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$\mathbf{GL}(2,\mathbb{R})$ geometry of ODE's

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ABSTRACT

We study five dimensional geometries associated with the 5-dimensional irreducible representation of $\mathbf{GL}(2,\mathbb{R})$. These are special Weyl geometries in signature $(3,2)$ having the structure group reduced from $\mathbf{CO}(3,2)$ to $\mathbf{GL}(2,\mathbb{R})$. The reduction is obtained by means of a conformal class of totally symmetric 3-tensors. Among all $\mathbf{GL}(2,\mathbb{R})$ geometries we distinguish a subclass which we term 'nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometries'. These define a unique $\mathfrak{gl}(2,\mathbb{R})$ connection which has totally skew symmetric torsion. This torsion splits onto the $\mathbf{GL}(2,\mathbb{R})$ irreducible components having respective dimensions three and seven.

We prove that on the solution space of every 5th order ODE satisfying certain three nonlinear differential conditions there exists a nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometry such that the skew symmetric torsion of its unique $\mathfrak{gl}(2,\mathbb{R})$ connection is very special. In contrast to an arbitrary nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometry, it belongs to the 3-dimensional irreducible representation of $\mathbf{GL}(2,\mathbb{R})$.

We provide nontrivial examples of 5th order ODEs satisfying the three nonlinear differential conditions, which in turn provide examples of inhomogeneous $\mathbf{GL}(2,\mathbb{R})$ geometries in dimension five, with torsion in $\mathbb{R}^3$.

We also outline the theory and the basic properties of $\mathbf{GL}(2,\mathbb{R})$ geometries associated with n-dimensional irreducible representations of $\mathbf{GL}(2,\mathbb{R})$ in $6\le n\le 9$. In particular we give conditions for an nth order ODE to possess this geometry on its solution space.

1. Introduction

Let us start with an elementary algebraic geometry in $\mathbb{R}^3$. Every point on a curve $(1,x,x^2)$ in $\mathbb{R}^3$ defines a straight line passing through the origin in the dual space $(\mathbb{R}^3)^*$ via the relation:

$$\begin{aligned}&\theta^0+2\theta^1x+\theta^2x^2=0\\&\theta^1+\theta^2x=0.\end{aligned}\tag{1.1}$$

Here $(\theta^0,\theta^1,\theta^2)$ parameterize points of $(\mathbb{R}^3)^*$. When moving along the curve $(1,x,x^2)$ in $\mathbb{R}^3$, the corresponding lines in the dual space $(\mathbb{R}^3)^*$ sweep out a ruled surface there, which is the cone

$$(\theta^1)^2-\theta^0\theta^2=0\tag{1.2}$$

with the tip in the origin. The points $(\theta^0,\theta^1,\theta^2)$ lying on this cone may be thought as those, and only those, which admit a common root x for the pair of Eqs. (1.1). A standard method for determining if two polynomials have a common root is to

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equate to zero their *resultant*. In the case of Eqs. (1.1) the resultant is:

$$R_3 = \det \begin{pmatrix} \theta^0 & 2\theta^1 & \theta^2 & 0 & 0 \\ 0 & \theta^0 & 2\theta^1 & \theta^2 & 0 \\ 0 & 0 & \theta^0 & 2\theta^1 & \theta^2 \\ \theta^1 & \theta^2 & 0 & 0 & 0 \\ 0 & \theta^1 & \theta^2 & 0 & 0 \end{pmatrix}.$$

It vanishes if and only if condition (1.2) holds.

Before passing to $\mathbb{R}^n$ with general $n \geq 3$, it is instructive to repeat the above considerations in the cases of $n = 4$ and $n = 5$.

A point on a curve $(1, x, x^2, x^3)$ in $\mathbb{R}^4$ defines a plane passing through the origin in the dual space $(\mathbb{R}^4)^*$ via the relation:

$$\begin{aligned} &\theta^0 + 3\theta^1 x + 3\theta^2 x^2 + \theta^3 x^3 = 0 \\ &\theta^1 + 2\theta^2 x + \theta^3 x^2 = 0. \end{aligned} \tag{1.3}$$

Now $(\theta^0, \theta^1, \theta^2, \theta^3)$ parameterize points of the dual $(\mathbb{R}^4)^*$ and when moving along the curve $(1, x, x^2, x^3)$ in $\mathbb{R}^4$, the corresponding planes in $(\mathbb{R}^4)^*$ sweep out a ruled hypersurface there, which is defined by the vanishing of the resultant of the two polynomials defined in (1.3). This is given by

$$-3(\theta^1)^2(\theta^2)^2 + 4\theta^0(\theta^2)^3 + 4(\theta^1)^3\theta^3 - 6\theta^0\theta^1\theta^2\theta^3 + (\theta^0)^2(\theta^3)^2 = 0, \tag{1.4}$$

as can be easily calculated.

For $n = 5$, a point on a curve $(1, x, x^2, x^3, x^4)$ in $\mathbb{R}^5$ defines a 3-plane passing through the origin in the dual space $(\mathbb{R}^5)^*$ via the relation:

$$\begin{aligned} &\theta^0 + 4\theta^1 x + 6\theta^2 x^2 + 4\theta^3 x^3 + \theta^4 x^4 = 0 \\ &\theta^1 + 3\theta^2 x + 3\theta^3 x^2 + \theta^4 x^3 = 0, \end{aligned} \tag{1.5}$$

where $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ parameterize points of the dual $(\mathbb{R}^5)^*$ as before. And now, when moving along the curve $(1, x, x^2, x^3, x^4)$ in $\mathbb{R}^5$, the corresponding 3-planes in $(\mathbb{R}^4)^*$ sweep out a ruled hypersurface there, which is again defined by the vanishing of the resultant of the two polynomials defined in (1.5). The algebraic expression for this hypersurface in terms of the θ coordinates is quite complicated:

$$\begin{aligned} &-36(\theta^1)^2(\theta^2)^2(\theta^3)^2 + 54\theta^0(\theta^2)^3(\theta^3)^2 + 64(\theta^1)^3(\theta^3)^3 - 108\theta^0\theta^1\theta^2(\theta^3)^3 \\ &\quad + 27(\theta^0)^2(\theta^3)^4 + 54(\theta^1)^2(\theta^2)^3\theta^4 - 81\theta^0(\theta^2)^4\theta^4 - 108(\theta^1)^3\theta^2\theta^3\theta^4 \\ &\quad + 180\theta^0\theta^1(\theta^2)^2\theta^3\theta^4 + 6\theta^0(\theta^1)^2(\theta^3)^2\theta^4 - 54(\theta^0)^2\theta^2(\theta^3)^2\theta^4 + 27(\theta^1)^4(\theta^4)^2 \\ &\quad - 54\theta^0(\theta^1)^2\theta^2(\theta^4)^2 + 18(\theta^0)^2(\theta^2)^2(\theta^4)^2 + 12(\theta^0)^2\theta^1\theta^3(\theta^4)^2 - (\theta^0)^3(\theta^4)^3 = 0, \end{aligned} \tag{1.6}$$

but easily calculable.

The beauty of the hypersurfaces (1.2), (1.4) and (1.6) consists of this that they are given by means of homogeneous equations, and thus they descend to the corresponding projective spaces. From the point of view of the present paper, even more important is the fact, that they are $\mathbf{GL}(2, \mathbb{R})$ *invariant*. By this we mean the following:

Consider a real polynomial of $(n-1)$-th degree

$$w(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} \theta^i x^i \tag{1.7}$$

in the real variable x with real coefficients $(\theta^0, \theta^1, \ldots, \theta^{n-1})$. The n-dimensional vector space $(\mathbb{R}^n)^*$ of such polynomials may be identified with the space of their coefficients. Now, replacing the variable x by a new variable x' such that

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{1.8}$$

we define a new covector $(\theta'^0, \theta'^1, \ldots, \theta'^{n-1})$ which is related to $(\theta^0, \theta^1, \ldots, \theta^{n-1})$ of (1.7) via

$$\sum_{i=0}^{n-1} \binom{n-1}{i} \theta'^i x'^i = (\gamma x' + \delta)^{n-1} w(x).$$

It is obvious that $\theta' = (\theta'^0, \theta'^1, \ldots, \theta'^{n-1})$ is linearly expressible in terms of $\theta = (\theta^0, \theta^1, \ldots, \theta^{n-1})$:

$$\theta' = \theta \cdot \rho_n(a), \quad a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{1.9}$$

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Here a corresponds to the $\mathbf{GL}(2,\mathbb{R})$ transformation (1.8), and the map

$$\rho_n : \mathbf{GL}(2,\mathbb{R}) \to \mathbf{GL}((\mathbb{R}^n)^*) \cong \mathbf{GL}(n,\mathbb{R})$$

defines the *real n-dimensional irreducible representation* of $\mathbf{GL}(2,\mathbb{R})$. For example, if $n=2$, we have $w(x)=\theta^0+2\theta^1x+\theta^2x^2$, and we easily get

$$\begin{pmatrix}\theta'^0 & \theta'^1 & \theta'^2\end{pmatrix} = \begin{pmatrix}\theta^0 & \theta^1 & \theta^2\end{pmatrix}\begin{pmatrix}\delta^2 & \gamma\delta & \gamma^2\\ 2\beta\delta & \alpha\delta+\beta\gamma & 2\alpha\gamma\\ \beta^2 & \alpha\beta & \alpha^2\end{pmatrix},$$

so that ρ_2 is given by

$$\rho_2\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\delta^2 & \gamma\delta & \gamma^2\\ 2\beta\delta & \alpha\delta+\beta\gamma & 2\alpha\gamma\\ \beta^2 & \alpha\beta & \alpha^2\end{pmatrix}.$$

Now, let us define $g(\theta,\theta)$, ${}^4I(\theta,\theta,\theta,\theta)$ and ${}^5I(\theta,\theta,\theta,\theta,\theta,\theta)$ by

$$\begin{aligned} &g(\theta,\theta) = \text{the left hand side of (1.2)}\\ &{}^4I(\theta,\theta,\theta,\theta) = \text{the left hand side of (1.4)}\\ &{}^5I(\theta,\theta,\theta,\theta,\theta,\theta) = \text{the left hand side of (1.6).}\end{aligned} \tag{1.10}$$

We will often abbreviate this notation to the respective: $g(\theta)$, ${}^4I(\theta)$ and ${}^5I(\theta)$.

To explain our comment about the $\mathbf{GL}(2,\mathbb{R})$ invariance of the respective hypersurfaces $g(\theta)=0$, ${}^4I(\theta)=0$ and ${}^5I(\theta)=0$ we calculate $g(\theta')$, ${}^4I(\theta')$ and ${}^5I(\theta')$ with θ' as in (1.9). The result is

$$\begin{aligned} &g(\theta') = (\alpha\delta-\beta\gamma)^2\, g(\theta)\\ &{}^4I(\theta') = (\alpha\delta-\beta\gamma)^4\, {}^4I(\theta)\\ &{}^5I(\theta') = (\alpha\delta-\beta\gamma)^6\, {}^5I(\theta).\end{aligned}$$

Thus the vanishing of the expressions $g(\theta)$, ${}^4I(\theta)$ and ${}^5I(\theta)$ is invariant under the action (1.9) of the irreducible $\mathbf{GL}(2,\mathbb{R})$ on $(\mathbb{R}^n)^*$.

We are now ready to discuss the general case $n\geq 3$ of the *rational normal curve* $(1,x,x^2,\ldots,x^{n-1})$ in $\mathbb{R}^n$. Associated with this curve is a pair of polynomials, namely $w(x)$ as in (1.7), and its derivative $\frac{\mathrm{d}w}{\mathrm{d}x}$. We consider the relation

$$w(x)=0 \quad \& \quad \frac{\mathrm{d}w}{\mathrm{d}x}=0. \tag{1.11}$$

This gives a correspondence between the points on the curve $(1,x,x^2,\ldots,x^{n-1})$ in $\mathbb{R}^n$ and the $(n-2)$-planes passing through the origin in the dual space $(\mathbb{R}^n)^*$ parameterized by $(\theta^0,\theta^1,\ldots,\theta^{n-1})$. When moving along the rational normal curve in $\mathbb{R}^n$, the corresponding $(n-2)$-planes in $(\mathbb{R}^n)^*$ sweep out a ruled hypersurface there. This is defined by the vanishing of the resultant, $R(w(x),\frac{\mathrm{d}w}{\mathrm{d}x})$, of the two polynomials in (1.11). The algebraic expression for this hypersurface is the vanishing of a homogeneous polynomial, let us call it $I(\theta)$, of order $2(n-2)$, in the coordinates $(\theta^0,\theta^1,\ldots,\theta^{n-1})$. The hypersurface $I(\theta)=0$ in $(\mathbb{R}^n)^*$ is $\mathbf{GL}(2,\mathbb{R})$ invariant, since the property of the two polynomials $w(x)$ and $\frac{\mathrm{d}w}{\mathrm{d}x}$ to have a common root is independent of the choice (1.8) of the coordinate x. Thus $\mathbf{GL}(2,\mathbb{R})$ is included in the stabiliser G_I of I under the action of the full $\mathbf{GL}(n,\mathbb{R})$ group. This stabiliser, by definition, is a subgroup of $\mathbf{GL}(n,\mathbb{R})$ with elements $b\in G_I\subset\mathbf{GL}(n,\mathbb{R})$ such that $I(\theta\cdot b)=(\det b)^{\frac{2(n-2)}{n}}I(\theta)$. Moreover, in $n=4,5$, it turns out that G_I is precisely the group $\mathbf{GL}(2,\mathbb{R})$ in the corresponding irreducible representation ρ_n. Thus if $n=4,5$ one can characterise the irreducible $\mathbf{GL}(2,\mathbb{R})$ in n dimensions as the stabiliser of the polynomial $I(\theta)$.

Crucial for the present paper is an observation of Karl Wünschmann that the algebraic geometry and the correspondences we were describing above, naturally appear in the analysis of solutions of the ODE $y^{(n)}=0$. Indeed, following Wünschmann[1] (see the Introduction in his Ph.D. thesis [1], pp. 5-6), we note the following:

Consider the third order ODE: $y'''=0$. Its general solution is $y=c_0+2c_1x+c_2x^2$, where c_0,c_1,c_2 are the integration constants. Thus, the solution space of the ODE $y'''=0$ is $\mathbb{R}^3$ with solutions identified with points $\mathbf{c}=(c_0,c_1,c_2)\in\mathbb{R}^3$. The solutions to the ODE $y'''=0$ may be also identified with curves $y(x)=c_0+2c_1x+c_2x^2$, actually parabolas, in the plane (x,y). Suppose now, that we take two solutions of $y'''=0$ corresponding to two neighbouring points $\mathbf{c}=(c_0,c_1,c_2)$ and $\mathbf{c}+\mathrm{d}\mathbf{c}=(c_0+\mathrm{d}c_0,c_1+\mathrm{d}c_1,c_2+\mathrm{d}c_2)$ in $\mathbb{R}^3$. Among all pairs of neighbouring solutions we choose only those, which have the property that their corresponding curves $y=y(x)$ and $y+\mathrm{d}y=y(x)+\mathrm{d}y(x)$ touch each other, at some point (x_0,y_0)

1 We are very grateful to Niels Schuman, who found a copy of Wünschmann's thesis in the *city* library of Berlin and sent it to us. It was this copy, which after translation from German by Denson Hill, led us to write this introduction.

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in the plane (x, y). If we do not require anything more about the properties of this incidence of the two curves, we say that solutions $\mathbf{c}$ and $\mathbf{c} + \mathrm{d}\mathbf{c}$ have *zero order contact* at (x_0, y_0).

In this 'baby' example everything is very simple: To get the criterion for the solutions to have zero order contact we first write the curves $y = c_0 + 2c_1x + c_2x^2$ and $y + \mathrm{d}y = c_0 + \mathrm{d}c_0 + 2(c_1 + \mathrm{d}c_1)x + (c_2 + \mathrm{d}c_2)x^2$ corresponding to $\mathbf{c}$ and $\mathbf{c} + \mathrm{d}\mathbf{c}$. Thus the solutions have zero order contact at $(x_0, y(x_0))$ provided that $\mathrm{d}y(x_0) = 0$, i.e. if and only if

$$\mathrm{d}c_0 + 2x_0\mathrm{d}c_1 + x_0^2\mathrm{d}c_2 = 0.$$

This shows that such a contact is possible if and only if the determinant

$$g(\mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}) = (\mathrm{d}c_1)^2 - \mathrm{d}c_0\mathrm{d}c_2$$

is non-negative, since otherwise the quadratic equation for x_0 has no solutions. Unexpectedly, we find that the requirement for the two neighbouring solution curves of $y''' = 0$ to have zero order contact at some point is equivalent to the requirement that the corresponding two neighbouring points $\mathbf{c}$ and $\mathbf{c} + \mathrm{d}\mathbf{c}$ in $\mathbb{R}^3$ be spacelike separated in the Minkowski metric g in $\mathbb{R}^3$. This is the discovery of Wünschmann that is quoted in Elie Cartan's 1941 year's paper[2] [4].

Now we consider the neighbouring solutions $\mathbf{c}$ and $\mathbf{c} + \mathrm{d}\mathbf{c}$ of $y''' = 0$ which are null separated in the metric $\mathrm{d}s^2$. What we can say about the corresponding curves in the plane (x, y)?

To answer this we need the notion of a *first order contact*: Two neighbouring solution curves $y = c_0 + 2c_1x + c_2x^2$ and $y + \mathrm{d}y = c_0 + 2c_1x + c_2x^2 + (\mathrm{d}c_0 + 2x\mathrm{d}c_1 + x^2\mathrm{d}c_2)$ of $y''' = 0$, corresponding to $\mathbf{c}$ and $\mathbf{c} + \mathrm{d}\mathbf{c}$ in $\mathbb{R}^3$, have first order contact at (x_0, y_0) iff they have zero order contact at (x_0, y_0) and, in addition, their curves of first derivatives, $y' = 2c_1 + 2c_2x$ and $y' + \mathrm{d}y' = 2(c_1 + \mathrm{d}c_1) + 2(c_2 + \mathrm{d}c_2)x$, have zero order contact at (x_0, y_0). Thus the condition of first order contact at $(x_0, y(x_0))$ is equivalent to $\mathrm{d}y(x_0) = 0$ and $\mathrm{d}y'(x_0) = 0$, i.e. to the condition that x_0 is a simultaneous root for

$$\begin{aligned} &\mathrm{d}c_0 + 2x_0\mathrm{d}c_1 + x_0^2\mathrm{d}c_2 = 0 \\ &\mathrm{d}c_1 + x_0\mathrm{d}c_2 = 0. \end{aligned} \tag{1.12}$$

Solving the second of these equations for x_0, and inserting it into the first, after an obvious simplification, we conclude that $(\mathrm{d}c_1)^2 - \mathrm{d}c_0\mathrm{d}c_2 = 0$. Thus we get the interpretation of the null separated neighbouring points in $\mathbb{R}^3$ as the solutions of $y''' = 0$ whose curves in the (x, y) plane are neighbouring and have first order contact at some point.

Wünschmann notes that the procedure described here for the equation $y''' = 0$ can be repeated for the equation $y^{(n)} = 0$ for arbitrary $n \geq 3$. In the cases of $n = 4$ and $n = 5$ he however passes to the discussion of the solutions that have contact of order $(n - 2)$ rather then one. This is an interesting possibility, complementary in a sense to the one in which the solutions have first order contact. Wünschmann spends rest of the thesis studying it. But we will not discuss it here.

Since Wünschmann does not discuss the first order contact of the solutions in $n = 4, 5$, let us look closer into these two cases:

The general solution to $y^{(4)} = 0$ is $y = c_0 + 3c_1x + 3c_2x^2 + c_3x^3$, and the general solution to $y^{(5)} = 0$ is $y = c_0 + 4c_1x + 6c_2x^2 + 4c_3x^3 + c_4x^4$. Thus now the solutions are points $\mathbf{c}$ in $\mathbb{R}^4$ and $\mathbb{R}^5$, respectively. The condition that the neighbouring solutions $\mathbf{c} = (c_0, c_1, c_2, c_3)$ and $\mathbf{c} + \mathrm{d}\mathbf{c} = (c_0 + \mathrm{d}c_0, c_1 + \mathrm{d}c_1, c_2 + \mathrm{d}c_2, c_3 + \mathrm{d}c_3)$ of $y^{(4)} = 0$ have first order contact at $(x_0, y(x_0))$ is equivalent to the requirement that the system

$$\begin{aligned} &\mathrm{d}c_0 + 3x_0\mathrm{d}c_1 + 3x_0^2\mathrm{d}c_2 + x_0^3\mathrm{d}c_3 = 0 \\ &\mathrm{d}c_1 + 2x_0\mathrm{d}c_2 + x_0^2\mathrm{d}c_3 = 0 \end{aligned} \tag{1.13}$$

have a common root x_0. Similarly, the condition that the neighbouring solutions $\mathbf{c} = (c_0, c_1, c_2, c_3, c_4)$ and $\mathbf{c} + \mathrm{d}\mathbf{c} = (c_0 + \mathrm{d}c_0, c_1 + \mathrm{d}c_1, c_2 + \mathrm{d}c_2, c_3 + \mathrm{d}c_3, c_4 + \mathrm{d}c_4)$ of $y^{(5)} = 0$ have first order contact at $(x_0, y(x_0))$ is equivalent to the requirement that the system

$$\begin{aligned} &\mathrm{d}c_0 + 4x_0\mathrm{d}c_1 + 6x_0^2\mathrm{d}c_2 + 4x_0^3\mathrm{d}c_3 + x_0^4\mathrm{d}c_4 = 0 \\ &\mathrm{d}c_1 + 3x_0\mathrm{d}c_2 + 3x_0^2\mathrm{d}c_3 + x_0^3\mathrm{d}c_4 = 0 \end{aligned} \tag{1.14}$$

have a common root x_0. Calculating the resultants for the systems (1.12)–(1.14) we get:

- $R_3 = g(\mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c})\mathrm{d}c_2 \quad$ if $n = 3$,
- $R_4 = {}^4I(\mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c})\mathrm{d}c_3 \quad$ if $n = 4$,
- $R_5 = {}^5I(\mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c})\mathrm{d}c_4 \quad$ if $n = 5$,

where g, ${}^4I$ and ${}^5I$ are as in (1.10).

2 It is worthwhile remarking, that Wünschmann thesis is dated '1905', the same year in which Einstein published his famous special relativity theory paper [2]. It was not until three years later when Minkowski gave the geometric interpretation of Einstein's theory in terms of his metric [3]. Perhaps Wünschmann was the first who ever wrote such a metric in a scientific paper. This is a very interesting feature of Wünschmann's thesis: he calls the expressions like $(\mathrm{d}c_1)^2 - \mathrm{d}c_0\mathrm{d}c_2 = 0$, a *Mongesche Gleichung* rather than a *cone in the metric*, because the notion of a metric with signature different than the Riemannian one was not yet abstracted!

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This confirms our earlier statement that two neighbouring solutions of $y''' = 0$ have first order contact iff $g(\mathrm{d}\mathbf{c}, \mathrm{d}\mathbf{c}) = 0$, since if $\mathrm{d}c_2 = 0$ the system (1.12) collapses to $\mathrm{d}c_1 = \mathrm{d}c_0 = 0$. Similarly, one can prove that two neighbouring solutions of $y^{(4)} = 0$ or $y^{(5)} = 0$ have first order contact if and only if they are null separated in the respective symmetric multilinear forms ${}^4I$ or ${}^5I$. Our previous discussion of the invariant properties of these forms, shows that in the solution space of an ODE $y^{(n)} = 0$, for $n \geq 4$, there is a naturally defined action of the the $\mathbf{GL}(2, \mathbb{R})$ group. This group is the stabiliser of the invariant polynomial $I(\mathrm{d}\mathbf{c})$ which distinguishes neighbouring solutions having first order contact.

First question one can ask in this context is if one can retain this $\mathbf{GL}(2, \mathbb{R})$ structure in the solution space for more complicated ODEs. In other words, one may asks the following: What does one have to assume about the function F, defining an ODE

$$y^{(n)} = F(x, y, y', \ldots, y^{(n-1)}),$$

in order to have a well defined conformal tensor g, ${}^4I$ or ${}^5I$, in the respective cases $n = 3, 4, 5$, on the solution space of the ODE? The same question can be repeated for any $n > 5$ and the invariant I.

The answer to this question in the $n = 4$ case was found by Robert Bryant in [5]. Our paper, among other things, gives a geometric background and an effective method for answering this question for $n > 4$. It follows that for every $n \geq 3$, one has $(n - 2)$ *contact invariant conditions* for F, whose vanishing is necessary and sufficient for defining a conformal tensor I on the solution space of the ODE. Each of these $(n - 2)$ conditions is of third order in the derivatives of F. In dimension $n = 4$ our $(4 - 2) = 2$ conditions agree with the Bryant ones. Since Wünschman was the first who obtained these types of conditions in $n = 3$, we call the $(n - 2)$ conditions for F the generalised Wünschmann's conditions, or *Wünschmann's conditions*, for short. We also mention that they are *lower order* equivalents of the conditions discussed recently in [6–8].

The main objective of the paper is a thorough study of the *irreducible* $\mathbf{GL}(2, \mathbb{R})$ *geometry in dimension five*. This is done from two points of view: first as a study of an *abstract* geometry on a manifold and, second, as a study of a *contact geometry of fifth order ODEs*. In the latter case we also describe the $\mathbf{GL}(2, \mathbb{R})$ geometry in the language of contact invariants of the ODE and construct the Wünschmann conditions.

We define an abstract 5-dimensional $\mathbf{GL}(2, \mathbb{R})$ geometry in two steps. First, in Section 2, we show how to construct the algebraic model for the $\mathbf{GL}(2, \mathbb{R})$ geometry in dimension five utilising the properties of a rational normal curve. Second, instead of obtaining the reduction from $\mathbf{GL}(5, \mathbb{R})$ to $\mathbf{GL}(2, \mathbb{R})$ by stabilising the 6-tensor ${}^5I$, we get the desired reduction by stabilising a conformal metric $g_{ij} \to e^{2\phi} g_{ij}$ of signature $(3, 2)$ and a conformal totally symmetric 3-tensor $\Upsilon_{ijk} \to e^{3\phi}\Upsilon_{ijk}$. These tensors are supposed to be related by the following algebraic constraint:

$$g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip}. \tag{1.15}$$

It is worthwhile noting that condition (1.15) is a non-Riemannian counterpart of the condition considered by Elie Cartan in the context of isoparametric surfaces [9,10]. Our main object of study is then defined in Section 3 as follows:

Definition. An irreducible $\mathbf{GL}(2, \mathbb{R})$ geometry in dimension five is a 5-dimensional manifold M^5 equipped with a class of triples $[g, \Upsilon, A]$ such that on M^5:

(a) g is a metric of signature $(3, 2)$,
(b) Υ is a traceless symmetric 3rd rank tensor,
(c) A is a 1-form,
(d) the metric g and the tensor Υ satisfy the identity (1.15),
(e) two triples (g, Υ, A) and (g', Υ', A') are in the same class $[g, \Upsilon, A]$ if and only if there exists a function $\phi : M^5 \to \mathbb{R}$ such that

$$g' = e^{2\phi}g, \qquad \Upsilon' = e^{3\phi}\Upsilon, \qquad A' = A - 2\mathrm{d}\phi.$$

This definition places $\mathbf{GL}(2, \mathbb{R})$ geometries in dimension five among the *Weyl geometries* $[g, A]$. They are special Weyl geometries i.e. such for which the structure group is reduced from $\mathbf{CO}(3, 2)$ to $\mathbf{GL}(2, \mathbb{R})$. A natural description of such geometries should be then obtained in terms of a certain $\mathfrak{gl}(2, \mathbb{R})$-valued connection. However, unlike in the usual Weyl case, the choice of a $\mathfrak{gl}(2, \mathbb{R})$ connection is ambiguous, due to the fact that such a connection has non-vanishing torsion in general, and one must find admissible conditions for the torsion that specifies a connection uniquely. Pursuing this problem in Section 3 we find an interesting subclass of $\mathbf{GL}(2, \mathbb{R})$ geometries.

Definition. A $\mathbf{GL}(2, \mathbb{R})$ geometry $[g, \Upsilon, A]$ is called nearly integrable if the Weyl connection $\overset{W}{\nabla}$ of $[g, A]$ satisfies

$$(\overset{W}{\nabla}_X \Upsilon)(X, X, X) = -\frac{1}{2}A(X)\Upsilon(X, X, X).$$

Next we prove the following property of nearly integrable geometries.

Proposition. *A nearly integrable* $\mathbf{GL}(2, \mathbb{R})$ *geometry uniquely defines a* $\mathfrak{gl}(2, \mathbb{R})$ *connection D. This is characterised by the two following requirements:*

- *D preserves the structural tensors:*

$$Dg_{ij} = -Ag_{ij},$$
$$D\Upsilon_{ijk} = -\frac{3}{2}A\Upsilon_{ijk},$$

- *and D has totally skew symmetric torsion.*

We call this unique connection the *characteristic connection* for the nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structure.

In Section 4 we briefly describe $\mathbf{GL}(2,\mathbb{R})$ geometry in the language of the bundle $\mathbf{GL}(2,\mathbb{R}) \to P \to M^5$. We also show how an appropriate coframe defined on a nine-dimensional manifold P turns this manifold into a bundle $\mathbf{GL}(2,\mathbb{R}) \to P \to M^5$ and generates the $\mathbf{GL}(2,\mathbb{R})$ geometry on M^5. The bundle approach is useful in proofs of results in Sections 5 and 6.

Section 5 is devoted to studying the algebraic structure of the torsion and the curvature of the characteristic connection of a nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structure. Since the tensor products of tangent spaces are reducible under the action of $\mathbf{GL}(2,\mathbb{R})$, we decompose the torsion and the curvature tensors into components belonging to the irreducible representations. In particular, the skew symmetric torsion T has two components, $T^{(3)}$ and $T^{(7)}$, lying in the three-dimensional and the seven-dimensional irreducible representations respectively. Likewise the Maxwell 2-form $\mathrm{d}A$ and the Ricci tensor R decompose according to $\mathrm{d}A = \mathrm{d}A^{(3)} + \mathrm{d}A^{(7)}$ and $R = R^{(1)} + R^{(3)} + R^{(5)} + R^{(7)} + R^{(9)}$. The last problem we address in Section 5 concerns with the properties of geometries whose characteristic connections have 'the smallest possible' torsion, that is the torsion of the pure three-dimensional type. In Theorems 5.4 and 5.5 we prove that the Ricci tensor for such structures satisfies the remarkable identities:

$$R^{(3)} = \frac{1}{4}\mathrm{d}A^{(3)}, \qquad R^{(7)} = \frac{3}{2}\mathrm{d}A^{(7)}, \qquad R^{(9)} = 0.$$

Here the third equation is equivalent to

$$R_{(ij)} = \frac{1}{5}Rg_{ij} + \frac{2}{7}R_{kl}\Upsilon^{klm}\Upsilon_{ijm}.$$

This closes the part of the paper that is devoted to abstract $\mathbf{GL}(2,\mathbb{R})$ geometries.

Section 6 contains the main result of this paper, Theorem 6.3, which links $\mathbf{GL}(2,\mathbb{R})$ geometry with the realm of ordinary differential equations. It can be encapsulated as follows.

Theorem. *A 5th order ODE $y^{(5)} = F(x, y, y', y'', y''', y^{(4)})$ that satisfies three Wünschmann conditions defines a nearly integrable irreducible $\mathbf{GL}(2,\mathbb{R})$ geometry $(M^5, [g, \Upsilon, A])$ on the space M^5 of its solutions. This geometry has the characteristic connection with torsion of the 'pure' type in the* 3*-dimensional irreducible representation of* $\mathbf{GL}(2,\mathbb{R})$*. Two* 5*th order ODEs which are equivalent under contact transformation of variables define equivalent* $\mathbf{GL}(2,\mathbb{R})$ *geometries.*

The theorem has numerous applications. For example, we use it to characterise various classes of Wünschmann 5th order ODEs, by means of the algebraic type of the tensors associated with the corresponding characteristic connection. For example iff $F_{y^{(4)}y^{(4)}} = 0$ the torsion of the characteristic connection vanishes, and iff $F_{y^{(4)}y^{(4)}y^{(4)}} = 0$ then we have $\mathrm{d}A^{(7)} = 0$.

The proof of the theorem consists of an application of the Cartan method of equivalence. We write an ODE, considered modulo contact transformation of variables, as a G-structure on the four-order jet space. Starting from this G-structure we explicitly construct a 9-dimensional manifold P, which is a $\mathbf{GL}(2,\mathbb{R})$ bundle over the solution space and carries a certain distinguished coframe. This construction is only possible provided that the ODE satisfies the Wünschmann conditions, which we write down explicitly. By means of Proposition 4.1 the coframe on P defines the nearly integrable geometry on the solution space of the ODE. It has the characteristic connection with torsion in the 3-dimensional representation.

Section 7 includes examples of 5th order equations in the Wünschmann class. We find equations generating all the structures with vanishing torsion, equations possessing at least a 6-dimensional group of contact symmetries and yielding geometries with $\mathrm{d}A = 0$. We also give highly nontrivial examples of equations for which $\mathrm{d}A \neq 0$, including a family of examples with function F being a solution of a certain second order ODE.

Finally, in Section 8 we consider ODEs of order $n > 5$. We apply results of the Hilbert theory of algebraic invariants, to define the tensors responsible for the reduction $\mathbf{GL}(n,\mathbb{R}) \to \mathbf{GL}(2,\mathbb{R})$. We also give the explicit formulae for the $(n-2)$ third order Wünschmann conditions for $n = 6$ and $n = 7$.

2. A peculiar third rank symmetric tensor

Consider $\mathbb{R}^n$ equipped with a *Riemannian* metric g and a 3rd rank tracefree symmetric tensor $\Upsilon \in S^3_0\mathbb{R}^n$ satisfying:

(i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$ (symmetry)
(ii) $g^{ij}\Upsilon_{ijk} = 0$ (tracefree)
(iii) $g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip}$.

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It turns out that the third condition is very restrictive. In particular Cartan has shown [9,10] that for (iii) to be satisfied the dimension n must be one of the following: $n = 5, 8, 14, 26$. Moreover Cartan constructed Υ in each of these dimensions and has shown that it is unique up to an $\mathbf{O}(n)$ transformation. Restricting to $n = 5, 8, 14, 26$, we consider the stabiliser H_n of Υ under the action of $\mathbf{GL}(n, \mathbb{R})$:

$$H_n = \{\mathbf{GL}(n, \mathbb{R}) \ni a : \Upsilon(aX, aY, aZ) = \Upsilon(X, Y, Z), \forall X, Y, Z \in \mathbb{R}^n\}.$$

Then, one finds that:

- $H_5 = \mathbf{SO}(3) \subset \mathbf{SO}(5)$ in the 5-dimensional irreducible representation,
- $H_8 = \mathbf{SU}(3) \subset \mathbf{SO}(8)$ in the 8-dimensional irreducible representation,
- $H_{14} = \mathbf{Sp}(3) \subset \mathbf{SO}(14)$ in the 14-dimensional irreducible representation,
- $H_{26} = \mathbf{F}_4 \subset \mathbf{SO}(26)$ in the 26-dimensional irreducible representation.

The relevance of conditions (i)–(iii) is that they are invariant under the $\mathbf{O}(n)$ action on the space of tracefree symmetric tensors $S^3_0\mathbb{R}^n$. Moreover they totally characterise the orbit $\mathbf{O}(n)/H_n \subset S^3_0$ of the tensor Υ under this action [11,12].

The question arises if one can construct tensors satisfying (i)–(iii) for metrics having non-Riemannian signatures. Below we show how to do it if $n = 5$ and the metric g has the signature $(3, 2)$. This construction described to us by Ferapontov [13,14] is as follows.

Consider $\mathbb{R}^5$ with coordinates $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$, and a curve

$$\gamma(x) = (1, x, x^2, x^3, x^4) \subset \mathbb{R}^5.$$

Associated to the curve γ there are two algebraic varieties in $\mathbb{R}^5$:

- *The bisecant variety*. This is defined to be a set consisting of all the points on all straight lines crossing the curve γ in precisely two points. It is given parametrically as

  $$B(x, s, u) = (1, x, x^2, x^3, x^4) + u(0, x - s, x^2 - s^2, x^3 - s^3, x^4 - s^4),$$

  where x, s, u are three real parameters.
- *The tangent variety*. This is defined to be a set consisting of all the points on all straight lines tangent to the curve γ. It is given parametrically as

  $$T(x, s) = (1, x, x^2, x^3, x^4) + s(0, 1, 2x, 3x^2, 4x^3).$$

One of many interesting features of these two varieties is that they define (up to a scale) a tri-linear symmetric form

$$\Upsilon(\theta) = 3\sqrt{3}(\theta^0\theta^2\theta^4 + 2\theta^1\theta^2\theta^3 - (\theta^2)^3 - \theta^0(\theta^3)^2 - \theta^4(\theta^1)^2) \tag{2.1}$$

and a bi-linear symmetric form

$$g(\theta) = \theta^0\theta^4 - 4\theta^1\theta^3 + 3(\theta^2)^2. \tag{2.2}$$

These forms are distinguished by the fact that the bisecant and tangent varieties are contained in their null cones. In the homogeneous coordinates $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ in $\mathbb{R}^5$ all the points θ of $B(x, s, u)$ satisfy

$$\Upsilon(\theta) = 0,$$

whereas all the points θ of $T(x, s)$ satisfy

$$\Upsilon(\theta) = 0 \quad \text{and} \quad g(\theta) = 0.$$

Writing the forms as $\Upsilon(\theta) = \Upsilon_{ijk}\theta^i\theta^j\theta^k$, $g(\theta) = g_{ij}\theta^i\theta^j$, $i, j, k = 0, 1, 2, 3, 4$ one can check that so defined g_{ij} and Υ_{ijk} satisfy relations (i)–(iii) of the previous section.

Although it is obvious we remark that the above defined metric g_{ij} has the signature $(3, 2)$.

As we have already noted the forms $\Upsilon(\theta)$ and $g(\theta)$ are defined only up to a scale. We were also able to find a factor, the $3\sqrt{3}$ in expression (2.1), that makes the corresponding g_{ij} and Υ_{ijk} to satisfy (i)–(iii). Note that these conditions are conformal under the simultaneous change:

$$g_{ij} \to e^{2\phi}g_{ij}, \qquad \Upsilon_{ijk} \to e^{3\phi}\Upsilon_{ijk}.$$

Thus it is interesting to consider in $\mathbb{R}^5$ a class of pairs $[g, \Upsilon]$, such that:

- in each pair (g, Υ)
  - g is a metric of signature $(3, 2)$,
  - Υ is a traceless symmetric 3rd rank tensor,
  - the metric g and the tensor Υ satisfy the identity

    $$g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip},$$

- two pairs (g, Υ) and (g', Υ') are in the same class $[g, \Upsilon]$ if and only if there exists $\phi \in \mathbb{R}$ such that

$$g' = e^{2\phi} g, \qquad \Upsilon' = e^{3\phi}\Upsilon. \tag{2.3}$$

Given a structure $(\mathbb{R}^5, [g, \Upsilon])$ we define a group CH to be a subgroup of the general linear group $\mathbf{GL}(5, \mathbb{R})$ preserving $[\Upsilon]$. This means that, choosing a representative Υ of the class $[\Upsilon]$, we define CH to be:

$$CH = \{\mathbf{GL}(5, \mathbb{R}) \ni a : \Upsilon(ax, ax, ax) = (\det a)^{(3/5)}\Upsilon(x, x, x)\}.$$

Note that the exponent $\frac{3}{5}$ in the above expression is caused by the fact that the r.h.s. of the equation defining the group elements must be homogeneous of degree 3 in a, similarly as the l.h.s. is.

This definition does not depend on the choice of a representative $\Upsilon \in [\Upsilon]$. We have the following

Proposition 2.1. *The set CH of 5×5 real matrices $a \in \mathbf{GL}(5, \mathbb{R})$ preserving $[\Upsilon]$ is the $\mathbf{GL}(2, \mathbb{R})$ group in its 5-dimensional irreducible representation. Moreover, we have natural inclusions*

$$CH = \mathbf{GL}(2, \mathbb{R}) \subset \mathbf{CO}(3, 2) \subset \mathbf{GL}(5, \mathbb{R}),$$

where $\mathbf{CO}(3, 2)$ is the 11-dimensional group of homotheties associated with the conformal class $[g]$.

Remark 2.2. According to our Introduction, there is another $\mathbf{GL}(2, \mathbb{R})$ invariant symmetric conformal tensor that stabilises $\mathbf{GL}(5, \mathbb{R})$ to the irreducible $\mathbf{GL}(2, \mathbb{R})$. This is the tensor ${}^5I_{ijklpq}$ defined via ${}^5I(\theta) = \frac{1}{120}{}^5I_{ijklpq}\theta^i\theta^j\theta^k\theta^l\theta^p\theta^q$ with ${}^5I$ as in (1.10). We prefer however to work with a pair (g_{ij}, Υ_{ijk}) rather then with ${}^5I_{ijklpq}$, because of the lower rank, and more importantly, because of the evident conformal metric properties of the (g_{ij}, Υ_{ijk}) approach. Also, it is worthwhile noting that the invariants g_{ij}, Υ_{ijk} and ${}^5I_{ijklpq}$ are not independent. Indeed, one can easily check that ${}^5I$ of (1.10), Υ of (2.1) and g of (2.2) are related by ${}^5I = \Upsilon^2 - g^3$. We interpret this relation as the definition of ${}^5I$ in terms of more primitive quantities g and Υ.

The isotropy condition for the group elements a of CH has its obvious counterpart at the level of the Lie algebra $\mathfrak{gl}(2, \mathbb{R}) = (\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})) \subset \mathfrak{co}(3, 2) \subset \mathfrak{gl}(5, \mathbb{R})$ of $CH = \mathbf{GL}(2, \mathbb{R})$. Writing $a = \exp(t\Gamma)$ we find that the infinitesimal version of the isotropy condition, written in terms of the 5×5 matrices $\Gamma = (\Gamma^i{}_j)$ is:

$$\Gamma^l{}_i\Upsilon_{ljk} + \Gamma^l{}_j\Upsilon_{ilk} + \Gamma^l{}_k\Upsilon_{ijl} = \frac{3}{5}Tr(\Gamma)\Upsilon_{ijk}, \tag{2.4}$$

where $Tr(\Gamma) = \Gamma^m{}_m$. Given Υ_{ijk}, these linear equations can be solved for Γ. Taking the most general matrix $\Gamma \in \mathbf{GL}(5, \mathbb{R})$ and Υ_{ijk} given by $\Upsilon(x, x, x) = \Upsilon_{ijk}x^ix^jx^k$ of (2.1) we find the explicit realisation of the 5-dimensional representation of the $\mathfrak{gl}(2, \mathbb{R})$ Lie algebra as:

$$\Gamma = \Gamma_- E_- + \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_1 E_1, \tag{2.5}$$

where $\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1$ are free real parameters, and (E_-, E_+, E_0, E_1) are 5×5 matrices given by:

$$E_+ = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \tag{2.6}$$

$$E_0 = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \qquad E_1 = -4\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The commutator in

$$\mathfrak{gl}(2, \mathbb{R}) = \mathrm{Span}_{\mathbb{R}}(E_-, E_+, E_0, E_1)$$

is the usual commutator of matrices. In particular, the non-vanishing commutators are:

$$[E_0, E_+] = -2E_+, \qquad [E_0, E_-] = 2E_-, \qquad [E_+, E_-] = -E_0.$$

Note that

$$\mathfrak{sl}(2, \mathbb{R}) = \mathrm{Span}_{\mathbb{R}}(E_-, E_+, E_0)$$

is a subalgebra of $\mathfrak{gl}(2, \mathbb{R})$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. It provides the 5-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$.

3. Irreducible GL(2, ℝ) geometries in dimension five

In this section we describe 5-dimensional manifolds whose tangent space at each point is equipped with the structure $[g, \Upsilon]$ of the previous section. We will analyse such manifolds in terms of an appropriately chosen connection. We will

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describe connections on a manifold M in terms of Lie-algebra-valued 1-forms on M. To be more specific, let $\dim M = n$ and let $\mathfrak{g}$ denote an n-dimensional representation of some Lie algebra. The connection 1-forms $\Gamma^i{}_j$ on M are the matrix entries of an element $\Gamma \in \mathfrak{g} \otimes \Lambda^1 M$. They define the covariant exterior derivative D. This acts on tensor-valued-forms via the extension to the higher order tensors of the formula:

$$Dv^i = \mathrm{d}v^i + \Gamma^i{}_j \wedge v^i.$$

Now suppose that we have a 5-dimensional manifold M^5 equipped with a class of pairs $[g, \Upsilon]$ such that g is a metric, Υ is a 3rd rank tensor related to the metric via properties (i)–(iii) of the previous section, and two pairs (g, Υ) and (g', Υ') are in the same pair iff they are related by (2.3), where ϕ is now a function on M^5. If we want to associate a connection with such a structure we have to specify how this connection is related to the pair $[g, \Upsilon]$. A possible approach is to choose a representative (g, Υ) of $[g, \Upsilon]$ and declare what is Dg and $D\Upsilon$. A first possible choice $Dg = 0$ or $D\Upsilon = 0$ is definitely not good since, in general, Dg' and $D\Upsilon'$ would not be vanishing for another choice of the representative of $[g, \Upsilon]$. A remedy for this situation comes from Weyl geometry where, given a conformal class $(M, [g])$, a 1-form A is introduced so that the connection satisfies $Dg_{ij} = -Ag_{ij}$. In our case we introduce a 1-form A on M^5 and require that $Dg_{ij} = -Ag_{ij}$ and $D\Upsilon_{ijk} = -\frac{3}{2}A\Upsilon_{ijk}$. Then, if we transform (g, Υ) according to (2.3), the transformed objects will satisfy $Dg'_{ij} = -A'g'_{ij}$ and $D\Upsilon'_{ijk} = -\frac{3}{2}A'\Upsilon'_{ijk}$ provided that $A' = A - 2\mathrm{d}\phi$. This motivates the following

Definition 3.1. An irreducible $\mathbf{GL}(2, \mathbb{R})$ structure in dimension five is a 5-dimensional manifold M^5 equipped with a class of triples $[g, \Upsilon, A]$ such that on M^5:

(a) g is a metric of signature $(3, 2)$,
(b) Υ is a traceless symmetric 3rd rank tensor,
(c) A is a 1-form,
(d) the metric g and the tensor Υ satisfy the identity

$$g^{lm}(\Upsilon_{ijl}\Upsilon_{kmp} + \Upsilon_{kil}\Upsilon_{jmp} + \Upsilon_{jkl}\Upsilon_{imp}) = g_{ij}g_{kp} + g_{kl}g_{jp} + g_{jk}g_{ip},$$

(e) two triples (g, Υ, A) and (g', Υ', A') are in the same class $[g, \Upsilon, A]$ if and only if there exists a function $\phi : M^5 \to \mathbb{R}$ such that

$$g' = \mathrm{e}^{2\phi}g, \qquad \Upsilon' = \mathrm{e}^{3\phi}\Upsilon, \qquad A' = A - 2\mathrm{d}\phi.$$

If M^5 was only equipped with a class of pairs $[g, A]$ satisfying conditions (a), (c) and (e) (with Υ, Υ' omitted), then $(M^5, [g, A])$ would define a Weyl geometry. Such a geometry, which has the structure group $\mathbf{CO}(3, 2)$, is usually studied in terms of the Weyl connection. This is the unique torsion-free connection preserving the conformal structure $[g, A]$. It is defined by the following two equations:

$$\overset{W}{D}\, g_{ij} = -Ag_{ij} \quad \text{(preservation of the class } [g, A]), \tag{3.1}$$

$$\overset{W}{D}\, \theta^i = 0 \quad \text{(no torsion)}, \tag{3.2}$$

where θ^i is a coframe related to the representative g of the class $[g]$ by $g = g_{ij}\theta^i\theta^j$. We describe the Weyl connection in terms of the Weyl connection 1-forms $\overset{W}{\Gamma}{}^i{}_j$, $i, j = 0, 1, 2, 3, 4$.

Take a representative (g, A) of the Weyl structure $[g, A]$ on M^5. Choose a coframe (θ^i), $i = 0, 1, 2, 3, 4$, such that $g = g_{ij}\theta^i\theta^j$, with all the metric coefficients g_{ij} being *constant*. Then the above two equations define $\overset{W}{\Gamma}{}^i{}_j$ together with $\overset{W}{\Gamma}{}_{ij} = g_{ik}\overset{W}{\Gamma}{}^k{}_j$ to be 1-forms on M^5 satisfying

$$\overset{W}{\Gamma}{}_{ij} + \overset{W}{\Gamma}{}_{ji} = Ag_{ij} \quad \text{(preservation of the class } [g, A]), \tag{3.3}$$

$$\mathrm{d}\theta^i + \overset{W}{\Gamma}{}^i{}_j \wedge \theta^j = 0 \quad \text{(no torsion)}. \tag{3.4}$$

It follows that once the representative (g, A) and the coframe θ^i is chosen the above equations uniquely determine the Weyl connection 1-forms $\overset{W}{\Gamma}{}^i{}_j$.

We note that, due to condition (3.3), matrix $\overset{W}{\Gamma} = (\overset{W}{\Gamma}{}^i{}_j)$ of the Weyl connection 1-forms belongs to the 5-dimensional defining representation of the Lie algebra $\mathfrak{co}(3, 2) \subset End(5, \mathbb{R})$ of the Lie group $\mathbf{CO}(3, 2) \subset \mathbf{GL}(5, \mathbb{R})$. Consequently, the Weyl connection coefficients $\overset{W}{\Gamma}{}^i{}_{jk}$, defined by $\overset{W}{\Gamma}{}^i{}_j = \overset{W}{\Gamma}{}^i{}_{jk}\theta^k$ belong to the tensor product $\mathfrak{co}(3, 2) \otimes \mathbb{R}^5$, the vector space of dimension $(1 + 10)5 = 55$.

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Now we assume that we have an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $[g,\Upsilon,A]$ on a 5-manifold M^5. Forgetting about Υ gives the Weyl geometry as before. In particular there is the unique Weyl connection $\overset{W}{\Gamma}$ associated with $[g,\Upsilon,A]$. But the existence of a metric compatible class of tensors Υ makes this Weyl geometry more special. To analyse it we introduce a new connection, which will be respecting the entire structure $[g,\Upsilon,A]$. This is rather a complicated procedure which we describe below.

First we require that the new connection preserves $[g]$ and $[\Upsilon]$:

$$Dg_{ij} = -Ag_{ij} \tag{3.5}$$

$$D\Upsilon_{ijk} = -\frac{3}{2}A\Upsilon_{ijk}. \tag{3.6}$$

This does not determine the connection uniquely — to have the uniqueness we need to specify what the torsion of D is. We need some preparations to discuss it.

Definition 3.2. Let (g,Υ,A) be a representative of an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure on a 5-dimensional manifold M^5. A coframe θ^i, $i=0,1,2,3,4$, on M^5 is called *adapted* to the representative (g,Υ,A) if

$$g = g_{ij}\theta^i\theta^j = \theta^0\theta^4 - 4\theta^1\theta^3 + 3(\theta^2)^2$$

and

$$\Upsilon = \Upsilon_{ijk}\theta^i\theta^j\theta^k = 3\sqrt{3}(\theta^0\theta^2\theta^4 + 2\theta^1\theta^2\theta^3 - (\theta^2)^3 - \theta^0(\theta^3)^2 - \theta^4(\theta^1)^2).$$

Locally such a coframe always exists and is given up to a $\mathbf{GL}(2,\mathbb{R})$ transformation.

Let us now choose an adapted coframe θ^i to a representative (g,Υ,A) of $[g,\Upsilon,A]$. In this coframe equations (3.5)–(3.6) can be rewritten in terms of the connection 1-forms $\Gamma^i{}_j$ as

$$\Gamma^l{}_i g_{lj} + \Gamma^l{}_j g_{li} = Ag_{ij} \tag{3.7}$$

$$\Gamma^l{}_i\Upsilon_{ljk} + \Gamma^l{}_j\Upsilon_{ilk} + \Gamma^l{}_k\Upsilon_{ijl} = \frac{3}{2}A\Upsilon_{ijk}. \tag{3.8}$$

When we contract the first equation in indices i and j we get

$$A = \frac{2}{5}\Gamma^l{}_l = \frac{2}{5}Tr(\Gamma). \tag{3.9}$$

Inserting this into (3.8) we get

$$\Gamma^l{}_i\Upsilon_{ljk} + \Gamma^l{}_j\Upsilon_{ilk} + \Gamma^l{}_k\Upsilon_{ijl} = \frac{3}{5}\Gamma^l{}_l\Upsilon_{ijk}. \tag{3.10}$$

Comparing this with (2.4) we see that the general solution for the connection 1-forms $\Gamma^i{}_j$ are given by (2.5), i.e.

$$\Gamma = \Gamma_- E_- + \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_1 E_1,$$

where $(\Gamma_-,\Gamma_+,\Gamma_0,\Gamma_1)$ are four 1-forms on M^5 such that

$$\Gamma_1 = -\frac{1}{8}A. \tag{3.11}$$

To fix the remaining three 1-forms $(\Gamma_-,\Gamma_+,\Gamma_1)$ we introduce an operator

$$\bar{\Upsilon} : \mathfrak{co}(3,2)\otimes\mathbb{R}^5 \to S^4\mathbb{R}^5$$

defined by:

$$\bar{\Upsilon}(\overset{W}{\Gamma})_{ijkm} = \Upsilon_{l(ij}\overset{W}{\Gamma}{}^l{}_{km)} - \frac{1}{5}\overset{W}{\Gamma}{}^l{}_{l(m}\Upsilon_{ijk)},$$

and analyse its kernel $\ker\bar{\Upsilon}$.

Writing Eq. (3.10) in terms of the coefficients $\Gamma^l{}_{im} \in \mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5$ and symmetrising it over the indices $\{imjk\}$, we see that the whole $\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5$ is included in $\ker\bar{\Upsilon}$.

We use the metric to identify $\mathbb{R}^5$ with $(\mathbb{R}^5)^*$, and more generally to identify tensor spaces $\bigotimes^k(\mathbb{R}^5)^*\bigotimes^l\mathbb{R}^5$ with $\bigotimes^{(k+l)}(\mathbb{R}^5)^*$. This enables us to identify the objects with upper indices with the corresponding objects with lower indices, e.g. $T_{ijk} = g_{il}T^l{}_{jk}$. Having in mind these identifications we easily see that, due to antisymmetry in last two indices, every 3-form $T_{ijk} = T_{[ijk]}$ is included in $\ker\bar{\Upsilon}$.

Thus we have:

$$\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5 \subset \ker\bar{\Upsilon},$$

$$\bigwedge^3\mathbb{R}^5 \subset \ker\bar{\Upsilon}.$$

The following proposition can be checked by a direct calculation involving the explicit form of the $\mathfrak{gl}(2,\mathbb{R})$ representation given in (2.5), (2.6).

Proposition 3.3. *The vector space* $\ker\bar{\Upsilon}$ *has the following properties:*

$$\ker\bar{\Upsilon} = (\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5)\oplus\bigwedge^3\mathbb{R}^5$$

and

$$\dim\ker\bar{\Upsilon} = 30.$$

Now we interpret the condition $\overset{W}{\Gamma}{}^l{}_{im}\in\ker\bar{\Upsilon}$, i. e. the equation

$$\Upsilon_{l(ij}\overset{W}{\Gamma}{}^l{}_{km)} = \frac{1}{5}\overset{W}{\Gamma}{}^l{}_{l(m}\Upsilon_{ijk)}, \tag{3.12}$$

as a *restriction* on possible Weyl connections. Let us assume that we have a structure $(M^5,[g,\Upsilon,A])$ with the Weyl connection coefficients $\overset{W}{\Gamma}{}^l{}_{im}$ satisfying (3.12). The coefficients $\overset{W}{\Gamma}{}^l{}_{im}$ are written in a coframe adapted to some choice (g,Υ,A). It is easy to see, using (3.3) and contracting (3.12) over all the free indices with a vector field X^i, that the restriction on the Weyl connection (3.12) in coordinate-free language is equivalent to

$$(\overset{W}{\nabla}_X\Upsilon)(X,X,X) = -\frac{1}{2}A(X)\Upsilon(X,X,X). \tag{3.13}$$

Here $\overset{W}{\nabla}$ denotes the Weyl connection in the Koszul notation.

Definition 3.4. An irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $(M^5,[g,\Upsilon,A])$ is called *nearly integrable* iff its Weyl connection $\overset{W}{\nabla}$ associated to the class $[g,A]$ satisfies (3.13).

3.1. *Nearly integrable* $\mathbf{GL}(2,\mathbb{R})$ *structures*

A nice feature of nearly integrable structures $(M^5,[g,\Upsilon,A])$ is that they define a unique $\mathfrak{gl}(2,\mathbb{R})$-valued connection Γ. This follows from the above discussion about the kernel of $\bar{\Upsilon}$. Indeed, given a nearly integrable structure $(M^5,[g,\Upsilon,A])$ it is enough to choose a representative (g,Υ,A) and to write the Eq. (3.13) for the Weyl connection $\overset{W}{\Gamma}$ in an adapted coframe θ^i. Then the uniquely given Weyl connection coefficients $\overset{W}{\Gamma}_{ijk}$ are by definition in $\ker\bar{\Upsilon} = (\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5)\oplus\bigwedge^3\mathbb{R}^5$, which means that they uniquely split onto $\Gamma_{ijk}\in\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5$ and $\frac{1}{2}T_{ijk}\in\bigwedge^3\mathbb{R}^5$. Thus, for all nearly integrable structures $(M^5,[g,\Upsilon,A])$, in a coframe adapted to (g,Υ,A), we have

$$\overset{W}{\Gamma}_{ijk} = \Gamma_{ijk}+\frac{1}{2}T_{ijk}, \tag{3.14}$$

and both $\Gamma_{ijk}\in\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5$ and $T_{ijk}\in\bigwedge^3\mathbb{R}^5$ are uniquely determined in terms of $\overset{W}{\Gamma}_{ijk}$. Now we rewrite the torsion-free condition (3.4) for the Weyl connection in the form

$$\mathrm{d}\theta^i+\Gamma^i{}_j\wedge\theta^j = \frac{1}{2}T^i_{jk}\theta^j\wedge\theta^k. \tag{3.15}$$

It can be interpreted as follows: The nearly integrable structure $(M^5,[g,\Upsilon,A])$, via (3.14), uniquely determines the $\mathfrak{gl}(2,\mathbb{R})$-valued connection Γ_{ijk} which respects the structure $[g,\Upsilon,A]$ due to (3.5), (3.6), and has totally skew symmetric torsion T_{ijk} due to (3.15). We summarise this part of our considerations in the following

Proposition 3.5. *Every nearly integrable* $\mathbf{GL}(2,\mathbb{R})$ *structure* $(M^5,[g,\Upsilon,A])$ *defines a unique* $\mathfrak{gl}(2,\mathbb{R})$*-valued connection which has totally skew symmetric torsion.*

Also the converse is true:

Proposition 3.6. *Let* $(M^5,[g,\Upsilon,A])$ *be an irreducible* $\mathbf{GL}(2,\mathbb{R})$ *structure and* $\overset{W}{\Gamma}_{ijk}$ *be the Weyl connection coefficients associated, in an adapted coframe* θ^i*, with the Weyl structure* $[g,A]$*. Assume that the Weyl structure* $[g,A]$ *admits a split*

$$\overset{W}{\Gamma}_{ijk} = \Gamma_{ijk}+\frac{1}{2}T_{ijk},$$

in which $\Gamma_{ijk}\in\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5$ *and* $T_{ijk}\in\bigwedge^3\mathbb{R}^5$*. Then* $[g,\Upsilon,A]$ *is nearly integrable, the split is unique and* $\Gamma_{ij}=\Gamma_{ijk}\theta^k$ *is a* $\mathfrak{gl}(2,\mathbb{R})$*-valued connection with the totally skew symmetric torsion* $\Theta_i=\frac{1}{2}T_{ijk}\theta^j\wedge\theta^k$*.*

Definition 3.7. The unique $\mathfrak{gl}(2,\mathbb{R})$-valued connection with totally skew symmetric torsion naturally associated with a nearly integrable structure $(M^5,[g,\Upsilon,A])$ is called the *characteristic connection*.

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3.2. Arbitrary $\mathbf{GL}(2,\mathbb{R})$ structures

So far we have only been able to introduce a unique $\mathfrak{gl}(2,\mathbb{R})$-valued connection for a nearly integrable $(M^5,[g,\Upsilon,A])$. Nevertheless such a connection can be always introduced. To see this consider a $\mathbf{GL}(2,\mathbb{R})$-invariant conformal pairing in $\mathfrak{co}(3,2)\otimes\mathbb{R}^5$ given by

$$(\overset{W}{\Gamma},\overset{W}{\Gamma}{}')=g^{il}g^{jm}g^{kp}\overset{W}{\Gamma}_{ijk}\overset{W}{\Gamma}{}'_{lmp},$$

where $\overset{W}{\Gamma},\overset{W}{\Gamma}{}'\in\mathfrak{co}(3,2)\otimes\mathbb{R}^5$. We use the orthogonal complement of $\ker\bar{\Upsilon}\subset\mathfrak{co}(3,2)\otimes\mathbb{R}^5$ with respect to this pairing:

$$\ker\bar{\Upsilon}^{\perp}=\{\overset{W}{\Gamma}\in\mathfrak{co}(3,2)\otimes\mathbb{R}^5 \text{ s.t } (\ker\bar{\Upsilon},\overset{W}{\Gamma})=0\}.$$

This vector space is 30-dimensional. It contains a 5-dimensional subspace spanned by $g_{ij}A_m$, which is related to the $\mathbb{R}$ factor in the split $\mathfrak{gl}(2,\mathbb{R})=\mathbb{R}\oplus\mathfrak{sl}(2,\mathbb{R})\subset\mathfrak{co}(3,2)=\mathbb{R}\oplus\mathfrak{so}(3,2)$. Thus it is reasonable to consider the intersection, say V_{25}, of this 30-dimensional space with $\mathfrak{so}(3,2)\otimes\mathbb{R}^5$. This 25-dimensional space

$$V_{25}=\ker\bar{\Upsilon}^{\perp}\cap(\mathfrak{so}(3,2)\otimes\mathbb{R}^5)$$

has, in turn, zero intersection with $(\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5)\oplus\bigwedge^3\mathbb{R}^5$ and provides the $\mathbf{GL}(2,\mathbb{R})$ invariant decomposition of $\mathfrak{co}(3,2)\otimes\mathbb{R}^5$:

$$\mathfrak{co}(3,2)\otimes\mathbb{R}^5=(\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5)\oplus\bigwedge^3\mathbb{R}^5\oplus V_{25}.$$

Therefore, if we choose a coframe adapted to a representative (g,Υ,A) we can uniquely decompose the Weyl connection coefficients $\overset{W}{\Gamma}_{ijk}\in\mathfrak{co}(3,2)\otimes\mathbb{R}^5$ of our arbitrary $\mathbf{GL}(2,\mathbb{R})$ structure according to

$$\overset{W}{\Gamma}_{ijk}=\Gamma_{ijk}+\frac{1}{2}B_{ijk}.$$

Now $\Gamma_{ijk}\in\mathfrak{gl}(2,\mathbb{R})\otimes\mathbb{R}^5$, and they are interpreted as new connection coefficients; the tensor B_{ijk} belongs to $\bigwedge^3\mathbb{R}^5\oplus V_{25}$ and its antisymmetrisation $T_{ijk}=B_{i[jk]}$ is now interpreted as the torsion of Γ. Thus, every $\mathbf{GL}(2,\mathbb{R})$ structure $(M^5,[g,\Upsilon,A])$ uniquely defines a $\mathfrak{gl}(2,\mathbb{R})$-valued connection with torsion in $\bigwedge^3\mathbb{R}^5\oplus V_{25}$. The torsion is not totally skew anymore. Space V_{25} further decomposes onto the $\mathbf{GL}(2,\mathbb{R})$-irreducible components according to $V_{25}=\bigodot_5\oplus\bigodot_9\oplus\bigodot_{11}$. The $\mathbf{GL}(2,\mathbb{R})$ structures equipped with the unique $\mathfrak{gl}(2,\mathbb{R})$ connection which has torsion in V_{25} find application in the theory of integrable equations of the hydrodynamic type [14].

4. GL(2, $\mathbb{R}$) bundle

First, we describe an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $[g,\Upsilon,A]$ on M^5 in the language of principal bundles.

Every irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $[g,\Upsilon,A]$ on a 5-manifold M^5 defines the 9-dimensional bundle $\mathbf{GL}(2,\mathbb{R})\to P\to M^5$, the $\mathbf{GL}(2,\mathbb{R})$ reduction of the bundle of linear frames $\mathbf{GL}(5,\mathbb{R})\to F(M^5)\to M^5$. If $[g,\Upsilon,A]$ is equipped with a $\mathfrak{gl}(2,\mathbb{R})$ connection Γ, then the structural equations on M^5 read

$$\mathrm{d}\omega^i+\Gamma^i{}_j\wedge\omega^j=\frac{1}{2}T^i{}_{jk}\omega^j\wedge\omega^k,$$
$$\mathrm{d}\Gamma^i{}_j+\Gamma^i{}_k\wedge\Gamma^k{}_j=\frac{1}{2}R^i{}_{jkl}\omega^k\wedge\omega^l.$$

Here (ω^i) is an adapted coframe and $\Gamma=(\Gamma^i{}_j)$ is written in the representation (2.5). We lift these structural equations to P obtaining:

$$\mathrm{d}\theta^0=4(\Gamma_1+\Gamma_0)\wedge\theta^0-4\Gamma_+\wedge\theta^1+\frac{1}{2}T^0{}_{ij}\theta^i\wedge\theta^j,$$
$$\mathrm{d}\theta^1=-\Gamma_-\wedge\theta^0+(4\Gamma_1+2\Gamma_0)\wedge\theta^1-3\Gamma_+\wedge\theta^2+\frac{1}{2}T^1{}_{ij}\theta^i\wedge\theta^j,$$
$$\mathrm{d}\theta^2=-2\Gamma_-\wedge\theta^1+4\Gamma_1\wedge\theta^2-2\Gamma_+\wedge\theta^3+\frac{1}{2}T^2{}_{ij}\theta^i\wedge\theta^j,$$
$$\mathrm{d}\theta^3=-3\Gamma_-\wedge\theta^2+(4\Gamma_1-2\Gamma_0)\wedge\theta^3-\Gamma_+\wedge\theta^4+\frac{1}{2}T^3{}_{ij}\theta^i\wedge\theta^j,$$
$$\mathrm{d}\theta^4=-4\Gamma_-\wedge\theta^3+4(\Gamma_1-\Gamma_0)\wedge\theta^4+\frac{1}{2}T^4{}_{ij}\theta^i\wedge\theta^j, \tag{4.1}$$

$$\mathrm{d}\Gamma_+ = 2\Gamma_0 \wedge \Gamma_+ + \frac{1}{2}R_{+ij}\theta^i \wedge \theta^j,$$
$$\mathrm{d}\Gamma_- = -2\Gamma_0 \wedge \Gamma_- + \frac{1}{2}R_{-ij}\theta^i \wedge \theta^j,$$
$$\mathrm{d}\Gamma_0 = \Gamma_+ \wedge \Gamma_- + \frac{1}{2}R_{0ij}\theta^i \wedge \theta^j,$$
$$\mathrm{d}\Gamma_1 = \frac{1}{2}R_{1ij}\theta^i \wedge \theta^j,$$

with the forms θ^i being the components of the canonical $\mathbb{R}^5$-valued form θ on P, c.f. [15]. In a coordinate system (x, a) on P, $x \in M^5$, $a \in \mathbf{GL}(2,\mathbb{R})$, which is compatible with the local trivialisation $P \cong M^5 \times \mathbf{GL}(2,\mathbb{R})$ they are given by

$$\theta^i(x, a) = (a^{-1})^i{}_j\omega^j(x).$$

The connection forms $(\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ are defined in terms of (2.6) via

$$\Gamma_-(E_-)^i{}_j + \Gamma_+(E_+)^i{}_j + \Gamma_0(E_0)^i{}_j + \Gamma_1(E_1)^i{}_j = (a^{-1})^i{}_k\Gamma^k{}_l(x)a^l{}_j + (a^{-1})^i{}_k\mathrm{d}a^k{}_j.$$

Note that $(\theta^1, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ is a coframe on P and the class of 1-forms $[A]$ lifts to a 1-form $\tilde{A} = -8\Gamma_1$. By some abuse of notation T and R denote torsion and curvature both on M^5 and on P.

Second, we change the point of view. Suppose that we are given a nine dimensional manifold P equipped with a coframe of nine 1-forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ on it. Suppose that these linearly independent forms, together with some functions $T^i{}_{jk}$, $R^l{}_{ijk}$, satisfy the system (4.1) on P. What we can say about such a 9-dimensional manifold P?

To answer this question consider a distribution $\mathfrak{h}$ on P which annihilates the forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$:

$$\mathfrak{h} = \{X \in \mathrm{T}P \text{ s.t. } X \lrcorner \theta^i = 0,\ i = 0, 1, 2, 3, 4\}.$$

Then the first five equations of the system (4.1) guarantee that the forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ satisfy the Fröbenius condition,

$$\mathrm{d}\theta^i \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0, \quad \forall\, i = 0, 1, 2, 3, 4$$

and that, in turn, the distribution $\mathfrak{h}$ is integrable. Thus manifold P is foliated by 4-dimensional leaves tangent to the distribution $\mathfrak{h}$.

Now on P we consider two multilinear symmetric forms. The bilinear one, defined by

$$\tilde{g} = \theta^0\theta^4 - 4\theta^1\theta^3 + 3(\theta^2)^2, \tag{4.2}$$

and the three-linear one given by

$$\tilde{\Upsilon} = 3\sqrt{3}(\theta^0\theta^2\theta^4 + 2\theta^1\theta^2\theta^3 - (\theta^2)^3 - \theta^0(\theta^3)^2 - \theta^4(\theta^1)^2). \tag{4.3}$$

Of course, since the 1-forms$(\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ are not present in the definitions (4.2) and (4.3), then $\tilde{g}$ and $\tilde{\Upsilon}$ are degenerate. For example, the signature of the bilinear form $\tilde{g}$ is $(+,+,+,-,-,0,0,0,0)$. The degenerate directions for these two forms are just the directions tangent to the leaves of the foliation generated by $\mathfrak{h}$. Let us denote by $(X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ the frame of vector fields on P dual to the 1-forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$. In particular (X_5, X_6, X_7, X_8) constitutes a basis for $\mathfrak{h}$, and we have $X_\mu \lrcorner \theta^i = 0$ for each $\mu = 5, 6, 7, 8$ and $i = 0, 1, 2, 3, 4$. Using this, and the exterior derivatives of θ^i given in the first five Eqs. (4.1), we easily find the Lie derivatives of $\tilde{g}$ and $\tilde{\Upsilon}$ along the directions tangent to the leaves of $\mathfrak{h}$. These are:

$$\mathcal{L}_{X_\mu}\tilde{g} = 8(X_\mu \lrcorner \Gamma_1)\tilde{g}, \qquad \mathcal{L}_{X_\mu}\tilde{\Upsilon} = 12(X_\mu \lrcorner \Gamma_1)\tilde{\Upsilon}, \quad \forall \mu = 5, 6, 7, 8.$$

Moreover, if we denote

$$\tilde{A} = -8\Gamma_1, \tag{4.4}$$

and we use the last of Eq. (4.1), we also find that

$$\mathcal{L}_{X_\mu}\tilde{A} = -8\mathrm{d}(X_\mu \lrcorner \Gamma_1), \quad \forall \mu = 5, 6, 7, 8.$$

This is enough to deduce that the objects $(\tilde{g}, \tilde{\Upsilon}, \tilde{A})$ descend to the 5-dimensional leaf space $M^5 = P/\mathfrak{h}$. There they define a conformal class of triples (g, Υ, A) with the transformation rules $g \to \mathrm{e}^{2\phi}g$, $\Upsilon \to \mathrm{e}^{3\phi}\Upsilon$, $A \to A - 2\mathrm{d}\phi$. Due to the fact that, when passing to the quotient $M^5 = P/\mathfrak{h}$, we reduced the degenerate directions of $\tilde{g}$ and $\tilde{\Upsilon}$ to points of M^5, the resulting descended triples (g, Υ, A) have non-degenerate g of signature $(3, 2)$ and non-degenerate Υ. It is clear that together with A they define an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure on M^5: a section $s: M^5 \to P$ is an adapted coframe on M^5, the triple $(s^*\tilde{g}, s^*\tilde{\Upsilon}, s^*\tilde{A})$ is a representative of the structure, the forms $s^*\Gamma_-$, $s^*\Gamma_+$, $s^*\Gamma_0$, $s^*\Gamma_1$ are $\mathfrak{gl}(2,\mathbb{R})$ connection 1-forms on M^5 and s^*T, s^*R are torsion and curvature of this connection, respectively. We have the following

Proposition 4.1. *Every* 9*-dimensional manifold* P *equipped with nine* 1*-forms* $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ *which*

- *are linearly independent at every point of* P*,*
- *satisfy system* (4.1) *with some functions* $T^i{}_{jk}$*,* $R^i{}_{jkl}$ *on* P*,*

is foliated by 4-dimensional leaves over a 5-dimensional space M^5, which is the base for the fibration $P \to M^5$. The manifold M^5 is equipped with a natural irreducible **GL**$(2,\mathbb{R})$ *structure $[g,\Upsilon,A]$ and a $\mathfrak{gl}(2,\mathbb{R})$ connection compatible with it. The torsion and the curvature of this connection is given by $T^i{}_{jk}$ and $R^i{}_{jkl}$.*

5. Torsion and curvature of characteristic connection

5.1. Torsion

Let $(M^5,[g,\Upsilon,A])$ be a nearly integrable **GL**$(2,\mathbb{R})$ structure and let Γ be its characteristic connection. Then the **GL**$(2,\mathbb{R})$ invariant information about $(M^5,[g,\Upsilon,A])$ is encoded in its totally skew symmetric torsion $\Theta_i = \frac{1}{2}T_{ijk}\theta^i\wedge\theta^k$ and its curvature

$$\Omega_{ij} = \frac{1}{2}R_{ijkl}\theta^k\wedge\theta^l = \mathrm{d}\Gamma_{ij} + \Gamma_{ik}\wedge\Gamma^k{}_j.$$

The spaces $\bigwedge^3\mathbb{R}^5$ and $\mathfrak{gl}(2,\mathbb{R})\otimes\bigwedge^2\mathbb{R}^5$ are reducible under the action of **GL**$(2,\mathbb{R})$. Their decompositions into the **GL**$(2,\mathbb{R})$ irreducible components may be used to classify the torsion types, in the case of $\bigwedge^3\mathbb{R}^5$, and the curvature types, in the case of $\mathfrak{gl}(2,\mathbb{R})\otimes\bigwedge^2\mathbb{R}^5$. In particular, to decompose $\bigwedge^3\mathbb{R}^5$ we use the Hodge star operation associated with one of the metrics g from the class $[g,\Upsilon,A]$. This identifies $\bigwedge^3\mathbb{R}^5$ with $\bigwedge^2\mathbb{R}^5$. The **GL**$(2,\mathbb{R})$ invariant decomposition of $\bigwedge^3\mathbb{R}^5$ is then transformed to the decomposition of $\bigwedge^2\mathbb{R}^5$. This is achieved in terms of the operator

$$Y_{ijkl} = 4\Upsilon_{ijm}\Upsilon_{klp}g^{mp}.$$

This, viewed as an endomorphism of $\bigotimes^2\mathbb{R}^5$ given by

$$Y(w)_{ik} = g^{mj}g^{pl}Y_{ijkl}w_{mp},$$

has the following eigenspaces:

$$\bigodot_1 = \left\{S\in\bigotimes^2\mathbb{R}^5 \mid Y(S) = 14\cdot S\right\} = \{S=\lambda\cdot g,\ \lambda\in\mathbb{R}\},$$
$$\bigwedge_3 = \left\{F\in\bigotimes^2\mathbb{R}^5 \mid Y(F) = 7\cdot F\right\} = \mathfrak{sl}(2,\mathbb{R}),$$
$$\bigodot_5 = \left\{S\in\bigotimes^2\mathbb{R}^5 \mid Y(S) = -3\cdot S\right\},$$
$$\bigwedge_7 = \left\{F\in\bigotimes^2\mathbb{R}^5 \mid Y(F) = -8\cdot F\right\},$$
$$\bigodot_9 = \left\{S\in\bigotimes^2\mathbb{R}^5 \mid Y(S) = 4\cdot S\right\}.$$

Here the index k in $\bigodot_k$ or $\bigwedge_k$ denotes the dimension of the eigenspace.

The decomposition

$$\bigotimes^2\mathbb{R}^5 = \bigodot_1\oplus\bigodot_5\oplus\bigodot_9\oplus\bigwedge_3\oplus\bigwedge_7 \tag{5.1}$$

is **GL**$(2,\mathbb{R})$ invariant. All the components in this decomposition are **GL**$(2,\mathbb{R})$-irreducible. We have the following

Proposition 5.1. *Under the action of* **GL**$(2,\mathbb{R})$ *the irreducible components of $\bigwedge^3\mathbb{R}^5 = *\bigwedge^2\mathbb{R}^5$ are*

$$\bigwedge^3\mathbb{R}^5 = \bigwedge_3\oplus\bigwedge_7.$$

At this stage an interesting question arises: Can we give examples of nearly integrable **GL**$(2,\mathbb{R})$ structures whose characteristic connection has torsion of a 'pure' type $T_{ijk}\in\bigwedge_3$?

In Section 6 we give an affirmative answer to this question. Here we only state a useful

Lemma 5.2. *The 3-dimensional vector space $\bigwedge_3$, when expressed in terms of an adapted coframe θ^i of Definition 3.2 is*

$$\bigwedge_3 = \mathrm{Span}_{\mathbb{R}}\left\{\theta^0\wedge\theta^3 - 3\theta^1\wedge\theta^2,\ \theta^0\wedge\theta^4 - 2\theta^1\wedge\theta^3,\ \theta^1\wedge\theta^4 - 3\theta^2\wedge\theta^3\right\}.$$

Similarly, in an adapted coframe θ^i, the Hodge dual $*\bigwedge_3$ of $\bigwedge_3$ is

$$*\bigwedge_3 = \mathrm{Span}_{\mathbb{R}}\left\{-\theta^0\wedge\theta^1\wedge\theta^4 + 2\theta^0\wedge\theta^2\wedge\theta^3, -\theta^0\wedge\theta^2\wedge\theta^4 + 8\theta^1\wedge\theta^2\wedge\theta^3, -\theta^0\wedge\theta^3\wedge\theta^4 + 2\theta^1\wedge\theta^2\wedge\theta^4\right\}.$$

In particular, torsion $T^i{}_{jk}$ of the characteristic connection Γ in system (3.15) is of pure type in $\bigwedge_3$ if and only if, in an adapted coframe θ^i, we have $g_{il}T^l{}_{jk} = T_{[ijk]}$, and its corresponding 3-form $T = \frac{1}{6}g_{il}T^l{}_{jk}\theta^i\wedge\theta^j\wedge\theta^k\in *\bigwedge_3$.

5.2. Curvature

Now we turn to analysis of curvature. The curvature tensor $R^i{}_{jkl}$ of a characteristic connection[3] defines the following objects:

$$\begin{aligned} R_{ij} &= R^k{}_{ikj} && \text{the Ricci tensor,}\\ R &= R_{ij}g^{ij} && \text{the Ricci scalar,}\\ R^i_{\rm v} &= \Upsilon^{ijk}R_{jk} && \text{the Ricci } \textit{vector},\\ (\mathrm{d}A)_{ij} &= \frac{2}{5}R^k{}_{kij} && \text{the Maxwell 2-form.} \end{aligned}$$

The Ricci tensor belongs to the space $\bigotimes^2 \mathbb{R}^5$ and decomposes according to (5.1). The Ricci symmetric tensor reads

$$R_{(ij)} = \frac{1}{5}Rg_{ij} + \frac{2}{7}R^k_{\rm v}\Upsilon_{ijk} + R^{(9)}_{ij}, \tag{5.2}$$

where $\frac{1}{5}Rg_{ij}$ is its $\bigodot_1$ part, $\frac{2}{7}R^k_{\rm v}\Upsilon_{ijk}$ is its $\bigodot_5$ part and $R^{(9)}_{ij}$ is its $\bigodot_9$ part defined by (5.2). The antisymmetric Ricci tensor decomposes into

$$R_{[ij]} = R^{(3)}_{ij} + R^{(7)}_{ij}$$

with the respective $\bigwedge_3$ and $\bigwedge_7$ components given by

$$R^{(3)}_{ij} = \frac{8}{15}R_{[ij]} + \frac{1}{15}Y(R_{[\,]})_{ij},$$
$$R^{(7)}_{ij} = \frac{7}{15}R_{[ij]} - \frac{1}{15}Y(R_{[\,]})_{ij}.$$

Here $Y(R_{[\,]})$ denotes the value of the operator Y on $R_{[ij]}$. Likewise, for the Maxwell form we have

$$(\mathrm{d}A)_{ij} = \mathrm{d}A^{(3)}_{ij} + \mathrm{d}A^{(7)}_{ij}$$

and

$$\mathrm{d}A^{(3)}_{ij} = \frac{8}{15}(\mathrm{d}A)_{ij} + \frac{1}{15}Y(\mathrm{d}A)_{ij},$$
$$\mathrm{d}A^{(7)}_{ij} = \frac{7}{15}(\mathrm{d}A)_{ij} - \frac{1}{15}Y(\mathrm{d}A)_{ij}.$$

The Ricci tensor and and the Maxwell 2-form have $25 + 10 = 35$ coefficients out of total number of 40 coefficients of the curvature. Since, c.f. [16],

$$\mathfrak{gl}(2,\mathbb{R}) \otimes \bigwedge^2 \mathbb{R}^5 = \bigodot_1 \oplus 2\bigwedge_3 \oplus 2\bigodot_5 \oplus 2\bigwedge_7 \oplus \bigodot_9,$$

the remaining 5 parameters are related to the coefficients of a vector field K^m, which is independent of the Ricci tensor. It is defined in terms of the totally skew symmetric part of the curvature. Using the volume form η^{ijklm}, we have

$$K^m = R_{ijkl}\eta^{ijklm},$$

and the so defined K^m yields the missing five components of the curvature. Thus we have the following

Proposition 5.3. *The irreducible components of the curvature R_{ijkl} of a characteristic connection are given by*

$$R, \quad R^i_{\rm v}, \quad R^{(9)}_{ij}, \quad R^{(3)}_{ij}, \quad R^{(7)}_{ij}, \quad \mathrm{d}A^{(3)}_{ij}, \quad \mathrm{d}A^{(7)}_{ij}, \quad K^i.$$

5.3. Curvature of characteristic connection with torsion of type $\bigwedge_3$

It is interesting to ask what is the decomposition of the curvature if the characteristic connection has torsion in the three-dimensional representation $\bigwedge_3$. It appears that it has a very special algebraic form. Its properties are summarised in:

Theorem 5.4. *Let Γ be a characteristic connection with torsion in $\bigwedge_3$. Then*

- *The Ricci tensor component $R^{(9)}_{ij} = 0$, which means that*

$$R_{(ij)} = \frac{1}{5}Rg_{ij} + \frac{2}{7}R^k_{\rm v}\Upsilon_{ijk}.$$

- *The skew symmetric Ricci tensor and the Maxwell 2-form are related by*

$$\mathrm{d}A^{(3)} = 4R^{(3)}, \qquad \mathrm{d}A^{(7)} = \frac{2}{3}R^{(7)}.$$

[3] Results of this section are also valid for an arbitrary $\mathfrak{gl}(2,\mathbb{R})$ connection.

- *The Ricci vector $R_{\rm v}$ is fully determined by T:*

$$R^i_{\rm v} = (40)^2 (*T)_{jk} (*T)_{lm} g^{kl} \Upsilon^{jmi}.$$

Thus, the curvature is fully described by tensors T, $\mathrm{d}A$, K and the scalar R.

Theorem 5.4 is implied by the following more detailed result.

Theorem 5.5. *Let M^5 be a nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometry such that torsion of the characteristic connection belongs to $\bigwedge_3$. Let $\mathbf{GL}(2,\mathbb{R}) \to P \to M^5$ be the reduction of the frame bundle defined by the geometry. Then:*

(1) *First structural equations for the characteristic connection on P are the following:*

$$\begin{aligned}
\mathrm{d}\theta^0 &= 4(\Gamma_1+\Gamma_0)\wedge\theta^0 - 4\Gamma_+\wedge\theta^1 - \frac{1}{3}t_1\theta^0\wedge\theta^1 - \frac{1}{3}t_2\theta^0\wedge\theta^2 - t_3\theta^0\wedge\theta^3 + 2t_3\theta^1\wedge\theta^2,\\
\mathrm{d}\theta^1 &= -\Gamma_-\wedge\theta^0 + (4\Gamma_1+2\Gamma_0)\wedge\theta^1 - 3\Gamma_+\wedge\theta^2 - \frac{1}{6}t_1\theta^0\wedge\theta^2 - \frac{1}{4}t_3\theta^0\wedge\theta^4 - \frac{2}{3}t_2\theta^1\wedge\theta^2,\\
\mathrm{d}\theta^2 &= -2\Gamma_-\wedge\theta^1 + 4\Gamma_1\wedge\theta^2 - 2\Gamma_+\wedge\theta^3 - \frac{1}{9}t_1\theta^0\wedge\theta^3 + \frac{1}{18}t_2\theta^0\wedge\theta^4 - \frac{4}{9}t_2\theta^1\wedge\theta^3 - \frac{1}{3}t_3\theta^1\wedge\theta^4,\\
\mathrm{d}\theta^3 &= -3\Gamma_-\wedge\theta^2 + (4\Gamma_1-2\Gamma_0)\wedge\theta^3 - \Gamma_+\wedge\theta^4 + \frac{1}{12}t_1\theta^0\wedge\theta^4 - \frac{2}{3}t_2\theta^2\wedge\theta^3 - \frac{1}{2}t_3\theta^2\wedge\theta^4,\\
\mathrm{d}\theta^4 &= -4\Gamma_-\wedge\theta^3 + 4(\Gamma_1-\Gamma_0)\wedge\theta^4 - \frac{1}{3}t_1\theta^1\wedge\theta^4 + \frac{2}{3}t_1\theta^2\wedge\theta^3 - \frac{1}{3}t_2\theta^2\wedge\theta^4 - t_3\theta^3\wedge\theta^4,
\end{aligned}$$

where t_1, t_2 and t_3 are coefficients of the torsion.

(2) *Second structural equations on P are the following:*

$$\begin{aligned}
\mathrm{d}\Gamma_+ \;=\; & 2\Gamma_0\wedge\Gamma_+ + \left(\frac{1}{6}b_2 - \frac{1}{81}t_1^2 + \frac{5}{3}f_5\right)\theta^0\wedge\theta^1 + \left(-\frac{2}{81}t_1t_2 - \frac{10}{3}f_4 + \frac{5}{12}b_3\right)\theta^0\wedge\theta^2\\
&+ \left(-\frac{1}{243}t_2^2 - \frac{1}{162}t_1t_3 + \frac{10}{3}f_3 - \frac{1}{30}R + b_4 - \frac{1}{4}a_2\right)\theta^0\wedge\theta^3\\
&+ \left(\frac{1}{54}t_2t_3 - \frac{1}{8}a_3 - \frac{5}{3}f_2 + \frac{1}{12}b_5\right)\theta^0\wedge\theta^4\\
&+ \left(-\frac{1}{27}t_2^2 - \frac{1}{18}t_1t_3 + \frac{1}{10}R + 2b_4 + \frac{3}{4}a_2\right)\theta^1\wedge\theta^2\\
&+ \left(-\frac{1}{9}t_2t_3 + \frac{1}{4}a_3 + \frac{2}{3}b_5\right)\theta^1\wedge\theta^3 + \left(\frac{1}{18}t_3^2 + \frac{5}{3}f_1 + \frac{1}{6}b_6\right)\theta^1\wedge\theta^4\\
&+ \left(-\frac{5}{18}t_3^2 - \frac{10}{3}f_1 + \frac{1}{3}b_6\right)\theta^2\wedge\theta^3 + \frac{1}{4}b_7\theta^2\wedge\theta^4,\\
\mathrm{d}\Gamma_- \;=\; & -2\Gamma_0\wedge\Gamma_- + \frac{1}{4}b_1\theta^0\wedge\theta^2 + \left(\frac{1}{6}b_2 - \frac{1}{162}t_1^2 - \frac{5}{3}f_5\right)\theta^0\wedge\theta^3\\
&+ \left(-\frac{1}{162}t_1t_2 + \frac{5}{3}f_4 + \frac{1}{12}b_3 - \frac{1}{8}a_1\right)\theta^0\wedge\theta^4\\
&+ \left(\frac{5}{162}t_1^2 + \frac{1}{3}b_2 + \frac{10}{3}f_5\right)\theta^1\wedge\theta^2 + \left(\frac{1}{27}t_1t_2 + \frac{2}{3}b_3 + \frac{1}{4}a_1\right)\theta^1\wedge\theta^3\\
&+ \left(b_4 - \frac{1}{4}a_2 + \frac{1}{162}t_1t_3 + \frac{1}{243}t_2^2 - \frac{10}{3}f_3 + \frac{1}{30}R\right)\theta^1\wedge\theta^4\\
&+ \left(\frac{1}{27}t_2^2 + \frac{1}{18}t_1t_3 - \frac{1}{10}R + 2b_4 + \frac{3}{4}a_2\right)\theta^2\wedge\theta^3\\
&+ \left(\frac{2}{27}t_2t_3 + \frac{10}{3}f_2 + \frac{5}{12}b_5\right)\theta^2\wedge\theta^4 + \left(\frac{1}{9}t_3^2 - \frac{5}{3}f_1 + \frac{1}{6}b_6\right)\theta^3\wedge\theta^4\\
\mathrm{d}\Gamma_0 \;=\; & \Gamma_+\wedge\Gamma_- - \frac{1}{4}b_1\theta^0\wedge\theta^1 + \left(-\frac{1}{6}b_2 - \frac{1}{162}t_1^2 + \frac{5}{6}f_5\right)\theta^0\wedge\theta^2 + \left(-\frac{1}{54}t_1t_2 - \frac{1}{12}b_3 + \frac{1}{8}a_1\right)\theta^0\wedge\theta^3\\
&- \left(\frac{1}{81}t_1t_3 + \frac{2}{243}t_2^2 + \frac{5}{6}f_3 + \frac{1}{60}R\right)\theta^0\wedge\theta^4 + \left(\frac{1}{162}t_1t_2 - \frac{20}{3}f_4 - \frac{1}{6}b_3 - \frac{3}{8}a_1\right)\theta^1\wedge\theta^2\\
&+ \left(-\frac{1}{81}t_1t_3 - \frac{2}{243}t_2^2 + \frac{20}{3}f_3 + \frac{1}{30}R\right)\theta^1\wedge\theta^3 + \left(-\frac{1}{18}t_2t_3 - \frac{1}{8}a_3 + \frac{1}{12}b_5\right)\theta^1\wedge\theta^4\\
&+ \left(\frac{1}{54}t_2t_3 + \frac{3}{8}a_3 - \frac{20}{3}f_2 + \frac{1}{6}b_5\right)\theta^2\wedge\theta^3 + \left(-\frac{1}{18}t_3^2 + \frac{5}{6}f_1 + \frac{1}{6}b_6\right)\theta^2\wedge\theta^4 + \frac{1}{4}b_7\theta^3\wedge\theta^4,
\end{aligned}$$

$$\mathrm{d}\Gamma_1 = -\frac{1}{8}b_1\theta^0\wedge\theta^1 - \frac{1}{8}b_2\theta^0\wedge\theta^2 - \frac{1}{8}(b_3+a_1)\theta^0\wedge\theta^3 - \frac{1}{8}(b_4+a_2)\theta^0\wedge\theta^4 + \left(\frac{3}{8}a_1 - \frac{1}{4}b_3\right)\theta^1\wedge\theta^2 + \left(\frac{1}{4}a_2 - b_4\right)\theta^1\wedge\theta^3 - \frac{1}{8}(a_3+b_5)\theta^1\wedge\theta^4 + \left(\frac{3}{8}a_3 - \frac{1}{4}b_5\right)\theta^2\wedge\theta^3 - \frac{1}{8}b_6\theta^2\wedge\theta^4 - \frac{1}{8}b_7\theta^3\wedge\theta^4.$$

(3) *Functions $t_1, t_2, t_3, a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, b_7, f_1, f_2, f_3, f_4, f_5$ and R above have the following properties.*
 (a) *R is the Ricci scalar of the connection.*
 (b) *The irreducible component $\mathrm{d}A^{(3)}$ is given by*

$$\mathrm{d}A^{(3)} = \begin{pmatrix} 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & -3a_1 & -2a_2 & a_3 \\ 0 & 3a_1 & 0 & -3a_3 & 0 \\ -a_1 & 2a_2 & 3a_3 & 0 & 0 \\ -a_2 & -a_3 & 0 & 0 & 0 \end{pmatrix}.$$

 (c) *The irreducible component $\mathrm{d}A^{(7)}$ is given by*

$$\mathrm{d}A^{(7)} = \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 \\ -b_1 & 0 & 2b_3 & 8b_4 & b_5 \\ -b_2 & -2b_3 & 0 & 2b_5 & b_6 \\ -b_3 & -8b_4 & -2b_5 & 0 & b_7 \\ -b_4 & -b_5 & -b_6 & -b_7 & 0 \end{pmatrix}.$$

 (d) *The irreducible component K is given by*

$$K = \frac{\sqrt{3}}{3}\begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \end{pmatrix}^T.$$

 (e) *The Ricci vector is given by*

$$R^i_v = \frac{7}{6}\sqrt{3}\left(t_3^2,\ -\frac{1}{3}t_2t_3,\ \frac{1}{9}t_1t_3 + \frac{2}{27}t_2^2,\ -\frac{1}{9}t_1t_2,\ \frac{1}{9}t_1^2\right).$$

(4) *All the components of the curvature but R are determined by the differentials of torsion:*

$$\mathrm{d}t_1 = 2t_2\Gamma_- - 2t_1\Gamma_0 - 4t_1\Gamma_1 + \frac{3}{2}b_1\theta^0 + \left(2b_2 - \frac{4}{27}t_1^2 + 20f_5\right)\theta^1 + \left(-\frac{4}{9}t_1t_2 - 60f_4 + 3b_3 - \frac{9}{2}a_1\right)\theta^2 + \left(-\frac{4}{9}t_1t_3 - \frac{8}{27}t_2^2 + 60f_3 + 6b_4 - 9a_2\right)\theta^3 + \left(-\frac{4}{9}t_2t_3 - \frac{9}{2}a_3 - 20f_2 + \frac{1}{2}b_5\right)\theta^4,$$

$$\mathrm{d}t_2 = 3t_3\Gamma_- + t_1\Gamma_+ - 4t_2\Gamma_1 + \left(\frac{1}{2}b_2 + \frac{2}{27}t_1^2 - 10f_5\right)\theta^0 + \left(\frac{4}{27}t_1t_2 + 20f_4 + 2b_3 + \frac{9}{2}a_1\right)\theta^1 + 9(a_2+b_4)\theta^2 + \left(-\frac{4}{9}t_2t_3 + \frac{9}{2}a_3 - 20f_2 + 2b_5\right)\theta^3 + \left(-\frac{2}{3}t_3^2 + 10f_1 + \frac{1}{2}b_6\right)\theta^4,$$

$$\mathrm{d}t_3 = 2t_3\Gamma_0 + \frac{2}{3}t_2\Gamma_+ - 4t_3\Gamma_1 + \left(\frac{4}{81}t_1t_2 + \frac{20}{3}f_4 + \frac{1}{6}b_3 - \frac{3}{2}a_1\right)\theta^0 + \left(\frac{4}{27}t_1t_3 + \frac{8}{81}t_2^2 - 20f_3 + 2b_4 - 3a_2\right)\theta^1 + \left(\frac{4}{9}t_2t_3 - \frac{3}{2}a_3 + 20f_2 + b_5\right)\theta^2 + \left(\frac{4}{9}t_3^2 - \frac{20}{3}f_1 + \frac{2}{3}b_6\right)\theta^3 + \frac{1}{2}b_7\theta^4.$$

Proof. Let indices i, j, k, l run through $\{0, 1, 2, 3, 4\}$, index A run through $\{-, 0, +, 1\}$ and μ run through $\{1, 2, 3\}$.

The structural equations for an arbitrary $\mathfrak{gl}(2,\mathbb{R})$-connection on P are given by Eqs. (4.1). Torsion T_{ijk} in $\bigwedge_3$ on P is given by (see Lemma 5.2)

$$T = \frac{1}{12}t_1(-\theta^0\wedge\theta^1\wedge\theta^4 + 2\theta^0\wedge\theta^2\wedge\theta^3) + \frac{1}{12}t_2(-\theta^0\wedge\theta^2\wedge\theta^4 + 8\theta^1\wedge\theta^2\wedge\theta^3) + \frac{1}{4}t_3(-\theta^0\wedge\theta^3\wedge\theta^4 + 2\theta^1\wedge\theta^2\wedge\theta^4),$$

and we obtain (1) by putting $T^i{}_{jk} = T_{ljk}g^{li}$ into Eqs. (4.1).

In order to obtain (2), (3) and (4) we must close first Bianchi identities, that is solve equations $\mathrm{d}^2\theta^i = 0$ with respect to torsion coefficients t_μ, curvature coefficients R_{Aij} (defined by (4.1)) and derivatives of torsion. We denote

$$\mathrm{d}t_\mu = \sum_A t_\mu{}^A\Gamma_A + \sum_i t_{\mu i}\theta^i.$$

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Next we calculate the exterior derivatives $\mathrm{d}^2\theta^i \equiv 0$ and insert in them the formulae for $\mathrm{d}t_\mu$, and the formulae for $\mathrm{d}\Gamma_A$ given by four last Eqs. (4.1). In this manner first Bianchi identities become a set of partial differential equations involving t_μ, $t_\mu^{\ A}$, $t_{\mu i}$ and R_{Akl}. These equations are linear in $t_\mu^{\ A}$ and $t_{\mu i}$, all of which we eliminate from the equations and express as functions of t_μ and R_{Aij}. However, after the elimination there are still many unsolved equations, which now are algebraic in R_{Aij} and t_μ. We solve them with respect to R_{Aij} (due to the fact that most of them are linear with respect to curvature coefficients) and we obtain that only 21 out of 40 functions R_{Aij} do not vanish. Among these 21 there are 5 which are quadratic functions of t_μ and remaining 16 are functionally independent of t_μ.

We decompose the so-obtained curvature into the irreducible components of Proposition 5.3. We notice that (i) one may choose a base $a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, b_7, f_1, f_2, f_3, f_4$ and f_5 for 16 the functionally independent curvature coefficients so that (3a)–3(d) hold, and (ii) five curvature coefficients which are quadratic functions of torsion constitute the Ricci vector as in (3e). But in this notation second structural equations become (2) and expressions for torsion coefficients are as in (4). □

Theorem 5.4 follows immediately from the structural equations of Theorem 5.5. We can also express the Ricci tensor $(Ric)^i_{\ j} = g^{ik}R_{kj}$ in terms of the endomorphisms E_-, E_0, E_+, E_1 of (2.5):

Corollary 5.6. *The Ricci tensor of a characteristic connection with torsion in $\bigwedge_3$ has the following form in any adapted coframe*

$$
\begin{aligned}
Ric = {} & \left(\frac{1}{54}t_2^2 + \frac{1}{36}t_1t_3 - \frac{1}{20}R\right)E_1 + \frac{1}{8}b_1E_-^3 + \frac{1}{108}t_1^2E_-^2 \\
& + \left(-\frac{1}{54}t_1t_2 + \frac{1}{8}a_1 - \frac{1}{2}b_3\right)E_- + \frac{5}{16}b_4E_0^3 + \left(\frac{1}{108}t_2^2 + \frac{1}{72}t_1t_3\right)E_0^2 \\
& + \left(\frac{1}{8}a_2 - \frac{17}{4}b_4\right)E_0 - \frac{1}{8}b_7E_+^3 + \frac{1}{12}t_3^2E_+^2 + \left(\frac{1}{2}b_5 - \frac{1}{8}a_3 - \frac{1}{18}t_2t_3\right)E_+ \\
& - \frac{5}{32}b_5E_0E_+E_0 + \frac{1}{8}b_6E_+E_0E_+ + \frac{1}{54}t_1t_2E_0E_- + \frac{5}{32}b_3E_0E_-E_0 + \frac{1}{8}b_2E_-E_0E_- - \frac{1}{18}t_2t_3E_0E_+.
\end{aligned}
$$

6. 5th order ODE as nearly integrable GL(2, ℝ) geometry with 'small' torsion. Main theorem

A large number of examples of nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures in dimension five is related to 5th order ODEs. This is mainly due to the following, well known,

Proposition 6.1. *An ordinary differential equation $y^{(5)} = 0$ has $\mathbf{GL}(2,\mathbb{R}) \times_{\rho_5} \mathbb{R}^5$ as its group of contact symmetries. Here $\rho_5 : \mathbf{GL}(2,\mathbb{R}) \to \mathbf{GL}(5,\mathbb{R})$ is the 5-dimensional irreducible representation of $\mathbf{GL}(2,\mathbb{R})$.*

To explain the above statement we consider a general 5th order ODE

$$y^{(5)} = F(x, y, y', y'', y^{(3)}, y^{(4)}) \tag{6.1}$$

for a real function $\mathbb{R} \ni x \mapsto y(x) \in \mathbb{R}$. Let us introduce the notation $y_1 = y'$, $y_2 = y''$, $y_3 = y^{(3)}$, $y_4 = y^{(4)}$ and $F_i = \frac{\partial F}{\partial y_i}$, $i = 1, 2, 3, 4$, $F_y = \frac{\partial F}{\partial y}$. The functions $(x, y, y_1, y_2, y_3, y_4)$ form a local coordinate system in the 4-order jet space J of curves in $\mathbb{R}^2$. Define the total derivative, which is a vector field in J

$$\mathcal{D} = \partial_x + y_1\partial_y + y_2\partial_{y_1} + y_3\partial_{y_2} + y_4\partial_{y_3} + F\partial_{y_4}. \tag{6.2}$$

With the help of $\mathcal{D}$ the derivatives are given by formulae $y_1 = \mathcal{D}y/\mathcal{D}x$, $y_2 = \mathcal{D}y_1/\mathcal{D}x$ and so on, up to $y_5 = \mathcal{D}y_4/\mathcal{D}x$.

A contact transformation of variables in a 5-order ODE is a transformation that mixes the independent variable x, the dependent variable y and the first derivative y_1 in such a way that the meaning of the first derivative is retained:

Definition 6.2. A contact transformation of variables is an invertible, sufficiently smooth transformation of the form

$$\begin{pmatrix} x \\ y \\ y_1 \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{y}_1 \end{pmatrix} = \begin{pmatrix} \bar{x}(x, y, y_1) \\ \bar{y}(x, y, y_1) \\ \bar{y}_1(x, y, y_1) \end{pmatrix} \tag{6.3}$$

satisfying the condition

$$\bar{y}_1 = \frac{\mathcal{D}\bar{y}}{\mathcal{D}\bar{x}}. \quad \text{(preservation of first derivative)}$$

The higher order derivatives are given by the iterative formula

$$y_{n+1} \mapsto \bar{y}_{n+1} = \frac{\mathcal{D}\bar{y}_n}{\mathcal{D}\bar{x}}, \quad i = 1, 2, 3, 4.$$

Let us now consider the equation $y^{(5)} = 0$. We show how the flat torsion-free 5-dimensional irreducible $\mathbf{GL}(2,\mathbb{R})$ structure is naturally generated on its space of solutions by means of the symmetry group. A solution to $y^{(5)} = 0$ is of the form

$$y(x) = c_4x^4 + 4c_3x^3 + 6c_2x^2 + 4c_1x + c_0 \tag{6.4}$$

with five integration constants c_0, c_1, c_2, c_3, c_4. Then a solution of $y^{(5)} = 0$ may be identified with a point $c = (c_0, c_1, c_2, c_3, c_4)^T$ in $\mathbb{R}^5$. A contact symmetry of $y^{(5)} = 0$ is a contact transformation of variables that transforms its solutions into solutions. The group of contact symmetries of $y^{(5)} = 0$ is generated by the following one-parameter groups of transformations on the xy-plane:

$$\begin{aligned}
&\varphi^0_t(x,\ y) = (x,\ y+t), && \varphi^1_t(x,\ y) = (x,\ y+4xt),\\
&\varphi^2_t(x,\ y) = (x,\ y+6x^2t), && \varphi^3_t(x,\ y) = (x,\ y+4x^3t),\\
&\varphi^4_t(x,\ y) = (x,\ y+x^4t), && \varphi^5_t(x,\ y) = (xe^{2t},\ ye^{4t}),\\
&\varphi^6_t(x,\ y) = (x,\ ye^{4t}), && \varphi^7_t(x,\ y) = (x+t,\ y),\\
&\varphi^8_t(x,y) = \left(\frac{x}{1+xt},\ \frac{y}{(1+xt)^4}\right)
\end{aligned}$$

and the transformation rules for y_1 are given by $\varphi^A(y_1) = \mathcal{D}(\varphi^A(y))/\mathcal{D}(\varphi^A(x))$, $A = 0,\dots,8$.

Transforming (6.4) according to the above formulae we find that $\varphi^0_t,\dots,\varphi^4_t$ are translations in the space of solutions:

$$\varphi^0_t(c) = (c_0 - t, c_1, c_2, c_3, c_4)^T, \dots, \qquad \varphi^4_t(c) = (c_0, c_1, c_2, c_3, c_4 - t)^T,$$

while transformations $\varphi^5_t,\dots,\varphi^8_t$ generate $\mathbf{GL}(2,\mathbb{R})$ and act through the 5-dimensional irreducible representation (2.6):

$$\begin{aligned}
&\varphi^5_t(c) = \exp(tE_0)c, && \varphi^6_t(c) = \exp(tE_1)c,\\
&\varphi^7_t(c) = \exp(tE_+)c, && \varphi^8_t(c) = \exp(tE_-)c.
\end{aligned}$$

Of course, $\mathbf{GL}(2,\mathbb{R})$ stabilises the origin $(0,0,0,0,0)$ in $\mathbb{R}^5$, thus the space of solutions is the homogeneous space $\mathbf{GL}(2,\mathbb{R}) \to \mathbf{GL}(2,\mathbb{R}) \times_{\rho_5} \mathbb{R}^5 \to \mathbb{R}^5$. The total space of this bundle is equipped with the Maurer–Cartan form ω_{MC} of $\mathbf{GL}(2,\mathbb{R}) \times_{\rho_5} \mathbb{R}^5$. Choosing an appropriate base in $\mathfrak{gl}(2,\mathbb{R})$ and writing explicitly the structural equations $d\omega_{MC} + \omega_{MC} \wedge \omega_{MC} = 0$ we get

$$\begin{aligned}
&d\theta^0 = 4(\Gamma_1 + \Gamma_0)\wedge\theta^0 - 4\Gamma_+\wedge\theta^1,\\
&d\theta^1 = -\Gamma_-\wedge\theta^0 + (4\Gamma_1 + 2\Gamma_0)\wedge\theta^1 - 3\Gamma_+\wedge\theta^2,\\
&d\theta^2 = -2\Gamma_-\wedge\theta^1 + 4\Gamma_1\wedge\theta^2 - 2\Gamma_+\wedge\theta^3,\\
&d\theta^3 = -3\Gamma_-\wedge\theta^2 + (4\Gamma_1 - 2\Gamma_0)\wedge\theta^3 - \Gamma_+\wedge\theta^4,\\
&d\theta^4 = -4\Gamma_-\wedge\theta^3 + 4(\Gamma_1 - \Gamma_0)\wedge\theta^4,\\
&d\Gamma_+ = 2\Gamma_0\wedge\Gamma_+,\\
&d\Gamma_- = -2\Gamma_0\wedge\Gamma_-,\\
&d\Gamma_0 = \Gamma_+\wedge\Gamma_-,\\
&d\Gamma_1 = 0,
\end{aligned}$$

which is the system (4.1) with all the torsion and curvature coefficients equal to zero. According to Proposition 4.1 it yields a flat and torsion-free irreducible $\mathbf{GL}(2,\mathbb{R})$ structure on the space of solutions of $y^{(5)} = 0$. Again, as in the case of the algebraic geometric realisation of Section 2, we learned about that from Ferapontow [13].

We now pass to a more general situation, namely to the Eq. (6.1) with a *general* F. The following questions are in order:

What shall one assume about F to be able to construct an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure on the solution space of the corresponding ODE? Is the case $F = 0$ very special, or there are other ODEs, contact nonequivalent to the $F = 0$ case, which define a $\mathbf{GL}(2,\mathbb{R})$ geometry on the solution space? If the answer is affirmative, how do we find such Fs and what can we say about the corresponding $\mathbf{GL}(2,\mathbb{R})$ structures?

The answer to these questions is given by the following

Theorem 6.3 (*Main Theorem*). *Every contact equivalence class of 5th order ODEs satisfying the Wünschmann conditions*

$$\begin{aligned}
&50\mathcal{D}^2F_4 - 75\mathcal{D}F_3 + 50F_2 - 60F_4\mathcal{D}F_4 + 30F_3F_4 + 8F_4^3 = 0,\\
&375\mathcal{D}^2F_3 - 1000\mathcal{D}F_2 + 350\mathcal{D}F_4^2 + 1250F_1 - 650F_3\mathcal{D}F_4 + 200F_3^2\\
&\quad -150F_4\mathcal{D}F_3 + 200F_2F_4 - 140F_4^2\mathcal{D}F_4 + 130F_3F_4^2 + 14F_4^4 = 0,\\
&1250\mathcal{D}^2F_2 - 6250\mathcal{D}F_1 + 1750\mathcal{D}F_3\mathcal{D}F_4 - 2750F_2\mathcal{D}F_4 - 875F_3\mathcal{D}F_3 + 1250F_2F_3 - 500F_4\mathcal{D}F_2 + 700(\mathcal{D}F_4)^2F_4\\
&\quad +1250F_1F_4 - 1050F_3F_4\mathcal{D}F_4 + 350F_3^2F_4 - 350F_4^2\mathcal{D}F_3 + 550F_2F_4^2 - 280F_4^3\mathcal{D}F_4 + 210F_3F_4^3 + 28F_4^5 + 18750F_y = 0
\end{aligned} \tag{6.5}$$

defines a nearly integrable irreducible $\mathbf{GL}(2,\mathbb{R})$ *geometry* $(M^5,[g,\Upsilon,A])$ *on the space* M^5 *of its solutions. This geometry has the characteristic connection with torsion* T *of the 'pure' type in the 3-dimensional irreducible representation* $\bigwedge_3$. *Properties of curvature and structural equations of this geometry are given by Theorems 5.4 and 5.5, with the torsion coefficients given by*

$$t_3=\frac{6(\alpha^5{}_5)^2}{5\alpha^1{}_1}F_{44},$$

$$t_2=\frac{9\alpha^5{}_5}{50(\alpha^1{}_1)^2}\left(\alpha^1{}_1(10\mathcal{D}F_{44}+3F_4F_{44})+5\alpha^1{}_0F_{44}\right),$$

$$t_1=\frac{1}{1000(\alpha^1{}_1)^3}\Big(225(\alpha^1{}_0)^2F_{44}+90\alpha^1{}_0\alpha^1{}_1(10\mathcal{D}F_{44}+3F_4F_{44})$$
$$-9(\alpha^1{}_1)^2\big(20(5\mathcal{D}F_{34}+20F_{24}-15F_{33}+3F_4\mathcal{D}F_{44}-11F_4F_{34})+F_{44}(-120\mathcal{D}F_4+340F_3+51F_4^2)\big)\Big),$$

where $(y,y_1,y_2,y_3,y_4,x,\alpha^1{}_1,\alpha^1{}_0,\alpha^5{}_5)$ *is a local coordinate system on* $\mathbf{GL}(2,\mathbb{R})\to P\to M^5$.

Before presenting the proof let us notice several facts.

The theorem guarantees that every equivalence class of ODEs satisfying conditions (6.5) has its corresponding nearly integrable $\mathbf{GL}(2,\mathbb{R})$ geometry $(M^5,[g,\Upsilon,A])$ with torsion in $\bigwedge_3$. It may happen, however, that there are contact non-equivalent classes of ODEs defining the same $\mathbf{GL}(2,\mathbb{R})$ geometries. (See also Remark 6.8).

The Wünschmann conditions, although very complicated, possess nontrivial solutions. For example the equation

$$y^{(5)}=c\Big(\frac{5y^{(3)3}(5-27cy''^2)}{9(1+cy''^2)^2}+10\frac{y''y^{(3)}y^{(4)}}{1+cy''^2}\Big),$$

where $c=\pm1$ satisfies the Wünschmann conditions and is not contact equivalent to $F=0$. Other examples are considered in Section 7.

Of course, since the geometry is constructed from an ODE determined by the choice of $F=F(x,y,y_1,y_2,y_3,y_4)$, the coefficients $a_1,\dots,a_3,b_1,\dots,b_7,R$ are expressible in terms of F and its derivatives. Given the connection of Theorem 6.3 we calculated the explicit formulae for these coefficients and obtained the following

Corollary 6.4. *A* $\mathbf{GL}(2,\mathbb{R})$ *geometry generated by a 5th order ODE satisfying Wünschmann conditions (6.5) has the following properties.*

The torsion T *vanishes iff*

$$F_{44}=0.$$

The 2-form $\mathrm{d}A^{(3)}$ *vanishes iff*

$$(\mathcal{D}F_4)_{34}-(\mathcal{D}F_3)_{44}-\frac{3}{5}(\mathcal{D}F_4)_4F_{44}-\frac{4}{5}\mathcal{D}F_4F_{444}+\frac{6}{25}F_{44}^2F_4+\frac{4}{25}F_4^2F_{444}$$
$$+\frac{3}{10}F_{34}F_{44}-\frac{1}{5}F_4F_{344}+\frac{3}{5}F_3F_{444}+F_{244}-\frac{1}{2}F_{433}=0.$$

The 2-form $\mathrm{d}A^{(7)}$ *vanishes iff*

$$F_{444}=0.$$

The Ricci vector R_{v} *is aligned with the vector* K, *i.e.* $K=uR_{\mathrm{v}}$, $u\in\mathbb{R}$, *iff*

$$(\mathcal{D}F_4)_{44}-\frac{1}{2}F_{344}-\frac{2}{5}F_4F_{444}-\frac{8}{15}F_{44}^2+7uF_{44}^2=0.$$

We skip writing the formula for the Ricci scalar since it is very complicated.

We now pass to the proof of Theorem 6.3. On doing this we will apply a variant of the Cartan method of equivalence. This will be a rather long and complicated procedure. Thus, for clarity of the presentation, we will divide the proof into three main steps, each of which will occupy its own respective Sections 6.1–6.3. First, in Section 6.1 we will prove Lemma 6.5, which assures that a class of contact equivalent 5th order ODEs is a G-structure on a 4-order jet space J. Thus, we will have a bundle $G\to J\times G\to J$, a reduction of the frame bundle $F(J)$. In the second step, in Section 6.2, we will use the Cartan method of equivalence in order to construct a submanifold $P\subset J\times G$ together with a coframe on P which fulfills the requirements of Proposition 4.1. This coframe, via Proposition 4.1, will define an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure for us and simultaneously will provide us with a $\mathfrak{gl}(2,\mathbb{R})$ connection on the space of solutions of the ODE. The obstructions for an ODE to possess this structure, Wünschmann's expressions for F, will appear automatically in the course of the construction. This part of considerations is summarised in Theorem 6.6. The $\mathbf{GL}(2,\mathbb{R})$ structure obtained in this way will turn out to be nearly integrable, but the connection constructed will differ from the characteristic one. Therefore, in Section 6.3, we will construct the characteristic connection associated with the $\mathbf{GL}(2,\mathbb{R})$ structure obtained. This will have torsion in $\bigwedge_3$. This construction is described by Lemma 6.7.

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6.1. 5th order ODE modulo contact transformations

Let us consider a general 5th order ODE (6.1). We define the following coframe

$$\begin{aligned}
\omega^0 &= \mathrm{d}y - y_1\mathrm{d}x,\\
\omega^1 &= \mathrm{d}y_1 - y_2\mathrm{d}x,\\
\omega^2 &= \mathrm{d}y_2 - y_3\mathrm{d}x,\\
\omega^3 &= \mathrm{d}y_3 - y_4\mathrm{d}x,\\
\omega^4 &= \mathrm{d}y_4 - F(x, y, y_1, y_2, y_3, y_4)\mathrm{d}x,\\
\omega_+ &= \mathrm{d}x
\end{aligned} \tag{6.6}$$

on J. We see that every solution of (6.1) is a curve $c(x) = (x, y(x), y_1(x), y_2(x), y_3(x), y_4(x)) \subset J$ and the vector field $\mathcal{D}$ on J has curves $c(x)$ as the integral curves. The 1-forms $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4)$ annihilate $\mathcal{D}$ whereas $\mathcal{D} \lrcorner \omega_+ = 1$. The 5-dimensional space M^5 of integral curves of $\mathcal{D}$ is clearly the space of solutions of (6.1) and we have a fibration $\mathbb{R} \to J \to M^5$.

Suppose now, that Eq. (6.1) undergoes a contact transformation (6.3), which brings it to $\bar{y}_5 = \bar{F}(\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4)$. Then the coframe transforms according to

$$\begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega_+\end{pmatrix} \mapsto \begin{pmatrix}\bar{\omega}^0\\ \bar{\omega}^1\\ \bar{\omega}^2\\ \bar{\omega}^3\\ \bar{\omega}^4\\ \bar{\omega}_+\end{pmatrix} = \begin{pmatrix}\alpha^0{}_0 & 0 & 0 & 0 & 0 & 0\\ \alpha^1{}_0 & \alpha^1{}_1 & 0 & 0 & 0 & 0\\ \alpha^2{}_0 & \alpha^2{}_1 & \alpha^2{}_2 & 0 & 0 & 0\\ \alpha^3{}_0 & \alpha^3{}_1 & \alpha^3{}_2 & \alpha^3{}_3 & 0 & 0\\ \alpha^4{}_0 & \alpha^4{}_1 & \alpha^4{}_2 & \alpha^4{}_3 & \alpha^4{}_4 & 0\\ \alpha^5{}_0 & \alpha^5{}_1 & 0 & 0 & 0 & \alpha^5{}_5\end{pmatrix} \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^3\\ \omega^4\\ \omega_+\end{pmatrix}. \tag{6.7}$$

Here $\alpha^i{}_j$, $i, j = 0, 1, 2, 3, 4, 5$, are real functions on J defined by the formulae (6.3). They satisfy the nondegeneracy condition

$$\alpha^0{}_0\alpha^1{}_1\alpha^2{}_2\alpha^3{}_3\alpha^4{}_4\alpha^5{}_5 \neq 0.$$

The transformed coframe encodes all the contact invariant information about the ODE. In particular, it preserves the simple ideal $(\omega^0, \ldots, \omega^4)$, from which we can recover solutions of the transformed equation. Hence we have

Lemma 6.5. *A 5th order ODE $y_5 = F(x, y, y_1, y_2, y_3, y_4)$ considered modulo contact transformations of variables is a G-structure on the 4-jet space J, such that the coframe $(\omega^0, \omega^1, \omega^2, \omega^3, \omega^4, \omega_+)$ of (6.6) belongs to it and the group G is given by the matrix in (6.7).*

6.2. $\mathbf{GL}(2, \mathbb{R})$ bundle over space of solutions

Using the Cartan method we explicitly construct a submanifold $P \subset J \times G$ and a coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$ on P satisfying Proposition 4.1. This part of the proof is divided into eight steps.

Step (1) We observe that there is a natural choice for the forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ of the coframe. Since we are going to build a $\mathbf{GL}(2, \mathbb{R})$ structure on the space of solutions, P must be a bundle over M^5, and the forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$ must annihilate vectors tangent to leaves of the projection $P \to M^5$. But on $J \times G$ there are six distinguished 1-forms given by

$$\begin{pmatrix}\theta^0\\ \theta^1\\ \theta^2\\ \theta^3\\ \theta^4\\ \theta_+\end{pmatrix} = \begin{pmatrix}\alpha^0{}_0\omega^0\\ \alpha^1{}_0\omega^0 + \alpha^1{}_1\omega^1\\ \alpha^2{}_0\omega^0 + \alpha^2{}_1\omega^1 + \alpha^2{}_2\omega^2\\ \alpha^3{}_0\omega^0 + \alpha^3{}_1\omega^1 + \alpha^3{}_2\omega^2 + \alpha^3{}_3\omega^3\\ \alpha^4{}_0\omega^0 + \alpha^4{}_1\omega^1 + \alpha^4{}_2\omega^2 + \alpha^4{}_3\omega^3 + \alpha^4{}_4\omega^4\\ \alpha^5{}_0\omega_0 + \alpha^5{}_1\omega^1 + \alpha^5{}_5\omega_+\end{pmatrix}. \tag{6.8}$$

These forms are the components of the canonical $\mathbb{R}^6$ valued 1-form on $J \times G$. Five among these forms, $\theta^0, \theta^1, \theta^2, \theta^3, \theta^4$ also annihilate vectors tangent to the projection $J \times G \to M^5$. We choose them to be the members of the sought coframe $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4, \Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1)$. Now we must construct a 9-dimensional submanifold P on which θ^i satisfy Eq. (4.1) with some linearly independent forms $\Gamma_-, \Gamma_+, \Gamma_0, \Gamma_1$.

Step (2) We calculate $\mathrm{d}\theta^0$ and get

$$\mathrm{d}\theta^0 = \left(\frac{\mathrm{d}\alpha^0{}_0}{\alpha^0{}_0} - \frac{\alpha^1{}_0}{\alpha^1{}_1\alpha^5{}_5}\theta_+\right) \wedge \theta^0 + \frac{\alpha^0{}_0}{\alpha^1{}_1\alpha^5{}_5}\theta_+ \wedge \theta^1 - \frac{\alpha^5{}_0}{\alpha^1{}_1\alpha^5{}_5}\theta^0 \wedge \theta^1.$$

For this equation to match (4.1) we define

$$\Gamma_+ = \theta_+ \tag{6.9}$$

$$4(\Gamma_1+\Gamma_0) = \frac{\mathrm{d}\alpha^0{}_0}{\alpha^0{}_0} - \frac{\alpha^1{}_0}{\alpha^1{}_1\alpha^5{}_5}\theta_+ \mod \theta^i, \tag{6.10}$$

with yet unspecified θ^i terms in (6.10), and set

$$\alpha^0{}_0 = -4\alpha^1{}_1\alpha^5{}_5 \tag{6.11}$$

to get -4 coefficient in the $\Gamma_+\wedge\theta^1$ term. Thereby

$$\mathrm{d}\theta^0 = 4(\Gamma_1+\Gamma_0)\wedge\theta^0 - 4\Gamma_+\wedge\theta^1 \mod \theta^i\wedge\theta^j$$

on the 23-dimensional subbundle of $J\times G\to M^5$ given by (6.11). We see that the form θ_+ plays naturally the role of the connection 1-form Γ_+.

Step (3) We calculate $\mathrm{d}\theta^1$ on the 23-dimensional bundle. In order to get

$$\mathrm{d}\theta^1 = -\Gamma_-\wedge\theta^0 + (4\Gamma_1+2\Gamma_0)\wedge\theta^1 - 3\Gamma_+\wedge\theta^2 \mod \theta^i\wedge\theta^j$$

we set

$$4\Gamma_1+2\Gamma_0 = \frac{\mathrm{d}\alpha^1{}_1}{\alpha^1{}_1} + \frac{\alpha^1{}_0\alpha^2{}_2-\alpha^1{}_1\alpha^2{}_1}{\alpha^1{}_1\alpha^2{}_2\alpha^5{}_5}\theta_+ \mod \theta^i, \tag{6.12}$$

$$\Gamma_- = -\frac{\mathrm{d}\alpha^1{}_0}{4\alpha^1{}_1\alpha^5{}_5} + \frac{\alpha^1{}_0\mathrm{d}\alpha^1{}_1}{4(\alpha^1{}_1)^2\alpha^5{}_5} + \frac{(\alpha^1{}_0)^2\alpha^2{}_2+(\alpha^1{}_1)^2\alpha^2{}_0-\alpha^1{}_1\alpha^2{}_1(\alpha^1{}_0)^2}{4(\alpha^1{}_1)^2\alpha^2{}_2(\alpha^5{}_5)^2}\theta_+ \mod \theta^i, \tag{6.13}$$

and

$$\alpha^2{}_2 = -\frac{\alpha^1{}_1}{3\alpha^5{}_5} \tag{6.14}$$

obtaining a 22-dimensional subbundle of $J\times G\to M^5$ on which $\mathrm{d}\theta^0$ and $\mathrm{d}\theta^1$ are in the desired form.

Step (4) At this point all four connection 1-forms Γ_-, Γ_+, Γ_0, Γ_1 are fixed up to the θ^i terms. They are determined by the Eqs. (6.9), (6.10), (6.12) and (6.13). Thus we cannot introduce any new 1-forms to bring $\mathrm{d}\theta^2$ into the desired form. Now to get $\mathrm{d}\theta^2$ in the form as in Theorem 6.3, we may only use the yet unspecified coefficients αs. That is why $\mathrm{d}\theta^2$ imposes more conditions on αs. It follows that for $\mathrm{d}\theta^0$, $\mathrm{d}\theta^1$ and $\mathrm{d}\theta^2$ to be of the form (4.1) the subbundle P must satisfy

$$\begin{aligned}
\alpha^0{}_0 &= -4\alpha^1{}_1\alpha^5{}_5,\\
\alpha^2{}_0 &= \frac{-75(\alpha^1{}_0)^2+(\alpha^1{}_1)^2(-20\mathcal{D}F_4+20F_3+7F_4^2)}{300\alpha^1{}_1\alpha^5{}_5},\\
\alpha^2{}_1 &= \frac{-15\alpha^1{}_0+\alpha^1{}_1F_4}{30\alpha^5{}_5},\\
\alpha^2{}_2 &= -\frac{\alpha^1{}_1}{3\alpha^5{}_5},\\
\alpha^3{}_0 &= [1800(\alpha^1{}_1\alpha^5{}_5)^2]^{-1}\\
&\quad\times[1125(\alpha^1{}_0)^3+45\alpha^1{}_0(\alpha^1{}_1)^2(20\mathcal{D}F_4-20F_3-7F_4^2)+2(\alpha^1{}_1)^3(100\mathcal{D}^2F_4-200F_2-30F_4\mathcal{D}F_4-60F_3F_4-11F_4^3)],\\
\alpha^3{}_1 &= \frac{225(\alpha^1{}_0)^2-30\alpha^1{}_0\alpha^1{}_1F_4+(\alpha^1{}_1)^2(80\mathcal{D}F_4-100F_3-31F_4^2)}{1200\alpha^1{}_1(\alpha^5{}_5)^2},\\
\alpha^3{}_2 &= \frac{5\alpha^1{}_0-\alpha^1{}_1F_4}{20(\alpha^5{}_5)^2},\\
\alpha^3{}_3 &= \frac{\alpha^1{}_1}{6(\alpha^5{}_5)^2},\\
\alpha^4{}_1 &= [18\,000(\alpha^1{}_1)^2(\alpha^5{}_5)^3]^{-1}\times[-1125(\alpha^1{}_0)^3+225(\alpha^1{}_0)^2\alpha^1{}_1F_4-15\alpha^1{}_0(\alpha^1{}_1)^2(80\mathcal{D}F_4-100F_3-31F_4^2)\\
&\quad+(\alpha^1{}_1)^3(-400\mathcal{D}^2F_4+1400F_2+240F_4\mathcal{D}F_4+180F_3F_4+11F_4^3)],\\
\alpha^4{}_2 &= \frac{-75(\alpha^1{}_0)^2+30\alpha^1{}_0\alpha^1{}_1F_4+(\alpha^1{}_1)^2(-40\mathcal{D}F_4+80F_3+17F_4^2)}{600\alpha^1{}_1(\alpha^5{}_5)^3},
\end{aligned} \tag{6.15}$$

$$\alpha^4{}_3 = \frac{-5\alpha^1{}_0 + 3\alpha^1{}_1 F_4}{30(\alpha^5{}_5)^3},$$
$$\alpha^4{}_4 = -\frac{\alpha^1{}_1}{6(\alpha^5{}_5)^3}.$$

The necessity of these conditions can be checked by direct, quite lengthy calculations. We performed these calculations using the symbolic computation programs Maple and Mathematica.

We stress that conditions (6.15) are only *necessary* for $d\theta^2$ to satisfy (4.1). It is because certain unwanted terms cannot be removed by any choice of subbundle P. Vanishing of these unwanted terms is a property of the ODE itself, and this is the reason for the Wünschmann conditions to appear.

More specifically, to achieve

$$d\theta^2 = -2\Gamma_- \wedge \theta^1 + 4\Gamma_1 \wedge \theta^2 - 2\Gamma_+ \wedge \theta^3 \mod \theta^i \wedge \theta^j$$

on the bundle defined by (6.11), (6.14) and (6.15) an ODE must satisfy

$$50\mathcal{D}^2 F_4 - 75\mathcal{D}F_3 + 50F_2 - 60F_4\mathcal{D}F_4 + 30F_3F_4 + 8F_4^3 = 0. \qquad (6.16)$$

It follows from the construction that this condition, the first of (6.5), is invariant under the contact transformation of variables.

From now on we restrict our considerations only to a contact equivalence class of ODEs satisfying (6.16). If (6.15) and (6.16) are satisfied then the three differentials $d\theta^0$, $d\theta^1$ and $d\theta^2$ are precisely in the form (4.1).

Step (5) The requirement that also $d\theta^3$ be in the form (4.1) is equivalent to the following equation for $\alpha^4{}_0$:

$$\alpha^4{}_0 = [120\,000(\alpha^1{}_1\alpha^5{}_5)^3]^{-1} \times \Big[-1875(\alpha^1{}_0)^4 - 150(\alpha^1{}_0\alpha^1{}_1)^2(20\mathcal{D}F_4 - 20F_3 - 7F_4^2)$$
$$- 40\alpha^1{}_0(\alpha^1{}_1)^3(50\mathcal{D}F_3 - 100F_2 + 30F_4\mathcal{D}F_4 - 40F_3F_4 - 9F_4^3) + (\alpha^1{}_1)^4$$
$$\times\Big(400(-5\mathcal{D}^2F_3 + 10\mathcal{D}F_2 - 6(\mathcal{D}F_4)^2 + 10F_3\mathcal{D}F_4 - 3F_3^2 + F_4\mathcal{D}F_3) + 120F_4^2(7\mathcal{D}F_4 - 5F_3) - 63F_4^4\Big)\Big]. \qquad (6.17)$$

Step (6) If condition (6.17) is also imposed we have

$$(d\theta^4 + 4\Gamma_- \wedge \theta^3 - 4(\Gamma - \Gamma_0) \wedge \theta^4) \wedge \theta^0 \wedge \theta^1 = 0 \mod \theta^i.$$

However,

$$d\theta^4 \wedge \theta^0 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0$$

if and only if second condition of (6.5) is satisfied:

$$375\mathcal{D}^2F_3 - 1000\mathcal{D}F_2 + 350\mathcal{D}F_4^2 + 1250F_1 - 650F_3\mathcal{D}F_4 + 200F_3^2$$
$$-150F_4\mathcal{D}F_3 + 200F_2F_4 - 140F_4^2\mathcal{D}F_4 + 130F_3F_4^2 + 14F_4^4 = 0. \qquad (6.18)$$

Again it follows from the construction that condition (6.18), considered simultaneously with (6.16), is invariant under contact transformations of the variables. From now on, we assume that all our 5th order ODEs (6.1) satisfy both conditions (6.16), (6.18). It follows that it is still not sufficient to force $d\theta^4$ to satisfy the system (4.1), since without further assumptions on F, we do *not* have $d\theta^4 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0$. To achieve this it is necessary and sufficient to impose the last restriction on F:

$$1250\mathcal{D}^2F_2 - 6250\mathcal{D}F_1 + 1750\mathcal{D}F_3\mathcal{D}F_4 - 2750F_2\mathcal{D}F_4 - 875F_3\mathcal{D}F_3 + 1250F_2F_3 - 500F_4\mathcal{D}F_2 + 700(\mathcal{D}F_4)^2F_4$$
$$+ 1250F_1F_4 - 1050F_3F_4\mathcal{D}F_4 + 350F_3^2F_4 - 350F_4^2\mathcal{D}F_3$$
$$+ 550F_2F_4^2 - 280F_4^3\mathcal{D}F_4 + 210F_3F_4^3 + 28F_4^5 + 18\,750F_y = 0. \qquad (6.19)$$

Step (7) Assuming that F satisfies conditions (6.5) and fixing coefficients $\alpha^i{}_j$ according to (6.15), (6.17) we are remained with a 11-dimensional subbundle of $J \times G \to M^5$ parameterised by $(x, y, y_1, y_2, y_3, y_4, \alpha^1{}_0, \alpha^1{}_1, \alpha^5{}_5, \alpha^5{}_0, \alpha^5{}_1)$. It follows that the forms $\Gamma_0, \Gamma_1, \Gamma_-, \Gamma_+$ on this bundle are

$$\Gamma_+ = \theta_+,$$
$$\Gamma_0 = \frac{d\alpha^5{}_5}{2\alpha^5{}_5} - \frac{5\alpha_0^1 + \alpha^1{}_1F_4}{20\alpha^1{}_1\alpha^5{}_5}\theta_+ \mod \theta^i,$$
$$\Gamma_1 = \frac{d\alpha^1{}_1}{4\alpha^1{}_1} - \frac{d\alpha^5{}_5}{4\alpha^5{}_5} + \frac{F_4}{20\alpha^5{}_5}\theta_+ \mod \theta^i, \qquad (6.20)$$
$$\Gamma_- = \frac{d\alpha^1{}_0}{4\alpha^1{}_1\alpha^5{}_5} - \frac{\alpha^1{}_0 d\alpha^1{}_1}{4(\alpha^1{}_1)^2\alpha^5{}_5} - \frac{25(\alpha^1{}_0)^2 + 10\alpha^1{}_0\alpha^1{}_1F_4 + (\alpha^1{}_1)^2(20\mathcal{D}F_4 - 20F_3 - 7F_4^2)}{400(\alpha^1{}_1\alpha^5{}_5)^2}\theta_+ \mod \theta^i.$$

Step (8) In order to construct a 9-dimensional bundle and find the θ^i terms in (6.20) we need to consider the $\mathrm{d}\Gamma_A$ part of Eqs. (4.1). Forcing $\mathrm{d}\Gamma_A$ not to have $\Gamma_A\wedge\theta^i$ terms we *uniquely* specify the θ^i terms in (6.20). This requirement, in particular, fixes the coefficients ${\alpha^5}_1$ and ${\alpha^5}_0$ to be:

$$\begin{aligned}{\alpha^5}_1&=\frac{{\alpha^5}_5(10\mathcal{D}F_{44}+5F_{34}+6F_4F_{44})}{50},\\ {\alpha^5}_0&=\frac{{\alpha^5}_5}{250}\,[50(\mathcal{D}F_{34}+7F_{24}-5F_{33})+5F_4(6\mathcal{D}F_{44}-37F_{34})+2F_{44}(-60\mathcal{D}F_4+145F_3+21F_4^2)].\end{aligned}\tag{6.21}$$

Now all the forms $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\Gamma_+,\Gamma_-,\Gamma_0,\Gamma_1)$ are well defined and independent on a 9-dimensional manifold P parameterised by $(y,y_1,y_2,y_3,y_4,x,{\alpha^1}_0,{\alpha^1}_1,{\alpha^5}_5)$. We calculate structural equations (4.1) for these forms and have the following

Theorem 6.6. *A 5th order ODE $y^{(5)}=F(x,y,y',y'',y^{(3)},y^{(4)})$ considered modulo contact transformation of variables has an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure on the space of its solution M^5 together with a $\mathfrak{gl}(2,\mathbb{R})$ connection Γ if and only if its defining function $F=F(x,y,y_1,y_2,y_3,y_4)$ satisfies the contact invariant Wünschmann conditions (6.5). The bundle $\mathbf{GL}(2,\mathbb{R})\to P\to M^5$ is given by the Eqs. (6.15), (6.17) and (6.21). The first structural equations for the connection $\Gamma=(\Gamma_+,\Gamma_-,\Gamma_0,\Gamma_1)$ on P read*

$$\begin{aligned}\mathrm{d}\theta^0&=4(\Gamma_1+\Gamma_0)\wedge\theta^0-4\Gamma_+\wedge\theta^1+t_1\theta^0\wedge\theta^1+t_2\theta^0\wedge\theta^2+t_3\theta^0\wedge\theta^3,\\ \mathrm{d}\theta^1&=-\Gamma_-\wedge\theta^0+(4\Gamma_1+2\Gamma_0)\wedge\theta^1-3\Gamma_+\wedge\theta^2+\frac12t_1\theta^0\wedge\theta^2\\ &\quad+\frac13t_2\theta^0\wedge\theta^3+\frac14t_3\theta^0\wedge\theta^4+t_2\theta^1\wedge\theta^2+t_3\theta^1\wedge\theta^3,\\ \mathrm{d}\theta^2&=-2\Gamma_-\wedge\theta^1+4\Gamma_1\wedge\theta^2-2\Gamma_+\wedge\theta^3+\frac29t_1\theta^0\wedge\theta^3+\frac1{18}t_2\theta^0\wedge\theta^4\\ &\quad+\frac13t_1\theta^1\wedge\theta^2+\frac89t_2\theta^1\wedge\theta^3+\frac23t_3\theta^1\wedge\theta^4+t_3\theta^2\wedge\theta^3,\\ \mathrm{d}\theta^3&=-3\Gamma_-\wedge\theta^2+(4\Gamma_1-2\Gamma_0)\wedge\theta^3-\Gamma_+\wedge\theta^4+\frac1{12}t_1\theta^0\wedge\theta^4+\frac13t_1\theta^1\wedge\theta^3\\ &\quad+\frac13t_2\theta^1\wedge\theta^4+t_2\theta^2\wedge\theta^3+\frac32t_3\theta^2\wedge\theta^4,\\ \mathrm{d}\theta^4&=-4\Gamma_-\wedge\theta^3+4(\Gamma_1-\Gamma_0)\wedge\theta^4+\frac13t_1\theta^1\wedge\theta^4+t_2\theta^2\wedge\theta^4+3t_3\theta^3\wedge\theta^4,\end{aligned}\tag{6.22}$$

with the torsion coefficients

$$\begin{aligned}t_3&=\frac{6({\alpha^5}_5)^2}{5{\alpha^1}_1}F_{44},\\ t_2&=\frac{9{\alpha^5}_5}{50({\alpha^1}_1)^2}\,[{\alpha^1}_1(10\mathcal{D}F_{44}+3F_4F_{44})+5{\alpha^1}_0F_{44}],\\ t_1&=[1000({\alpha^1}_1)^3]^{-1}\times\Big(225({\alpha^1}_0)^2F_{44}+90{\alpha^1}_0{\alpha^1}_1(10\mathcal{D}F_{44}+3F_4F_{44})\\ &\quad-9({\alpha^1}_1)^2[20(5\mathcal{D}F_{34}+20F_{24}-15F_{33}+3F_4\mathcal{D}F_{44}-11F_4F_{34})+F_{44}(-120\mathcal{D}F_4+340F_3+51F_4^2)]\Big).\end{aligned}$$

Also the second structural equations are easily calculable but we skip them due to their complexity.

It is remarkable that the above $\mathfrak{gl}(2,\mathbb{R})$ connection has torsion with not more than three functionally independent coefficients t_1,t_2,t_3. This suggests that the $\mathbf{GL}(2,\mathbb{R})$ geometry on the 5-dimensional solution space M^5 of the ODE is nearly integrable with torsion in the irreducible part $\bigwedge_3$ only. That it is really the case will be shown below.

6.3. Characteristic connection with torsion in $\bigwedge_3$

As we know from Section 3, given an irreducible $\mathbf{GL}(2,\mathbb{R})$-structure $(M^5,[g,\Upsilon,A])$, we can ask if such a structure is nearly integrable. According to Propositions 3.5 and 3.6, the necessary and sufficient condition for nearly integrability is that the structure admits a $\mathfrak{gl}(2,\mathbb{R})$-valued connection with totally skew symmetric torsion.

In our case of ODEs satisfying Wünschmann conditions we have a $\mathfrak{gl}(2,\mathbb{R})$-valued connection of Theorem 6.6, whose torsion is expressible in terms of three independent functions. This torsion, however, has quite a complicated algebraic structure, in particular it is not totally skew symmetric.

It appears that an irreducible $\mathbf{GL}(2,\mathbb{R})$ structure $(M^5,[g,\Upsilon,A])$ associated with any 5th order ODE satisfying conditions (6.5) admits another $\mathfrak{gl}(2,\mathbb{R})$-valued connection that *has* totally skew symmetric torsion. Thus all structures $(M^5,[g,\Upsilon,A])$

originating from Wünschmann 5th order ODEs are nearly integrable; the new connection is their characteristic connection. Even more interesting is the fact that its torsion is still more special: it is always in $\bigwedge_3$.

One way of seeing this is to calculate the Weyl connection $\overset{W}{\Gamma}$ for the corresponding $(M^5,[g,\Upsilon,A])$ and to decompose it according to (3.14). Here we prefer another method — the analysis in terms of the Cartan bundle P of Theorem 6.6.

Lemma 6.7. *Consider a contact equivalence class of 5th order ODEs satisfying conditions (6.5). Let $\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\Gamma_+$, $\Gamma_-,\Gamma_0,\Gamma_1$ and t_1,t_2,t_3 be the objects of Theorem 6.6. Then there is a $\mathfrak{gl}(2,\mathbb{R})$ connection $\tilde\Gamma=(\tilde\Gamma_+,\tilde\Gamma_-,\tilde\Gamma_0,\tilde\Gamma_1)$ whose torsion $\tilde T^i{}_{jk}$ is totally skew symmetric and has its associated 3-form in $\tilde T\in *\bigwedge_3$. Explicitly:*

$$\tilde T=\frac{1}{12}t_1(-\theta^0\wedge\theta^1\wedge\theta^4+2\theta^0\wedge\theta^2\wedge\theta^3)+\frac{1}{12}t_2(-\theta^0\wedge\theta^2\wedge\theta^4+8\theta^1\wedge\theta^2\wedge\theta^3)$$
$$+\frac{1}{4}t_3(-\theta^0\wedge\theta^3\wedge\theta^4+2\theta^1\wedge\theta^2\wedge\theta^4).$$

Proof. Any $\mathfrak{gl}(2,\mathbb{R})$ connection $\tilde\Gamma=(\tilde\Gamma_+,\tilde\Gamma_-,\tilde\Gamma_0,\tilde\Gamma_1)$ compatible with the $\mathbf{GL}(2,\mathbb{R})$ structure of Theorem 6.6 is given by

$$\tilde\Gamma_A=\Gamma_A+\sum_i\gamma_{Ai}\theta^i,\quad A\in\{+,0,-\},\ i=0,\dots,4,\qquad (6.23)$$
$$\tilde\Gamma_1=\Gamma_1$$

with arbitrary functions γ_{Ai}. We calculate the structural equations $d\theta+\tilde\Gamma\wedge\theta=\tilde T$ for $\tilde\Gamma$ utilising Eqs. (6.22), and ask if there exists a choice of γ_{Ai} such that the new torsion $\tilde T^i{}_{jk}$ satisfies $g_{il}\tilde T^l{}_{jk}=\tilde T_{[ijk]}$ and $\tilde T=\frac16 g_{il}\tilde T^l{}_{jk}\theta^i\wedge\theta^j\wedge\theta^k\in *\bigwedge_3$. Using Lemma 5.2 we easily find that the unique solution is given by

$$\tilde\Gamma_+=\Gamma_+-\frac16 t_1\theta^0-\frac13 t_2\theta^1-\frac12 t_3\theta^2,$$
$$\tilde\Gamma_-=\Gamma_-+\frac16 t_1\theta^2+\frac13 t_2\theta^3+\frac12 t_3\theta^4,$$
$$\tilde\Gamma_0=\Gamma_0-\frac16 t_1\theta^1-\frac13 t_2\theta^2-\frac12 t_3\theta^3,$$
$$\tilde\Gamma_1=\Gamma_1,\quad\square$$

Lemma 6.7 together with Proposition 4.1 and Theorem 5.5 prove Theorem 6.3.

Remark 6.8. Note that a passage from Γ_+ to

$$\tilde\Gamma_+=\Gamma_+-\frac16 t_1\theta^0-\frac13 t_2\theta^1-\frac12 t_3\theta^2$$

belongs to a larger class of transformations than the contact transformations (6.7), (6.8); it involves a forbidden θ^2 term. Thus it may happen that there are nonequivalent classes of ODEs which define the same $(M^5,[g,\Upsilon,A])$. To distinguish between nonequivalent ODEs one has to use the connection of Theorem 6.6.

7. Examples of nearly integrable GL(2, ℝ) structures from 5th order ODEs

In this section we provide examples of Wünschmann ODEs and nearly integrable $\mathbf{GL}(2,\mathbb{R})$ structures related to them. Since such structures have the torsions of their characteristic connections in $\bigwedge_3$, then via Theorem 5.4, they are characterised by the torsion T, the Ricci scalar R, the components of Maxwell 2-forms $dA^{(3)}$, $dA^{(7)}$, and the vector K; all these objects being associated to the characteristic connection Γ. There is also the unique Weyl connection $\overset{W}{\Gamma}$ associated with these structures.

7.1. Torsion-free structures

We see from Corollary 6.4 that

$$T\equiv 0\quad\Longleftrightarrow\quad F_{44}\equiv 0.$$

Then $\overset{W}{\Gamma}=\Gamma$ and all the curvature components but the Ricci scalar necessarily vanishes. The following proposition can be checked by direct calculation.

Proposition 7.1. *The three nonequivalent differential equations*

$$y^{(5)}=c\Big(\frac{5y^{(3)3}(5-27cy''^2)}{9(1+cy''^2)^2}+10\frac{y''y^{(3)}y^{(4)}}{1+cy''^2}\Big),$$

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with $c = +1, 0, -1$, represent the only three contact nonequivalent classes of 5th order ODEs having the corresponding nearly integrable $\mathbf{GL}(2,\mathbb{R})$ *structures* $(M^5, [g, \Upsilon, A])$ *with the characteristic connection with vanishing torsion. In all three cases the holonomy of the Weyl connection* $\overset{W}{\Gamma}$ *of structures* $(M^5, [g, \Upsilon, A])$ *is reduced to the* $\mathbf{GL}(2,\mathbb{R})$*. For all the three cases the Maxwell 2-form* $\mathrm{d}A \equiv 0$*. The corresponding Weyl structure is flat for* $c = 0$*. If* $c = \pm 1$*, then in the conformal class* $[g]$ *there is an Einstein metric of positive* $(c = +1)$ *or negative* $(c = -1)$ *Ricci scalar. In the case* $c = 1$ *the manifold* M^5 *can be identified with the homogeneous space* $\mathbf{SU}(1,2)/\mathbf{SL}(2,\mathbb{R})$ *with an Einstein g descending from the Killing form on* $\mathbf{SU}(1,2)$*. Similarly in* $c = -1$ *case the manifold* M^5 *can be identified with the homogeneous space* $\mathbf{SL}(3,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})$ *with an Einstein g descending from the Killing form on* $\mathbf{SL}(3,\mathbb{R})$*. In both cases with* $c \neq 0$ *the metric g is not conformally flat.*

7.2. Structures with vanishing Maxwell form

From now on we assume that

$$F_{44} \neq 0.$$

In this section we additionally assume that the Maxwell 2-form vanishes

$$\mathrm{d}A = 0.$$

For such structures both torsion and curvature have at most 9 independent coefficients contained in T, K and the scalar R. The simplest geometries in this class are those satisfying the additional equality

$$K^i = uR^i_{\mathrm{v}}, \quad u \in \mathbb{R}.$$

Putting $\mathrm{d}A = 0$ and $K^i = uR^i_{\mathrm{v}}$ into structural equations of Theorem 5.5 and using Bianchi identities we find that either

$$u = -\frac{1}{420}, \qquad R = \frac{35}{54}(t_2^2 - 3t_1t_3)$$

or

$$u = \frac{2}{105}, \qquad R = \frac{10}{27}(t_2^2 - 3t_1t_3).$$

Thus in these cases R is functionally dependent on t_1, t_2, t_3 and the only invariants for such $\mathbf{GL}(2,\mathbb{R})$ structures are u and the sign of R. For each possible values of u and $\operatorname{sgn} R$ we found a generating ODE.

Proposition 7.2. *Consider the equations*

$$F = \frac{5y_4^2}{3y_3} + \epsilon y_3^{5/3}, \quad \epsilon = -1, 0, 1, \tag{7.1}$$

$$F = \frac{5y_4^2}{4y_3}, \tag{7.2}$$

and

$$F = \frac{5(8y_3^3 - 12y_2y_3y_4 + 3y_1y_4^2)}{6(2y_1y_3 - 3y_2^2)}, \tag{7.3}$$

where the sign of expression $(2y_1y_3 - 3y_2^2)$ *is an invariant, and the singular locus* $2y_1y_3 - 3y_2^2 = 0$ *separates nonequivalent equations with* $\pm$ *signs. The equations generate all the six* $\mathbf{GL}(2,\mathbb{R})$ *structures satisfying* $\mathrm{d}A = 0$ *and* $K^i = uR^i_{\mathrm{v}}$, $u \in \mathbb{R}$.

For (7.1) $\quad u = -\dfrac{1}{420}$ *and* $\operatorname{sgn} R = \epsilon$,

for (7.2) $\quad u = \dfrac{2}{105}$ *and* $R = 0$,

for (7.3) $\quad u = \dfrac{2}{105}$ *and* $\operatorname{sgn} R = \operatorname{sgn}(3y_2^2 - 2y_1y_3)$.

Moreover, the above ODEs can be also described in a geometric way by means of the symmetry group.

Proposition 7.3. *The Eqs.* (7.1)–(7.3) *are the only 5th order Wünschmann ODEs satisfying* $F_{44} \neq 0$, $F_{444} = 0$ *and possessing the maximal group of transitive contact symmetries of dimension greater than five. Equations* $F = \frac{5y_4^2}{3y_3}$ *and* $F = \frac{5y_4^2}{4y_3}$ *have 7-dimensional groups of symmetries, all the remaining have 6-dimensional ones.*

Proof. The proof is based on a further application of the Cartan method of equivalence. Let us return to the coframe of Theorem 6.6, which encodes all the contact invariant information about the ODE. If there are any nonconstant coefficients in the structural equations for this coframe we can use them for further reduction of the group $\mathbf{GL}(2,\mathbb{R})$ and of the bundle P. For an ODE satisfying $F_{44}\neq 0$ we normalise $t_3=1$, $t_2=0$, which implies

$$\alpha^1{}_1=\frac{6}{5}(\alpha^5{}_5)^2F_{44},\qquad \alpha^1{}_0=-\frac{6}{25}(\alpha^5{}_5)^2(10\mathcal{D}F_{44}+3F_4F_{44}).$$

Now the coframe of Theorem 6.6 is reduced to a 7-dimensional manifold P_7 parameterised by $(x,y,y_1,y_2,y_3,y_4,\alpha^5{}_5)$, three 1-forms $(\Gamma_0,\Gamma_-,\Gamma_1)$ become dependent on each other and we can use only one of them, our choice is Γ_0, to supplement $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\Gamma_+)$ to an invariant coframe on P_7. Next we calculate structural equations for the new coframe. The coefficients in these equations are built from $\alpha^5{}_5$ and 16 functions $\lambda_1,\ldots,\lambda_{16}$ of $x,y,y_1,\ldots,y_4$. In particular

$$\mathrm{d}\theta^0=6\Gamma_0\wedge\theta^0-4\Gamma_+\wedge\theta^1+\frac{\lambda_1}{(\alpha^5{}_5)^2}\theta^0\wedge\theta^1+\frac{\lambda_2}{\alpha^5{}_5}\theta^0\wedge\theta^2+\lambda_3\theta^0\wedge\theta^3+\lambda_4\alpha^5{}_5\theta^0\wedge\theta^4,$$

where for example

$$\lambda_3=-\frac{5F_{344}F_{44}+10DF_{44}F_{444}+6F_4F_{44}F_{444}}{F_{44}^3},\qquad \lambda_4=5\frac{F_{444}}{F_{44}^2}.$$

Let us assume $F_{444}=0$ and consider two possibilities: $\lambda_3\neq const$ and $\lambda_3=const$. If $\lambda_3\neq const$ then it follows from the equations $\mathrm{d}^2\theta^i=0$, $\mathrm{d}^2\Gamma_A=0$ that λ_2 may not be a constant. Thus $\lambda_2/\alpha^5{}_5$ and λ_3 are *two* functionally independent coefficients in structural equations for the 7-dimensional coframe $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\Gamma_+,\Gamma_0)$. According to the procedure of finding symmetries of ODEs, which is described in [17], the dimension of the group of contact symmetries of a corresponding 5-order ODE is not larger than the dimension of the coframe minus the number of the independent coefficients in the structural equations, that is $7-2=5$. It follows that ODEs possessing a contact symmetry group greater than 5-dimensional necessarily satisfy $\lambda_3=const$. Let us assume $\lambda_3=const$ then, and we get from identities $\mathrm{d}^2\theta^i=0$, $\mathrm{d}^2\Gamma_A=0$ that (i) either $\lambda_3=2$ or $\lambda_3=\frac{3}{2}$ and (ii) for both admissible values of λ_3 all the remaining nonvanishing functions λ_j are expressible by λ_1. For example, the system corresponding to $\lambda_3=\frac{3}{2}$ is the following

$$\mathrm{d}\theta^0=6\Gamma_0\wedge\theta^0-4\Gamma_+\wedge\theta^1+\frac{\lambda_1}{(\alpha^5{}_5)^2}\theta^0\wedge\theta^1+\frac{3}{2}\theta^0\wedge\theta^3$$

$$\mathrm{d}\theta^1=4\Gamma_0\wedge\theta^1+\frac{2\lambda_1}{7(\alpha^5{}_5)^2}\Gamma_+\wedge\theta^0-3\Gamma_+\wedge\theta^2+\frac{3\lambda_1}{7(\alpha^5{}_5)^2}\theta^0\wedge\theta^2+\frac{3}{2}\theta^1\wedge\theta^3$$

$$\mathrm{d}\theta^2=2\Gamma_0\wedge\theta^2+\frac{4\lambda_1}{7(\alpha^5{}_5)^2}\Gamma_+\wedge\theta^1-2\Gamma_+\wedge\theta^3-\frac{2\lambda_1^2}{49(\alpha^5{}_5)^4}\theta^0\wedge\theta^1+\frac{4\lambda_1}{21(\alpha^5{}_5)^2}\theta^0\wedge\theta^3+\frac{\lambda_1}{7(\alpha^5{}_5)^2}\theta^1\wedge\theta^2+\frac{1}{6}\theta^1\wedge\theta^4+\frac{3}{2}\theta^2\wedge\theta^3$$

$$\mathrm{d}\theta^3=\frac{6\lambda_1}{7(\alpha^5{}_5)^2}\Gamma_+\wedge\theta^2-\Gamma_+\wedge\theta^4-\frac{3\lambda_1^2}{49(\alpha^5{}_5)^4}\theta^0\wedge\theta^2+\frac{\lambda_1}{14(\alpha^5{}_5)^2}\theta^0\wedge\theta^4+\frac{\lambda_1}{7(\alpha^5{}_5)^2}\theta^1\wedge\theta^3+\frac{3}{4}\theta^2\wedge\theta^4$$

$$\mathrm{d}\theta^4=-2\Gamma_0\wedge\theta^4+\frac{8\lambda_1}{7(\alpha^5{}_5)^2}\Gamma_+\wedge\theta^3-\frac{4\lambda_1^2}{49(\alpha^5{}_5)^4}\theta^0\wedge\theta^3+\frac{\lambda_1}{7(\alpha^5{}_5)^2}\theta^1\wedge\theta^4+\frac{3}{2}\theta^3\wedge\theta^4$$

$$\mathrm{d}\Gamma_+=2\Gamma_0\wedge\Gamma_++\frac{3\lambda_1^2}{98(\alpha^5{}_5)^4}\theta^0\wedge\theta^1+\frac{\lambda_1}{14(\alpha^5{}_5)^2}\theta^0\wedge\theta^3+\frac{1}{8}\theta^1\wedge\theta^4$$

$$\mathrm{d}\Gamma_0=\frac{\lambda_1^2}{49(\alpha^5{}_5)^4}\Gamma_+\wedge\theta^0-\frac{1}{4}\Gamma_+\wedge\theta^4+\frac{3\lambda_1^2}{196(\alpha^5{}_5)^4}\theta^0\wedge\theta^2+\frac{\lambda_1}{56(\alpha^5{}_5)^2}\theta^0\wedge\theta^4+\frac{\lambda_1}{14(\alpha^5{}_5)^2}\theta^1\wedge\theta^3+\frac{3}{16}\theta^2\wedge\theta^4.$$

If $\lambda_1=0$ then to this system there corresponds a unique equivalence class of ODEs satisfying the Wünschmann conditions and having a 7-dimensional transitive contact symmetry group. The class is represented by

$$F=\frac{5y_4^2}{3y_3}.$$

In the case $\lambda_1\neq 0$ we have next *two* nonequivalent classes of ODEs enumerated by the sign of λ_1 and possessing 6-dimensional transitive contact symmetry groups. Representatives of these classes are

$$F=\frac{5y_4^2}{3y_3}\pm y_3^{5/3},$$

where $\pm 1=\operatorname{sgn}\lambda_1$.

In the similar vein we find that the only ODEs related to the case $\lambda_1=2$ are (7.2) and (7.3). □

7.3. Simple structures with nonvanishing Maxwell form

All the previous examples satisfy the contact invariant condition $F_{444} = 0$. In this paragraph we give examples of Wünschmann ODEs with $F_{444} \neq 0$. As such they will lead to the $\mathbf{GL}(2, \mathbb{R})$ structures with the Maxwell form having a nonzero $\mathrm{d}A^{(7)}$ part. First and the simplest example of such equations is

$$F = (y_4)^{(5/4)}. \tag{7.4}$$

The $\mathbf{GL}(2, \mathbb{R})$ structure associated with this ODE has the following properties

$$\mathrm{d}A^{(3)} = 0, \qquad \mathrm{d}A^{(7)} \neq 0 \qquad R = 0, \qquad K = \frac{2}{105} R_{\mathrm{v}}.$$

It is then an example of a structure with nonvanishing $\mathrm{d}A$ belonging to the 7-dimensional irreducible representation.

Next example is the ODE given by the formula

$$\begin{aligned} F = {} & \frac{1}{9(y_1^2 + y_2)^2} \times \Big(5w\big(y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12 y_1 y_2 y_3 + 4y_3^2 - 3y_4(y_1^2 + y_2)\big) \\ & + 45 y_4 (y_1^2 + y_2)(2y_1 y_2 + y_3) - 4y_1^9 - 18 y_1^7 y_2 - 54 y_1^5 y_2^2 - 90 y_1^3 y_2^3 + 270 y_1 y_2^4 \\ & + 15 y_1^6 y_3 + 45 y_1^4 y_2 y_3 - 405 y_1^2 y_2^2 y_3 + 45 y_2^3 y_3 + 60 y_1^3 y_3^2 - 180 y_1 y_2 y_3^2 - 40 y_3^3\Big), \end{aligned} \tag{7.5}$$

where[4]

$$w^2 = y_1^6 + 3y_1^4 y_2 + 9y_1^2 y_2^2 - 9y_2^3 - 4y_1^3 y_3 + 12 y_1 y_2 y_3 + 4y_3^2 - 3y_1^2 y_4 - 3y_2 y_4.$$

Torsion and curvature for the corresponding $\mathbf{GL}(2, \mathbb{R})$ structure are complicated and are of general algebraic form. Both these examples have 6-dimensional transitive group of contact symmetries.

7.4. A remarkable nonhomogeneous example

Finally, we present an example of 5th order ODEs satisfying Wünschmann conditions (6.5), which are generic, in a sense that the function F representing it satisfies $F_{444} \neq 0$, but which have the corresponding group of transitive symmetries of dimension $D < 6$. We consider an ansatz in which function F depends in a special way on only two coordinates y_3 and y_4. Explicitly:

$$F = (y_3)^{5/3}\, q\Big(\frac{y_4^3}{y_3^4}\Big), \tag{7.6}$$

where $q = q(z)$ is a sufficiently differentiable real function of its argument

$$z = \frac{y_4^3}{y_3^4}.$$

It is remarkable that the above F satisfies all Wünschmann conditions provided that

- either $q(z) = \frac{5}{3} z^{2/3}$
- or function $q(z)$ satisfies the following second order ODE:

$$90 z^{4/3}(3q - 4z^{2/3}) q'' - 54 z^{4/3} q'^2 + 30 z^{1/3}(6q - 5z^{2/3}) q' - 25q = 0. \tag{7.7}$$

In the first case $F = \frac{5}{3}\frac{y_4^2}{y_3}$, and we recover function (7.1) with 7-dimensional group of symmetries. Note that one of the solutions of Eq. (7.7) is $q = \frac{5}{4} z^{2/3}$, which corresponds to $F = \frac{5}{4}\frac{y_4^2}{y_3}$. Thus also the other solution with seven symmetries, the solution (7.2), is covered by this ansatz.

We observe that if function $q(z)$ satisfies

$$25q - 60 z q' + 27 z^{4/3} q'^2 = 0, \tag{7.8}$$

then it also satisfies the reduction (7.7) of conditions (6.5). Eq. (7.8) can be solved by first putting it in the form

$$q' = \frac{5(2z^{1/3} \pm \sqrt{(4z^{2/3} - 3q)})}{9z^{2/3}}$$

[4] Note that $w = 0$ also gives rise to F satisfying conditions (6.5). But since such F has only linear y_4-dependence it is equivalent to one of Proposition 7.1. Actually the one with $c < 0$.

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and then by integrating, according to the sign ± 1. In the upper sign case the integration gives q in an implicit form:

$$\frac{(2z^{1/3}+\sqrt{(4z^{2/3}-3q)})^{24}(2\sqrt{(4z^{2/3}-3q)}-z^{1/3})^3}{(2\sqrt{(4z^{2/3}-3q)}+z^{1/3})^3(5z^{2/3}-4q)^3}=\text{const.}$$

In the lower sign case the implicit equation for q is:

$$\frac{(2z^{1/3}+\sqrt{(4z^{2/3}-3q)})^{24}(2\sqrt{(4z^{2/3}-3q)}-z^{1/3})^3(5z^{2/3}-4q)^3}{(2\sqrt{(4z^{2/3}-3q)}+z^{1/3})^3q^{24}}=\text{const.}$$

Inserting these qs into (7.6) we have a quite nontrivial Wünschmann ODE $F=F_{\pm}$. We close this section with a remark that other solutions to the second order ODE (7.7) also provide examples of 5th order Wünschmann ODEs.

8. Higher order ODEs

All our considerations about $\mathbf{GL}(2,\mathbb{R})$ structures associated with ODEs of 5th order can be repeated for other orders. This is due to the following well known fact generalising Proposition 6.1:

Proposition 8.1. *For every $n\geq 4$, the ordinary differential equation*

$$y^{(n)}=0$$

has $\mathbf{GL}(2,\mathbb{R})\times_{\rho_n}\mathbb{R}^n$ as its group of contact symmetries. Here $\rho_n:\mathbf{GL}(2,\mathbb{R})\to\mathbf{GL}(n,\mathbb{R})$ is the n-dimensional irreducible representation of $\mathbf{GL}(2,\mathbb{R})$.

The representation ρ_n, at the level of Lie algebra $\mathfrak{gl}(2,\mathbb{R})$, is given in terms of the Lie algebra generators

$$E_+=\begin{pmatrix}0&n-1&0&\dots&0&0&0\\0&0&n-2&\dots&0&0&0\\&&&\dots&&&\\0&0&0&\dots&3&0&0\\0&0&0&\dots&0&2&0\\0&0&0&\dots&0&0&1\\0&0&0&\dots&0&0&0\end{pmatrix},\qquad E_-=\begin{pmatrix}0&0&0&\dots&0&0&0\\1&0&0&\dots&0&0&0\\0&2&0&\dots&0&0&0\\0&0&3&\dots&0&0&0\\&&&\dots&&&\\0&0&0&\dots&n-2&0&0\\0&0&0&\dots&0&n-1&0\end{pmatrix},$$

$$E_0=\begin{pmatrix}1-n&0&0&\dots&0&0&0\\0&3-n&0&\dots&0&0&0\\0&0&5-n&\dots&0&0&0\\&&&\dots&&&\\0&0&0&\dots&n-5&0&0\\&&&\dots&0&n-3&0\\0&0&0&\dots&0&0&n-1\end{pmatrix},\qquad E_1=(1-n)\mathbf{1},$$

where $\mathbf{1}$ is the $n\times n$ identity matrix. In case of dimension $n=5$ these matrices coincide with (2.6). They also satisfy the same commutation relations

$$[E_0,E_+]=-2E_+,\qquad [E_0,E_-]=2E_-,\qquad [E_+,E_-]=-E_0,$$

where the commutator in the $\mathfrak{gl}(2,\mathbb{R})=\text{Span}_{\mathbb{R}}(E_-,E_+,E_0,E_1)\subset End(\mathbb{R}^n)$ is the usual commutator of matrices.

Now, we consider a general n-th order ODE

$$y^{(n)}=F(x,y,y',y'',y^{(3)},\dots,y^{(n-1)}),\tag{8.1}$$

and as before, to simplify the notation, we introduce the coordinates $x,y,y_1=y',y_2=y'',y_3=y^{(3)},\dots,y_{n-1}=y^{(n-1)}$ on the $(n+1)$-dimensional jet space J. Introducing the n *contact forms*

$$\begin{aligned}\omega^0&=\mathrm{d}y-y_1\mathrm{d}x,\\ \omega^1&=\mathrm{d}y_1-y_2\mathrm{d}x,\\ &\vdots\\ \omega^i&=\mathrm{d}y_i-y_{i+1}\mathrm{d}x,\\ &\vdots\\ \omega^{n-2}&=\mathrm{d}y_{n-2}-y_{n-1}\mathrm{d}x,\\ \omega^{n-1}&=\mathrm{d}y_{n-1}-F(x,y,y_1,y_2,\dots,y_{n-1})\mathrm{d}x\end{aligned}\tag{8.2}$$

and the additional 1-form

$$w_+ = \mathrm{d}x,$$

we define a *contact transformation* to be a diffeomorphism $\phi : J \to J$ which transforms the above $n+1$ one-forms via:

$$\phi^*\omega^i = \sum_{k=0}^{i} \alpha^i{}_k \omega^k, \quad i = 0, 1, \ldots n-1,$$
$$\phi^* w_+ = \alpha^n{}_0\omega^0 + \alpha^n{}_1\omega^1 + \alpha^n{}_n w_+.$$

Here $\alpha^i{}_j$ are functions on J such that $\prod_{i=0}^n \alpha^i{}_i \neq 0$ at each point of J.

Therefore, as in the case of $n = 5$, the contact equivalence problem for the nth order ODEs (8.1) can be studied in terms of the invariant forms $(\theta^0, \theta^1, \ldots, \theta^{n-1}, \Gamma_+)$ defined by

$$\theta^i = \sum_{k=0}^{i} \alpha^i{}_k \omega^k, \quad i = 0, 1, \ldots n-1, \tag{8.3}$$
$$\Gamma_+ = \alpha^n{}_0\omega^0 + \alpha^n{}_1\omega^1 + \alpha^n{}_n w_+.$$

These forms initially live on an $\frac{n^2+3n+8}{2}$-dimensional manifold $G \to J \times G \to J$, with the G-factor parameterised by $\alpha^i{}_j$, such that $\prod_{i=0}^{n} \alpha^i{}_i \neq 0$.

Introducing $\mathfrak{gl}(2, \mathbb{R})$-valued forms

$$\Gamma = \Gamma_- E_- + \Gamma_+ E_+ + \Gamma_0 E_0 + \Gamma_1 E_1, \tag{8.4}$$

where $(\Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$ are 1-forms on $J \times G$, we can specialise to $F \equiv 0$, and reformulate Proposition 8.1 to

Proposition 8.2. *If $F \equiv 0$ then one can chose $\frac{n(n+1)}{2}$ parameters $\alpha^i{}_j$, as functions of $x, y, y_1, \ldots, y_{n-1}$ and the remaining three αs, say $\alpha^{i_1}{}_{j_1}, \alpha^{i_2}{}_{j_2}, \alpha^{i_3}{}_{j_3}$, so that the $(n+4)$-dimensional manifold P parameterised by $(x, y, y_1, \ldots, y_{n-1}, \alpha^{i_1}{}_{j_1}, \alpha^{i_2}{}_{j_2}, \alpha^{i_3}{}_{j_3})$ is locally the contact symmetry group, $P \cong \mathbf{GL}(2, \mathbb{R}) \times_{\rho_n} \mathbb{R}^n$, of equation $y^{(n)} = 0$. Forms (8.3), after restriction to P, can be supplemented by three additional 1-forms $(\Gamma_-, \Gamma_0, \Gamma_1)$, so that $(\theta^0, \theta^1, \ldots, \theta^{n-1}, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$ constitute a basis of the left invariant forms on the Lie group P. The choice of αs and Ωs is determined by the requirement that the basis $(\theta^0, \theta^1, \ldots, \theta^{n-1}, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$ satisfies*

$$\mathrm{d}\theta + \Gamma \wedge \theta = 0, \tag{8.5}$$
$$\mathrm{d}\Gamma + \Gamma \wedge \Gamma = 0,$$

where $\theta = (\theta^0, \theta^1, \ldots, \theta^{n-1})^T$ is a column n-vector, and Γ is given by (8.4).

The defining Eqs. (8.5) of the left invariant basis, when written explicitly in terms of θ^is and Γs, read

$$\begin{aligned}
\mathrm{d}\theta^0 &= (n-1)(\Gamma_1 + \Gamma_0) \wedge \theta^0 + (1-n)\Gamma_+ \wedge \theta^1,\\
\mathrm{d}\theta^1 &= -\Gamma_- \wedge \theta^0 + [(n-1)\Gamma_1 + (n-3)\Gamma_0] \wedge \theta^1 + (2-n)\Gamma_+ \wedge \theta^2,\\
&\vdots\\
\mathrm{d}\theta^k &= -k\Gamma_- \wedge \theta^{k-1} + [(n-1)\Gamma_1 + (n-2k-1)\Gamma_0] \wedge \theta^k + (1+k-n)\Gamma_+ \wedge \theta^{k+1},\\
&\vdots\\
\mathrm{d}\theta^{n-1} &= (1-n)\Gamma_- \wedge \theta^{n-2} + (n-1)(\Gamma_1 - \Gamma_0) \wedge \theta^{n-1},\\
\mathrm{d}\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+,\\
\mathrm{d}\Gamma_- &= -2\Gamma_0 \wedge \Gamma_-,\\
\mathrm{d}\Gamma_0 &= \Gamma_+ \wedge \Gamma_-,\\
\mathrm{d}\Gamma_1 &= 0.
\end{aligned} \tag{8.6}$$

This system can be analysed in the same spirit as system (4.1) of Section 4. Thus, we first consider the distribution

$$\mathfrak{h} = \{X \in \mathrm{T}P \text{ s.t. } X \lrcorner \theta^i = 0,\ i = 0, 1, 2, \ldots, n-1\}$$

annihilating θ.

Then the first n equations of the system (8.6) guarantee that forms $(\theta^0, \theta^1, \theta^2, \ldots, \theta^{n-1})$ satisfy the Fröbenius condition,

$$\mathrm{d}\theta^i \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{n-1} = 0, \quad \forall\, i = 0, 1, 2, \ldots n-1$$

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and that, in turn, the distribution $\mathfrak{h}$ is integrable. Thus manifold P is foliated by 4-dimensional leaves tangent to the distribution $\mathfrak{h}$. The space of leaves of this distribution $P/\mathfrak{h}$ can be identified with the solution space $M^n = P/\mathfrak{h}$ of equation $y^{(n)} = 0$. This in particular means, that all Eqs. (8.5) can be interpreted respectively as the first and the second structure equations for a $\mathfrak{gl}(2,\mathbb{R})$-valued connection Γ having vanishing torsion and and vanishing curvature. This $\mathfrak{gl}(2,\mathbb{R})$-valued connection originates from a certain $\mathbf{GL}(2,\mathbb{R})$ (conformal) structure on the solution space M^n.

To make this last statement more precise we have to invoke a few results from Hilbert's theory of algebraic invariants [18] adapted to our situation of ODEs.

8.1. Results from Hilbert's theory of algebraic invariants

First we ask if for a given order $n \geq 4$ of an ODE (8.1) with $F = 0$ there exists a bilinear form $\tilde{g}$ on P of Proposition 8.2 such that it projects to a nondegenerate conformal metric on M^n. This is answered, in a bit more general form, by applying the *reciprocity law of Hermite* (see [18], p. 60), and its corollaries, due to Hilbert (see [18], p. 60).

To adapt Hilbert's results to our paper we introduce a definition of an *invariant of degree q*. Let $\tilde{t}$ be a *totally symmetric* covariant tensor field of rank q defined on the group manifold P of Proposition 8.2.

Definition 8.3. The tensor field $\tilde{t}$ is called a $\mathbf{GL}(2,\mathbb{R})$-*invariant* of degree q, if and only if, it is degenerate on $\mathfrak{h}$ and if for every $X \in \mathfrak{h}$, there exists a function $c(X)$ on P such that

$$\mathcal{L}_X \tilde{t} = c(X)\tilde{t}.$$

The degeneracy condition means that $\tilde{t}(X,\ldots) = 0$, for all $X \in \mathfrak{h}$.

In the following we will usually abbreviate the term 'a $\mathbf{GL}(2,\mathbb{R})$-invariant' to: 'an invariant'.

The first result from Hilbert's theory, adapted to our situation, is given by the following

Proposition 8.4. *For every $n = 2m+1$, $m = 2, 3, \ldots$ there exists a unique, up to a scale, invariant $\tilde{g}$ of second degree on P. This invariant, a degenerate symmetric conformal bilinear form $\tilde{g}$ of signature $(m+1, m, 0, 0, 0, 0)$ on P, satisfies*

$$\mathcal{L}_X \tilde{g} = 2(n-1)(X \lrcorner \Gamma_1)\tilde{g},$$

for all $X \in \mathfrak{h}$.

In case of even orders $n = 2m$, Hilbert's theory gives the following

Proposition 8.5. *For $n = 2m$ every $\mathbf{GL}(2,\mathbb{R})$-invariant has degree $q \geq 4$.*

Thus, if $n = 2m$, we do not have a conformal metric on the solution space M^n.

Returning to odd orders, we present the quadratic invariants $\tilde{g}$, of Proposition 8.4, for $n < 10$:

$$\begin{aligned}
{}^5\tilde{g} &= 3(\theta^2)^2 - 4\theta^1\theta^3 + \theta^0\theta^4, \quad \text{if } n = 5,\\
{}^7\tilde{g} &= -10(\theta^3)^2 + 15\theta^2\theta^4 - 6\theta^1\theta^5 + \theta^0\theta^6 \quad \text{if } n = 7,\\
{}^9\tilde{g} &= 35(\theta^4)^2 - 56\theta^3\theta^5 + 28\theta^2\theta^6 - 8\theta^1\theta^7 + \theta^0\theta^8 \quad \text{if } n = 9.
\end{aligned} \tag{8.7}$$

These expressions can be generalised to higher (odd) ns. We have the following

Proposition 8.6. *If $n = 2m+1$ and $m \geq 2$, the invariant $\tilde{g}$ of Proposition 8.4 is given by:*

$$\tilde{g} = \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} \theta^j\theta^{2m-j} + \frac{1}{2}(-1)^m \binom{2m}{m} (\theta^m)^2.$$

Remark 8.7. This proposition is also valid for $m = 1$. For such m, the value of n is $n = 3$, and we are in the regime of third order ODEs. Such ODEs were considered by Wünschmann [1]. Since ${}^3\tilde{g} = \theta^0\theta^2 - (\theta^1)^2$ is the only invariant in this case, the counterpart of the bundle P of Proposition 8.2 is a 10-dimensional bundle $P \cong \mathbf{O}(2,3)$, the full conformal group in Lorentzian signature $(1,2)$. The counterpart of system (8.5)/ (8.6) is given by Maurer–Cartan equations for $\mathbf{O}(3,2)$:

$$\begin{aligned}
d\theta^0 &= 2(\Gamma_1 + \Gamma_0) \wedge \theta^0 - 2\Gamma_+ \wedge \theta^1,\\
d\theta^1 &= -\Gamma_- \wedge \theta^0 + 2\Gamma_1 \wedge \theta^1 - \Gamma_+ \wedge \theta^2,\\
d\theta^2 &= -2\Gamma_1\theta^1 + (2\Gamma_1 - 2\Gamma_0) \wedge \theta^2,\\
d\Gamma_+ &= 2\Gamma_0 \wedge \Gamma_+ + \frac{1}{2}\Gamma_3 \wedge \theta^0 + \Gamma_4 \wedge \theta^1,\\
d\Gamma_- &= -2\Gamma_0 \wedge \Gamma_- + \Gamma_2 \wedge \theta^1 + \frac{1}{2}\Gamma_3 \wedge \theta^2,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\Gamma_0 &= \Gamma_+ \wedge \Gamma_- - \frac{1}{2}\Gamma_2 \wedge \theta^0 + \frac{1}{2}\Gamma_4 \wedge \theta^2,\\
\mathrm{d}\Gamma_1 &= -\frac{1}{2}\Gamma_2 \wedge \theta^0 - \frac{1}{2}\Gamma_3 \wedge \theta^1 - \frac{1}{2}\Gamma_4 \wedge \theta^2,\\
\mathrm{d}\Gamma_2 &= -\Gamma_3 \wedge \Gamma_- + 2\Gamma_2 \wedge \Gamma_0 + 2\Gamma_2 \wedge \Gamma_1,\\
\mathrm{d}\Gamma_3 &= -2\Gamma_2 \wedge \Gamma_+ - 2\Gamma_4 \wedge \Gamma_- + 2\Gamma_3 \wedge \Gamma_1,\\
\mathrm{d}\Gamma_4 &= -2\Gamma_4 \wedge \Gamma_0 - \Gamma_3 \wedge \Gamma_+ + 2\Gamma_4 \wedge \Gamma_1.
\end{aligned}$$

Here, apart from $\theta^0, \theta^1, \theta^2$ and $\Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1$ we have also left invariant forms $\Gamma_2, \Gamma_3, \Gamma_4$.

Now we pass to the invariants of degree $q = 3$. The question of their existence was again determined by Hilbert (see [18], p. 60), in terms of the reciprocity law of Hermite. In the language of our paper we have the following

Proposition 8.8. *An invariant of third degree $\tilde{\Upsilon}$ exists on P if and only if*

$$n = 4\mu + 1, \quad \mu \in \mathbb{N}.$$

Hilbert's theory, [18], p. 60, implies also the following:

Proposition 8.9. *In low dimensions $n = 4\mu + 1$, the unique up to a scale cubic invariant is given by*

- $n = 5$:

$${}^5\tilde{\Upsilon} = (\theta^2)^3 - 2\theta^1\theta^2\theta^3 + \theta^0(\theta^3)^2 - \theta^0\theta^2\theta^4 + (\theta^1)^2\theta^4$$

- $n = 9$:

$$\begin{aligned}
{}^9\tilde{\Upsilon} = {} & 15(\theta^4)^3 - 36\theta^3\theta^4\theta^5 + 24\theta^2(\theta^5)^2 + 24(\theta^3)^2\theta^6 - 22\theta^2\theta^4\theta^6\\
& - 8\theta^1\theta^5\theta^6 + 3\theta^0(\theta^6)^2 - 8\theta^2\theta^3\theta^7 + 12\theta^1\theta^4\theta^7 - 4\theta^0\theta^5\theta^7 + 3(\theta^2)^2\theta^8 - 4\theta^1\theta^3\theta^8 + \theta^0\theta^4\theta^8.
\end{aligned}$$

The rough statement about the even orders, $n = 2m$, described in Proposition 8.5, can be again refined in terms of the reciprocity law of Hermite. Following Hilbert we have

Proposition 8.10. *If $4 \le n = 2m$ the lowest order invariant tensor $\tilde{\Upsilon}$ on P has degree four. This is unique (up to a scale) only if $n = 4, 6, 8, 12$. If $n = 10$ or $n = 14$ we have two independent quartic invariants $\tilde{\Upsilon}$; if $n = 16, 18, 20$ we have three independent quartic invariants; and so on.*

Proposition 8.11. *In low dimensions $n = 2m$, the quartic invariant tensor $\tilde{\Upsilon}$ on P is given by*

- $n = 4$:

$${}^4\tilde{\Upsilon} = -3(\theta^1)^2(\theta^2)^2 + 4\theta^0(\theta^2)^3 + 4(\theta^1)^3\theta^3 - 6\theta^0\theta^1\theta^2\theta^3 + (\theta^0)^2(\theta^3)^2$$

- $n = 6$:

$$\begin{aligned}
{}^6\tilde{\Upsilon} = {} & -32(\theta^2)^2(\theta^3)^2 + 48\theta^1(\theta^3)^3 + 48(\theta^2)^3\theta^4 - 76\theta^1\theta^2\theta^3\theta^4 - 12\theta^0(\theta^3)^2\theta^4 + 9(\theta^1)^2(\theta^4)^2\\
& + 16\theta^0\theta^2(\theta^4)^2 - 12\theta^1(\theta^2)^2\theta^5 + 16(\theta^1)^2\theta^3\theta^5 + 4\theta^0\theta^2\theta^3\theta^5 - 10\theta^0\theta^1\theta^4\theta^5 + (\theta^0)^2(\theta^5)^2.
\end{aligned}$$

- $n = 8$:

$$\begin{aligned}
{}^8\tilde{\Upsilon} = {} & -375(\theta^3)^2(\theta^4)^2 + 600\theta^2(\theta^4)^3 + 600(\theta^3)^3\theta^5 - 990\theta^2\theta^3\theta^4\theta^5 - 240\theta^1(\theta^4)^2\theta^5 + 81(\theta^2)^2(\theta^5)^2\\
& + 360\theta^1\theta^3(\theta^5)^2 - 240\theta^2(\theta^3)^2\theta^6 + 360(\theta^2)^2\theta^4\theta^6 + 50\theta^1\theta^3\theta^4\theta^6 + 40\theta^0(\theta^4)^2\theta^6 - 234\theta^1\theta^2\theta^5\theta^6\\
& - 60\theta^0\theta^3\theta^5\theta^6 + 25(\theta^1)^2(\theta^6)^2 + 24\theta^0\theta^2(\theta^6)^2 + 40\theta^1(\theta^3)^2\theta^7\\
& - 60\theta^1\theta^2\theta^4\theta^7 - 10\theta^0\theta^3\theta^4\theta^7 + 24(\theta^1)^2\theta^5\theta^7 + 18\theta^0\theta^2\theta^5\theta^7 - 14\theta^0\theta^1\theta^6\theta^7 + (\theta^0)^2(\theta^7)^2.
\end{aligned}$$

Among the small dimensions $n = 7$ is quite special, since here the next invariant linearly and functionally independent of the metric $\tilde{g}$ has $q = 4$. We have the following

Proposition 8.12. *In dimension $n = 7$, the invariant of the lowest degree is the metric ${}^7\tilde{g}$. There are no invariants of degree $q = 3$ and only two linearly independent, invariants of degree $q = 4$. The first of them is ${}^7\tilde{g}^2$. The second can be chosen to be*

$$\begin{aligned}
{}^7\tilde{\Upsilon} = {} & 160(\theta^3)^4 - 480\theta^2(\theta^3)^2\theta^4 + 1035(\theta^2)^2(\theta^4)^2 - 1080\theta^1\theta^3(\theta^4)^2 + 540\theta^0(\theta^4)^3\\
& - 1080(\theta^2)^2\theta^3\theta^5 + 1920\theta^1(\theta^3)^2\theta^5 - 180\theta^1\theta^2\theta^4\theta^5\\
& - 1080\theta^0\theta^3\theta^4\theta^5 - 288(\theta^1)^2(\theta^5)^2 + 540\theta^0\theta^2(\theta^5)^2 + 540(\theta^2)^3\theta^6 - 1080\theta^1\theta^2\theta^3\theta^6 + 400\theta^0(\theta^3)^2\theta^6\\
& + 540(\theta^1)^2\theta^4\theta^6 - 330\theta^0\theta^2\theta^4\theta^6 - 84\theta^0\theta^1\theta^5\theta^6 + 7(\theta^0)^2(\theta^6)^2.
\end{aligned}$$

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8.2. Stabilizers of the irreducible $\mathbf{GL}(2,\mathbb{R})$ *in dimensions* $n<10$

In dimensions $n\le 10$ the $\mathbf{GL}(2,\mathbb{R})$ invariant tensors of low order $q\le 4$ turn out to be sufficient to reduce the $\mathbf{GL}(n,\mathbb{R})$ group to $\mathbf{GL}(2,\mathbb{R})$ in its irreducible n-dimensional representation.

Given an invariant tensor

$$\tilde t=\frac{1}{q!}t_{i_1i_2\dots i_q}\theta^{i_1}\dots\theta^{i_q}$$

of degree q on P and a $\mathbf{GL}(n,\mathbb{R})$-valued function $a=(a^i{}_j)$ on P, at every point $p\in P$, we have a $\mathbf{GL}(n,\mathbb{R})$-action

$$(a^i{}_j,\tilde t_{i_1i_2\dots i_q})\mapsto(\rho_n(a)\tilde t)_{j_1j_2\dots j_q}=a^{i_1}{}_{j_1}a^{i_2}{}_{j_2}\dots a^{i_q}{}_{j_q}\tilde t_{i_1i_2\dots i_q}.$$

A subgroup $G_{\tilde t}$ of $\mathbf{GL}(n,\mathbb{R})$ consisting of $a=(a^i{}_j)$ such that

$$\rho_n(a)\tilde t=(\det a)^{q/n}\,\tilde t,$$

is the stabiliser of $\tilde t$ at $p\in P$. Since $\tilde t$ is an invariant then, obviously $\mathbf{GL}(2,\mathbb{R})\subset G_{\tilde t}$.

This leads to the following question: how many invariants is needed in dimension n so that its common stabiliser is precisely $\mathbf{GL}(2,\mathbb{R})$ in its n dimensional irreducible representation?

Inspecting Hilbert's results we checked that in dimensions $4\le n\le 9$ we have

Theorem 8.13. *For each $n=4,5,6,7,8,9$, the full stabiliser group of the respective invariant tensor ${}^n\tilde\Upsilon$ of Propositions 8.9, 8.11 and 8.12, is the group $\mathbf{GL}(2,\mathbb{R})$ in the n-dimensional irreducible representation ρ_n. In particular, if $n=5,7,9$ these stabilisers are subgroups of the respective pseudohomothetic groups $\mathbf{CO}(3,2)$, $\mathbf{CO}(4,3)$ and $\mathbf{CO}(5,4)$, each in its defining representation.*

Thus in each of these dimensions it is the lowest order non-quadratic invariant what is responsible for the full reduction from $\mathbf{GL}(n,\mathbb{R})$ to $\mathbf{GL}(2,\mathbb{R})$.

Remark 8.14. In dimension $n=5$, using (8.7) and Proposition 8.9 we define a conformal metric $[{}^5g_{ij}]$ represented by

$${}^5g_{ij}=\frac{1}{2}\frac{\partial^2}{\partial\theta^i\partial\theta^j}\big({}^5\tilde g\big),\quad i,j=0,1,2,3,4$$

and a conformal symmetric tensor of third degree $[{}^5\Upsilon_{ijk}]$ represented by

$${}^5\Upsilon_{ijk}=-\frac{\sqrt3}{8}\frac{\partial^3}{\partial\theta^i\partial\theta^j\partial\theta^k}\big({}^5\tilde\Upsilon\big),\quad i,j,k,l=0,1,2,3,4.$$

The convenient factor $-\frac{\sqrt3}{8}$ in the expression for ${}^5\Upsilon_{ijk}$ was chosen so that the pair $({}^5g_{ij},{}^5\Upsilon_{ijk})$ satisfies Cartan's identities (i)–(iii) of Section 2. This leads to the $\mathbf{GL}(2,\mathbb{R})$ geometries in dimension 5 considered in Sections 3–6.

Remark 8.15. In the next odd dimension situation is quite similar, but now we have a quartic invariant ${}^7\tilde\Upsilon$. Thus apart from the conformal metric $[{}^7g_{ij}]$ represented by

$${}^7g_{ij}=\frac{1}{2}\frac{\partial^2}{\partial\theta^i\partial\theta^j}\big({}^7\tilde g\big),\quad i,j=0,1,2,3,4,5,6$$

we have a conformal symmetric tensor of fourth degree $[{}^7\Upsilon_{ijkl}]$ represented by

$${}^7\Upsilon_{ijkl}=\frac{1}{24}\frac{\partial^4}{\partial\theta^i\partial\theta^j\partial\theta^k\partial\theta^l}\big({}^7\tilde\Upsilon\big),\quad i,j,k,l=0,1,2,3,4,5,6. \tag{8.8}$$

Note that ${}^7\tilde\Upsilon$ of Proposition 8.12 was chosen in such a way that the fourth order ${}^7\Upsilon_{ijkl}$ satisfied

$${}^7g^{ij}\,{}^7\Upsilon_{ijkl}=0,\quad\text{where }{}^7g^{ij}\,{}^7g_{jk}=\delta^i{}_k.$$

This choice of the fourth order invariant is nevertheless arbitrary, since we can always get another invariant of the fourth order by replacing ${}^7\Upsilon$ with

$${}^7\bar\Upsilon_{ijkl}=c_1{}^7\tilde\Upsilon_{ijkl}+c_2{}^7\tilde g_{(ij}{}^7\tilde g_{kl)}.$$

It is interesting to note that the choice

$$c_1=\frac{2\sqrt5}{\sqrt{3147}},\qquad c_2=\frac{34}{\sqrt{15\,735}}$$

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applied to ${}^7\bar{\Upsilon}$, leads, via formula like (8.8), to ${}^7\bar{\Upsilon}_{ijkl}$ satisfying the Cartan-like identity:

$${}^7g^{ih}\,{}^7g^{ef}\,{}^7\bar{\Upsilon}_{ie(jk}\,{}^7\bar{\Upsilon}_{lm)fh} = {}^7g_{(jk}\,{}^7g_{lm)}$$

and

$${}^7g^{ij}\,{}^7\bar{\Upsilon}_{ijkl} = \frac{3}{2}c_2\,{}^7g_{kl}, \quad \text{where } {}^7g^{ij}\,{}^7g_{jk} = \delta^i{}_k.$$

Note also that the above Cartan-like identities are preserved under the conformal transformation

$$({}^7g_{ij}, {}^7\bar{\Upsilon}_{ijkl}) \mapsto ({}^7g'_{ij}, {}^5\bar{\Upsilon}'_{ijkl}) = (e^{2\phi}\,{}^7g_{ij}, e^{4\phi}\,{}^7\bar{\Upsilon}_{ijkl}),$$

where $\phi \in \mathbb{R}$.

Thus the $\mathbf{GL}(2,\mathbb{R})$ geometries in dimension $n = 7$ may be defined by a conformal class of pairs of tensors $[{}^7g_{ij}, {}^7\bar{\Upsilon}_{ijkl}]$ with the properties and transformations as above.

Remark 8.16. By analogy, in dimensions $n = 4, 6, 8$, the irreducible $\mathbf{GL}(2,\mathbb{R})$ geometries may be described in terms of a conformal tensor $[{}^n\Upsilon_{ijkl}]$ represented by

$${}^n\Upsilon_{ijkl} = \frac{1}{24}\frac{\partial^4}{\partial\theta^i\partial\theta^j\partial\theta^k\partial\theta^l}\left({}^n\tilde{\Upsilon}\right), \quad i,j,k,l = 0,1,2,\ldots,n-1,$$

and obtained in terms of the respective quartic invariants ${}^n\tilde{\Upsilon}$ of Proposition 8.11.

Remark 8.17. Dimension $n = 9$ is similar to dimension $n = 5$. A periodicity with period four is a remarkable feature of Hilbert's theory of algebraic invariants [18], p. 60.

8.3. Wünschmann conditions for the existence of $\mathbf{GL}(2,\mathbb{R})$ *geometries on the solution space of ODEs*

An invariant tensor $\tilde{t}$, by its very definition, has a property that it descends to a nondegenerate conformal tensor $[t]$ on the solutions space $M^n = P/\mathfrak{h}$ of the equation $y^{(n)} = 0$. In particular in dimensions $4 \le n \le 9$ the conformal class $[{}^n\Upsilon]$, corresponding to invariant tensors ${}^n\tilde{\Upsilon}$ reduces the structure group of M^n to $\mathbf{GL}(2,\mathbb{R})$ defining an irreducible $\mathbf{GL}(2,\mathbb{R})$ geometry there. We do not know how many invariant tensors are needed to achieve this reduction for $n > 9$, but it is obvious that for a given n this number is finite, say w_n. Thus for each $n \ge 3$ we have a finite number of invariants ${}^n\tilde{\Upsilon}_I, I = 1,2,\ldots w_n$, which descend to the solution space M^n of the equation $y^{(n)} = 0$ equipping it with a $\mathbf{GL}(2,\mathbb{R})$ structure. It is important that each of the invariants ${}^n\tilde{\Upsilon}_I$ has only *constant* coefficients when expressed in terms of the invariant coframe $(\theta^0,\ldots,\theta^{n-1})$ on P (see, for example, every ${}^n\tilde{\Upsilon}$ of the preceding section).

Now, we return to a *general* n-th order ODE (8.1). Thus we now have a general function $F(x,y,y',y'',y^{(3)},\ldots,y^{(n-1)})$, which determines the contact forms $(\omega^0,\omega^1,\ldots,\omega^{n-1},w_+)$ by (8.2). Corresponding to these forms we have the invariant forms $(\theta^0,\ldots,\theta^{n-1},\Gamma_+)$ of (8.3), which live on bundle $J\times G$ over J. We can now ask the following question (this generalises to arbitrary $n > 3$ the similar question of Section 6): What shall we assume about F defining the contact equivalence class of ODEs (8.1) that there exists a $(4+n)$-dimensional subbundle P of $J\times G$ on which the forms $(\theta^0,\ldots,\theta^{n-1},\Gamma_+)$ satisfy:

$$\begin{aligned}
d\theta^0 &= (n-1)(\Gamma_1+\Gamma_0)\wedge\theta^0 + (1-n)\Gamma_+\wedge\theta^1 + \frac{1}{2}T^0{}_{ij}\theta^i\wedge\theta^j,\\
d\theta^1 &= -\Gamma_-\wedge\theta^0 + [(n-1)\Gamma_1 + (n-3)\Gamma_0]\wedge\theta^1 + (2-n)\Gamma_+\wedge\theta^2 + \frac{1}{2}T^1{}_{ij}\theta^i\wedge\theta^j,\\
&\vdots\\
d\theta^k &= -k\Gamma_-\wedge\theta^{k-1} + [(n-1)\Gamma_1 + (n-2k-1)\Gamma_0]\wedge\theta^k + (1+k-n)\Gamma_+\wedge\theta^{k+1} + \frac{1}{2}T^k{}_{ij}\theta^i\wedge\theta^j,\\
&\vdots\\
d\theta^{n-1} &= (1-n)\Gamma_-\wedge\theta^{n-2} + (n-1)(\Gamma_1-\Gamma_0)\wedge\theta^{n-1} + \frac{1}{2}T^{n-1}_{ij}\theta^i\wedge\theta^j,\\
d\Gamma_+ &= 2\Gamma_0\wedge\Gamma_+ + \frac{1}{2}R_{+ij}\theta^i\wedge\theta^j,\\
d\Gamma_- &= -2\Gamma_0\wedge\Gamma_- + \frac{1}{2}R_{-ij}\theta^i\wedge\theta^j,\\
d\Gamma_0 &= \Gamma_+\wedge\Gamma_- + \frac{1}{2}R_{0ij}\theta^i\wedge\theta^j,\\
d\Gamma_1 &= \frac{1}{2}R_{ij}\theta^i\wedge\theta^j.
\end{aligned} \tag{8.9}$$

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As first observed by Wünschmann [1] and then successively used by Newman and collaborators [19] this question can be reformulated into a nicer one. To make this reformulation we repeat our arguments from Proposition 8.1.

Suppose that we are able to satisfy system (8.9) by the forms (8.3). Consider the distribution

$$\mathfrak{h} = \{X \in \mathrm{T}P \text{ s.t. } X \lrcorner \theta^i = 0,\ i = 0, 1, 2, \dots, n-1\}$$

annihilating θs. Despite of the fact that system (8.9) involves new terms, when compared with system (8.6), they do not destroy the integrability of the distribution $\mathfrak{h}$; the first n Eqs. (8.9) still guarantee that $\mathfrak{h}$ is integrable. Thus manifold P is foliated by 4-dimensional leaves tangent to the distribution $\mathfrak{h}$. The space of leaves of this distribution $P/\mathfrak{h}$ can be identified with the solution space $M^n = P/\mathfrak{h}$ of Eq. (8.1). Now, on the manifold P of system (8.9), we define w_n tensors ${}^n\tilde{\Upsilon}_I$, which formally are given by the same formulae that defined the w_n invariants ${}^n\tilde{\Upsilon}_I$ of the flat system (8.6) needed to get the full reduction to $\mathbf{GL}(2, \mathbb{R})$. So, when defining the present ${}^n\tilde{\Upsilon}_I$, we use the same formulae as for the $y^{(n)} = 0$ case, replacing forms θ of the flat case, with forms θ satisfying system (8.9). It is now easy to verify that the question about the conditions on F to admit P with system (8.9) is equivalent to the requirement that all w_n tensors ${}^n\tilde{\Upsilon}_I$ transform *conformally* when Lie transported along the leaves of distribution $\mathfrak{h}$. Infinitesimally this condition is equivalent to the existence of functions $c_I(X)$ on P such that

$$\mathcal{L}_X({}^n\tilde{\Upsilon}_I) = c_I(X)\, {}^n\tilde{\Upsilon}_I,$$

$\forall X \in \mathfrak{h}$, and $\forall I = 1, 2, \dots w_n$. If this is satisfied then tensors ${}^n\tilde{\Upsilon}_I$ descend to a conformal class of tensors $[{}^n\Upsilon_1, {}^n\Upsilon_2, \dots, {}^n\Upsilon_{w_n}]$ on the solution space M^n defining a $\mathbf{GL}(2, \mathbb{R})$ there.

We know that in dimension $n = 5$ the conformal preservation of ${}^5\tilde{g}$ and ${}^5\tilde{\Upsilon}$ is equivalent to the requirement on function $F = F(x, y, y_1, y_2, y_3, y_4)$ to satisfy Wünschmann conditions (6.5). Also in higher dimensions the Wünschmann conditions are obtained in this way. They are obstructions for the conformal preservation of tensors $[{}^n\Upsilon_1, {}^n\Upsilon_2, \dots, {}^n\Upsilon_{w_n}]$ along the distribution $\mathfrak{h}$. In particular, if $4 \le n < 10$ they are given by the following

Theorem 8.18. *Let M^n be the solution space of nth order ODE*

$$y^{(n)} = F(x, y, y', y'', y^{(3)}, \dots, y^{(n-1)}), \tag{8.10}$$

with $4 \le n < 10$, and let

$$\mathcal{D} = \partial_x + y_1\partial_y + y_2\partial_{y_1} + \dots + y_{n-1}\partial_{y_{n-2}} + F\partial_{y_{n-1}}$$

be the total derivative. The necessary conditions for a contact equivalence class of ODEs (8.10) to define a principal $\mathbf{GL}(2, \mathbb{R})$-bundle $\mathbf{GL}(2, \mathbb{R}) \to P \to M^n$ with invariants forms $(\theta^0, \dots, \theta^{n-1}, \Gamma_+, \Gamma_-, \Gamma_0, \Gamma_1)$ satisfying system (8.9) is that the defining function F of (8.10) satisfies $n-2$ Wünschmann conditions given below:

- $n = 4$*:*

$$4\mathcal{D}^2F_3 - 8\mathcal{D}F_2 + 8F_1 - 6\mathcal{D}F_3F_3 + 4F_2F_3 + F_3^3 = 0,$$
$$160\mathcal{D}^2F_2 - 640\mathcal{D}F_1 + 144(\mathcal{D}F_3)^2 - 352\mathcal{D}F_3F_2 + 144F_2^2 - 80\mathcal{D}F_2F_3 + 160F_1F_3$$
$$-72\mathcal{D}F_3F_3^2 + 88F_2F_3^2 + 9F_3^4 + 16000F_y = 0,$$

- $n = 5$*:*

$$50\mathcal{D}^2F_4 - 75\mathcal{D}F_3 + 50F_2 - 60F_4\mathcal{D}F_4 + 30F_3F_4 + 8F_4^3 = 0$$
$$375\mathcal{D}^2F_3 - 1000\mathcal{D}F_2 + 350\mathcal{D}F_4^2 + 1250F_1 - 650F_3\mathcal{D}F_4$$
$$+ 200F_3^2 - 150F_4\mathcal{D}F_3 + 200F_2F_4 - 140F_4^2\mathcal{D}F_4 + 130F_3F_4^2 + 14F_4^4 = 0$$
$$1250\mathcal{D}^2F_2 - 6250\mathcal{D}F_1 + 1750\mathcal{D}F_3\mathcal{D}F_4 - 2750F_2\mathcal{D}F_4 - 875F_3\mathcal{D}F_3 + 1250F_2F_3 - 500F_4\mathcal{D}F_2 + 700(\mathcal{D}F_4)^2F_4$$
$$+ 1250F_1F_4 - 1050F_3F_4\mathcal{D}F_4 + 350F_3^2F_4 - 350F_4^2\mathcal{D}F_3 + 550F_2F_4^2 - 280F_4^3\mathcal{D}F_4 + 210F_3F_4^3 + 28F_4^5 + 18750F_y = 0.$$

- $n = 6$*:*

$$45\mathcal{D}^2F_5 - 54\mathcal{D}F_4 + 27F_3 - 45\mathcal{D}F_5F_5 + 18F_4F_5 + 5F_5^3\ 945\mathcal{D}^2F_4 - 1890\mathcal{D}F_3 + 900(\mathcal{D}F_5)^2 + 1575F_2 - 1350\mathcal{D}F_5F_4$$
$$+ 333F_4^2 - 315\mathcal{D}F_4F_5 + 315F_3F_5 - 300\mathcal{D}F_5F_5^2 + 225F_4F_5^2 + 25F_5^4 = 0$$
$$2835\mathcal{D}^2F_3 - 9450\mathcal{D}F_2 + 4320\mathcal{D}F_4\mathcal{D}F_5 + 14175F_1 - 5130\mathcal{D}F_5F_3 - 1728\mathcal{D}F_4F_4 + 1863F_3F_4 - 945\mathcal{D}F_3F_5$$
$$+ 1800(\mathcal{D}F_5)^2F_5 + 1575F_2F_5 - 2160\mathcal{D}F_5F_4F_5 + 576F_4^2F_5 - 720\mathcal{D}F_4F_5^2 + 855F_3F_5^2$$
$$- 600\mathcal{D}F_5F_5^3 + 360F_4F_5^3 + 50F_5^5 = 0$$
$$14175\mathcal{D}^2F_2 - 85050\mathcal{D}F_1 + 6480(\mathcal{D}F_4)^2 + 16200\mathcal{D}F_3\mathcal{D}F_5 - 31050\mathcal{D}F_5F_2 - 9720\mathcal{D}F_4F_3 + 3645F_3^2 - 6480\mathcal{D}F_3F_4$$
$$+ 5400\mathcal{D}F_5^2F_4 + 11475F_2F_4 - 4320\mathcal{D}F_5F_4^2 + 864F_4^3 - 4725\mathcal{D}F_2F_5 + 10800\mathcal{D}F_4\mathcal{D}F_5F_5 + 14175F_1F_5$$
$$- 10800\mathcal{D}F_5F_3F_5 - 6480\mathcal{D}F_4F_4F_5 + 5940F_3F_4F_5 - 2700\mathcal{D}F_3F_5^2 + 4500(\mathcal{D}F_5)^2F_5^2 + 5175F_2F_5^2$$
$$-7200\mathcal{D}F_5F_4F_5^2 + 2340F_4^2F_5^2 - 1800\mathcal{D}F_4F_5^3 + 1800F_3F_5^3 - 1500\mathcal{D}F_5F_5^4 + 1050F_4F_5^4 + 125F_5^6 + 297675F_y = 0$$

• $n = 7$:

$$245\mathcal{D}^2F_6 - 245\mathcal{D}F_5 + 98F_4 - 210\mathcal{D}F_6F_6 + 70F_5F_6 + 20F_6^3 = 0$$

$$6860\mathcal{D}^2F_5 - 10976\mathcal{D}F_4 + 6615(\mathcal{D}F_6)^2 + 6860F_3 - 8330\mathcal{D}F_6F_5 + 1715F_5^2 - 1960\mathcal{D}F_5F_6 + 1568F_4F_6 - 1890\mathcal{D}F_6F_6^2 + 1190F_5F_6^2 + 135F_6^4 = 0$$

$$9604\mathcal{D}^2F_4 - 24010\mathcal{D}F_3 + 15435\mathcal{D}F_5\mathcal{D}F_6 + 24010F_2 - 14749\mathcal{D}F_6F_4 - 5145\mathcal{D}F_5F_5 + 4459F_4F_5 - 2744\mathcal{D}F_4F_6 + 6615(\mathcal{D}F_6)^2F_6 + 3430F_3F_6 - 6615\mathcal{D}F_6F_5F_6 + 1470F_5^2F_6 - 2205\mathcal{D}F_5F_6^2 + 2107F_4F_6^2 - 1890\mathcal{D}F_6F_6^3 + 945F_5F_6^3 + 135F_6^5 = 0$$

$$336140\mathcal{D}^2F_3 - 1344560\mathcal{D}F_2 + 180075(\mathcal{D}F_5)^2 + 432180\mathcal{D}F_4\mathcal{D}F_6 + 2352980F_1 - 624260\mathcal{D}F_6F_3 - 216090\mathcal{D}F_5F_4 + 64827F_4^2 - 144060\mathcal{D}F_4F_5 + 154350(\mathcal{D}F_6)^2F_5 + 192080F_3F_5 - 102900\mathcal{D}F_6F_5^2 + 17150F_5^3 - 96040\mathcal{D}F_3F_6 + 308700\mathcal{D}F_5\mathcal{D}F_6F_6 + 192080F_2F_6 - 246960\mathcal{D}F_6F_4F_6 - 154350\mathcal{D}F_5F_5F_6 + 113190F_4F_5F_6 - 61740\mathcal{D}F_4F_6^2 + 132300(\mathcal{D}F_6)^2F_6^2 + 89180F_3F_6^2 - 176400\mathcal{D}F_6F_5F_6^2 + 47775F_5^2F_6^2 - 44100\mathcal{D}F_5F_6^3 + 35280F_4F_6^3 - 37800\mathcal{D}F_6F_6^4 + 22050F_5F_6^4 + 2700F_6^6 = 0$$

$$2352980\mathcal{D}^2F_2 - 16470860\mathcal{D}F_1 + 1512630\mathcal{D}F_4\mathcal{D}F_5 + 2268945\mathcal{D}F_3\mathcal{D}F_6 - 5126135\mathcal{D}F_6F_2 - 1512630\mathcal{D}F_5F_3 - 907578\mathcal{D}F_4F_4 + 648270(\mathcal{D}F_6)^2F_4 + 907578F_3F_4 - 756315\mathcal{D}F_3F_5 + 1080450\mathcal{D}F_5\mathcal{D}F_6F_5 + 1596665F_2F_5 - 1080450\mathcal{D}F_6F_4F_5 - 360150\mathcal{D}F_5F_5^2 + 288120F_4F_5^2 - 672280\mathcal{D}F_2F_6 + 540225(\mathcal{D}F_5)^2F_6 + 1296540\mathcal{D}F_4\mathcal{D}F_6F_6 + 2352980F_1F_6 - 1620675\mathcal{D}F_6F_3F_6 - 864360\mathcal{D}F_5F_4F_6 + 324135F_4^2F_6 - 648270\mathcal{D}F_4F_5F_6 + 926100(\mathcal{D}F_6)^2F_5F_6 + 756315F_3F_5F_6 - 771750\mathcal{D}F_6F_5^2F_6 + 154350F_5^3F_6 - 324135\mathcal{D}F_3F_6^2 + 926100\mathcal{D}F_5\mathcal{D}F_6F_6^2 + 732305F_2F_6^2 - 926100\mathcal{D}F_6F_4F_6^2 - 617400\mathcal{D}F_5F_5F_6^2 + 524790F_4F_5F_6^2 - 185220\mathcal{D}F_4F_6^3 + 396900(\mathcal{D}F_6)^2F_6^3 + 231525F_3F_6^3 - 661500\mathcal{D}F_6F_5F_6^3 + 209475F_5^2F_6^3 - 132300\mathcal{D}F_5F_6^4 + 119070F_4F_6^4 - 113400\mathcal{D}F_6F_6^5 + 75600F_5F_6^5 + 8100F_6^7 + 65883440F_y = 0.$$

Remark 8.19. If $n = 3$ we have only one Wünschmann condition [20,1]:

$$9\mathcal{D}^2F_2 - 27\mathcal{D}F_1 - 18\mathcal{D}F_2F_2 + 18F_1F_2 + 4F_2^3 + 54F_y = 0.$$

and, if it satisfied, a conformal Lorentzian geometry associated with a metric

$$^3g = \theta^0\theta^2 - (\theta^1)^2$$

is naturally defined on the solution space.

Remark 8.20. If $n = 4$ the ODEs satisfying the two Wünschmann conditions lead to very *nontrivial* geometries on 4-dimensional solution spaces. These are a sort of conformal Weyl geometries, which instead of a metric are define in terms of the conformal rank four tensor $^4\Upsilon$. These geometries define a characteristic connection, which is $\mathfrak{gl}(2,\mathbb{R})$ valued and has an exotic holonomy [5]. By this we mean that the holonomy of this nonmetric but torsionless connection does not appear on the Berger's list [5]. See also our account on this subject in [21].

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