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Exceptional Lie algebras f_4 and e_6, accidental CR structures and the triality

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‘Geometric Structures’ Seminar

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Simple Lie algebras

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- One of the most spectacular achievements of *algebra*, and perhaps, the *whole mathematics*, is the *classification of simple Lie groups*.
- This is the result of **Wilhelm Killing** (1847-1923), which he published in 1887, while being a teacher and the rector of the *Lyceum Hosianum* in Braniewo, a city in Warmia, PL.
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List of pairwise nonequivalent simple Lie algebras

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- First, we have the classical simple Lie algebras:

Lie algebra		dimension	notes
$\mathfrak{a}_\ell = \mathfrak{sl}(\ell+1)$	$\ell \geq 1$	$\ell(\ell+2)$	all these Lie algebras correspond to Lie groups preserving multilinear forms in $\mathbb{R}^N$ and $\mathbb{C}^N$.
$\mathfrak{b}_\ell = \mathfrak{so}(2\ell+1)$	$\ell \geq 2$	$\ell(2\ell+1)$	
$\mathfrak{c}_\ell = \mathfrak{sp}(\ell)$	$\ell \geq 3$	$\ell(2\ell+1)$	
$\mathfrak{d}_\ell = \mathfrak{so}(2\ell)$	$\ell \geq 4$	$\ell(2\ell-1)$	

- There are infinitely many of them; they are grouped in *infinite* series, parametrized by the *Lie theoretic invariant* number ℓ, called the *rank* of the simple Lie algebra.

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Lie algebra	dimension	notes
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$\mathfrak{f}_4$	52	but it was not clear
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Realizations of exceptional Lie algebras

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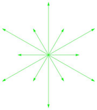

- Killing discovered the **root diagram** for $\mathfrak{g}_2$ and found also diagrams for the other exceptional Lie algebras. But he did **not obtained realizations** of these Lie algebras (nor the corresponding Lie groups).
- In particular, Killing did **not obtained realization of the smallest** exceptional simple Lie algebra $\mathfrak{g}_2$, but he claimed in **1887** that **it should be realized** as a Lie algebra of a **transformation group in dimension 5**.

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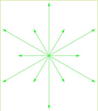

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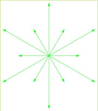

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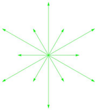

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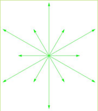

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Why dimension 5?

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- The simplest way of realizing a Lie group G geometrically, is to provide a space M, which is G-**homogeneous.** Then the group G is the **symmetry group** of the space M and, in particular, it is a **symmetry** of the **entire structure** with which M is naturally equipped. Such space is **always locally equivalent** to one of the **coset spaces** G/P, with P being some Lie subgroup of G.
- The dimension of the homogeneous space $M = G/P$ is $dim(M) = dim(G) - dim(P)$, so to find a realisation of G in the **lowest** dimension, one has to take a subgroup P in G of the **largest** dimension.
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Cartan enters into the stage

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- Now, due to **Élie Cartan**'s PhD thesis (written in **1894**, when he was 24), we know that the group G_2 has **two geometrically different** realizations as a transformation group in dimension 5; I discussed this story many times, in particular at this seminar.
- So, today I want to discuss lowest possible realizations of other exceptionals.
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- Cartan, in his thesis, established what are the dimensions for the **lowest dimensional irreducible and faithful representations of the exceptionals.** He has shown that the lowest dimensional irreducible representation of
  - G_2 is in dimension 7,
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  - E_7 is in dimension 56,
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- However these are **not the lowest dimensions** for the **realizations** of these groups. If one **drops the linearity** of the action the **dimensions are lower.** And here is the proper start of my seminar.

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- However these are **not the lowest dimensions** for the **realizations** of these groups. If one **drops the linearity** of the action the **dimensions are lower.** And here is the proper start of my seminar.

I mean realizations and NOT representations

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Killing had been led to expect that G_2 could be realized as a transformation group in $\mathbf{R}^5$, but not in a lower-dimensional space. Engel and Cartan showed that it can be realized as the stability group of the system

$$dx_3 + x_1dx_2 - x_2dx_1 = 0,$$
$$dx_4 + x_3dx_1 - x_1dx_3 = 0,$$
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Cartan represented F_4 similarly by the Pfaffian system in $\mathbf{R}^{15}$ given by

$$dz = \sum_1^4 y_i dx_i, \qquad dx_{ij} = x_i dx_j - x_j dx_i + y_h dy_k - y_k dy_h, \tag{14}$$

where $z, x_i, y_j, x_{ij} = -x_{ji}$ $(i \neq j, i, j = 1, 2, 3, 4)$ are coordinates in $\mathbf{R}^{15}$ and in (14) i, j, h, k is an even permutation [**1a**, p. 418].

And this made it clear to me how to realize $\mathbf{F}_4$. It is the symmetry group of a **rank eight distribution** in $\mathbb{R}^{15}$ with coordinates (x_i, y_j, z, z_{kl}), $i, j = 1, 2, 3, 4$, $1 \leq k < l \leq 4$, which annihilates the **seven** 1-forms in equation (14). This is similar to $\mathbf{G}_2$ being the symmetry group of a **rank two distribution** in $\mathbb{R}^5$, with coordinates $(x_1, x_2, x_3, x_4, x_5)$ annihilating the **three** 1-forms above equation (14).

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Cartan-Helgason story

Helgason continues:

> Similar results for E_6 in $\mathbf{R}^{16}$, E_7 in $\mathbf{R}^{27}$ and E_8 in $\mathbf{R}^{29}$ as contact transformations are indicated in [1a]. Unfortunately, detailed proofs of these remarkable representations of the exceptional groups do not seem to be available.

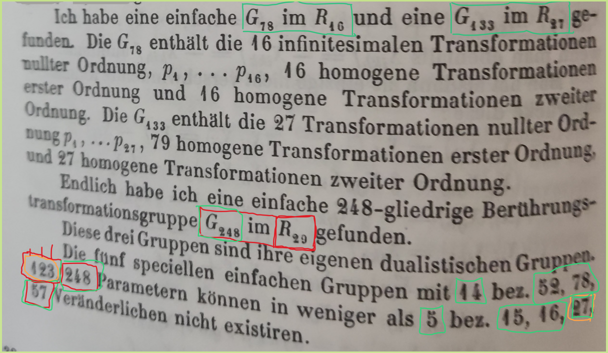
S. Helgason, Invariant differential equations on homogeneous manifolds, *BAMS* **83**, 751-756, (1977).

This agrees with the **'last minus 2'** line in the **German version** of Cartan's PhD thesis (which was published in **1893**, one year before the French version):

> Ich habe eine einfache G_{78} im R_{16} und eine G_{133} im R_{27} gefunden. Die G_{78} enthält die 16 infinitesimalen Transformationen nullter Ordnung, $p_1, \ldots p_{16}$, 16 homogene Transformationen erster Ordnung und 16 homogene Transformationen zweiter Ordnung. Die G_{133} enthält die 27 Transformationen nullter Ordnung $p_1, \ldots p_{27}$, 79 homogene Transformationen erster Ordnung und 27 homogene Transformationen zweiter Ordnung.
>
> Endlich habe ich eine einfache 248-gliedrige Berührungstransformationsgruppe G_{248} im R_{29} gefunden.

Cartan-Helgason story

But here is the **full** Cartan's text; with the last two sentences, which Helgason seemingly did not read:



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Diese drei Gruppen sind ihre eigenen dualistischen Gruppen.

Die fünf speciellen einfachen Gruppen mit 14 bez. 52, 78, 133, 248 Parametern können in weniger als 5 bez. 15, 16, 27, 57 Veränderlichen nicht existiren.

Cartan-Helgason story

Norway grants

Conclusion #1: Cartan's thesis in German has two **misprints**, and Helgason amplified **one** of these misprints in his influencial **review article** in the *Bulletin of the AMS*; this made me to believe for many years that the **lowest dimension in which the exceptional Lie group E_8 is realized is** 29.
This is wrong! And Cartan is innocent here. If Helgason was patient enough, he would get the correct dimension from the last sentence of Cartan's German thesis.

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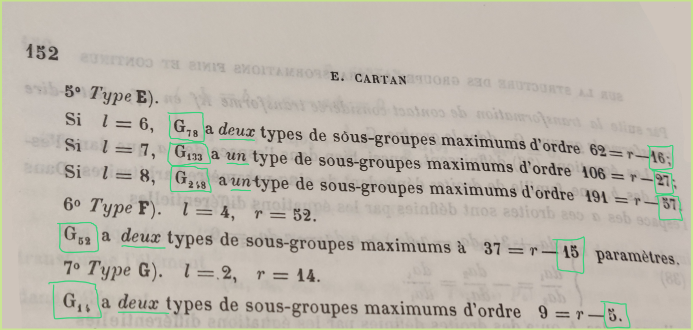
Cartan's thesis **in French**, was published one year later, that is to say in 1894 (I have a copy of the second edition from 1933). It **must have** obviously **been available in 1976**. And it does not leave any doubt about **Cartan's accuracy in determining the lowest dimensions of the realizations**:



152 E. CARTAN

5° *Type* E).

Si $l = 6$, G_{78} a *deux* types de sous-groupes maximums d'ordre $62 = r - 16$;

Si $l = 7$, G_{133} a *un* type de sous-groupes maximums d'ordre $106 = r - 27$;

Si $l = 8$, G_{248} a *un* type de sous-groupes maximums d'ordre $191 = r - 57$.

6° *Type* F). $l = 4$, $r = 52$.

G_{52} a *deux* types de sous-groupes maximums à $37 = r - 15$ paramètres.

7° *Type* G). $l = 2$, $r = 14$.

G_{14} a *deux* types de sous-groupes maximums d'ordre $9 = r - 5$.

Here are coordinates of the French copy of Cartan's thesis I have: É. Cartan, 'Sur la structure des grupes de transformations finis et continus', *Thése*, Paris, Nony, 2e édition, Vuibert, (1933).

Cleaning the mess

Norway grants

Recall that as far as the lowest dimensional **faithful representaions** are concerned we have:

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And Cartan in his Thesis claims that the lowest dimensional **realizations** are in dimension:

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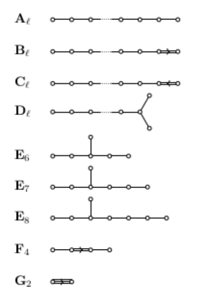

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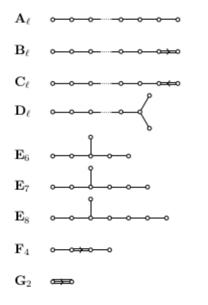

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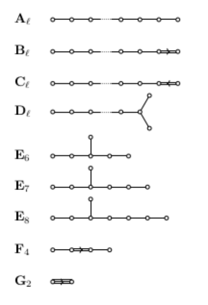

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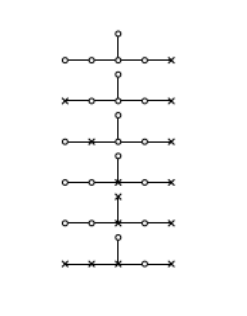
Example of nonequivalent crossings in E_6...



... and equivalent pairs in $\mathbf{E}_6$

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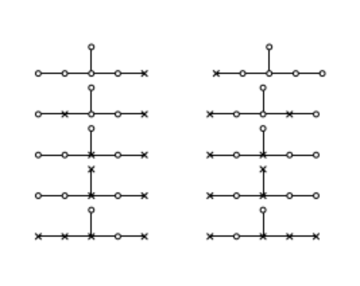

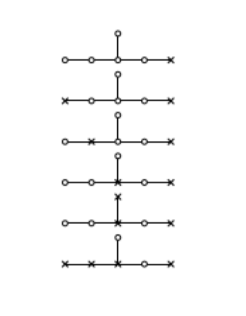

- Each crossed Dynkin diagram corresponds to a particular choice of a parabolic subalgebra $\mathfrak{p}$ in $\mathfrak{g}$, and as such defines a **gradation** $\mathfrak{p} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ in $\mathfrak{g}$.

- What crossed diagram visualises is the $\mathfrak{g}_0$ part of this gradation:
  - It follows that $\mathfrak{g}_0$ is a **direct sum** of the **semisimple part** $\mathfrak{g}_{ss}$ and the **center**, which is a direct sum of a number of $\mathbb{C}$s, $\mathfrak{g}_0 = \mathfrak{g}_{ss} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$.
  - The semisimple $\mathfrak{g}_{ss}$ is the **uncrossed** remnant of the original crossed diagram, and the number of $\mathbb{C}$ factors in $\mathfrak{g}_0$ is the number of crosses.
- Thus, in our picture on the left, the $\mathfrak{g}_0$s of the corresponding gradations in $\mathfrak{e}_6$ are respectively: $\mathfrak{so}(10) \oplus \mathbb{C}$, $\mathfrak{so}(8) \oplus 2\mathbb{C}$, $\mathfrak{sl}(4) \oplus \mathfrak{sl}(2) \oplus 2\mathbb{C}$, $\mathfrak{sl}(3) \oplus 2\mathfrak{sl}(2) \oplus 2\mathbb{C}$, $\mathfrak{sl}(2) \oplus \mathfrak{sl}(3) \oplus 3\mathbb{C}$, and $2\mathfrak{sl}(2) \oplus 4\mathbb{C}$.

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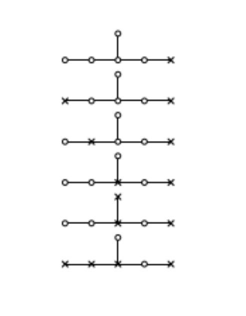

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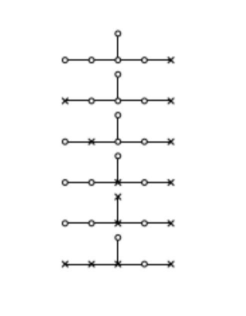

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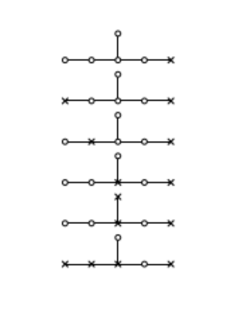

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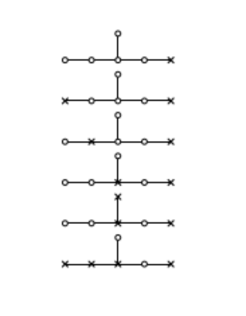

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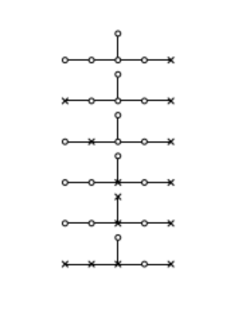

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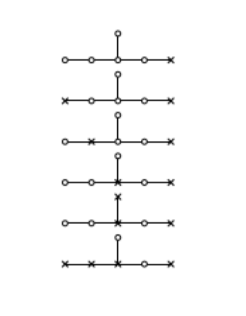
Explaining diagram with crossings



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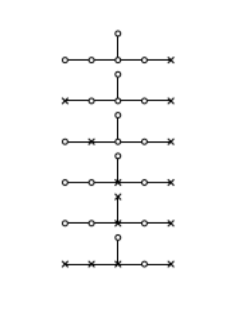

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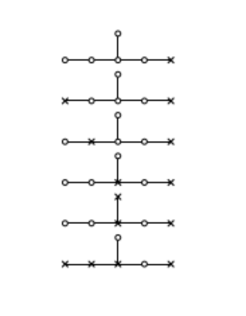

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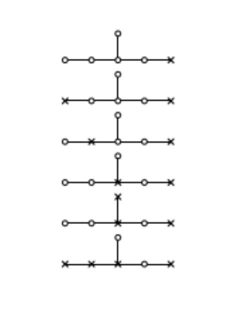

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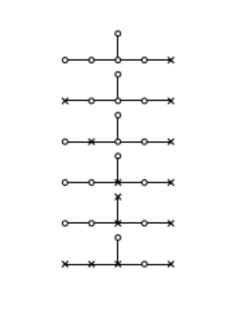

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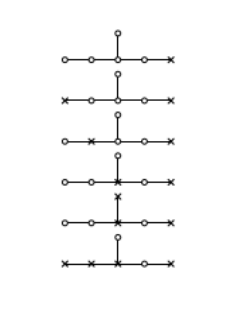
Explaining diagram with crossings



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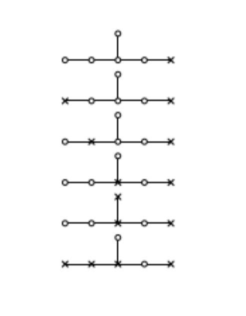

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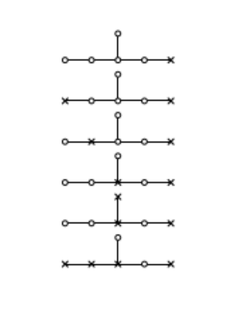

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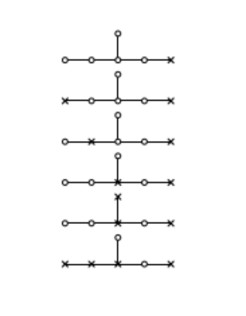

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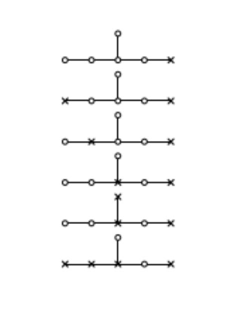

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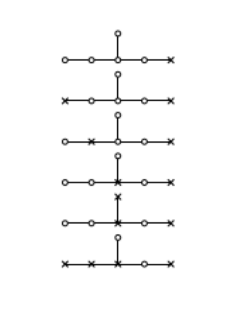

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Explaining diagram with crossings

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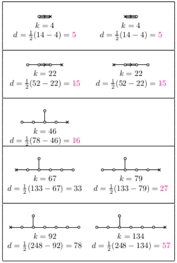

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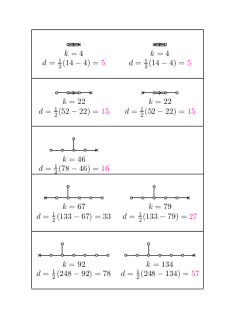

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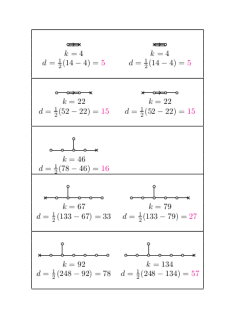

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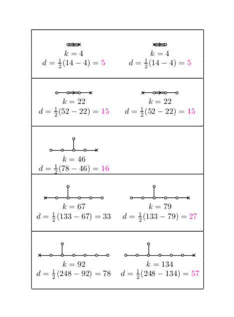

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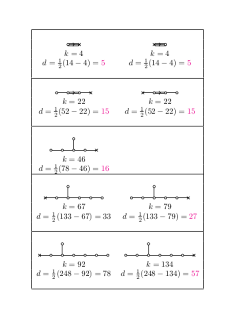

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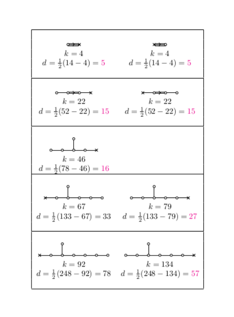

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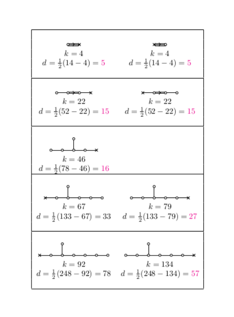

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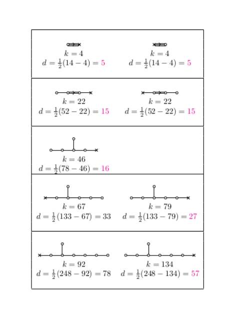

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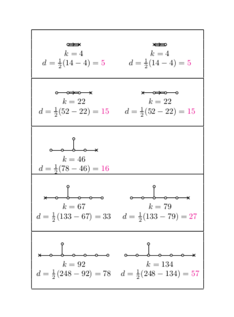

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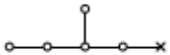
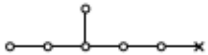
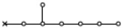
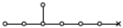
Cartan's lowest dimensions for realizations of the exceptionals

 $k = 4$ $d = \frac{1}{2}(14-4) = 5$	 $k = 4$ $d = \frac{1}{2}(14-4) = 5$
 $k = 22$ $d = \frac{1}{2}(52-22) = 15$	 $k = 22$ $d = \frac{1}{2}(52-22) = 15$
 $k = 46$ $d = \frac{1}{2}(78-46) = 16$	
 $k = 67$ $d = \frac{1}{2}(133-67) = 33$	 $k = 79$ $d = \frac{1}{2}(133-79) = 27$
 $k = 92$ $d = \frac{1}{2}(248-92) = 78$	 $k = 134$ $d = \frac{1}{2}(248-134) = 57$

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- I discussed **complex** simple exceptional Lie groups, and my realizations were in $\mathbb{C}^N$. All my numerical dimensions were **complex** dimensions. My homogeneous spaces were portions of $\mathbb{C}^N$s and **not** $\mathbb{R}^N$s.
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- **Cartan realized** $\mathbf{F}_4$ and its **split real form** denoted by F_I as an automorphism group of a certain $(8, 15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I of $\mathbf{F}_4$ is a **2-step parablic geometry**.
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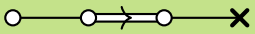
- **Cartan realized** $\mathbf{F}_4$ and its **split real form** denoted by F_I as an automorphism group of a certain $(8,15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I of $\mathbf{F}_4$ is a **2-step parablic geometry**.
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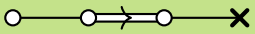
- **Cartan realized $\mathbf{F}_4$** and its **split real form** denoted by F_I as an automorphism group of a certain $(8, 15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I of $\mathbf{F}_4$ is a **2-step parablic geometry**.
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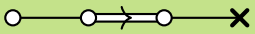
- **Cartan realized** $\mathbf{F}_4$ and its **split real form** denoted by F_I as an automorphism group of a certain $(8, 15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I of $\mathbf{F}_4$ is a **2-step parablic geometry**.
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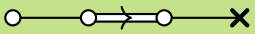
- **Cartan realized $\mathbf{F}_4$** and its **split real form** denoted by F_I as an automorphism group of a certain $(8, 15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I **of** $\mathbf{F}_4$ **is a 2-step parablic geometry**.
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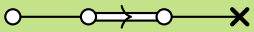
- **Cartan realized $\mathbf{F}_4$** and its **split real form** denoted by F_I as an automorphism group of a certain $(8, 15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I of $\mathbf{F}_4$ is a **2-step parablic geometry**.
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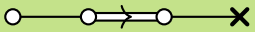
- **Cartan realized** $\mathbf{F}_4$ and its **split real form** denoted by F_I as an automorphism group of a certain $(8, 15)$ distribution. This can be realized on the **nilpotent** Lie gorup G_-, whose Lie algebra is $\mathfrak{g}_-$, related to the choice of parabolic . This nilpotent Lie algebra is **2-step**: $\mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, with the respective dimensions $\dim(\mathfrak{g}_{-2}) = 7$ and $\dim(\mathfrak{g}_{-1}) = 8$, corresponding to the **rank** 8 of the distribution and its **co-rank** 7. So Cartan's **realization of the split real form** F_I of $\mathbf{F}_4$ is a **2-step parablic geometry**.
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Real forms, on an example of $\mathbf{E}_6$

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Satake diagrams for $\mathbf{E}_6$

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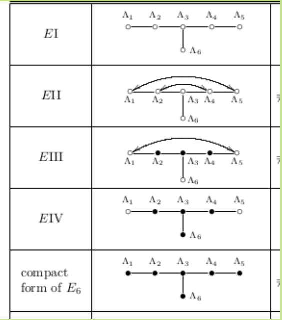
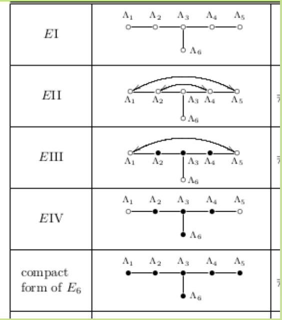


Rules for crossings/ chosing parabolics



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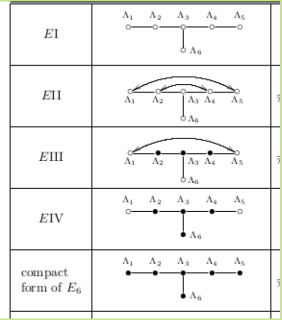

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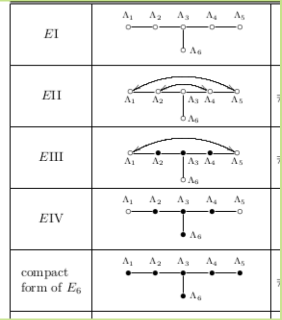

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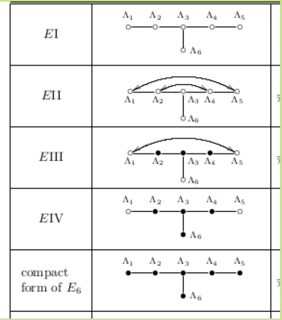

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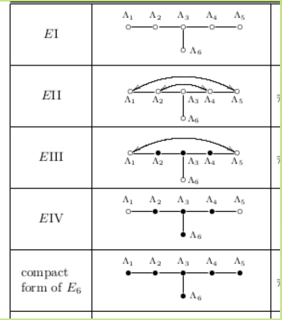

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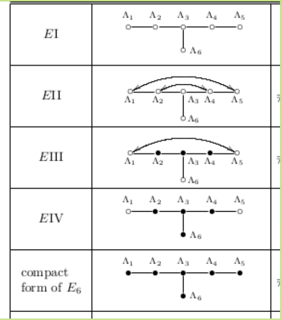

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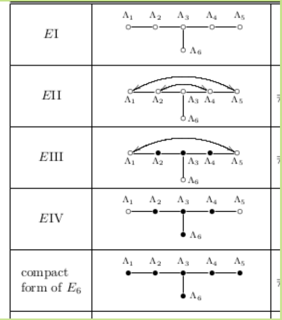

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Crosing only the first node is not possible for E_{II} and E_{III}

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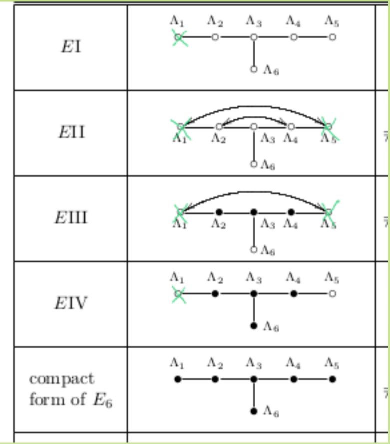

- Because of the second of the selection rules, we see that if we cross node Λ_1 in E_{II} or E_{III} we must also cross the node Λ_5 on these diagrams.
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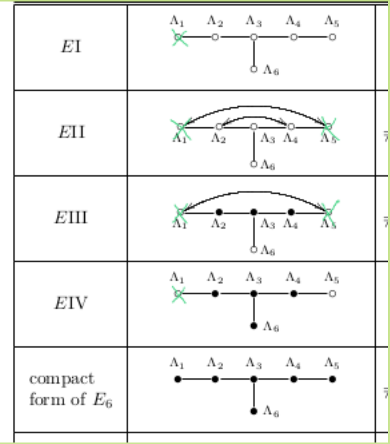

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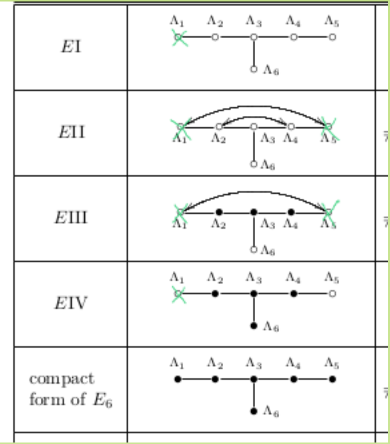

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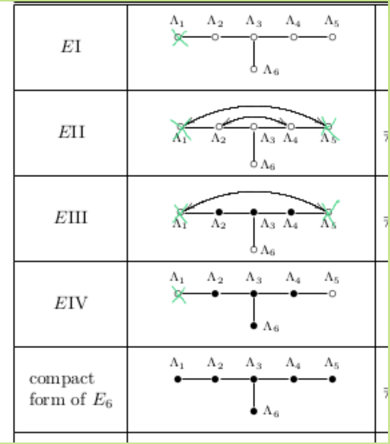

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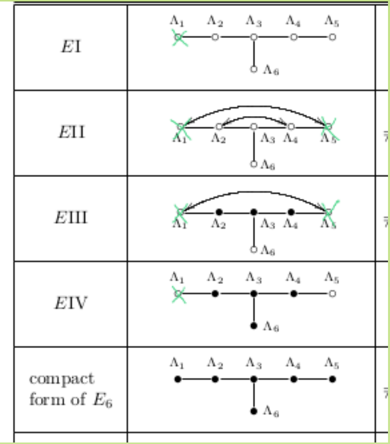

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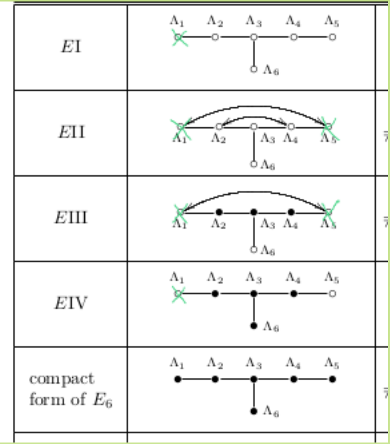

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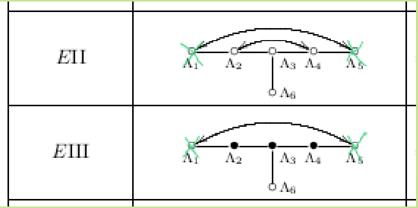

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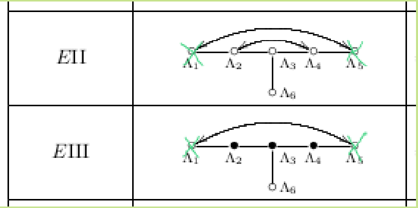

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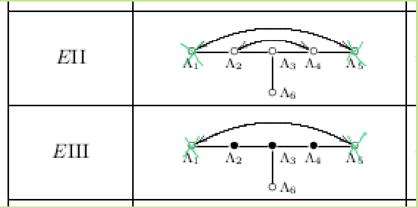

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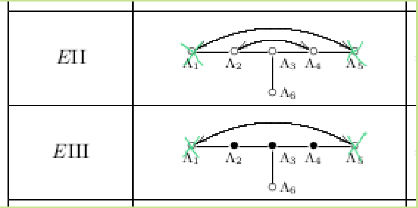

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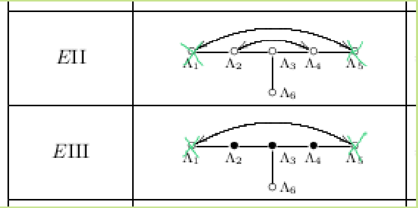

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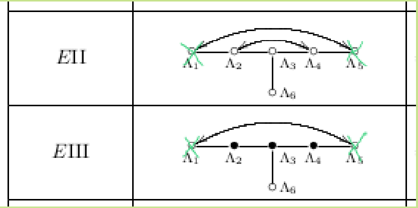

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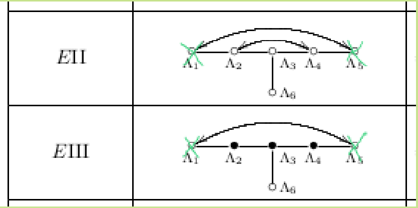

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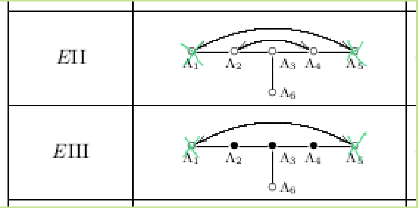

- Wanting to be as close as possible to Cartan's realization of E_I in dimension 16, which coresponds to cross on the node Λ_1 in the E_I diagram, together with **Denny Hill**, **Joël Merker**, and **Zhoahu Nie**, we crossed nodes Λ_1 on the Satake diagrams for E_{II} and E_{III}.
- This, by the second selection rule, forced us to cross nodes Λ_5 on both of these diagrams.
- We got two $\mathbf{E}_6$-homogenoeus 24-dimensional manifolds E_{II}/P_{II} and E_{III}/P_{III}.
- It turns out that each of these **real** manifolds is equipped with a $(16, 24)$ **distribution**, and this distribution has the E_{II} or E_{III} symmetry, respectively.
- So we have two realizations of the two **exceptionals**, E_{II} and E_{III}, which **are similar** to Cartan's realization of F_I.
- What is more interesting, the rank 16-distributions in both of these realizations admit an **integrable complex structure** J, s.t. $J^2 = -\mathrm{Id}$, which is respectively E_{II} or E_{III} **compatible**.
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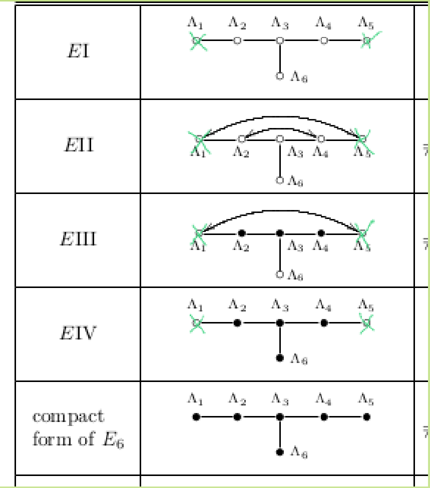

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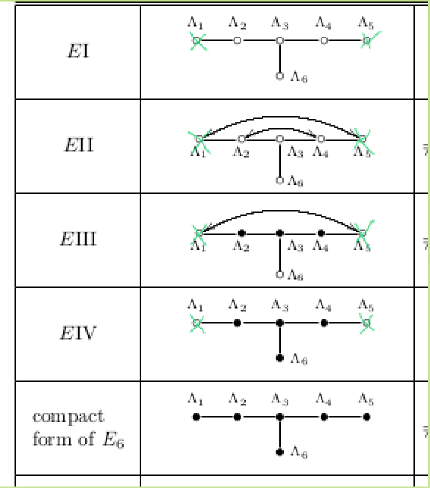

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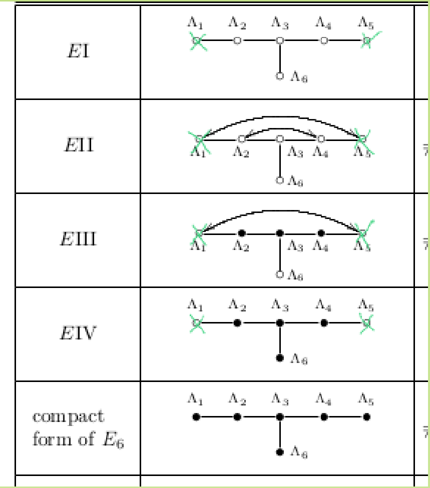

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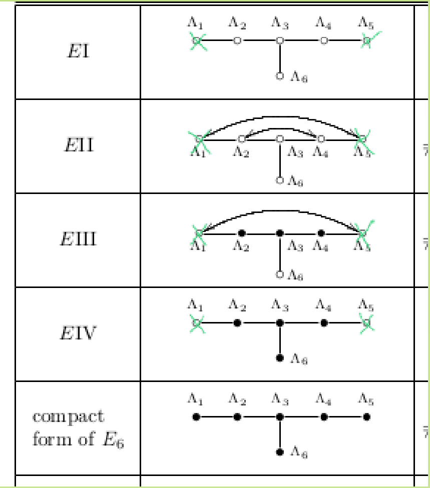

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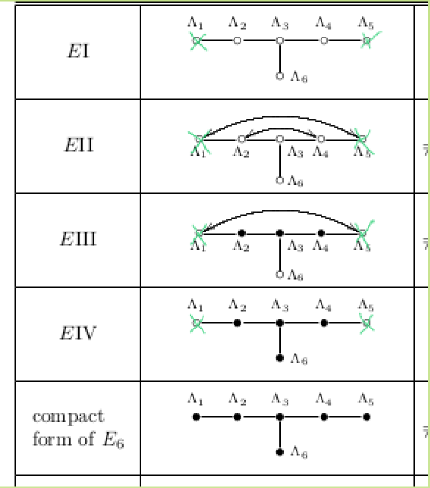

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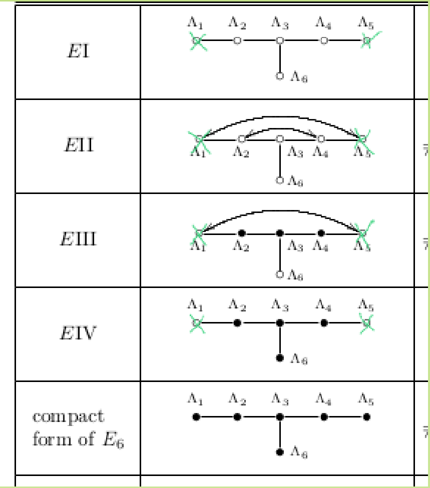

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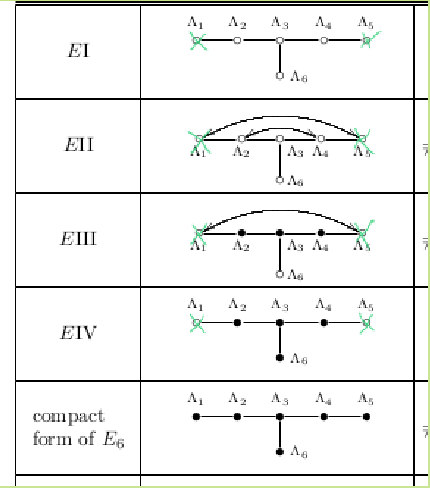

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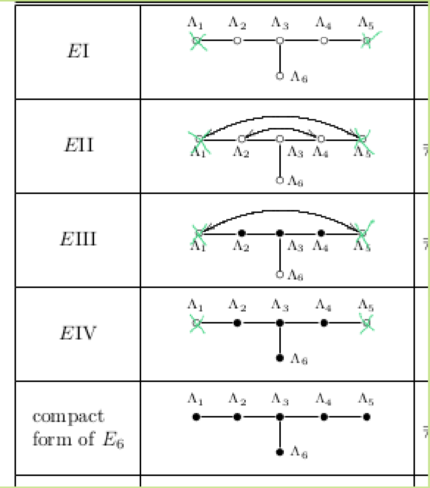

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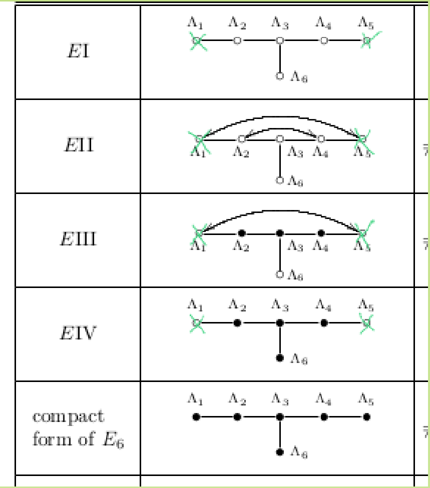

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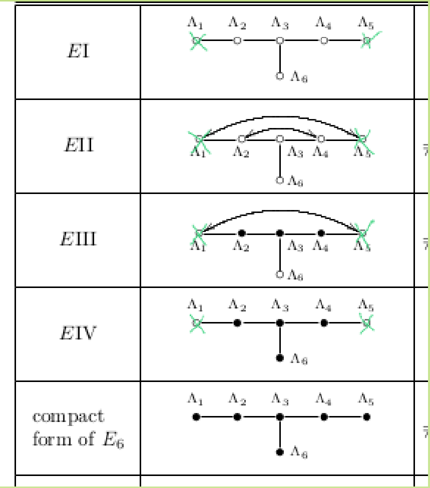

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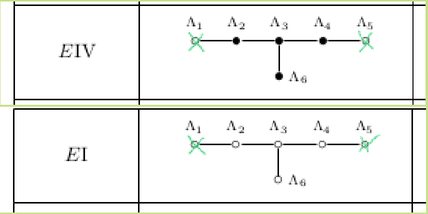

Compare the semisimple parts of these two diagrams with the real forms of $\mathfrak{d}_4$:

which respectively correspond to $\mathbf{SO}(8,0)$, $\mathbf{SO}(7,1)$, $\mathbf{SO}(6,2)$, $\mathbf{SO}^*(8)$, $\mathbf{SO}(4,4)$ and $\mathbf{SO}(5,3)$.

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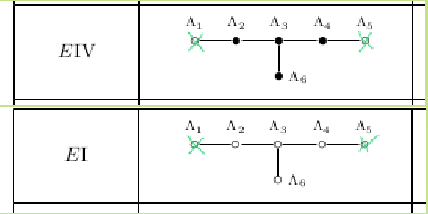

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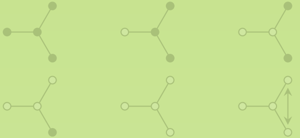

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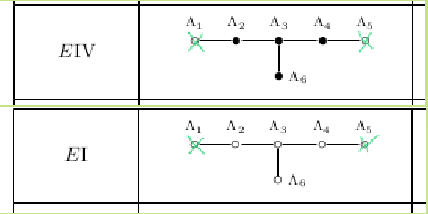

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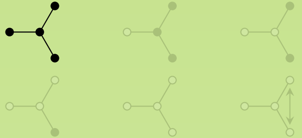

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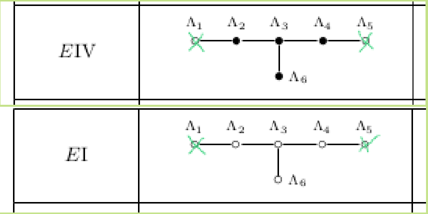

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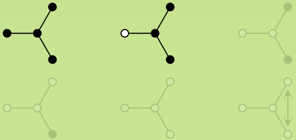

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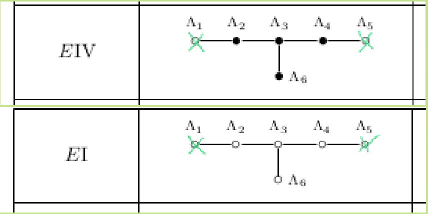

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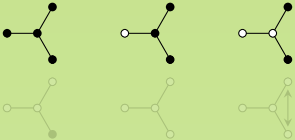

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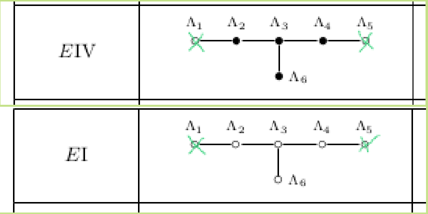

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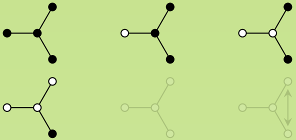

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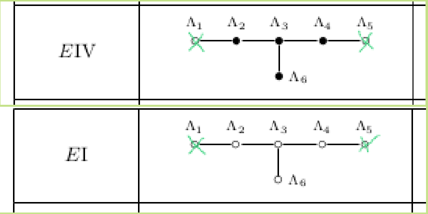

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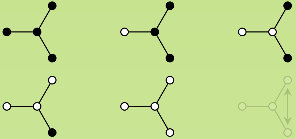

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Look at the simple part of $\mathfrak{g}_0$ in E_I and E_{IV} cases

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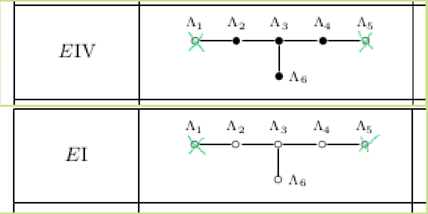

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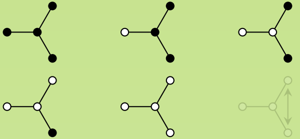

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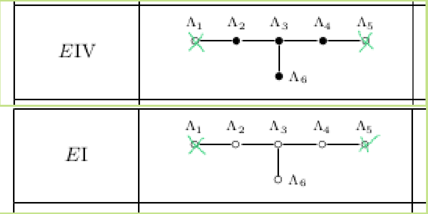

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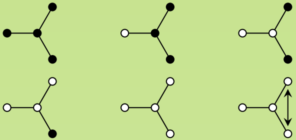

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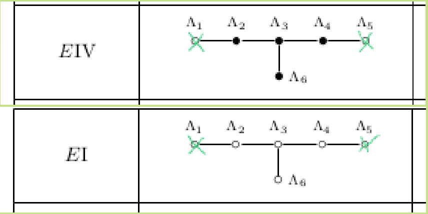

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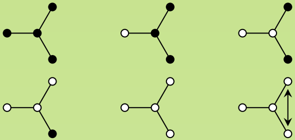

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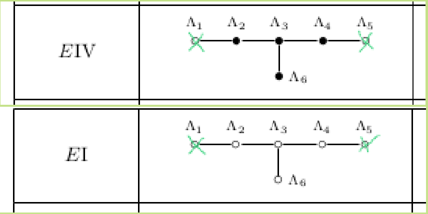

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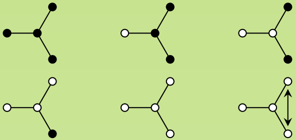

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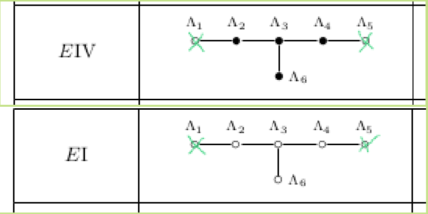

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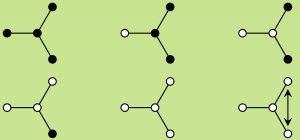

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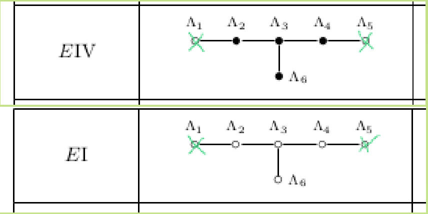

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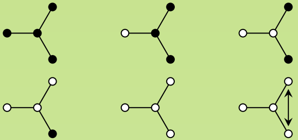

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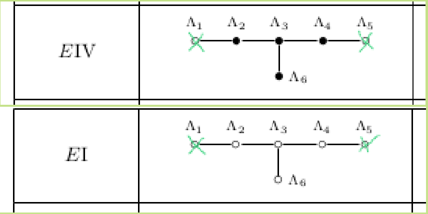

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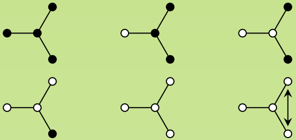

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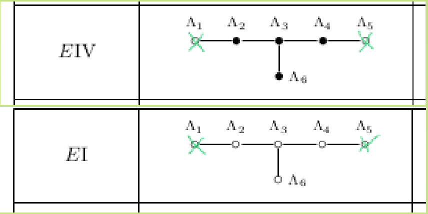

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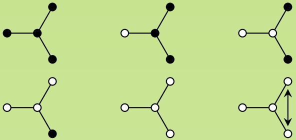

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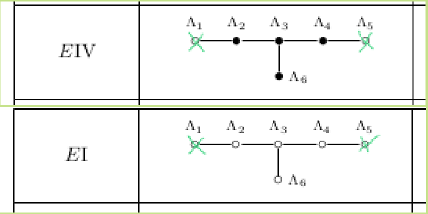

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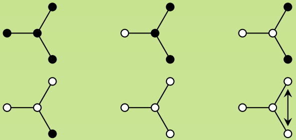

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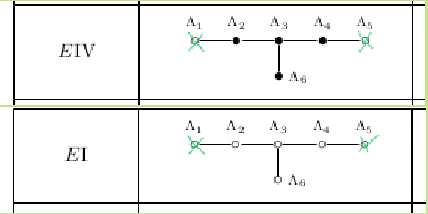

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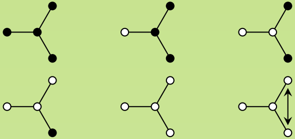

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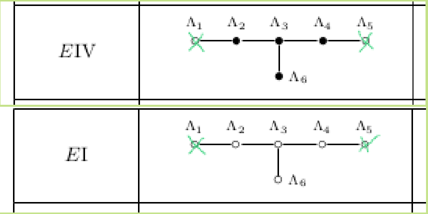

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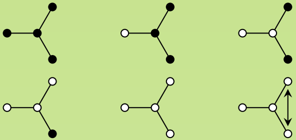

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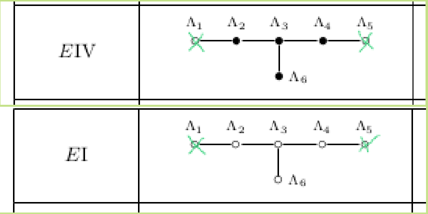

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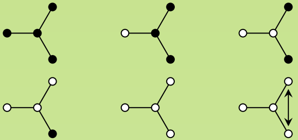

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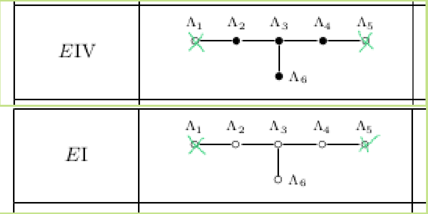

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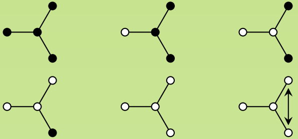

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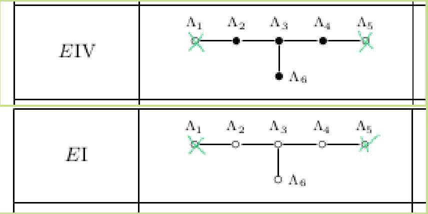

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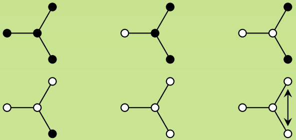

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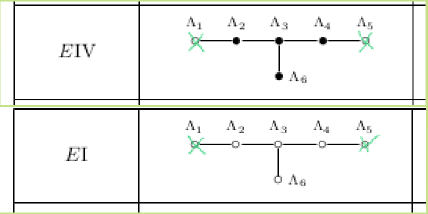

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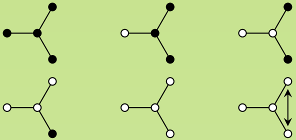

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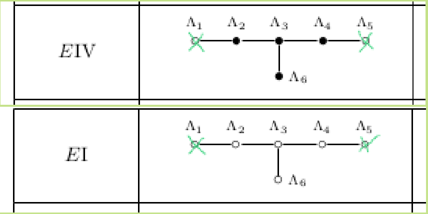

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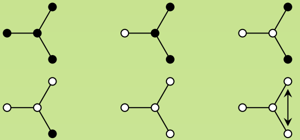

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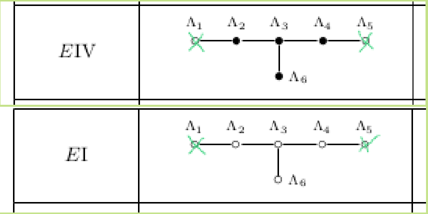

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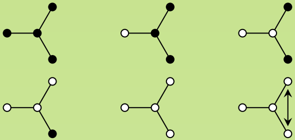

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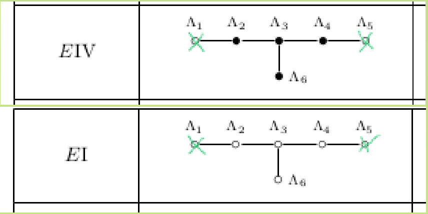
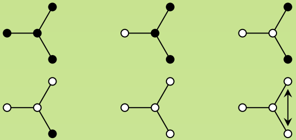


Compare the semisimple parts of these two diagrams with the real forms of $\mathfrak{d}_4$:



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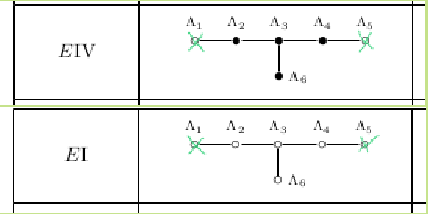

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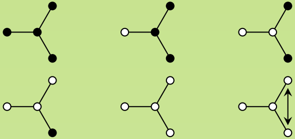

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Look at the simple part of $\mathfrak{g}_0$ in E_I and E_{IV} cases

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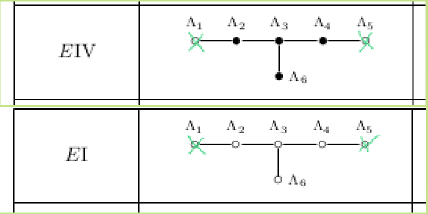

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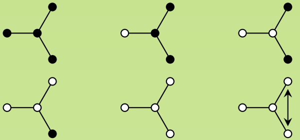

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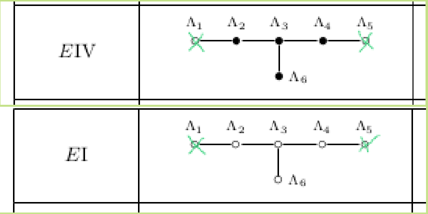

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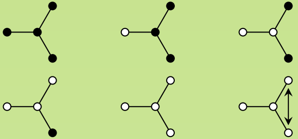

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