized by $(p, A, \varphi, y, \bar{y}, w, \bar{w})$. Thus $\mathcal{C}'$ is 12-dimensional and the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ are well defined on it.

Calculating the differential of Λ on $\mathcal{C}'$ we find that

$$\mathrm{d}\Lambda = -\mathrm{d}\log A \wedge \Lambda + A^{-1}\{E' \wedge \bar{F}' + \bar{E}' \wedge F' + [\gamma - \bar{\omega}_z]\Lambda' \wedge F' + [\bar{\gamma} - \omega_{\bar{z}}]\Lambda' \wedge \bar{F}' + [\omega + \bar{\omega}]\Lambda' \wedge T'\}.$$

This equation suggests the introduction of an auxiliary real-valued 1-form

$$\Omega + \bar{\Omega} = -\mathrm{d}\log A + s_1F' + \bar{s}_1\bar{F}' + s_3T' + s_4\Lambda' + s_5E' + \bar{s}_5\bar{E}'. \tag{43}$$

The functional coefficients s_3, s_4 (real) and s_1, s_5 (complex) are for the moment arbitrary. They may be used to eliminate some of the terms in the differential of Λ. Indeed, using $\Omega + \bar{\Omega}$ we can rewrite the differential of Λ in the form

$$\mathrm{d}\Lambda = (\Omega + \bar{\Omega}) \wedge \Lambda + \frac{w\mathrm{e}^{\mathrm{i}\varphi}}{A}E \wedge \bar{F} + \frac{\bar{w}\mathrm{e}^{-\mathrm{i}\varphi}}{A}\bar{E} \wedge F + \Lambda \wedge \{[\gamma - \bar{\omega}_z + s_1]F' + [\bar{\gamma} - \omega_{\bar{z}} + \bar{s}_1]\bar{F}' + [\omega + \bar{\omega} + s_3]T' + [y + s_5]E' + [\bar{y} + \bar{s}_5]\bar{E}'\}.$$

This suggests the following choice of s_1, s_2 and s_5:

$$s_1 = -\gamma + \bar{\omega}_z, \qquad s_3 = -\omega - \bar{\omega}, \qquad s_5 = -y. \tag{44}$$

With this choice the differential of Λ assumes the form

$$\mathrm{d}\Lambda = (\Omega + \bar{\Omega}) \wedge \Lambda + \frac{w\mathrm{e}^{\mathrm{i}\varphi}}{A} E \wedge \bar{F} + \frac{\bar{w}\mathrm{e}^{-\mathrm{i}\varphi}}{A} \bar{E} \wedge F.$$

Now, we can make the first gauge-fixing condition

$$w = A\mathrm{e}^{-\mathrm{i}\varphi}. \tag{45}$$

This brings the differential of Λ' into the simplest possible form

$$\mathrm{d}\Lambda = (\Omega + \bar{\Omega}) \wedge \Lambda + E \wedge \bar{F} + \bar{E} \wedge F. \tag{46}$$

Note that the choice (45) uniquely subordinates A and $\mathrm{e}^{\mathrm{i}\varphi}$ to w. Explicitly, $A = |w|$, $\mathrm{e}^{\mathrm{i}\varphi} = |w|/w$. Thus, after this choice, we have

$$\Lambda = \frac{1}{|w|}\Lambda' \tag{47}$$

$$F = \frac{|w|}{w}(F' + \bar{y}\Lambda') \tag{48}$$

$$T = |w|(T' + \bar{y}\bar{F}' + yF' + y\bar{y}\Lambda') \tag{49}$$

$$E = \frac{1}{w}E'. \tag{50}$$

It is now clear that the set of all the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ is well defined on the 10-dimensional manifold $\mathcal{C}$ parametrized by $(p, w, \bar{w}, y, \bar{y})$. The price paid for the passage from $\mathcal{C}'$ to $\mathcal{C}$ is the introduction of a form

$$\Omega + \bar{\Omega} = -\mathrm{d}\log|w| + s_4\Lambda' - (\omega + \bar{\omega})T' + (\bar{\omega}_z - \gamma)F' + (\omega_{\bar{z}} - \bar{\gamma})\bar{F}' - yE' - \bar{y}\bar{E}', \tag{51}$$

which is still not fully determined, since the real function s_4 is still arbitrary.

We now pass to the analysis of $\mathrm{d}F$, with F being given by (48).

One easily calculates that

$$\mathrm{d}F = \mathrm{d}\log\frac{|w|}{w} \wedge F + \bar{w}\,\mathrm{d}\bar{y} \wedge \Lambda' + \frac{|w|}{w}(\mathrm{d}F' + \bar{y}\,\mathrm{d}\Lambda'). \tag{52}$$

This suggests the introduction of two other auxiliary forms, $\Omega - \bar{\Omega}$ and Γ, on $\mathcal{C}$, which are given by

$$\Omega - \bar{\Omega} = \mathrm{d}\log\frac{|w|}{w} + b_1F' - \bar{b}_1\bar{F}' + b_3T' + b_4\Lambda' + b_5E' - \bar{b}_5\bar{E}', \tag{53}$$

$$\Gamma = w[\mathrm{d}y + c_1F' + c_2\bar{F}' + c_3T' + c_4\Lambda' + c_5E' + c_6\bar{E}']. \tag{54}$$

Here b_3, b_4 (purely imaginary) and b_1, b_5, c_i $(i = 1, 2, \ldots, 6)$ (complex) are functions on $\mathcal{C}$ which should be determined.

Lengthy, but straightforward calculations lead to the following lemma.

Lemma 1. *If equation (46) for* $\mathrm{d}\Lambda$ *is satisfied then the conditions*

$$\mathrm{d}F = (\Omega - \bar{\Omega}) \wedge F + \bar{\Gamma} \wedge \Lambda + E \wedge T \tag{55}$$

$$\mathrm{d}T = T \wedge (\Omega + \bar{\Omega}) + \bar{\Gamma} \wedge \bar{F} + \Gamma \wedge F \tag{56}$$

uniquely determine s_4, b_μ, c_i $(\mu = 1, 3, 4, 5$*;* $i = 1, 2, \ldots, 6)$ *and, thus, the forms* Ω *and* Γ*. Explicitly, if* Λ, F, T, E *are given by (47)–(50), respectively, then*

$$\Omega = -\frac{1}{2}\frac{\mathrm{d}w}{w} + \gamma_{\bar{z}}\Lambda' - \gamma F' + \omega_{\bar{z}}\bar{F}' - \omega T' - yE' \tag{57}$$

$$\begin{aligned}\Gamma = w[&\mathrm{d}y + (\gamma_{z\bar{z}} + 2y\gamma_{\bar{z}})\Lambda' + (\gamma_{\bar{z}} + \omega_{z\bar{z}} + 2y\omega_{\bar{z}})\bar{F}' \\ &-(\gamma_z + 2y\gamma)F' - (\gamma + \omega_z + 2y\omega)T' - y^2E'].\end{aligned} \tag{58}$$

Thus, on $\mathcal{C}$ the 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E}, \Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ given by (47)–(50), (25), (57), (58) are well defined. They satisfy the differential equations (46), (55), (56). From now on we analyse these forms.

Straightforward calculations lead to the following expression for the differential of $\mathrm{d}E$:

$$\begin{aligned}\mathrm{d}E = 2\Omega \wedge E &+ \frac{1}{|w|^2}\Phi T \wedge F + \frac{\bar{w}}{w}\big[\tfrac{1}{2}\Phi_{\bar{z}\bar{z}} + \Phi_{\bar{z}}\bar{y} + \Phi\bar{y}^2\big]\bar{F} \wedge \Lambda \\ &+\frac{1}{w}\big[\Psi y - \Phi\bar{y} + \tfrac{1}{4}\Psi_z - \tfrac{1}{2}\Phi_{\bar{z}}\big]\bar{F} \wedge F - \frac{1}{w}\big[\Psi y + \Phi\bar{y} + \tfrac{1}{4}\Psi_z + \tfrac{1}{2}\Phi_{\bar{z}}\big]T \wedge \Lambda \\ &+\frac{1}{w^2}\Psi T \wedge \bar{F} + \big[\tfrac{1}{12}\Psi_{zz} + \tfrac{1}{12}R + \tfrac{1}{2}\Psi_z y + \Psi y^2\big]F \wedge \Lambda.\end{aligned} \tag{59}$$

The following three cases are of particular interest.

(**A**) The metric g of the 4-manifold $\mathcal{M}$ satisfies Einstein equations $R_{ij} = \lambda g_{ij}$ and is conformally non-flat. This case is characterized by $\Phi \equiv 0$ and $\Psi \not\equiv 0$.
(**B**) The metric g is conformally flat but not Einstein. This case corresponds to $\Psi \equiv 0$, $\Phi \not\equiv 0$.
(**C**) The metric g is of constant curvature. This means that $\Psi \equiv \Phi \equiv 0$.

Only in the first two cases is there a unique way of fixing the gauge for $(F, \bar{F}, T, \Lambda, E, \bar{E})$. Thus in these two cases it is possible to reduce the system of 1-forms from $\mathcal{C}$ back to $\mathcal{P}$. Such a reduction corresponds to an appropriate choice of y and w. As usual this choice will be such that it implies the vanishing of certain well defined terms in (59). Such an approach is impossible in case (**C**), since in this case there is an immediate reduction of (59) to

$$\mathrm{d}E = 2\Omega \wedge E + \tfrac{1}{12}R\ F \wedge \Lambda. \tag{60}$$

From now on we consider the case (**A**) where the metric is not conformally flat and satisfies Einstein's equations. Imposing the restrictions (**A**) on (59) we immediately see that

$$\begin{aligned}\mathrm{d}E = 2\Omega \wedge E &- \frac{1}{w}\big[\Psi y + \tfrac{1}{4}\Psi_z\big][T \wedge \Lambda + F \wedge \bar{F}] - \frac{1}{w^2}\Psi T \wedge \bar{F} \\ &+\big[\tfrac{1}{12}\Psi_{zz} + \tfrac{1}{12}R + \tfrac{1}{2}\Psi_z y + \Psi y^2\big]F \wedge \Lambda.\end{aligned}$$

Assuming that $\Psi \not\equiv 0$ and making the choice

$$y = -\frac{1}{4}\frac{\Psi_z}{\Psi} \tag{61}$$

we bring $\mathrm{d}E$ to the form

$$\mathrm{d}E = 2\Omega \wedge E - \frac{1}{w^2}\Psi\bar{F} \wedge T + \left[\frac{1}{12}\Psi_{zz} - \frac{1}{16}\frac{\Psi_z^2}{\Psi} + \frac{1}{12}R\right]F \wedge \Lambda.$$

Now the last gauge-fixing condition can be made by demanding that

$$\omega^2 = -\Psi. \tag{62}$$

This determines w up to a sign

$$w = \pm \mathrm{i}(\Psi)^{1/2}. \tag{63}$$

Now, the expressions (61) and (63) can be substituted into the 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E}, \Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ given by (47)–(50), (25), (57) and (58). After such a substitution the dependence of $y, \bar{y}, w, \bar{w}$ disappears from the forms. Thus they project to $\mathcal{P}$ where they are defined uniquely up to signs. This shows that in the case of the Einstein 4-metric we are able to fix the freedom in the choice of our initial 1-forms of (26)–(29), everywhere on $\mathcal{P}$ except at points where Ψ vanishes. As we know such vanishing occurs on sections of $\mathcal{P}$ corresponding to the principal null directions on $\mathcal{M}$. It is possible to overcome this difficulty by changing the topology of each fibre of $\mathcal{P}$. This possibility was studied by one of us in [9]. Summing up we have the following theorem.

Theorem 2. *Let $\mathcal{M}$ be a four-dimensional Lorentzian manifold and let $\mathcal{P}$ be its corresponding bundle of null directions. Suppose that the metric g on $\mathcal{M}$ satisfies the Einstein equations $R_{ij} = \lambda g_{ij}$ and is not conformally flat. Then on $\mathcal{P}$, apart the points that correspond to principal null directions, there exist preferred forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$, which are in the class (26)–(29), forms $(\Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ and a function α such that*

$$\begin{aligned} \mathrm{d}\Lambda &= (\Omega + \bar{\Omega}) \wedge \Lambda + E \wedge \bar{F} + \bar{E} \wedge F \\ \mathrm{d}F &= (\Omega - \bar{\Omega}) \wedge F + \bar{\Gamma} \wedge \Lambda + E \wedge T \\ \mathrm{d}T &= T \wedge (\Omega + \bar{\Omega}) + \bar{\Gamma} \wedge \bar{F} + \Gamma \wedge F \\ \mathrm{d}E &= 2\Omega \wedge E + \bar{F} \wedge T + \alpha \Lambda \wedge F. \end{aligned} \tag{64}$$

The forms are given by

$$\Lambda = \frac{1}{|\Psi|^{1/2}} \Lambda',$$

$$F = \varepsilon \mathrm{i} \left(\frac{\bar{\Psi}}{\Psi} \right)^{1/4} \left[F' - \tfrac{1}{4} (\log \bar{\Psi})_{\bar{z}} \Lambda' \right], \qquad \varepsilon = \pm 1,$$

$$T = |\Psi|^{1/2} \left[T' - \tfrac{1}{4} (\log \Psi)_z F' - \tfrac{1}{4} (\log \bar{\Psi})_{\bar{z}} \bar{F}' + \tfrac{1}{16} |(\log \Psi)_z|^2 \Lambda' \right],$$

$$E = \frac{\varepsilon \mathrm{i}}{|\Psi|^{1/2}} \left(\frac{\bar{\Psi}}{\Psi} \right)^{1/4} E',$$

$$\Omega = -\tfrac{1}{4} \mathrm{d} \log \Psi + \gamma_{\bar{z}} \Lambda' - \gamma F' + \omega_{\bar{z}} \bar{F}' - \omega T' + \tfrac{1}{4} (\log \Psi)_z E',$$

$$\begin{aligned} \Gamma = -\varepsilon \mathrm{i} |\Psi|^{1/2} \left(\frac{\Psi}{\bar{\Psi}} \right)^{1/4} \Big[& -\tfrac{1}{4} \mathrm{d} (\log \Psi)_z + \big(\gamma_{z\bar{z}} - \tfrac{1}{2} (\log \Psi)_z \gamma_{\bar{z}} \big) \Lambda' \\ & + \big(\gamma_{\bar{z}} + \omega_{z\bar{z}} - \tfrac{1}{2} (\log \Psi)_z \omega_{\bar{z}} \big) \bar{F}' - \big(\gamma_z - \tfrac{1}{2} (\log \Psi)_z \gamma \big) F' \\ & - \big(\gamma + \omega_z - \tfrac{1}{2} (\log \Psi)_z \omega \big) T' - \tfrac{1}{16} (\log \Psi)_z{}^2 E' \Big], \end{aligned}$$

where (Λ', F', T', E') are given by (25), γ, ω, Ψ are those of (34)–(37) and α is given by

$$\alpha = \frac{1}{16} \frac{\Psi_z^2}{\Psi} - \frac{1}{12} \Psi_{zz} - \frac{1}{12} R.$$

The forms $(F, \bar{F}, T, \Lambda, E, \bar{E}, \Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ that appear in the above theorem will be called the Cartan-invariant 1-forms for a Lorentzian conformally non-flat Einstein manifold. Together with the function α, which we call the Cartan-invariant function, they may be used to determine whether two given metrics are locally isometrically equivalent.

6. Remarks on the equivalence problem

Cartan's approach to the question of determining whether or not two given metrics are isometrically equivalent can be given a useful formulation in the context of the previous section. Here we outline the way in which theorem 2 can be used to do this.

Suppose that we are given two Lorentzian metrics g and $\hat{g}$ on two 4-manifolds $\mathcal{M}$ and $\hat{\mathcal{M}}$. The metrics are assumed to be Einstein and conformally non-flat. Suppose now that there exists a local isometry between g and $\hat{g}$, that is a local diffeomorphism $\phi\colon \mathcal{M} \to \hat{\mathcal{M}}$ such that $\phi^*\hat{g} = g$. Taking an ordered null cotetrad $(\hat{M}, \bar{\hat{M}}, \hat{K}, \hat{L})$ on $\hat{\mathcal{M}}$ and applying to it ϕ^* we obtain the 1-forms

$$M = \phi^*(\hat{M}), \qquad \bar{M} = \phi^*(\bar{\hat{M}}), \qquad K = \phi^*(\hat{K}), \qquad L = \phi^*(\hat{L}). \tag{65}$$

Due to the isometric property of ϕ we find that $(M, \bar{M}, K, L)$ constitutes a null cotetrad for g on $\mathcal{M}$. Now we use the cotetrads $(M, \bar{M}, K, L)$ on $\mathcal{M}$ and $(\hat{M}, \bar{\hat{M}}, \hat{K}, \hat{L})$ on $\hat{\mathcal{M}}$ to calculate the Cartan invariants on the corresponding bundles of null directions $\mathcal{P}$ and $\hat{\mathcal{P}}$. Let z and $\hat{z}$ be fibre coordinates on $\mathcal{P}$ and $\hat{\mathcal{P}}$ related to the cotetrads $(M, \bar{M}, K, L)$ and $(\hat{M}, \bar{\hat{M}}, \hat{K}, \hat{L})$ by the formulae analogous to (24). Then theorem 2 yields two sets of the Cartan-invariant 1-forms: $(F, \bar{F}, T, \Lambda, E, \bar{E}, \Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ on $\mathcal{P}$ and $(\hat{F}, \bar{\hat{F}}, \hat{T}, \hat{\Lambda}, \hat{E}, \bar{\hat{E}}, \hat{\Omega}, \bar{\hat{\Omega}}, \hat{\Gamma}, \bar{\hat{\Gamma}})$ on $\hat{\mathcal{P}}$. In addition, a pair of Cartan invariants α and $\hat{\alpha}$ may be easily calculated. It follows from the definition of the Cartan invariants and (65) that the map $\hat{p}\colon \mathcal{P} \to \hat{\mathcal{P}}$ defined by $\hat{p}(x^i, z, \bar{z}) = (\phi(x^i), z, \bar{z})$ has the property that

$$\hat{p}^*(\hat{\Lambda}) = \Lambda, \qquad \hat{p}^*(\hat{T}) = T, \qquad \hat{p}^*(\hat{F}) = \pm F, \qquad \hat{p}^*(\hat{E}) = \pm E, \tag{66}$$

$$\hat{p}^*(\hat{\Omega}) = \Omega, \qquad \hat{p}^*(\hat{\Gamma}) = \pm\Gamma, \tag{67}$$

$$\hat{p}^*(\hat{\alpha}) = \alpha. \tag{68}$$

This proves the following proposition.

Proposition 3. *Any (local) diffeomorphism $\phi\colon \mathcal{M} \to \hat{\mathcal{M}}$ which is an isometry between g and $\hat{g}$ generates a (local) diffeomorphism $\hat{p}\colon \mathcal{P} \to \hat{\mathcal{P}}$ which satisfies (66)–(68).*

To prove the converse we need the following lemma.

Lemma 2. *A diffeomorphism $\hat{p}\colon \mathcal{P} \to \hat{\mathcal{P}}$ satisfying (66)–(68) induces a diffeomorphism $\phi\colon \mathcal{M} \to \hat{\mathcal{M}}$ such that the following diagram:*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\hat{p}} & \hat{\mathcal{P}} \\ \downarrow \pi & & \downarrow \hat{\pi} \\ \mathcal{M} & \xrightarrow{\phi} & \hat{\mathcal{M}} \end{array} \tag{69}$$

commutes.

Proof. $\mathcal{P}$ is foliated by the fibres Σ_x of the fibration $\pi\colon \mathcal{P} \to \mathcal{M}$. This foliation, which we denote by $\mathcal{V}$, is such that each of the forms $(F, \bar{F}, T, \Lambda)$ vanishes when restricted to its leaves Σ_x†. An analogous foliation $\hat{\mathcal{V}}$ exists on $\hat{\mathcal{P}}$. Our aim is to prove that two points from the same leaf of $\mathcal{V}$ cannot be transformed by $\hat{p}$ to different leaves of $\hat{\mathcal{V}}$. To do this, observe that the diffeomorphism property of $\hat{p}$ implies that the leaves of the foliation $\mathcal{V}$ are transformed by $\hat{p}$ to non-intersecting two-dimensional submanifolds of $\hat{\mathcal{P}}$. Moreover, equations (66) guarantee

† This is consistent, since the forms $(F, \bar{F}, T, \Lambda)$ constitute a closed differential ideal due to the equations (64).

that each of the forms $(\hat{F}, \bar{\hat{F}}, \hat{T}, \hat{\Lambda})$ identically vanishes when restricted to any of these 2-manifolds. Thus, the set of all $\hat{p}(\Sigma_x)$, $x \in \mathcal{M}$, defines a foliation $\hat{p}(\mathcal{V})$ of $\hat{\mathcal{P}}$ which possesses all the properties of $\hat{\mathcal{V}}$. Due to the uniqueness of such a foliation, which follows from the Frobenius theorem, $\hat{p}(\mathcal{V}) \equiv \hat{\mathcal{V}}$. This, in particular, means that points from a given leaf of $\mathcal{V}$ are transformed by $\hat{p}$ to the same leaf of $\hat{\mathcal{V}}$. Thus the map $\hat{p}$: $\mathcal{P} \to \hat{\mathcal{P}}$ projects to the map ϕ: $\mathcal{M} \to \hat{\mathcal{M}}$. This map, by definition, has the property (69). This proves the lemma. □

Now, conversely to proposition 3, we have

Proposition 4. *If there exists a diffeomorphism $\hat{p}$: $\mathcal{P} \to \hat{\mathcal{P}}$, which satisfies equations (66)–(68), then the metrics g and $\hat{g}$ are isometrically equivalent.*

To prove this, consider ϕ of lemma 2 and observe that diagram (69) implies that

$$\hat{p}^*\hat{\pi}^*\hat{g} = \pi^*\phi^*\hat{g}. \tag{70}$$

On the other hand, since $\hat{\pi}^*\hat{g} = 2(\hat{F}\bar{\hat{F}} - \hat{T}\hat{\Lambda})$ and $\pi^*g = 2(F\bar{F} - T\Lambda)$, applying (66) gives $\hat{p}^*\hat{\pi}^*\hat{g} = \pi^*g$. Comparison of this with (70) yields $\phi^*\hat{g} = g$. This proves proposition 4.

It follows that we have the following algorithm for checking the local isometric equivalence of 4-metrics.

(a) Calculate the Petrov types of the metrics g and $\hat{g}$. If the Petrov types are different then the metrics are not equivalent.
(b) If the Petrov types are the same calculate the Cartan-invariant 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E}, \Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ on $\mathcal{P}$ and $(\hat{F}, \bar{\hat{F}}, \hat{T}, \hat{\Lambda}, \hat{E}, \bar{\hat{E}}, \hat{\Omega}, \bar{\hat{\Omega}}, \hat{\Gamma}, \bar{\hat{\Gamma}})$ on $\hat{\mathcal{P}}$. Also calculate the Cartan-invariant functions α and $\hat{\alpha}$.
(c) Search for a diffeomorphism $\hat{p}$: $\mathcal{P} \to \hat{\mathcal{P}}$ which satisfies (66)–(68). The metrics are (locally) equivalent if and only if such a $\hat{p}$ exists.

To perform step (c) one needs to solve differential equations such as, for example, $\hat{p}^*(\hat{T}) = T$ for $\hat{p}$. This may be not easy. To avoid this difficulty the following alternative procedure can be used. Recall that the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ (respectively, $(\hat{F}, \bar{\hat{F}}, \hat{T}, \hat{\Lambda}, \hat{E}, \bar{\hat{E}})$) are linearly independent at each point of $\mathcal{P}$ (respectively, $\hat{\mathcal{P}}$). We can therefore use the basis $(F, \bar{F}, T, \Lambda, E, \bar{E})$ (respectively, $(\hat{F}, \bar{\hat{F}}, \hat{T}, \hat{\Lambda}, \hat{E}, \bar{\hat{E}})$) to decompose the forms $(\Omega, \bar{\Omega}, \Gamma, \bar{\Gamma})$ (respectively, $(\hat{\Omega}, \bar{\hat{\Omega}}, \hat{\Gamma}, \bar{\hat{\Gamma}})$) onto them. These decompositions

$$\Omega = \omega_1 F + \omega_2 \bar{F} + \omega_3 T + \omega_4 \Lambda + \omega_5 E + \omega_6 \bar{E}$$
$$\Gamma = \gamma_1 F + \gamma_2 \bar{F} + \gamma_3 T + \gamma_4 \Lambda + \gamma_5 E + \gamma_6 \bar{E}$$

define coefficients ω_i, γ_i, and the analogous coefficients $\hat{\omega}_i$, $\hat{\gamma}_i$ for $\hat{\Omega}$, $\hat{\Gamma}$. The functions ω_i, γ_i, $i = 1, 2, \ldots, 6$ will be called the higher-order Cartan-invariant functions for the Lorentzian conformally non-flat Einstein metric g.

It is easy to see that some of the higher order Cartan-invariant functions vanish identically. Indeed, from the definitions of Ω and Γ given in theorem 2 one easily finds that

$$\omega_5 \equiv \omega_6 \equiv \gamma_6 \equiv 0.$$

It is also straightforward to see that

$$\gamma_5 + 3\alpha + \tfrac{1}{4}R \equiv 0.$$

By using the Bianchi identities for Ψ one also obtains the equations

$$4\omega_4 + \gamma_2 \equiv 4\omega_1 + \gamma_3 \equiv 0.$$

Hence we can always write Ω and Γ in the form

$$\Omega = \omega_1 F + \omega_2 \bar{F} + \omega_3 T + \omega_4 \Lambda,$$
$$\Gamma = \gamma_1 F - 4\omega_4 \bar{F} - 4\omega_1 T + \gamma_4 \Lambda - \left(3\alpha + \tfrac{1}{4}R\right)E,$$

(cf equation (14)).

Thus the relevant Cartan-invariant functions are: α, ω_i, $i = 1, 2, 3, 4$, γ_1, γ_3 and their complex conjugates. R is a constant invariant. In terms of these invariants the conditions (67) and (68) for $\hat{p}$ may be rewritten in the form

$$\begin{aligned} &\hat{p}^*(\hat{\alpha}) = \alpha, \\ &\hat{p}^*(\hat{\omega}_1) = \pm\omega_1, \qquad \hat{p}^*(\hat{\omega}_2) = \pm\omega_2, \qquad \hat{p}^*(\hat{\omega}_3) = \omega_3, \qquad \hat{p}^*(\hat{\omega}_4) = \omega_4, \\ &\hat{p}^*(\hat{\gamma}_1) = \gamma_1, \qquad \hat{p}^*(\hat{\gamma}_4) = \pm\gamma_4 \\ &R = \hat{R}. \end{aligned} \tag{71}$$

It follows that (as is well known) metrics with different scalar curvatures R and $\hat{R}$ are always non-isometric. If $R = \hat{R}$ then equations (66) and (71) are equivalent to the system (66)–(68). However, the system (66)–(68) includes only two non-differential equations (68) and $\overline{(68)}$ for $\hat{p}$. On the other hand, the system (66) and (71) includes 14 non-differential equations. These are precisely (71) and their complex conjugates.

Now, suppose that six independent real functions, say $\hat{f}_1, \hat{f}_2, \hat{f}_3, \hat{f}_4, \hat{f}_5, \hat{f}_6$ ($\mathrm{d}\hat{f}_1 \wedge \mathrm{d}\hat{f}_2 \wedge \mathrm{d}\hat{f}_3 \wedge \mathrm{d}\hat{f}_4 \wedge \mathrm{d}\hat{f}_5 \wedge \mathrm{d}\hat{f}_6 \neq 0$), of the real and imaginary parts of the Cartan invariants $\hat{\alpha}, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{\omega}_4, \hat{\gamma}_1, \hat{\gamma}_4$ can be chosen near a point $\hat{p}_0 \in \hat{\mathcal{P}}$†. Taking the corresponding functions $f_1, f_2, f_3, f_4, f_5, f_6$ on $\mathcal{P}$ and using equations (71) we find that the map $\hat{p}$ must satisfy the six independent non-differential equations

$$\hat{p}^* \hat{f}_i = \varepsilon f_i, \qquad i = 1, 2, \ldots, 6. \tag{72}$$

Here ε may be either 1 or -1 depending on which of the Cartan invariants we have used. It follows from the implicit function theorem that the six equations (72) uniquely determine the desired map $\hat{p}^i = \hat{p}^i\left(p^j\right)$. Thus, in this case, to solve the equivalence problem for the two metrics we have to check whether $\hat{p}$ thus determined satisfies all the remaining equations (71) and the differential equations (66). If it does then the two metrics are isometrically equivalent, otherwise they are not. This solves the equivalence problem for the Lorentzian conformally non-flat Einstein metrics in the generic case. The discussion does not apply to the case when the number of independent functions among the Cartan invariants is less than six. We call such cases degenerate. These are more subtle and will be presented elsewhere‡.

7. Elliptic fibrations

Suppose now that we have six 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ defined on an open set $\mathcal{P}_0$ of $\mathbf{R}^6$ which satisfy the differential system $(\mathcal{I}, \mathcal{P}_0)$ of theorem 1. According to this theorem $\mathcal{P}_0$ is

† In such cases it is convenient to use them as a coordinate system on $\hat{\mathcal{P}}$.

‡ Here we only note that if the number of independent Cartan-invariant functions is less than six, two cases may occur. Either all the Cartan-invariant functions are constant or there exists at least one which is not constant. The former case will be totally analysed in section 8. In the latter case one takes the differential of the non-constant Cartan invariant and decomposes it onto a basis of the Cartan-invariant forms $(\hat{F}, \hat{\bar{F}}, \hat{T}, \hat{\Lambda}, \hat{E}, \hat{\bar{E}})$. This produces new Cartan-invariant functions of the next order. In generic cases one can use these to obtain new algebraic equations for $\hat{p}$. If in this way we are able to produce six independent algebraic equations for $\hat{p}$ then we return to the already discussed case. If not, the procedure can be applied once more. There will be, of course, cases in which it is not possible to construct six independent algebraic equations for $\hat{p}$. This may occur if, for example, all the Cartan invariants depend only on one variable. In such cases there are symmetries and they may be analysed by using group-theoretical methods.

foliated by two-dimensional leaves in such a way that it can be considered a fibration over the Einstein conformally non-flat spacetime $\mathcal{M}$. Theorem 1 says nothing about the topology of the fibres of $\mathcal{P}_0$ since it deals with local solutions to differential equations (10). Thus, given a solution $(F, \bar{F}, T, \Lambda, E, \bar{E})$ of the system $\mathcal{I}$ on $\mathcal{P}_0$ we know only that $\mathcal{P}_0$ is foliated by leaves that in the generic case have the topology of an open disc in $\mathbf{R}^2$. The question arises as to whether we can extend the solution $(F, \bar{F}, T, \Lambda, E, \bar{E})$ to a larger fibration $\tilde{\mathcal{P}}$ over $\mathcal{M}$ in such a way that its fibres contain fibres of $\mathcal{P}_0$, and have a more interesting topology than that of open discs. It follows that given a solution there may be several such extensions. In this section we describe the most natural one, making more explicit the considerations of [9]. The other possibility is discussed in section 8.

A natural way of extending the fibres of $\mathcal{P}_0$ of theorem 1 is as follows. Given $\mathcal{P}_0$ with the system $\mathcal{I}$ on it, one passes to the spacetime $\mathcal{M}$ that is associated with it via theorem 1. Then, using theorem 2, one considers the bundle of null directions $\mathcal{P}$ for $\mathcal{M}$ and defines the Cartan-invariant 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ on it. These forms satisfy again the system of equations (10). Thus, we have an extension map ϕ from $\mathcal{P}_0$ with its fibres (say, of open disc topology) to $\mathcal{P}$ with fibres being spheres of null directions. The only problem is that some of the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ on $\mathcal{P}$ are defined only up to a sign (see theorem 2). To avoid this double-valuedness of F and E we again need to extend the fibres of $\mathcal{P}$. This is done as follows.

Recall that the Cartan-invariant 1-forms of theorem 2 are defined on $\mathcal{P}$ by the gauge-fixing conditions (61) and (62). The first of these conditions makes no sense if

$$\Psi = \Psi_4 - 4\Psi_3 z + 6\Psi_2 z^2 - 4\Psi_1 z^3 + \Psi_0 z^4$$

is zero. Here $z \in \mathbf{C} \cup \{\infty\}$ is a coordinate on a given fibre of $\mathcal{P}$. Thus in each fibre of $\mathcal{P}$ there are at most four points (which via (24) correspond to principal null directions at the spacetime point) at which the above expression vanishes. Consider now the function $w = \sqrt{-\Psi}$ defined by condition (62). We analyse how w changes when we pass along a small loop around a zero of Ψ in a fibre.

We write $\Psi = c(z - z_1)(z - z_2)(z - z_3)(z - z_4)$, where the roots z_i, $i = 1, 2, 3, 4$ are, in general, different. As is well known, if some of the roots z_i coincide then the 4-metric is algebraically special. The resulting Petrov types of the 4-metric are:

z_1, z_2, z_3, z_4 all different $\longleftrightarrow$ Petrov type I

$z_1 = z_2, z_3, z_4$ different $\longleftrightarrow$ Petrov type II

$z_1 = z_2 \neq z_3 = z_4 \longleftrightarrow$ Petrov type D

$z_1 = z_2 = z_3, z_4$ different $\longleftrightarrow$ Petrov type III

$z_1 = z_2 = z_3 = z_4 \longleftrightarrow$ Petrov type N.

Suppose now that we consider a spacetime point for which the metric is of Petrov type I. Consider a loop

$$z(t) = z_i + \rho \mathrm{e}^{\mathrm{i}t}, \tag{73}$$

$t \in [0, 2\pi]$ around one of the roots z_i (say z_1) of Ψ. The loop is supposed to be small ($\rho \ll 1$), so that the value of $\Psi = \Psi(t)$ at the points of this loop may be approximated by $\Psi = c(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)\rho \mathrm{e}^{\mathrm{i}t}$. We can write this fact as $\Psi = -\sigma_0 \mathrm{e}^{\mathrm{i}(t+t_0)}$, where $\sigma_0 > 0$ and t_0 are real constants. Substituting such a Ψ in $w = \sqrt{-\Psi}$ we find that $w = \sqrt{\sigma_0}\, \mathrm{e}^{\mathrm{i}t_0/2} \mathrm{e}^{\mathrm{i}t/2}$. Suppose now that we change t from 0 to 2π. This corresponds to the passage from a point $z(0)$ to itself along the loop. Since $t/2$ changes its value only by π, then w changes sign. To return to the initial value of w we need to change t from 0 to 4π. This shows that the point z_1 is a double-branch point for the function w. If the metric of the spacetime point under consideration is algebraically general the same is true for z_2, z_3, z_4. To make the function w single-valued we need to extend the z-space (a sphere) to a torus. This is obtained by considering two copies of the z-spaces (two spheres—the domains on which w has a plus and minus sign, respectively) each having two cuts between its points z_1–z_2 and z_3–z_4, say, with an appropriate identification of the cut edges.

The situation is a bit different if some of the roots z_i of Ψ are multiple. It is easy to see that after a passage along a loop around a double or quadruple root the function w does not change sign, and that after a passage around a triple root w changes sign and returns to its original value only after the second turn. Considering two copies of spheres with appropriate cuts and identifications for all the possible situations we arrive to the following conclusion.

Let x be a spacetime point and $\mathcal{T}_x$ its space on which the function w is single-valued. If the metric at x is:

- algebraically general, then $\mathcal{T}_x$ has topology of a 2-torus;
- of Petrov type II, then $\mathcal{T}_x$ has topology of a 2-torus with one vanishing cycle;
- of Petrov type D, then $\mathcal{T}_x$ has topology of two 2-spheres touching each other in two different points;
- of Petrov type III, then $\mathcal{T}_x$ has topology of a 2-sphere with one singular point;
- of Petrov type N, then $\mathcal{T}_x$ has topology of two 2-spheres touching each other in one point.

Recall that the origin of the double-valuedness of the Cartan-invariant forms was the double-valuedness of function w which we used in their definition via (62). Thus it is clear that on a fibration $\tilde{\mathcal{P}}$ whose fibre over a spacetime point x is $\mathcal{T}_x$ the 1-forms of theorem 2 are single valued, and can have only simple singularities in at most four points at the fibre. This shows that the fibration $\mathcal{P}_0$ on which the system $\mathcal{I}$ of theorem 1 is satisfied can be naturally extended to the fibration $\tilde{\mathcal{P}}$ whose fibres over a spacetime point x are tori or their degenerate counterparts, depending on the algebraic type of the metric at x. Since fibres with the topology of tori are defined by an algebraic equation $w^2 = \Psi_4 - 4\Psi_3 z + 6\Psi_2 z^2 - 4\Psi_1 z^3 + \Psi_0 z^4$, which can be identified with an elliptic curve (possibly degenerate) in $\mathbf{C}^2$, we call $\mathcal{P}$ an elliptic fibration [9]. It is clear that it constitutes a double-branched cover of the bundle of null directions over the spacetime.

8. Generalized bundles of null directions and their symmetries

It is now clear that given the system $(\mathcal{P}_0, \mathcal{I})$ of theorem 1 one has not enough information to determine the topology of the fibres of $\mathcal{P}_0$. We know that locally $\mathcal{P}_0$ has all the properties of the bundle of null directions over the associated Einstein spacetime and that it can be further extended to the elliptic fibration of the preceding section. However, other extensions are possible. In this section we consider examples of systems $(\mathcal{P}_0, \mathcal{I})$ of theorem 1 that can have fibres with topologies different from those discussed so far. They correspond to the known Einstein spaces with six-dimensional groups of symmetries.

Because of local equivalence of $\mathcal{P}_0$ and $\mathcal{P}$ we introduce the following definition.

Definition 1. *A real six-dimensional manifold $\mathcal{P}$ equipped with forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ which satisfy the differential system (10) will be called a generalized Einstein bundle of null directions.*

In this section the letter $\mathcal{P}$ always denotes such bundles.

Via theorems 1 and 2 the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ on $\mathcal{P}$, as well as the derived quantities α, Ω and Γ, may be identified with the Cartan invariants for the associated Einstein spacetime. Thus we will also call them the Cartan invariants for $\mathcal{P}$.

Definition 2. *We say that $\mathcal{P}$ is (locally) symmetric iff there exists a (local) diffeomorphism ϕ: $\mathcal{P} \to \mathcal{P}$ which preserves the 1-forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$, i.e. iff*

$$\phi^*(F) = \pm F, \qquad \phi^*(T) = T, \qquad \phi^*(\Lambda) = \Lambda, \qquad \phi^*(E) = \pm E. \tag{74}$$

A real vector field $\tilde{X}$ on $\mathcal{P}$ is a symmetry iff

$$\mathcal{L}_{\tilde{X}} F = 0, \qquad \mathcal{L}_{\tilde{X}} T = 0, \qquad \mathcal{L}_{\tilde{X}} \Lambda = 0, \qquad \mathcal{L}_{\tilde{X}} E = 0. \tag{75}$$

The following lemma shows that all of the Cartan invariants are preserved by a symmetry.

Lemma 3. *If $\tilde{X}$ is a symmetry of $\mathcal{P}$ then*

$$\tilde{X}(\alpha) = 0, \qquad \mathcal{L}_{\tilde{X}} \Omega = 0, \qquad \mathcal{L}_{\tilde{X}} \Gamma = 0, \tag{76}$$

where Ω, Γ and α are the Cartan invariants of equations (10).

To prove this we observe that (75) implies $\mathcal{L}_{\tilde{X}}\, \mathrm{d}\Lambda = 0$, $\mathcal{L}_{\tilde{X}}\, \mathrm{d}F = 0$, $\mathcal{L}_{\tilde{X}}\, \mathrm{d}T = 0$, $\mathcal{L}_{\tilde{X}}\, \mathrm{d}E = 0$. Now, combining these equations with (10) we find that

$$\begin{aligned}
&[\mathcal{L}_{\tilde{X}}(\Omega + \bar{\Omega})] \wedge \Lambda = 0,\\
&[\mathcal{L}_{\tilde{X}}(\Omega - \bar{\Omega})] \wedge F + [\mathcal{L}_{\tilde{X}} \bar{\Gamma}] \wedge \Lambda = 0,\\
&-\mathcal{L}_{\tilde{X}}(\Omega + \bar{\Omega}) \wedge T + \mathcal{L}_{\tilde{X}} \bar{\Gamma} \wedge \bar{F} + \mathcal{L}_{\tilde{X}} \Gamma \wedge F = 0,\\
&2[\mathcal{L}_{\tilde{X}} \Omega] \wedge E + \tilde{X}(\alpha) \Lambda \wedge F = 0.
\end{aligned}$$

Due to the independence of $(F, \bar{F}, T, \Lambda, E, \bar{E})$ the above equations imply (76). This finishes the proof of lemma 3.

Definition 3. *A symmetry is called vertical if it has the form $\tilde{X} = se + \bar{s}\bar{e}$, where s is any complex-valued function on $\mathcal{P}$, and $(f, \bar{f}, t, l, e, \bar{e})$ constitute a basis of vector fields on $\mathcal{P}$ dual to $(F, \bar{F}, T, \Lambda, E, \bar{E})$.*

Lemma 4. *The only vertical symmetry of $\mathcal{P}$ is $\tilde{X} = 0$.*

Proof. Let $\tilde{X}$ be a symmetry of $\mathcal{P}$. Its general form is

$$\tilde{X} = x_1 f + \bar{x}_1 \bar{f} + x_3 t + x_4 l + x_5 e + \bar{x}_5 \bar{e}, \tag{77}$$

where x_3, x_4 are real functions. Using the symmetry conditions (75) and the differentials (10), (13) and (14) we find that

$$\mathrm{d}x_1 = (\bar{x}_7 - x_7)F - x_5 T - x_8 \Lambda + x_3 E + x_1(\Omega - \bar{\Omega}) + x_4 \bar{\Gamma}, \tag{78}$$

$$\mathrm{d}x_3 = -\bar{x}_8 F - x_8 \bar{F} + (x_7 + \bar{x}_7)T - x_3(\Omega + \bar{\Omega}) + \bar{x}_1 \bar{\Gamma} + x_1 \Gamma, \tag{79}$$

$$\mathrm{d}x_4 = -\bar{x}_5 F - x_5 \bar{F} - (x_7 + \bar{x}_7)\Lambda + \bar{x}_1 E + x_1 \bar{E} + x_4(\Omega + \bar{\Omega}), \tag{80}$$

$$\mathrm{d}x_5 = -\alpha x_4 F + x_3 \bar{F} - \bar{x}_1 T + \alpha x_1 \Lambda - 2x_7 E + 2x_5 \Omega, \tag{81}$$

where x_7 and x_8 are functions whose form is not relevant here. Now, if the symmetry is vertical then $x_1 = x_3 = x_4 = 0$. It follows from equation (78) that in such case $x_5 = 0$. This implies $\tilde{X} = 0$ which completes the proof. □

Lemma 5. *Any symmetry $\tilde{X}$ of $\mathcal{P}$ generates a Killing symmetry X of the metric g on the quotient 4-manifold $\mathcal{M}$ of leaves of the foliation $\{\mathcal{S}_x\}$. Moreover, the Lie algebra of symmetries $\{\tilde{X}_i\}$ is isomorphic to the algebra $\{X_i\}$ of the corresponding Killing symmetries.*

Proof. We write $\tilde{X}$ in its general form (77). Using the conditions (78)–(81) we find $e(x_i)$. The differentials (10) imply the form of commutators $[e, l]$, $[e, f]$, etc. Combining this with $e(x_i)$ we find that $[e, \tilde{X}] = Ue + U'\bar{e}$, where U, U' are certain complex functions on $\mathcal{P}$. This implies that vectors of $\tilde{X}$ calculated at points of the same leaf of the foliation $\mathcal{V}$ differ by a vertical part $V = U''e + \bar{U}''\bar{e}$ only. Thus $\tilde{X}$ uniquely projects to X on $\mathcal{M}$. X is not zero since any non-zero symmetry $\tilde{X}$ has always a non-zero $(f, \bar{f}, t, l)$ part. Consider now two non-zero symmetries $\tilde{X} = X + x_5 e + \bar{x}_5\bar{e}$ and $\tilde{Y} = Y + y_5 e + \bar{y}_5\bar{e}$. Here X, Y denote $(f, \bar{f}, t, l)$ parts of $\tilde{X}$ and $\tilde{Y}$, respectively. The commutator of $\tilde{X}$ and $\tilde{Y}$ has the form

$$[\tilde{X}, \tilde{Y}] = [X, Y] + x_5[e, Y] + \bar{x}_5[\bar{e}, Y] - y_5[e, X] + \bar{y}_5[\bar{e}, X] \\ +(x_5\bar{y}_5 - \bar{x}_5 y_5)[e, \bar{e}] \quad \text{modulo } e \text{ and } \bar{e}.$$

Since for any symmetry $\tilde{Z}$ we have $[e, Z] = 0$ modulo e and $\bar{e}$, then $[\tilde{X}, \tilde{Y}] = [X, Y] + (x_5\bar{y}_5 - \bar{x}_5 y_5)[e, \bar{e}]$ modulo e and $\bar{e}$. It follows from equations (10), (13) that $[e, \bar{e}] = 0$. Thus $[\tilde{X}, \tilde{Y}] = [X, Y]$ modulo terms which vanish under the projection $\pi\colon \mathcal{P} \to \mathcal{M}$. Thus the Lie algebra of $\{\tilde{X}_i\}$ is the same as that of $\{X_i\}$.

Finally, we note that if $\tilde{X} = X + x_5 e + \bar{x}_5\bar{e}$ then the symmetry equations (75) and the differentials (10), (14) imply that

$$\mathcal{L}_X \Lambda = -x_5\bar{F} - \bar{x}_5 F,$$
$$\mathcal{L}_X F = -x_5 T + \bar{x}_5(3\bar{\alpha} + \lambda)\Lambda,$$
$$\mathcal{L}_X T = \bar{x}_5(3\bar{\alpha} + \lambda)\bar{F} + x_5(3\alpha + \lambda)F.$$

These equations imply that on $\mathcal{P}$ we have $\mathcal{L}_X G = 0$. This equation projects to the equation $\mathcal{L}_X g = 0$ on $\mathcal{M}$, since both X and G have unique projections to X and g on $\mathcal{M}$, respectively. This finishes the proof of the lemma. □

Definition 4. *A generalized Einstein bundle of null direction is called (locally) homogeneous if it possesses six symmetries, which generate a (local) transitive group of transformations of* $\mathcal{P}$.

It is clear from lemma 3 that on homogeneous generalized Einstein bundles of null directions all the Cartan-invariant functions are constant. In such a case we may interpret equations (10) as the Cartan structure equations for the left-invariant forms on a certain Lie group. This shows that (locally) homogeneous generalized Einstein bundles of null directions are (local) Lie groups whose structure constants may be read from equations (10) and (14). To determine all the possible groups that are homogeneous generalized Einstein bundles of null directions we need to check which constants $\alpha, \gamma_1, \gamma_4, \omega_1, \omega_2, \omega_3, \omega_4$ are compatible with equations (10), (13) and (14). We find that if $\alpha, \gamma_1, \gamma_3, \omega_1, \omega_2, \omega_3, \omega_4$ are constants then equations (13) imply that $a = \alpha_1 = \alpha_4 = a_1 = a_4 = h_1 = h_4 = \gamma_1 = \gamma_4 = 0$ and $h = -(3\alpha + \lambda)^2$. Moreover, the combination of equations (13) and (10) leads to two possibilities only.

$$\text{(i)} \qquad \alpha = \lambda = h = a = 0, \qquad \Omega = \omega_3 T, \qquad \Gamma = 0.$$
$$\text{(ii)} \qquad \alpha = -\tfrac{1}{2}\lambda, \qquad h = -\tfrac{1}{4}\lambda^2, \qquad a = 0, \qquad \Omega = 0, \qquad \Gamma = \tfrac{1}{2}\lambda E. \tag{82}$$

Thus we have two families of homogeneous Einstein bundles. To characterize the corresponding groups we note that the family (i) leads to the vacuum ($\lambda = 0$) Einstein 4-manifold $\mathcal{M}$ which has a metric of Petrov type N (cf equation (16)). The family (ii) leads to a 4-metric of Petrov type D (cf equation (15)) satisfying the Einstein equations with cosmological constant λ. The following two examples deal explicitly with cases (i) and (ii). They correspond to all possible conformally non-flat solutions to the vacuum Einstein equations (with or without cosmological constant) that have a six-dimensional group of symmetries.

Example 6 (Example 2 continued). Since example 2 corresponds to all possible vacuum Einstein spaces with six symmetries, it therefore also exhausts all the possibilities for homogeneous generalized vacuum Einstein bundles of null directions $\mathcal{P}$. The simplest among them is the bundle $\mathcal{P}$ of example 1. It may be identified with a six-dimensional group, say G_0. Interpreting the forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ as the left-invariant forms on G_0 we easily read the structure constants for G_0 from the equations

$$\begin{aligned}
&\mathrm{d}F - E \wedge T = 0\\
&\mathrm{d}T = 0\\
&\mathrm{d}\Lambda - \bar{E} \wedge F - E \wedge \bar{F} = 0\\
&\mathrm{d}E - \bar{F} \wedge T = 0
\end{aligned}$$

obtained from (10) after insertion of conditions (i) with $\omega_3 = 0$.

Analysis of the structure constants shows that G_0, therefore $\mathcal{P}$, is a six-dimensional solvable group. This group is isomorphic to the group of symmetries of the corresponding plane wave.

Example 7. In case (ii) $\mathcal{P}$ also may be identified with a six-dimensional group, say G_λ, and its forms $(F, \bar{F}, T, \Lambda, E, \bar{E})$ can be identified with the left invariant forms on G_λ. To be more explicit we insert conditions (ii) to (10) obtaining

$$\begin{aligned}
&\mathrm{d}F - E \wedge T - \tfrac{\lambda}{2}\bar{E} \wedge \Lambda = 0\\
&\mathrm{d}T - \tfrac{1}{2}\lambda(E \wedge F + \bar{E} \wedge \bar{F}) = 0\\
&\mathrm{d}\Lambda - \bar{E} \wedge F - E \wedge \bar{F} = 0\\
&\mathrm{d}E - \bar{F} \wedge T + \tfrac{1}{2}\lambda\Lambda \wedge F = 0.
\end{aligned} \tag{83}$$

Since $\lambda = 0$ corresponds to the equations discussed in the previous example we assume that $\lambda \neq 0$. It is convenient to introduce real 1-forms $(A_1, A_2, A_3, A_1', A_2', A_3')$ defined by

$$\begin{aligned}
&F = -\frac{1}{\sqrt{2|\lambda|}}(A_1 + \mathrm{i}A_1')\\
&T = \tfrac{1}{2}(A_2 - A_2')\\
&\Lambda = \frac{1}{\lambda}(A_2 + A_2')\\
&E = \frac{1}{\sqrt{2|\lambda|}}(A_3 + \mathrm{i}A_3').
\end{aligned} \tag{84}$$

Equations (83) written in terms of these forms then become

$$\begin{aligned}
&\mathrm{d}A_1 = A_2 \wedge A_3 &&\mathrm{d}A_1' = -A_2' \wedge A_3'\\
&\mathrm{d}A_3 = -A_1 \wedge A_2 &&\mathrm{d}A_3' = -A_1' \wedge A_2'\\
&\mathrm{d}A_2 = -\varepsilon A_3 \wedge A_1 &&\mathrm{d}A_2' = -\varepsilon A_3' \wedge A_1',
\end{aligned} \tag{85}$$

where $\varepsilon = \pm 1 = (\text{sign of } \lambda)$. These equations show that the group G_λ is a direct product of two groups H and H'. The Lie algebra of H is isomorphic to $\mathbf{sl}(2, \mathbf{R})$ and the Lie algebra of H' depends on the sign of λ and is isomorphic to $\mathbf{su}(2)$ if $\lambda > 0$ and to $\mathbf{sl}(2, \mathbf{R})$ if $\lambda < 0$. Thus in this case $\mathcal{P} \equiv G_\lambda \equiv H \times H'$. In terms of the variables A_i, A_i' the degenerate metric G of theorem 1 has the form $G = \frac{1}{\lambda}\big[A_1^2 - \varepsilon A_2^2 + A_1'^2 + \varepsilon A_2'^2\big]$. According to this theorem the space of leaves of the fibration is equipped with the Einstein metric g with non-vanishing cosmological constant λ.

(iia) Assume that $\lambda > 0$.

In this case the Lie algebra of H is isomorphic to $\mathbf{sl}(2, \mathbf{R})$ and the Lie algebra of H' is isomorphic to $\mathbf{su}(2)$. In the following we concentrate on the case when $H = \mathbf{SO}(1, 2)$ and $H' = \mathbf{SO}(3)$, but one can also consider cases in which H and/or H' are double covers of these groups.

If $H = \mathbf{SO}(1, 2)$ and $H' = \mathbf{SO}(3)$ then a coordinate system (x_1, x_2, x_3) on H and a coordinate system (x'_1, x'_2, x'_3) on H' may be chosen such that

$$
\begin{aligned}
A_1 &= \cosh x_2 \cosh x_3\, \mathrm{d}x_1 - \sinh x_3\, \mathrm{d}x_2 \\
A_2 &= -\cosh x_2 \sinh x_3\, \mathrm{d}x_1 + \cosh x_3\, \mathrm{d}x_2 \\
A_3 &= \sinh x_2\, \mathrm{d}x_1 + \mathrm{d}x_3 \\
A'_1 &= \cos x'_2 \cos x'_3\, \mathrm{d}x'_1 + \sin x'_3\, \mathrm{d}x'_2 \\
A'_2 &= -\cos x'_2 \sin x'_3\, \mathrm{d}x'_1 + \cos x'_3\, \mathrm{d}x'_2 \\
A'_3 &= \sin x'_2\, \mathrm{d}x'_1 + \mathrm{d}x'_3.
\end{aligned}
\tag{86}
$$

Since the above forms satisfy equations (85) for $\varepsilon = 1$, then for each value of $\lambda > 0$, via (84), they define a solution to the system (10) with α, Ω, Γ given by (82) (ii).

To obtain a better insight into this solution and its corresponding spacetime consider a generic point $P \in \mathcal{P} = \mathbf{SO}(1, 2) \times \mathbf{SO}(3)$ which in coordinates $(x_1, x_2, x_3, x'_1, x'_2, x'_3)$ can be represented by a 6×6 matrix of the form

$$
P = \begin{pmatrix} p & 0 \\ 0 & p' \end{pmatrix},
$$

where $p = p_1 p_2 p_3$, $p' = p'_1 p'_2 p'_3$ and the one-parameter groups p_i, p'_i are given by

$$
p_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x_1 & -\sin x_1 \\ 0 & \sin x_1 & \cos x_1 \end{pmatrix}, \qquad
p_2 = \begin{pmatrix} \cosh x_2 & 0 & -\sinh x_2 \\ 0 & 1 & 0 \\ -\sinh x_2 & 0 & \cosh x_2 \end{pmatrix},
$$
$$
p_3 = \begin{pmatrix} \cosh x_3 & \sinh x_3 & 0 \\ \sinh x_3 & \cosh x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad
p'_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos x'_1 & -\sin x'_1 \\ 0 & \sin x'_1 & \cos x'_1 \end{pmatrix},
$$
$$
p'_2 = \begin{pmatrix} \cos x'_2 & 0 & \sin x'_2 \\ 0 & 1 & 0 \\ -\sin x'_2 & 0 & \cos x'_2 \end{pmatrix}, \qquad
p'_3 = \begin{pmatrix} \cos x'_3 & -\sin x'_3 & 0 \\ \sin x'_3 & \cos x'_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Then the forms A_i, A'_i can be identified with the components of the Maurer–Cartan form $A = P^{-1}\, \mathrm{d}P$ on $\mathcal{P}$ by

$$
A = P^{-1}\, \mathrm{d}P = \begin{pmatrix}
0 & A_3 & -A_2 & & & \\
A_3 & 0 & -A_1 & & & \\
-A_2 & A_1 & 0 & & & \\
& & & 0 & -A'_3 & A'_2 \\
& & & A'_3 & 0 & -A'_1 \\
& & & -A'_2 & A'_1 & 0
\end{pmatrix}.
$$

Consider now the degenerate metric $G = \frac{1}{\lambda}\left[A_1^2 - A_2^2 + A_1'^2 + A_2'^2\right]$ on $\mathcal{P}$ and a subgroup

$\mathbf{SO}(1,1)\times\mathbf{SO}(2)$ of $\mathbf{SO}(1,2)\times\mathbf{SO}(3)$ given by those elements which have the form

$$g_* = \begin{pmatrix} \cosh t_3 & \sinh t_3 & 0 & & & \\ \sinh t_3 & \cosh t_3 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & \cos t_3' & -\sin t_3' & 0 \\ & & & \sin t_3' & \cos t_3' & 0 \\ & & & 0 & 0 & 1 \end{pmatrix},$$

where $t_3 \in \mathbf{R}$, $t_3' \in [0, 2\pi]$. It follows that the left action ($P \to g_* P$) of this group on $\mathcal{P} = \mathbf{SO}(1,2)\times\mathbf{SO}(3)$ leaves the form A invariant. The right action ($P \to P g_*$) transforms A according to $A \to g_*^{-1} A g_*$. These relations imply that the metric G is invariant under the right action so that it projects to the homogeneous space $\mathcal{M} = \mathbf{SO}(1,2)\times\mathbf{SO}(3)/\mathbf{SO}(1,1)\times\mathbf{SO}(2)$ equipped with the Einstein ($\lambda > 0$) metric

$$g = \frac{1}{\lambda}\left[\cosh^2 x_2\, \mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \cos^2 x_2'\, \mathrm{d}x_1'^2 + \mathrm{d}x_2'^2\right].$$

This shows that $\mathcal{P} = \mathbf{SO}(1,2)\times\mathbf{SO}(3)$ is fibred over the Einstein spacetime $\mathcal{M} = \mathbf{H}_{+-}\times\mathbf{S}^2$, which is a Cartesian product of a neutral-signature hyperbolic space and a 2-sphere, both with their natural metrics†. It is clear that the fibres of $\mathcal{P}$, being homeomorphic to $\mathbf{SO}(1,1)\times\mathbf{SO}(2)$ have the topology of a cylinder.

(iib) Assume that $\lambda < 0$.

Now the Lie algebras of both H and H' are isomorphic to $\mathbf{sl}(2,\mathbf{R})$. We again concentrate on the case when $H = \mathbf{SO}(1,2)$ and $H' = \mathbf{SO}(1,2)$. Introducing $(x_1, x_2, x_3, x_1', x_2', x_3')$ as local coordinates on $\mathcal{P} = H \times H'$ we can represent any point P of $\mathcal{P}$ as

$$P = \begin{pmatrix} p & 0 \\ 0 & p' \end{pmatrix},$$

where $p = p_1 p_2 p_3$, $p' = p_1' p_2' p_3'$ and the one-parameter groups p_i, p_i' are given by

$$p_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh x_1 & \sinh x_1 \\ 0 & \sinh x_1 & \cosh x_1 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} \cosh x_2 & 0 & -\sinh x_2 \\ 0 & 1 & 0 \\ -\sinh x_2 & 0 & \cosh x_2 \end{pmatrix},$$

$$p_3 = \begin{pmatrix} \cos x_3 & \sin x_3 & 0 \\ -\sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad p_1' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh x_1' & \sinh x_1' \\ 0 & \sinh x_1' & \cosh x_1' \end{pmatrix},$$

$$p_2' = \begin{pmatrix} \cos x_2' & 0 & \sin x_2' \\ 0 & 1 & 0 \\ -\sin x_2' & 0 & \cos x_2' \end{pmatrix}, \qquad p_3' = \begin{pmatrix} \cosh x_3' & -\sinh x_3' & 0 \\ -\sinh x_3' & \cosh x_3' & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the Maurer–Cartan form $A = P^{-1}\,\mathrm{d}P$ defines forms A_i and A_i' by

$$A = P^{-1}\,\mathrm{d}P = \begin{pmatrix} 0 & A_3 & -A_2 & & & \\ -A_3 & 0 & A_1 & & & \\ -A_2 & A_1 & 0 & & & \\ & & & 0 & -A_3' & A_2' \\ & & & -A_3' & 0 & A_1' \\ & & & -A_2' & A_1' & 0 \end{pmatrix}.$$

† $\mathbf{H}_{+-} = \{(z_1, z_2, z_3) \in \mathbf{R}^{1,2} | -z_1^2 + z_2^2 + z_3^2 = 1\}$, a quadric in $\mathbf{R}^3$ equipped with the flat Lorentzian metric of signature $(-, +, +)$. A parametrization of $\mathbf{H}_{+-}$ used above is: $z_1 = \sinh x_2$, $z_2 = \cosh x_2 \sin x_1$, $z_3 = \cosh x_2 \cos x_1$. The sphere is parametrized by $z_1 = \cos x_1' \cos x_2'$, $z_2 = \sin x_1' \cos x_2'$ and $z_3' = \sin x_2'$.

The subgroup $\mathbf{SO}(2) \times \mathbf{SO}(1,1)$ of $\mathbf{SO}(1,2) \times \mathbf{SO}(1,2)$ consisting of elements g_* of the form

$$g_* = \begin{pmatrix} \cos t_3 & \sin t_3 & 0 & & & \\ -\sin t_3 & \cos t_3 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & \cosh t_3' & -\sinh t_3' & 0 \\ & & & -\sinh t_3' & \cos t_3' & 0 \\ & & & 0 & 0 & 1 \end{pmatrix},$$

where $t_3 \in [0, 2\pi]$, $t_3' \in \mathbf{R}$, acts on $\mathcal{P}$ from the right. The form A transforms by $A \to g_*^{-1} A g_*$ under this action which implies that the metric $G = \frac{1}{|\lambda|}\left[A_1^2 + A_2^2 + A_1'^2 - A_2'^2\right]$ is invariant. The quotient space $\mathcal{M} = \mathbf{SO}(1,2) \times \mathbf{SO}(1,2)/\mathbf{SO}(2) \times \mathbf{SO}(1,1)$ is naturally equipped with the projected Einstein ($\lambda < 0$) metric

$$g = \frac{1}{|\lambda|}\left[\cosh^2 x_2\, \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cos^2 x_2'\, \mathrm{d}x_1'^2 - \mathrm{d}x_2'^2\right].$$

It follows that now $\mathcal{M} = \mathbf{H}_{++} \times \mathbf{H}_{+-}$, that is it is a Cartesian product of Euclidean- and neutral-signature hyperbolic 2-spaces†. The metric g is a sum of the natural metric on $\mathbf{H}_{++}$ minus the natural metric on $\mathbf{H}_{+-}$. The fibres of $\mathcal{P}$ again have the topology of a cylinder‡.

Example 7 shows that if $\lambda \neq 0$ then the *homogeneous* generalized bundles of null directions $\mathcal{P}$ are principal fibre bundles over the spacetime with the structure group $G_* = \mathbf{SO}(2) \times \mathbf{SO}(1,1)$. The system of 1-forms of theorem 1 on these bundles equips them with a 1-form A which is valued in the Lie algebra of group G_λ such that $\dim G_\lambda = \dim \mathcal{P}$. Moreover, A has the following properties:

- if X is a vector field tangent to the flow of the one parameter subgroup of G_* generated by ξ then $A(X) = \xi$;
- $A(X) \neq 0$ on each vector tangent to $\mathcal{P}$;
- under the right action of G_* the form A transforms as $g_*^{-1} A g_*$.

Thus A can be understood as a Cartan connection on $\mathcal{P}$ (cf [5], pp 127–30). Note that in example 7 the form A is always a Maurer–Cartan form on G_λ which implies that its curvature is zero. This suggests that a generic (nonhomogeneous) generalized Einstein bundle of null directions with non-zero cosmological constant can find a useful formulation in terms of curvature conditions on Cartan connections on principal fibre bundles with group G_* over the spacetime. Such Cartan connections on $\mathcal{P}$ can be further understood as the usual connections on fibre bundles $\mathcal{P} \times_{G_*} G$ (cf [5], pp 127–8). This possibility will be studied elsewhere.

9. Concluding remarks

The study of null objects in general relativity has led to many important advances in the understanding of Einstein's equations. In this paper this general line of enquiry has been developed by employing the bundle of null directions, over a four-dimensional Lorentzian spacetime, as a tool in the investigation of Einstein spaces. It has been shown that a Lorentzian

† $\mathbf{H}_{++} = \{(z_1, z_2, z_3) \in \mathbf{R}^{1,2} | -z_1^2 + z_2^2 + z_3^2 = -1\}$, a quadric in $\mathbf{R}^3$ equipped with flat Lorentzian metric of signature $(-,+,+)$. The parametrization of $\mathbf{H}_{++}$ used here is: $z_1 = \cosh x_2$, $z_2 = \sinh x_2 \sin x_1$, $z_3 = \sinh x_2 \cos x_1$. $\mathbf{H}_{+-}$ is parametrized by $z_1 = \cos x_2' \sinh x_1'$, $z_2 = \cos x_2' \cosh x_1'$, $z_3 = \sin x_2'$.

‡ Note that the metrics $\lambda^{-1}(-g_{H+-} + g_S)$, and $|\lambda|^{-1}(g_{H++} + g_{H+-})$, where g_{H++}, g_{H+-} and g_S denote the natural metrics on $\mathbf{H}_{++}$, $\mathbf{H}_{+-}$ and $\mathbf{S}^2$, respectively, do not satisfy the vacuum Einstein equations with cosmological constant. It is interesting to note that the first of these is a solution to the Einstein–Maxwell equations with vanishing cosmological constant known as the Bertotti–Robinson solutions. The second metric has an energy–momentum with negative energy and thus cannot be interpreted as the Einstein–Maxwell solution.

4-metric can be defined by a differential system on a 6-manifold over a 4-manifold. In fact, a G-structure on the 6-manifold (the total space of the bundle of null directions) encodes the requirement that the 4-manifold be Einstein. This structure, as has been demonstrated, can be used to study spacetimes. An extension of this structure leads to the construction of a generalized bundle of null directions over a conformally non-flat Einstein spacetime. An effective algorithm for the equivalence problem for Lorentzian 4-metrics has been constructed by making use of this generalized bundle. Finally, it has been observed that the Petrov-type Weyl tensor of a conformally non-flat Einstein metric can be encoded in the fibration of a 6-manifold over a 4-manifold. Different fibrations provide interesting insights into Einstein spacetimes.

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