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Sharp version of the Goldberg–Sachs theorem

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Abstract We reexamine from first principles the classical Goldberg–Sachs theorem from General Relativity. We cast it into the form valid for complex metrics, as well as real metrics of any signature. We obtain the sharpest conditions on the derivatives of the curvature that are sufficient for the implication (integrability of a field of alpha planes)⇒(algebraic degeneracy of the Weyl tensor). With every integrable field of alpha planes, we associate a natural connection, in terms of which these conditions have a very simple form.

Keywords Goldberg–Sachs theorem · Algebraically special fields · Newman–Penrose formalism

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1 Introduction

The original Goldberg–Sachs theorem of General Relativity [4] is a statement about Ricci flat 4-dimensional Lorentzian manifolds. Nowadays, it is often stated in the following, slightly stronger, form:

Theorem 1.1 *Let* $(\mathcal{M}, g)$ *be a 4-dimensional Lorentzian manifold which satisfies the Einstein equations* $Ric(g) = \Lambda g$. *Then it locally admits a congruence of null and shearfree geodesics if and only if its Weyl tensor is algebraically special.*

If $(\mathcal{M}, g)$ is conformally flat, then such a spacetime admits infinitely many congruences of null and shearfree geodesics.

This theorem proved to be very useful in General Relativity, especially during the 'golden era' of General Relativity in the 1960s, when the important Einstein spacetimes, such as Kerr-Newman, were constructed.

Remarkably, years after the Lorentzian version was first stated, it was pointed out that the theorem has a Riemannian analog [21]. This gives a very powerful local result in 4-dimensional Riemannian geometry, which can be stated as follows [15,16]:

Theorem 1.2 *Let* $(\mathcal{M}, g)$ *be a 4-dimensional Riemannian manifold which satisfies the Einstein equations* $Ric(g) = \Lambda g$. *Then it is locally a hermitian manifold if and only if its Weyl tensor is algebraically special.*

Note that the notion of a congruence of null and shearfree geodesics, in the Lorentzian case, is replaced by the notion of a complex surface with an orthogonal complex structure, in the Riemannian case. Also in this case, if $(\mathcal{M}, g)$ is conformally flat, it admits infinitely many local hermitian structures.

Theorem 1.2 was in particular used by LeBrun [11] to obtain all compact complex surfaces, which admit an Einstein metric that is hermitian but not Kähler, (see also [3,12]).

The only other signature which, in addition to the Lorentzian and Euclidean signatures, a four-dimensional metric may have, is the 'split signature': $(+, +, -, -)$. It is again remarkable, that the Goldberg–Sachs theorem has also its split signature version. Here, however, the situation is more complicated, and the theorem should be split into two statements:

Theorem 1.3 *Let* $(\mathcal{M}, g)$ *be a 4-dimensional manifold equipped with a split signature metric which satisfies the Einstein equations* $Ric(g) = \Lambda g$. *If in addition* $(\mathcal{M}, g)$ *is either locally a pseudohermitian manifold, or it is locally foliated by real 2-dimensional totally null submanifolds, then* $(\mathcal{M}, g)$ *has an algebraically special Weyl tensor.*

Theorem 1.4 *Let* $(\mathcal{M}, g)$ *be a 4-dimensional manifold equipped with a split signature metric which satisfies the Einstein equations* $Ric(g) = \Lambda g$ *and which is conformally non-flat. If in addition* $(\mathcal{M}, g)$ *has an algebraically special Weyl tensor with a multiple principal totally null field of 2-planes having locally constant real index, then it is either locally a pseudohermitian manifold, or it is locally foliated by real 2-dimensional totally null submanifolds.*

In these two theorems, the term 'pseudohermitian manifold' means: 'a complex manifold with a complex structure which is an orthogonal transformation for the split signature metric g'. The more complicated terms such as 'multiple principal totally null field of 2-planes having locally constant real index' will be explained in Sect. 3.

All four theorems have in common the part concerned with the Einstein assumption and algebraic speciality of the Weyl tensor. But they look quite different on the other side of the equivalence. The similarity in the first part suggests that also the second part should have a unified description. This is indeed the case. As will be shown in the sequel, these theorems are consequences, or better said, appropriate interpretations, of the following complex theorem [19,20]:

Theorem 1.5 *Let* $(\mathcal{M}, g)$ *be a 4-dimensional manifold equipped with a complex valued metric* g *which is Einstein. Then the following two conditions are equivalent:*

(i) $(\mathcal{M}, g)$ *admits a complex two-dimensional totally null distribution* $\mathcal{N} \subset \mathrm{T}^{\mathbb{C}}\mathcal{M}$, *which is integrable in the sense that* $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$.
(ii) *The Weyl tensor of* $(\mathcal{M}, g)$ *is algebraically special.*

2 Convenient sharper versions

Our motivation for reexamining these theorems is as follows:

First, as remarked e.g. by Trautman [26], all the theorems have an *aesthetic* defect. This is due to the fact that both equivalence conditions, such as (i) and (ii) in Theorem 1.5, are *conformal* properties of $(\mathcal{M}, g)$; the Einstein assumption does not share this symmetry. Of course, a way out is to replace the Einstein assumption by an assumption about $(\mathcal{M}, g)$ being *conformal* to Einstein, see e.g. [5]. Thus, in the complex version of the theorem the assumption should be: $(\mathcal{M}, g)$ is conformal to Einstein.

This leads to the question about the weakest conformal assumption involving (the derivatives of) the Ricci part of the curvature that is sufficient to ensure the thesis of the Goldberg–Sachs theorem. Several authors have proposed their assumptions here (see [9,18,22–24]). For example, the authors of [9,18,24] use an assumption, which involves contractions of (the derivatives of) the Ricci tensor with the vectors spanning the totally null distribution $\mathcal{N}$.

Trautman in [26] has a different point of view. He proposes that there should be a conformally invariant assumption which does not refer to the thesis of the theorem. Trautman conjectures that a proper replacement for the assumption is: $(\mathcal{M}, g)$ is *Bach flat*. This, in four dimensions, is certainly conformal, does not refer to $\mathcal{N}$, and is necessary for g to be conformal to Einstein.

In this paper, among other things, we show that the approach of [9,18,24] is the proper one. In particular in Sect. 7.4, we show that in the case of a Riemannian signature metric, Trautman's conjecture is not true.

Our new analysis of the Goldberg–Sachs theorem starts with Theorem 5.10. Its proof shows that it is rather hard to find a single curvature condition, different than the conformally Einstein one, which would guarantee equivalence in the thesis of Goldberg and Sachs. This proof also clearly shows that it is the implication $(algebraical\ speciality) \Rightarrow (integrability\ of\ totally\ null\ 2-planes)$ that causes the difficulties. Then in Sect. 5.2 we give various generalizations of the Goldberg–Sachs theorem to the conformal setting, starting with the conformal replacement of the assumption of Theorem 5.10 which implies $(algebraical\ speciality) \Rightarrow (integrability\ of\ totally\ null\ \ 2\text{-}planes)$. This culminates in a slight improvement of the theorem of Penrose and Rindler [18], which we give in our Theorem 5.28, and in Theorems 5.31 and 5.32, which treat more special cases. These three theorems we consider as the sharpest conformal improvement of the classical Goldberg–Sachs theorem, in a sense that they include both implications $(algebraical\ speciality) \Rightarrow (integrability\ of\ totally\ null\ 2-planes)$ and $(algebraical\ speciality) \Leftarrow (integrability\ of\ totally\ null\ 2\text{-}planes)$. In Sect. 7, the real versions of theorems from Sect. 5.2 are considered, the most striking of them being:

Theorem 2.1 *Let $\mathcal{M}$ be a 4-dimensional oriented manifold with a (real) metric g of Riemannian signature, whose self-dual part of the Weyl tensor is non-vanishing. Let J be a metric compatible almost complex structure on $\mathcal{M}$ such that its holomorphic distribution $\mathcal{N} = \mathrm{T}^{(1,0)}\mathcal{M}$ is self-dual. Then any two of the following imply the third:*

(0) *The Cotton tensor of g is degenerate on $\mathcal{N}$, $A_{|\mathcal{N}} \equiv 0$.*
(i) *J has vanishing Nijenhuis tensor on $\mathcal{M}$, meaning that $(\mathcal{M}, g, J)$ is a hermitian manifold.*
(ii) *The self-dual part of the Weyl tensor is algebraically special on $\mathcal{M}$ with $\mathcal{N}$ as a field of multiple principal self-dual totally null 2-planes.*

This theorem in its (more complicated) Lorentzian version is present in [9,18,24]. The Riemannian version is implicit there, once one understands the relation between fields of totally null 2-planes and almost hermitian structures, as for example, explained in [15,16], (see also [1] where these developments are related to global issues on compact Riemannian manifolds.)

When one is only interested in the implication $(algebraical\ speciality) \Rightarrow (integrabilityof\ totally\ null\ 2\text{-}planes)$, our proposal for the sharpest version of the Goldberg–Sachs theorem, is given in Theorem 5.21. This gets its final version in Theorem 6.5. This last theorem utilizes a new object which we introduce in this paper, namely a connection, which is naturally associated with each *integrable* field of totally null 2-planes $\mathcal{N}$. We call this connection the *characteristic connection* of a field of totally null 2-planes.

If $\mathcal{N}$ satisfies the integrability conditions $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, we prove in Theorem 6.1 the existence of a class of connections $\overset{W}{\nabla}$, which are characterized by the following two conditions:

$$\left.\begin{array}{l}\overset{W}{\nabla}_X\mathcal{N} \subset \mathcal{N}\\ \overset{W}{\nabla}_X g = -B(X)g\end{array}\right. \qquad \text{for all } X \in \mathrm{T}\mathcal{N}.$$

These connections are *not* canonical—they define the 1-form B only partially. However, they naturally restrict to a *unique* (partial) connection $\check{\nabla}$ on $\mathcal{N}$. This by definition is the characteristic connection of $\mathcal{N}$. In general this connection is complex. It is defined everywhere on

$\mathcal{M}$, but it only enables one to differentiate vectors from $\mathcal{N}$ along vectors from $\mathcal{N}$. Thus, the connection $\check{\nabla}$ is effectively 2-dimensional, and as such, its curvature $\check{R}^A_{BCD}$ has only one independent component. It follows that

$$\check{R}^A_{BCD} = 4\Psi_1 \delta^A_B \epsilon_{CD},$$

where Ψ_1 is the Weyl tensor component whose non-vanishing is the obstruction to the algebraic speciality of the metric. The symbol δ^A_B is the Kronecker delta (i.e. the identity) on $\mathcal{N}$ and the ϵ_{CD} is the 2-dimensional antisymmetric tensor. The Ricci tensor $\check{R}_{AB} = \check{R}^C_{ACB}$ for $\check{\nabla}$ is then $\check{R}_{AB} = 4\Psi_1\epsilon_{AB}$ and is *antisymmetric*.

Now the replacement for the Einstein condition in the Goldberg–Sachs theorem, in its $(integrability\ of\ \mathcal{N}) \Rightarrow (algebraical\ speciality)$ part, is

$$\check{\nabla}_{[A}\check{\nabla}_{B]}\check{R}_{CD} \equiv 0,$$

as is explained in Theorem 6.5.

An interesting situation occurs in the Riemannian (and also in the split signature) case. There, the reality conditions imposed on the 1-form B defining the class of connections $\overset{W}{\nabla}$, choose a prefered connection from the class. This connection yields more information than the partial connection. Using this connection we get Theorem 7.16, which is a slightly more elegant (pseudo)hermitian version of the signature independent Theorem 6.5.

3 Totally null 2-planes in four dimensions

To discuss the geometrical meaning of the complex version of the Goldberg–Sachs theorem, we recall the known [7] properties of totally null 2-planes as we range over the possible signatures of 4-dimensional metrics.

Let V be a 4-dimensional *real* vector space equipped with a metric g, of some signature. Given V and g, we consider their complexifications. Thus, we have $V^{\mathbb{C}}$ and the metric g which is extended to act on complexified vectors of the form $v_1 + iv_2$, $v_1, v_2 \in V$, via: $g(v_1 + iv_2, v_1' + iv_2') = g(v_1, v_1') - g(v_2, v_2') + i(g(v_1, v_2') + g(v_2, v_1'))$.

Let $\mathcal{N}$ be a 2-complex-dimensional vector subspace in $V^{\mathbb{C}}$, $\mathcal{N} \subset V^{\mathbb{C}}$, with the property that g identically vanishes on $\mathcal{N}$, $g_{|\mathcal{N}} \equiv 0$. In other words: $\mathcal{N}$ is a 2-complex-dimensional vector subspace of $V^{\mathbb{C}}$ such that for all n_1 and n_2 from $V^{\mathbb{C}}$ we have $g(n_1, n_2) = 0$. This is the definition of $\mathcal{N}$ being *totally null*.

Such $\mathcal{N}$s exist irrespectively of the signature of g. In fact, let (e_1, e_2, e_3, e_4) be an orthonormal basis for g in V. Then, if the metric has signature $(+, +, +, +)$, an example of $\mathcal{N}$ is given by

$$\mathcal{N}_E = \mathrm{Span}_{\mathbb{C}}(e_1 + ie_2, e_3 + ie_4).$$

If the metric has Lorentzian signature $(+, +, +, -)$ then we chose the basis so that $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1 = -g(e_4, e_4)$, and as an example of $\mathcal{N}$ we take

$$\mathcal{N}_L = \mathrm{Span}_{\mathbb{C}}(e_1 + ie_2, e_3 + e_4).$$

In the case of split signature $(+, +, -, -)$, we have $g(e_1, e_1) = g(e_2, e_2) = 1$, $g(e_3, e_3) = g(e_4, e_4) = -1$, and we distinguish two different classes of 2-dimensional totally null $\mathcal{N}$s. As an example of the first class, we take

$$\mathcal{N}_{S_c} = \mathrm{Span}_{\mathbb{C}}(e_1 + ie_2, e_3 + ie_4),$$

and as an example of the second class we take

$$\mathcal{N}_{S_r} = \mathrm{Span}_{\mathbb{C}}(e_1 + e_3, e_2 + e_4).$$

If V is a *complex* 4-dimensional vector space with a *complex* metric g, the notion of a totally null 2-dimensional vector subspace $\mathcal{N}$ still makes sense: these are simply 2-dimensional complex vector subspaces $\mathcal{N} \subset V$ for which $g_{|\mathcal{N}} \equiv 0$.

Irrespective of the fact if the 2-dimensional totally null vector space $\mathcal{N}$ is defined in terms of a complex vector space V with a complex metric, or in terms of $(V^{\mathbb{C}}, g)$ in which V is real and g is the complexified real metric g, choosing an orientation in V, one can check that $\mathcal{N}$ is always either *self-dual* or *anti-self-dual* (see e.g. [17]). By this we mean that we always have

- either: $*(n_1 \wedge n_2) = n_1 \wedge n_2$ for all $n_1, n_2 \in \mathcal{N}$,
- or: $*(n_1 \wedge n_2) = -n_1 \wedge n_2$ for all $n_1, n_2 \in \mathcal{N}$,

where $*$ denotes the Hodge star operator. Thus, the property of being self-dual or anti-self-dual (partially) characterizes totally null 2-planes.

In case of *real* V, irrespective of the metric signature, totally null spaces in $V^{\mathbb{C}}$ may be further characterized by their *real index* [7]. This is defined as follows:

Given a vector subspace $\mathcal{N} \subset V^{\mathbb{C}}$ one considers its *complex conjugate*

$$\bar{\mathcal{N}} = \{w \in V^{\mathbb{C}} \mid \bar{w} \in \mathcal{N}\}.$$

Then the intersection $\mathcal{N} \cap \bar{\mathcal{N}}$ is the complexification of a real vector space, say $\mathcal{K}$, and the real index of $\mathcal{N}$ is by definition the real dimension of $\mathcal{K}$, or the complex dimension of $\mathcal{N} \cap \bar{\mathcal{N}}$, which is the same.

In our examples above, $\mathcal{N}_E$ and $\mathcal{N}_{S_c}$ have real index *zero*, $\mathcal{N}_L$ has real index *one* and $\mathcal{N}_{S_r}$ has real index *two*. These are examples of a general fact, discussed in any dimension in [7], which when specialized to a *four* dimensional V, reads:

- If g has Euclidean signature, $(+, +, +, +)$, then every 2-dimensional totally null space $\mathcal{N}$ in the complexification $V^{\mathbb{C}}$ has real index *zero*;
- If g has Lorentzian signature, $(+, +, +, -)$, then every 2-dimensional totally null space $\mathcal{N}$ in the complexification $V^{\mathbb{C}}$ has real index *one*;
- If g has split signature, $(+, +, -, -)$, then a 2-dimensional totally null space $\mathcal{N}$ in the complexification $V^{\mathbb{C}}$ has either real index *zero* or *two*;
- In either signature, the spaces of all $\mathcal{N}$s with indices *zero* or *one* are generic—they form real 2-dimensional manifolds; in the split signature the spaces of all $\mathcal{N}$s with index *two* are special—they form a real manifold of dimension *one*.

If we have a 2-dimensional totally null $\mathcal{N}$ with real index *zero* then $V^{\mathbb{C}} = \mathcal{N} \oplus \bar{\mathcal{N}}$. This enables us to equip the real vector space V with a complex structure J, by declaring that the holomorphic vector space $V^{(1,0)}$ of this complex structure is $\mathcal{N}$. In other words, J is defined as a linear operator in V such that, after complexification, $J(\mathcal{N}) = i\mathcal{N}$. Due to the fact that $\mathcal{N}$ is totally null, the so defined J is hermitian, $g(Jv_1, Jv_2) = g(v_1, v_2)$ for all $v_1, v_2 \in V$. Thus, *a totally null $\mathcal{N}$ of real index zero in dimension four defines a hermitian structure J in the corresponding 4-dimensional real vector space* (V, g). Also the converse is true. For if we have (V, g, J) in real dimension four, we define $\mathcal{N}$ by $\mathcal{N} = V^{(1,0)}$, i.e. we declare that $\mathcal{N}$ is just the holomorphic vector space for J. Due to the fact that J is hermitian, and because of the assumed Euclidean or split signature of the metric, $\mathcal{N}$ is totally null and has real index zero. This proves the following

Proposition 3.1 *There is a one to one correspondence between (pseudo)hermitian structures J in a four-dimensional real vector space (V, g), equipped with a metric of either Euclidean or split signature, and 2-dimensional totally null planes $\mathcal{N} \subset V^{\mathbb{C}}$ with real index zero.*

In the Lorentzian case, where all $\mathcal{N}$s have index one, every $\mathcal{N}$ defines a 1-real-dimensional vector space $\mathcal{K}$. This is spanned by a real vector, say k, which is *null*, as it is a vector from $\mathcal{N}$. The space $\mathcal{K}^{\perp}$ orthogonal to $\mathcal{K}$ includes $\mathcal{K}$, $\mathcal{K} \subset \mathcal{K}^{\perp}$. Its complexification $(\mathcal{K}^{\perp})^{\mathbb{C}} = \mathcal{N} + \bar{\mathcal{N}}$. The *quotient space* $\mathcal{H} = \mathcal{K}^{\perp}/\mathcal{K}$ has real dimension two, and acquires a complex structure in a similar way as V did in the Euclidean/split case. Indeed, we define J in $\mathcal{H}$ by declaring that its holomorphic space $\mathcal{H}^{(1,0)}$ coincides with the 2-dimensional complex vector space $(\mathcal{N} + \bar{\mathcal{N}})/(\mathcal{N} \cap \bar{\mathcal{N}})$. This shows that *a 2-dimensional totally null $\mathcal{N}$, in the complexification of a Lorentzian 4-dimensional (V, g), defines a real null direction k in V together with a complex structure J in the quotient space $\mathcal{K}^{\perp}/\mathcal{K}$, $\mathcal{K} = \mathbb{R}k$*. One can easily see that also the converse is true, and we have the following

Proposition 3.2 *There is a one to one correspondence between 2-dimensional totally null planes $\mathcal{N}$, in the complexification of a four-dimensional oriented and time oriented Lorentzian vector space (V, g), and null directions $\mathcal{K} = \mathbb{R}k$ in V together with their associated complex structures J in $\mathcal{K}^{\perp}/\mathcal{K}$.*

The last case, in which the signature of g is split, $(+, +, -, -)$, and in which the $\mathcal{N}$s have real index 2, provides us with a *real* 2-dimensional totally null plane in V. Thus, we have

Proposition 3.3 *There is a one to one correspondence between 2-dimensional totally null planes $\mathcal{N}$ with real index two, in the complexification of a four-dimensional split signature vector space (V, g), and real totally null 2-planes in V.*

We now pass to the analogous considerations on 4-manifolds. Thus, we consider a 4-dimensional manifold $\mathcal{M}$, with a metric g, equipped in addition with a smooth distribution $\mathcal{N}$ of complex totally null 2-planes $\mathcal{N}_x$, $x \in \mathcal{M}$, of a *fixed* index. Applying the above propositions we see that, depending on the index of $\mathcal{N}$, such an $\mathcal{M}$ is equipped either with an *almost hermitian structure* $(\mathcal{M}, g, J)$ (in case of index 0), or with an *almost optical structure* $(\mathcal{M}, g, \mathcal{K}, J_{\mathcal{K}^{\perp}/\mathcal{K}})$ (in case of index 1), or with a *real distribution of totally null 2-planes* (in case of index 2). The interesting question about the integrability conditions for these three different real structures has a uniform answer in terms of the integrability of the complex distribution $\mathcal{N}$. Actually, by inspection of the three cases determined by the real indices of $\mathcal{N}$, one proves the following [16]

Proposition 3.4 *Let M be a 4-dimensional real manifold and g be a real metric on it. Let $\mathcal{N}$ be a complex 2-dimensional distribution on $\mathcal{M}$ such that $g_{|\mathcal{N}} \equiv 0$. Then, the integrability condition,*

$$[\mathcal{N}, \mathcal{N}] \subset \mathcal{N},$$

for the distribution $\mathcal{N}$ is equivalent to

- *the Newlander–Nirenberg integrability condition for the corresponding J, if $\mathcal{N}$ has index zero;*
- *the geodesic and shear-free condition for the corresponding real null direction field k, if $\mathcal{N}$ has index one. In this case the 3-dimensional space of integral curves of k has (locally) the structure of 3-dimensional CR manifold.*
- *the classical Fröbenius integrability for the real distribution corresponding to $\mathcal{N}$, if $\mathcal{N}$ has index two. In this case we have a foliation of $\mathcal{M}$ by 2-dimensional real manifolds corresponding to the leaves of $\mathcal{N}$.*

Returning to the complex Goldberg–Sachs Theorem 1.5, we see that one part of its thesis, which is concerned with the integrabilty condition $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, has a very nice geometric interpretation in each of the real signatures. In particular, in the real index zero case, the theorem gives if and only if conditions for the local existence of a hermitian structure on a 4-manifold [15,16].

4 Signature independent Newman–Penrose formalism

The purpose of this section is to establish a version of the Newman–Penrose formalism [14]—a very convenient tool to study the properties of 4-dimensional manifolds equipped with a metric—in such a way that it will be usable in the following different settings. These are as follows:

(a) $\mathcal{M}$ is a *complex* 4-dimensional manifold, and g is a *holomorphic* metric on $\mathcal{M}$,
(b) $\mathcal{M}$ is a *real* 4-dimensional manifold, and g is a *complex valued* metric on $\mathcal{M}$,
(c) $\mathcal{M}$ is a *real* 4-dimensional manifold, and g is:

- (ci) real of Lorentzian signature,
- (cii) real of Euclidean signature,
- (ciii) real of split signature,
- (civ) a complexification of a real metric having one of the above signatures.

The classical Newman–Penrose formalism was devised for the case where $\mathcal{M}$ is real, and g is Lorentzian. Although the generalization of the formalism, applicable to all the above settings, is implicit in the formulation given in the Penrose and Rindler monograph [18], one needs to have some experience to use it in the cases (cii) and (ciii). For this reason, we decided to derive the formalism from first principles, emphasizing from the very beginning how to apply it to the above different situations. To achieve our goal of very easy applicability of this formalism to these different situations, we have introduced a convenient notation, in various instances quite different from the Newman–Penrose original. Since the Newman–Penrose formalism proved to be a great tool in the study of Lorenztian 4-manifolds, we believe that our formulation, explained here from the basics, will help the community of mathematicians working with 4-manifolds having metrics of Euclidean or split signature to appreciate this tool.

From now on $(\mathcal{M}, g)$ is a 4-dimensional *real* or *complex* manifold equipped with a *complex* valued metric. This means that the metric g is a non-degenerate symmetric bilinear form, $g : \mathrm{T}^{\mathbb{C}}\mathcal{M} \times \mathrm{T}^{\mathbb{C}}\mathcal{M} \to \mathbb{C}$, with values in the complex numbers [17].

Given g, we use a (local) *null* coframe $(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)$ on $\mathcal{M}$ in which g is

$$g = g_{ab}\theta^a\theta^b = 2(MP + NK). \tag{1}$$

Here, and in the following, formulae like $\theta^a\theta^b$ denote the symmetrized tensor product of the complex valued 1-forms θ^a and θ^b: $\theta^a\theta^b = \frac{1}{2}(\theta^a \otimes \theta^b + \theta^b \otimes \theta^a)$.

Remark 4.1 *Note that our setting, although in general complex, includes all the* real *cases. These cases correspond to metrics g such that $g(X, Y)$ is* real *for all* real *vector fields $X, Y \in \mathrm{T}\mathcal{M}$. In other words, in such cases the metric g restricted to the tangent space $\mathrm{T}\mathcal{M}$ of $\mathcal{M}$ is real. If $\mathcal{M}$ is equipped with a metric g satisfying this condition, then we always locally have a null coframe $(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)$ in which*

(E) $P = \bar{M}$ *and* $K = \bar{N}$ *if the metric* $g_{|T\mathcal{M}}$ *has* Euclidean *signature,*
(S$_c$) $P = \bar{M}$ *and* $K = -\bar{N}$*, if the metric* $g_{|T\mathcal{M}}$ *has* split *signature,*
(L) $P = \bar{M}$*,* $N = \bar{N}$ *and* $K = \bar{K}$*, if the metric* $g_{|T\mathcal{M}}$ *has* Lorentzian *signature.*

Remark 4.2 *The main statement above about the cases (E), (S$_c$) and (L) can be rephrased as follows: In the complexification of the cotangent space of* $T^{*\mathbb{C}}\mathcal{M}$*, one can introduce three different* real structures *by appropriate* conjugation operators*: 'bar'. On the basis of the 1-forms* $(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)$ *these are defined according to:*

(E) $\bar{M} = P$*,* $\bar{P} = M$*,* $\bar{N} = K$ *and* $\bar{K} = N$*. With this choice of the conjugation,* $g_{|T\mathcal{M}}$ *is real and has* Euclidean *signature.*
(S$_c$) $\bar{M} = P$*,* $\bar{P} = M$*,* $\bar{N} = -K$ *and* $\bar{K} = -N$*. With this choice of the conjugation,* $g_{|T\mathcal{M}}$ *is real and has* split *signature.*
(L) $\bar{M} = P$*,* $\bar{P} = M$*,* $\bar{N} = N$ *and* $\bar{K} = K$*. With this choice of the conjugation,* $g_{|T\mathcal{M}}$ *is real and has* Lorentzian *signature.*

Note also that the labels $a = 1, 2, 3, 4$ *of the null coframe components* θ^a*, behave in the following way under these conjugations:*

(E) $\bar{1} \to 2$, $\bar{2} \to 1$, $\bar{3} \to 4$, $\bar{4} \to 3$ *in the Euclidean case,*
(S$_c$) $\bar{1} \to 2$, $\bar{2} \to 1$, $\bar{3} \to -4$, $\bar{4} \to -3$ *in the split case,*
(L) $\bar{1} \to 2$, $\bar{2} \to 1$, $\bar{3} \to 3$, $\bar{4} \to 4$ *in the Lorentzian case.*

These transformations of indices under the respective complex conjugations will be important when we perform complex conjugations on multiindexed quantities, such as for example, R_{abcd}*. In particular, the above transformation of indices imply, for example, that in the (S$_c$) case* $\bar{R}_{1323} = R_{2414}$*,* $\bar{R}_{1321} = -R_{2412}$*, and so on.*

Remark 4.3 *We denoted the split signature case by the letter S with a subscript c to distinguish this case from the case* S_r *in which the field of 2-planes annihilating the coframe 1-forms P and K in (S$_c$) is* totally real. *It is well known [7], that if the metric* $g_{|\mathrm{T}\mathcal{M}}$ *has split signature, one can choose a* totally real *null coframe on* $\mathcal{M}$*, such that*

$$\bar{M} = M, \quad \bar{P} = P, \quad \bar{N} = N, \quad \bar{K} = K. \qquad (S_r)$$

This situation, although less generic [7] than (S$_c$) is worthy of consideration, since in the integrable case of the Goldberg–Sachs theorem it leads to the foliation of $\mathcal{M}$ *by* real *2-dimensional leaves, corresponding to the distribution of totally null 2-planes.*

Given a null coframe (θ^a), we calculate the differentials of its components

$$\mathrm{d}\theta^a = -\tfrac{1}{2} c^a_{bc} \theta^b \wedge \theta^c. \qquad (2)$$

Following Newman and Penrose [14], and the tradition in General Relativity literature [8], we will assign Greek letter names to the coefficient functions c^a_{bc}. As is well known, these coefficients naturally split into two groups with 12 complex coefficients in each group. They correspond to two spin connections associated with the metric g. The 12 coefficients from the first group will be denoted by $\alpha, \beta, \gamma, \lambda, \mu, \nu, \rho, \sigma, \tau, \varepsilon, \kappa, \pi$. The 12 coefficients from the second group will be denoted by putting primes on the same Greek letters. The 'primed' and 'unprimed' quantities, as describing two different spinorial connections, will be treated as independent objects in the complex setting. Their relations to the complex conjugation in the real settings will be described in Reamark 4.4. This said, we write the four equations (2) as:

$$
\begin{aligned}
\mathrm{d}\theta^1 &= (\alpha-\beta')\theta^1\wedge\theta^2 + (\gamma-\gamma'-\mu)\theta^1\wedge\theta^3 + (\varepsilon-\varepsilon'-\rho')\theta^1\wedge\theta^4\\
&\quad -\lambda\theta^2\wedge\theta^3 - \sigma'\theta^2\wedge\theta^4 + (\pi-\tau')\theta^3\wedge\theta^4\\
\mathrm{d}\theta^2 &= (\beta-\alpha')\theta^1\wedge\theta^2 - \lambda'\theta^1\wedge\theta^3 - \sigma\theta^1\wedge\theta^4\\
&\quad +(\gamma'-\gamma-\mu')\theta^2\wedge\theta^3 + (\varepsilon'-\varepsilon-\rho)\theta^2\wedge\theta^4 + (\pi'-\tau)\theta^3\wedge\theta^4\\
\mathrm{d}\theta^3 &= (\rho'-\rho)\theta^1\wedge\theta^2 + (\alpha'+\beta-\tau)\theta^1\wedge\theta^3 - \kappa\theta^1\wedge\theta^4\\
&\quad +(\alpha+\beta'-\tau')\theta^2\wedge\theta^3 - \kappa'\theta^2\wedge\theta^4 - (\varepsilon'+\varepsilon)\theta^3\wedge\theta^4\\
\mathrm{d}\theta^4 &= (\mu-\mu')\theta^1\wedge\theta^2 - \nu'\theta^1\wedge\theta^3 - (\alpha'+\beta+\pi')\theta^1\wedge\theta^4\\
&\quad -\nu\theta^2\wedge\theta^3 - (\alpha+\beta'+\pi)\theta^2\wedge\theta^4 - (\gamma'+\gamma)\theta^3\wedge\theta^4.
\end{aligned}
\tag{3}
$$

This notation for the coefficient functions c^a_{bc}, although ugly at first sight, has many advantages. One of them is the already mentioned property of separating the two spin connections associated with the metric g by associating them with the respective 'primed' and 'unprimed' objects. More explicitly, defining the Levi-Civita connection 1-forms Γ^a_b by

$$
\begin{gathered}
\mathrm{d}\theta^a + \Gamma^a_b\wedge\theta^b = 0\\
\Gamma_{ab} = -\Gamma_{ba}, \qquad \Gamma_{ab} = g_{ac}\Gamma^c_b,
\end{gathered}
\tag{4}
$$

we get the following expressions for Γ_{ab}:

$$
\begin{aligned}
\tfrac{1}{2}(\Gamma_{12}+\Gamma_{34}) &= \alpha'\theta^1 + \beta'\theta^2 + \gamma'\theta^3 + \varepsilon'\theta^4\\
\Gamma_{13} &= \lambda'\theta^1 + \mu'\theta^2 + \nu'\theta^3 + \pi'\theta^4\\
\Gamma_{24} &= \rho'\theta^1 + \sigma'\theta^2 + \tau'\theta^3 + \kappa'\theta^4.
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\tfrac{1}{2}(-\Gamma_{12}+\Gamma_{34}) &= \beta\theta^1 + \alpha\theta^2 + \gamma\theta^3 + \varepsilon\theta^4\\
\Gamma_{23} &= \mu\theta^1 + \lambda\theta^2 + \nu\theta^3 + \pi\theta^4\\
\Gamma_{14} &= \sigma\theta^1 + \rho\theta^2 + \tau\theta^3 + \kappa\theta^4.
\end{aligned}
\tag{6}
$$

The two spin connections correspond to $\chi' = (\Gamma_{24}, \frac{1}{2}(\Gamma_{12}+\Gamma_{34}), \Gamma_{13})$ and $\chi = (\Gamma_{14}, \frac{1}{2}(-\Gamma_{12}+\Gamma_{34}), \Gamma_{23})$, respectively.

Remark 4.4 *The above notation is an adaptation of the* Lorentzian version *of the Newman–Penrose formalism. This can be easily seen, taking into account the reality conditions discussed in Remarks 4.1, 4.2. In particular, in the Lorentzian case (L), the complex conjugation defined in Remark 4.2, applied to the quantities $\alpha, \beta, \gamma, \ldots$, yields:*

$$
\begin{pmatrix} \bar\alpha & \bar\beta & \bar\gamma & \bar\varepsilon\\ \bar\lambda & \bar\mu & \bar\nu & \bar\pi\\ \bar\rho & \bar\sigma & \bar\tau & \bar\kappa \end{pmatrix} = \begin{pmatrix} \alpha' & \beta' & \gamma' & \varepsilon'\\ \lambda' & \mu' & \nu' & \pi'\\ \rho' & \sigma' & \tau' & \kappa' \end{pmatrix}. \tag{L}
$$

Thus, in the Lorentzian case, the complex conjugation changes 'unprimed' Greek letters into 'primed' ones and vice versa*. Therefore, in this signature the 'primed' Greek letter quantities are totally determined by the 'unprimed' ones. The situation is drastically different in the two other real signatures. There the 'primed' Greek letter quantities are* independent *of the 'unprimed' ones. On the other hand in these two cases, there are some relations between the quantities* within *each of the 'primed' and 'unprimed' family. In the Euclidean case, they are given by*

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\varepsilon} \\ \bar{\lambda} & \bar{\mu} & \bar{\nu} & \bar{\pi} \\ \bar{\rho} & \bar{\sigma} & \bar{\tau} & \bar{\kappa} \end{pmatrix} = \begin{pmatrix} -\beta & -\alpha & -\varepsilon & -\gamma \\ \sigma & \rho & \kappa & \tau \\ \mu & \lambda & \pi & \nu \end{pmatrix}, \tag{E}$$

with the same relations after the replacement of all 'unprimed' quantities by their 'primed' counterparts on both sides.

In the split signature cases, we have

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\varepsilon} \\ \bar{\lambda} & \bar{\mu} & \bar{\nu} & \bar{\pi} \\ \bar{\rho} & \bar{\sigma} & \bar{\tau} & \bar{\kappa} \end{pmatrix} = \begin{pmatrix} -\beta & -\alpha & \varepsilon & \gamma \\ -\sigma & -\rho & \kappa & \tau \\ -\mu & -\lambda & \pi & \nu \end{pmatrix}, \tag{S_c}$$

and

$$\begin{pmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} & \bar{\varepsilon} \\ \bar{\lambda} & \bar{\mu} & \bar{\nu} & \bar{\pi} \\ \bar{\rho} & \bar{\sigma} & \bar{\tau} & \bar{\kappa} \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma & \varepsilon \\ \lambda & \mu & \nu & \pi \\ \rho & \sigma & \tau & \kappa \end{pmatrix}, \tag{S_r}$$

again with the identical relations for the 'primed' quantities.

Now we pass to the 'prime'–'unprime' decomposition of the curvature. The Riemann tensor coefficients $R^a_{\ bcd}$ are defined by Cartan's second structure equations:

$$\mathrm{d}\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b} = \tfrac{1}{2} R^a_{\ bcd} \theta^c \wedge \theta^d. \tag{7}$$

Due to our conventions, modulo symmetry, the only non-zero components of the metric are $g_{12} = g_{34} = 1$. The inverse of the metric, g^{ab}, again modulo symmetry, has $g^{12} = g^{34} = 1$ as the only non-vanishing components. The Ricci tensor is defined as $R_{ab} = R^c_{\ acb}$. Its scalar is: $R = R_{ab} g^{ab}$, and its tracefree part is: $\check{R}_{ab} = R_{ab} - \frac{1}{4} R g_{ab}$. Using the metric g_{ab} we also define $R_{abcd} = g_{ae} R^e_{\ bcd}$. This is further used to define the covariant components of the Weyl tensor $C^a_{\ bcd}$ via:

$$C_{abcd} = R_{abcd} - \tfrac{1}{12} R (g_{ac} g_{db} - g_{ad} g_{cb}) + \tfrac{1}{2} (g_{ad} \check{R}_{cb} - g_{ac} \check{R}_{db} + g_{bc} \check{R}_{da} - g_{bd} \check{R}_{ca}).$$

In the context of the present paper, in which the conformal properties matter, it is convenient to use the Schouten tensor P, with help of which we can write the above displayed equality as

$$C_{abcd} = R_{abcd} + g_{ad} \mathsf{P}_{cb} - g_{ac} \mathsf{P}_{db} + g_{bc} \mathsf{P}_{da} - g_{bd} \mathsf{P}_{ca}. \tag{8}$$

The Schouten tensor P is a 'trace-corrected' Ricci tensor, with the explicit relation given by

$$\mathsf{P}_{ab} = \tfrac{1}{2} R_{ab} - \tfrac{1}{12} R g_{ab}.$$

In the Newman–Penrose formalism, the 10 components of the Weyl tensor are encoded in 10 complex quantities $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ and $\Psi'_0, \Psi'_1, \Psi'_2, \Psi'_3, \Psi'_4$. Five of them have 'primes', to emphasize that they are associated with the 'primed' spin connection. Another way of understanding this notation is to say that the 'unprimed' Ψs are five components of the self-dual part of the Weyl tensor, and the 'primed' Ψs are the components of the anti-self-dual part of the Weyl.

The Ricci and Schouten tensors are mixed 'prime'–'unprime' objects, and as such are not very nicely denoted in the 'prime' vs. 'unprime' setting. For this reason, when referring to R_{ab}, $\check{R}_{ab}$ and P_{ab}, we will not use the Newman–Penrose notation, and will express these objects using the standard four-dimensional indices $a = 1, 2, 3, 4$, as e.g. in $12(\mathsf{P}_{12}+\mathsf{P}_{34}) = 2(R_{12} + R_{34}) = R$.

Having said all of this we express Cartan's second structure equation (7), and in particular the curvature coefficients R^a_{bcd}, in terms of Ψs, Ψ's, P and the null coframe (θ^a) as follows:

$$
\begin{aligned}
&\tfrac{1}{2}\mathrm{d}(\Gamma_{12} + \Gamma_{34}) + \Gamma_{24} \wedge \Gamma_{13}\\
&= -\Psi'_3\theta^1 \wedge \theta^3 + \Psi'_1\theta^2 \wedge \theta^4 + \tfrac{1}{2}(2\Psi'_2 - \mathsf{P}_{12} - \mathsf{P}_{34})(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)\\
&\quad + \mathsf{P}_{23}\theta^2 \wedge \theta^3 - \mathsf{P}_{14}\theta^1 \wedge \theta^4 - \tfrac{1}{2}(\mathsf{P}_{12} - \mathsf{P}_{34})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)\\[2ex]
&\mathrm{d}\Gamma_{13} + (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{13}\\
&= \Psi'_4\theta^1 \wedge \theta^3 + (\Psi'_2 + \mathsf{P}_{12} + \mathsf{P}_{34})\theta^2 \wedge \theta^4 - \Psi'_3(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)\\
&\quad + \mathsf{P}_{33}\theta^2 \wedge \theta^3 + \mathsf{P}_{11}\theta^1 \wedge \theta^4 - \mathsf{P}_{13}(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)\\[2ex]
&\mathrm{d}\Gamma_{24} + \Gamma_{24} \wedge (\Gamma_{12} + \Gamma_{34})\\
&= (\Psi'_2 + \mathsf{P}_{12} + \mathsf{P}_{34})\theta^1 \wedge \theta^3 + \Psi'_0\theta^2 \wedge \theta^4 + \Psi'_1(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)\\
&\quad + \mathsf{P}_{22}\theta^2 \wedge \theta^3 + \mathsf{P}_{44}\theta^1 \wedge \theta^4 + \mathsf{P}_{24}(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4),
\end{aligned}
\tag{9}
$$

with analogous equations for the 'unprimed' objects:

$$
\begin{aligned}
&\tfrac{1}{2}\mathrm{d}(-\Gamma_{12} + \Gamma_{34}) + \Gamma_{14} \wedge \Gamma_{23}\\
&= -\Psi_3\theta^2 \wedge \theta^3 + \Psi_1\theta^1 \wedge \theta^4 - \tfrac{1}{2}(2\Psi_2 - \mathsf{P}_{12} - \mathsf{P}_{34})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)\\
&\quad + \mathsf{P}_{13}\theta^1 \wedge \theta^3 - \mathsf{P}_{24}\theta^2 \wedge \theta^4 + \tfrac{1}{2}(\mathsf{P}_{12} - \mathsf{P}_{34})(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)\\[2ex]
&\mathrm{d}\Gamma_{23} + (-\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{23}\\
&= \Psi_4\theta^2 \wedge \theta^3 + (\Psi_2 + \mathsf{P}_{12} + \mathsf{P}_{34})\theta^1 \wedge \theta^4 + \Psi_3(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)\\
&\quad + \mathsf{P}_{33}\theta^1 \wedge \theta^3 + \mathsf{P}_{22}\theta^2 \wedge \theta^4 + \mathsf{P}_{23}(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4)\\[2ex]
&\mathrm{d}\Gamma_{14} + \Gamma_{14} \wedge (-\Gamma_{12} + \Gamma_{34})\\
&= (\Psi_2 + \mathsf{P}_{12} + \mathsf{P}_{34})\theta^2 \wedge \theta^3 + \Psi_0\theta^1 \wedge \theta^4 - \Psi_1(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4)\\
&\quad + \mathsf{P}_{11}\theta^1 \wedge \theta^3 + \mathsf{P}_{44}\theta^2 \wedge \theta^4 - \mathsf{P}_{14}(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4).
\end{aligned}
\tag{10}
$$

Note that in the first part (9) of the structure equations, the full traceless part of the Schouten tensor P, represented by its nine components P_{11}, P_{13}, P_{14}, P_{22}, P_{23}, P_{24}, P_{33}, P_{44} and $\mathsf{P}_{12} - \mathsf{P}_{34}$, stays with the basis of the self-dual 2-forms:

$$\Sigma = (\theta^2 \wedge \theta^3, \theta^1 \wedge \theta^4, \theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4). \tag{11}$$

In the second part (10) of the structure equations, the full traceless part of the Schouten tensor P appears again, but now at the basis of the anti-self-dual 2-forms:

$$\Sigma' = (\theta^1 \wedge \theta^3, \theta^2 \wedge \theta^4, \theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4). \tag{12}$$

On the other hand, the self-dual and the anti-self-dual parts of the Weyl tensor, corresponding to the respective Ψs and Ψ's, are separated: in Eq. (9) we only have Ψ's, whereas in (10) we only have Ψs. The trace of the Schouten tensor $2(\mathsf{P}_{12} + \mathsf{P}_{34})$, proportional to the Ricci scalar R, appears in both sets of equations, always together with the respective Weyl tensor components Ψ_2 and Ψ'_2. It is also worthwhile to mention that if one uses the following basis

$$E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

of the Lie algebra $\mathfrak{sl}(2)$, and if one defines

$$\Gamma = \Gamma_{14} E_- + \tfrac{1}{2}(-\Gamma_{12} + \Gamma_{34}) E_0 + \Gamma_{23} E_+,$$
$$\Gamma' = \Gamma_{24} E_- + \tfrac{1}{2}(\Gamma_{12} + \Gamma_{34}) E_0 + \Gamma_{13} E_+,$$

then the left hand sides of Eqs. (9,10) appear in the formulae

$$\mathrm{d}\Gamma + \Gamma \wedge \Gamma = \begin{pmatrix} \frac{1}{2}\mathrm{d}(-\Gamma_{12} + \Gamma_{34}) + \Gamma_{14} \wedge \Gamma_{23} & -\mathrm{d}\Gamma_{23} - (-\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{23} \\ \mathrm{d}\Gamma_{14} + \Gamma_{14} \wedge (-\Gamma_{12} + \Gamma_{34}) & -\frac{1}{2}\mathrm{d}(-\Gamma_{12} + \Gamma_{34}) - \Gamma_{14} \wedge \Gamma_{23} \end{pmatrix},$$

$$\mathrm{d}\Gamma' + \Gamma' \wedge \Gamma' = \begin{pmatrix} \frac{1}{2}\mathrm{d}(\Gamma_{12} + \Gamma_{34}) + \Gamma_{24} \wedge \Gamma_{13} & -\mathrm{d}\Gamma_{13} - (\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{13} \\ \mathrm{d}\Gamma_{24} + \Gamma_{24} \wedge (\Gamma_{12} + \Gamma_{34}) & -\frac{1}{2}\mathrm{d}(\Gamma_{12} + \Gamma_{34}) - \Gamma_{24} \wedge \Gamma_{13} \end{pmatrix}.$$

This explains the term ‘spin connections’ assigned to the previously defined quantities χ and χ'. It also justifies the ‘prime’–‘unprime’ notation, which is rooted in the speciality of 4-dimensions, stating that for $n \geq 3$ the Lie algebra $\mathfrak{so}(n, \mathbb{C})$ is not simple only when $n = 4$, and in that case it has the symmetric split: $\mathfrak{so}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. This enables us to split the $\mathfrak{so}(4, \mathbb{C})$-valued Levi-Civita connection into the well defined $\mathfrak{sl}(2, \mathbb{C})$-valued ‘primed’ and ‘unprimed’ parts, which are totally independent. In real signatures, we have an analogous split for $\mathfrak{so}(4 - p, p) = \mathfrak{g} \oplus \mathfrak{g}'$, $p = 0, 1, 2$, where now $\mathfrak{g}$ and $\mathfrak{g}'$ are two copies of the appropriate real form of $\mathfrak{sl}(2, \mathbb{C})$. This again enables us to split the Levi-Civita connection into the ‘primed’ and ‘unprimed’ connections, with the appropriate reality conditions, as in (E), (S_c), (S_r) or (L).

Comparing Eqs. (5, 6) with (9, 10), one finds relations between the curvature quantities P, Ψ and Ψ' and the first derivatives of the connection coefficients $\alpha, \beta, \ldots, \alpha', \beta', \ldots$. These relations are called the Newman–Penrose equations [14]. We present them in the “Appendix”. In these equations, and in the rest of the paper, we denote the vector fields dual on $\mathcal{M}$ to the null coframe (M, P, N, K) by the respective symbols $(\delta, \partial, \triangle, D)$. Thus, we have e.g. $\delta \lrcorner M = 1$, and zero on all the other coframe components, $D \lrcorner N = 0$, etc. Also when applying these vector fields to functions on $\mathcal{M}$ we omit parentheses. Thus, instead of writing $D(\alpha)$ to denote the derivative of a connection coefficient α in the direction of the basis vector field D, we simply write $D\alpha$.

In addition to the Newman–Penrose equations, we will also need the commutators of the basis vector fields. These are given by the formulae dual to Eq. (3), and read:

$$\begin{aligned}
[\delta, \partial] &= (\beta' - \alpha)\delta + (\alpha' - \beta)\partial + (\rho - \rho')\triangle + (\mu' - \mu)D \\
[\delta, \triangle] &= (\mu + \gamma' - \gamma)\delta + \lambda'\partial + (\tau - \alpha' - \beta)\triangle + \nu' D \\
[\partial, \triangle] &= \lambda\delta + (\mu' + \gamma - \gamma')\partial + (\tau' - \alpha - \beta')\triangle + \nu D \\
[\delta, D] &= (\rho' + \varepsilon' - \varepsilon)\delta + \sigma\partial + \kappa\triangle + (\alpha' + \beta + \pi')D \\
[\partial, D] &= \sigma'\delta + (\rho + \varepsilon - \varepsilon')\partial + \kappa'\triangle + (\alpha + \beta' + \pi)D \\
[\triangle, D] &= (\tau' - \pi)\delta + (\tau - \pi')\partial + (\varepsilon' + \varepsilon)\triangle + (\gamma' + \gamma)D
\end{aligned} \tag{13}$$

The Newman–Penrose equations are supplemented by the second Bianchi identities, which are crucial for the proof of the Goldberg–Sachs theorem. These are relations between the first derivatives of the curvature quantities Ψ, Ψ' and P and the connection coefficients. These Bianchi identities are also presented in the "Appendix".

5 Generalizations of the Goldberg–Sachs theorem for complex metrics

The thesis of the Goldberg–Sachs theorem can be restated in the language of the Newman–Penrose formalism as follows:

To interpret the integrability condition $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$ on the totally null distribution $\mathcal{N}$, we align the Newman–Penrose coframe $(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)$ in such a way that the two *null* and *mutually orthogonal* frame vectors $e_1 = m = \delta$ and $e_4 = k = D$ span $\mathcal{N}$, $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(\delta, D)$. Such a coframe on $(\mathcal{M}, g)$ will be called a coframe *adapted to* $\mathcal{N}$.

Then, the integrability of $\mathcal{N}$ is totally determined by the commutator $[\delta, D]$ of these basis vectors. Looking at this commutator in (13), we see that the condition that $[\delta, D]$ is in the span of δ and D is equivalent to $\kappa \equiv \sigma \equiv 0$. Thus, we have

Proposition 5.1 *Let $\mathcal{N}$ be a field of self-dual totally null 2-planes on a 4-dimensional manifold $\mathcal{M}$ with the metric g. Let (m, p, n, k) be a null frame in $\mathcal{U} \subset \mathcal{M}$ adapted to $\mathcal{N}$. Then, the field $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, in $\mathcal{U}$ if and only if the frame connection coefficients $\Gamma_{144} = \kappa$ and $\Gamma_{141} = \sigma$ vanish identically, $\kappa \equiv \sigma \equiv 0$, in $\mathcal{U}$.*

To interpret the algebraic speciality of the self-dual part of the Weyl tensor, we focus on the condition

$$C(m, k, m, k) \equiv 0. \tag{14}$$

Here we consider the Weyl tensor C_{abcd} as a linear map $C : \bigotimes^4 \mathrm{T}^{\mathbb{C}}\mathcal{M} \to \mathbb{C}$. Note that, since the so understood Weyl tensor is *antisymmetric* in the first two arguments, as well as, independently, in the last two arguments, the vanishing in Eq. (14), although defined on a particular basis of $\mathcal{N}$, is basis independent. Actually, if we think of C as a linear map $C : (\bigwedge^2 \mathrm{T}^{\mathbb{C}}\mathcal{M}) \odot (\bigwedge^2 \mathrm{T}^{\mathbb{C}}\mathcal{M}) \to \mathbb{C}$, and identify a 2-dimensional totally null distribution $\mathcal{N}$ with the complex line bundle

$$\mathcal{N}_\wedge = \{w \in \textstyle\bigwedge^2 \mathrm{T}^{\mathbb{C}}\mathcal{M} \mid w = v_1 \wedge v_2,\ v_1, v_2 \in \mathcal{N}\},$$

then we say that $\mathcal{N}$ is a *principal* totally null distribution iff

$$C(\mathcal{N}_\wedge, \mathcal{N}_\wedge) \equiv 0. \tag{15}$$

Remark 5.2 *The quantity $C(m, k, m, k)$ is a null counterpart of the sectional curvature from Riemannian geometry. In fact, given a 2-dimensional vector space $V = \mathrm{Span}_{\mathbb{R}}(X, Y)$, the sectional curvature associated with V is*

$$K = K(X, Y) = \frac{g(R(X, Y)X, Y)}{|X \wedge Y|^2}.$$

The appearance of the denominator $|X \wedge Y|^2 = g(X, X)g(Y, Y) - g(X, Y)^2$ in this expression makes this quantity independent of the choice of X, Y in V. The notion of sectional curvature loses its meaning for vector spaces V which are totally null, since for them the metric g when restricted to V vanishes, making the denominator $|X \wedge Y|^2 \equiv 0$ for all $X, Y \in V$.

To incorporate totally null vector spaces V, one needs to generalize the notion of sectional curvature, removing the denominator from its definition. This leads to the quantity

$$K_0 = K_0(X, Y) = g(R(X, Y)X, Y).$$

This, although basis dependent, transforms in a homogeneous fashion,

$$K_0(X, Y) \to (ad - bc)^2 K_0(X, Y),$$

under the change of basis $X \to aX + bY$, $Y \to cX + dY$. Thus, vanishing or not of K_0 is an invariant property of any 2-dimensional vector space $V \subset T_x\mathcal{M}$. This property of having K_0 equal or not equal to zero, characterizes V and is well defined regardless of the fact if the metric is real or complex, including the cases when V is totally null.

Now, passing to the specific situation of 4-dimensional manifolds, we can choose V to be a field of self-dual totally null 2-planes $\mathcal{N}$. More specifically, if $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$, we easily check (see 10) that $K_0(m, k) = C(m, k, m, k) = \Psi_0$. Thus $K_0(m, k)$ is the Ψ_0 component of the self-dual *part of the Weyl tensor. For an* anti-self-dual *totally null plane $\mathcal{N}' = \mathrm{Span}_{\mathbb{C}}(p, k)$, we have $K_0(p, k) = C(p, k, p, k) = \Psi'_0$, which is the corresponding component of the* anti-self-dual *part of the Weyl tensor. This shows that the principal self-dual totally null 2-planes are just those for which the quantity Ψ_0 vanishes. Thus, in a sense, the principal self-dual totally null 2-planes have vanishing sectional curvature. (We have also an analogous statement for the principal anti-self-dual 2-planes; they are related to the anti-self-dual part of the Weyl tensor, and are defined by the vanishing of the quantity Ψ'_0.)*

Let us now choose a Newman–Penrose coframe (M, P, N, K) which is not related to any particular choice of $\mathcal{N}$. Thus, we have $g = 2(MP + NK)$. Then, at every point of $\mathcal{M}$, we have two families $\mathcal{N}_z$ and $\mathcal{N}_{z'}$ of 2-dimensional totally null planes [17]. These two families are parameterized by a complex parameter z or z', respectively, and the 2-planes parameterized by z are self-dual, and those parameterized by z' are anti-self-dual. In terms of the frame $(e_1, e_2, e_3, e_4) = (m, p, n, k) = (\delta, \partial, \triangle, D)$ dual to (M, P, N, K), they are given by

$$\mathcal{N}_z = \mathrm{Span}_{\mathbb{C}}(m + zn, k - zp), \qquad z \in \mathbb{C}, \tag{16}$$

and

$$\mathcal{N}_{z'} = \mathrm{Span}_{\mathbb{C}}(p + z'n, k - z'm), \qquad z' \in \mathbb{C}. \tag{17}$$

Adding a totally null plane $\mathcal{N}_\infty = \mathrm{Span}_{\mathbb{C}}(n, p)$ to the first family, and $\mathcal{N}_{\infty'} = \mathrm{Span}_{\mathbb{C}}(n, m)$ to the second family, we have two *spheres* of 2-dimensional totally null planes at each point of $\mathcal{M}$. The first sphere consists of the self-dual 2-planes, the second of the anti-self-dual 2-planes.

Now we find the *principal* 2-planes in each of these spheres. The principal 2-planes in the first sphere correspond to those z such that

$$C(m + zn, k - zp, m + zn, k - zp) = 0. \tag{18}$$

The left hand side of this equation is a *fourth* order polynomial in the *complex* variable z, thus (18) treated as an equation for z, has *four* roots, some of which may be *multiple* roots. Moreover, Eq. (18) written explicitly in terms of the Newman–Penrose Weyl coeffcients Ψs and Ψ's, involves only the ‘unprimed’ quantities. Explicitly:

$$C(m + zn, k - zp, m + zn, k - zp) = \Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0,$$

where we have used the conventions of the previous section, such as $C(m, k, m, k) = \Psi_0$, etc. Similar considerations for the second sphere lead to the following proposition:

Proposition 5.3 *A self-dual totally null 2-plane* $\mathcal{N}_z = \mathrm{Span}_{\mathbb{C}}(m + zn, k - zp)$ *is principal at* $x \in \mathcal{M}$ *iff* z *is a root of the equation*

$$\Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0. \tag{19}$$

An anti-self-dual totally null 2-plane $\mathcal{N}_{z'} = \mathrm{Span}_{\mathbb{C}}(m + z'k, n - z'p)$ *is principal at* $x \in \mathcal{M}$ *iff* z' *is a root of the equation*

$$\Psi'_4 z'^4 - 4\Psi'_3 z'^3 + 6\Psi'_2 z'^2 + 4\Psi'_1 z' + \Psi'_0 = 0. \tag{20}$$

Thus at every point of $\mathcal{M}$, we have at most four self-dual principal null 2-planes and at most four anti-self-dual principal null 2-planes. If a principal null 2-plane corresponds to a multiple root of (19) or (20), then such a 2-plane is called a *multiple* principal null 2-plane. A self-dual or anti-self-dual part of the Weyl tensor with multiple principal 2-planes at a point is called *algebraically special* at this point.

We also note that the number and the multiplicity of the roots in (19) or (20) is a *conformal invariant* of the metric at a point. Thus, the algebraically special cases can be further stratified according to the number of the roots and their multiplicities.

The possibilities here for (19) are as follows: (a) three distinct roots, (b) two distinct roots, with one of multiplicity three, (c) two distinct roots, each with multiplicity two, (d) one root of multiplicity four, (e) self-dual part of the Weyl tensor is zero. We have also the corresponding possibilities (a′), (b′) (c′), (d′) and (e′) for (20).

Definition 5.4 The self-dual part of the Weyl tensor *is of Petrov type* II, III, D, N*, or* 0 *at a point*, if Eq. (19) has roots as in the respective cases (a), (b), (c), (d) and (e) at this point. If the Petrov type of the self-dual part of the Weyl tensor varies in $\mathcal{M}$, from point to point, but only between the types II and D, we say that it is of type $\overline{\mathrm{II}}$. The analogous classification holds also for the anti-self-dual part of the Weyl tensor.

Remark 5.5 *Suppose that the self-dual part of the Weyl tensor of* $(\mathcal{M}, g)$ *does not vanish at each point of a neighborhood* $\mathcal{U}' \subset \mathcal{M}$*. Thus at every point of* $\mathcal{U}'$*, we have at least one principal totally null 2-plane. We now take the principal null 2-plane which at* $x \in \mathcal{U}'$ *has the smallest multiplicity* $1 \leq q \leq 4$*. There always exists a neighborhood* $\mathcal{U} \subset \mathcal{U}'$ *of* x *in which this principal totally null 2-plane extends to a field* $\mathcal{N}$ *of principal totally null 2-planes of multiplicity not bigger than* q*. In* $\mathcal{U}$ *we choose a null frame* (m, p, n, k) *in such a way that* $\mathrm{Span}_{\mathbb{C}}(m, k) = \mathcal{N}$*. In this frame, the definition (16) shows that* $\mathcal{N} = \mathcal{N}_0$*, i.e. that the corresponding* $z = 0$ *in* $\mathcal{U}$*. Moreover since* $\mathcal{N}$*, as a field of principal null 2-planes in* $\mathcal{U}$ *satisfies (19), then* $\Psi_0 \equiv 0$ *everywhere in this frame.*

This proves the following

Proposition 5.6 *Around every point* x *of a manifold* $(\mathcal{M}, g)$ *with nowhere vanishing self-dual part of the Weyl tensor, there exists a neighborhood* $\mathcal{U}$ *and a null frame* (m, p, n, k) *in* $\mathcal{U}$ *in which* $\Psi_0 \equiv 0$ *everywhere.*

Now if the self-dual part of the Weyl tensor is algebraically special of type II in $\mathcal{U}$, with $\mathcal{N}$ the corresponding principal multiple field of totally null 2-planes, then in $\mathcal{U}$ we choose a null frame (m, p, n, k) adapted to $\mathcal{N}$. In this frame $\mathcal{N} = \mathcal{N}_0 = \mathrm{Span}(m, k)$, the value $z = 0$ is a double root of (19), and since this is true at every point of $\mathcal{U}$, we have $\Psi_0 \equiv \Psi_1 \equiv 0$. Performing similar considerations for types III and N, and forcing $z = 0$ to be a root of the Eq. (19) with the respective locally constant multiplicity $q = 1, 2, 3$ and 4, we get the following

Proposition 5.7 *Let $\mathcal{N}$ be a field of principal totally null 2-planes for the self-dual part of the Weyl tensor of a metric g on a 4-dimensional manifold $\mathcal{M}$. Assume that $\mathcal{N}$ has a constant multiplicity q in a neighborhood $\mathcal{U}$ in $\mathcal{M}$. Then one can choose a null frame (m, p, n, k) in $\mathcal{U}$, with $\mathcal{N} = \mathrm{Span}(m, k)$ and $g = 2(MP + NK)$, so that*

- *if $q = 1$ then in this frame $\Psi_0 \equiv 0$ and $\Psi_1 \neq 0$,*
- *if $q = 2$ then in this frame $\Psi_0 \equiv \Psi_1 \equiv 0$ and $\Psi_2 \neq 0$,*
- *if $q = 3$ then in this frame $\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0$ and $\Psi_3 \neq 0$,*
- *if $q = 3$ then in this frame $\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv \Psi_3 \equiv 0$ and $\Psi_4 \neq 0$.*

Conversely, if we have a null frame in $\mathcal{U}$ in which

- *$\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv \Psi_3 \equiv 0$ and $\Psi_4 \neq 0$ then $\mathcal{N} = \mathrm{Span}(m, k)$ is a field of multiple principal 2-planes in $\mathcal{U}$ with multiplicity $q = 4$,*
- *$\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0$ and $\Psi_3 \neq 0$ then $\mathcal{N} = \mathrm{Span}(m, k)$ is a field of multiple principal 2-planes in $\mathcal{U}$ with multiplicity $q = 3$,*
- *$\Psi_0 \equiv \Psi_1 \equiv 0$ and $\Psi_2 \neq 0$ then $\mathcal{N} = \mathrm{Span}(m, k)$ is a field of multiple principal 2-planes in $\mathcal{U}$ with multiplicity $q = 2$,*
- *$\Psi_0 \equiv 0$ and $\Psi_1 \neq 0$ then $\mathcal{N} = \mathrm{Span}(m, k)$ is a field of multiple principal 2-planes in $\mathcal{U}$ with multiplicity $q = 1$.*

This immediately implies

Corollary 5.8 *The self-dual part of the Weyl tensor of a metric g on a 4-dimensional manifold $\mathcal{M}$ is algebraically special in neighborhood $\mathcal{U}$, with $\mathcal{N}$ being a field of multiple principal 2-planes in $\mathcal{U}$ if and only if there exists a null frame (m, p, n, k) in $\mathcal{U}$ in which $\Psi_0 \equiv \Psi_1 \equiv 0$ in $\mathcal{U}$. In this frame, $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$.*

5.1 Generalizing the Przanowski–Plebański version

The starting point for our generalizations of the Goldberg–Sachs theorem is to replace the Ricci flat condition from the classical version [4], by a condition on only that part of the Ricci tensor, which is 'visible' to the integrable totally null 2-plane $\mathcal{N}$.

For this we consider the Ricci tensor of $(\mathcal{M}, g)$ as a *symmetric*, possibly *degenerate*, *bilinear form* on $\mathcal{M}$. We denote it by Ric and extend it to the complexification $\mathrm{T}^{\mathbb{C}}\mathcal{M}$ by linearity. Now given a complex distribution $\mathcal{Z} \subset \mathrm{T}^{\mathbb{C}}\mathcal{M}$ we say that the Ricci tensor is degenerate on $\mathcal{Z}$,

$$Ric_{|\mathcal{Z}} = 0, \qquad \text{iff} \qquad Ric(Z_1, Z_2) = 0, \quad \forall Z_1, Z_2 \in \mathcal{Z}.$$

Then we have the following theorem:

Theorem 5.9 *Let $\mathcal{N} \subset \mathrm{T}^{\mathbb{C}}\mathcal{M}$ be a field of totally null 2-planes on a 4-dimensional manifold $(\mathcal{M}, g)$ equipped with a real metric g of any signature. Assume that the Ricci tensor Ric of $(\mathcal{M}, g)$, considered as a symmetric bilinear form on $\mathrm{T}^{\mathbb{C}}\mathcal{M}$, is degenerate on $\mathcal{N}$,*

$$Ric_{|\mathcal{N}} = 0.$$

If in addition the field $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, everywhere on $\mathcal{M}$, then $(\mathcal{M}, g)$ is algebraically special at every point, with a field of multiple principal totally null 2-planes tangent to $\mathcal{N}$.

To prove it, we fix a null frame (m, p, n, k) on $\mathcal{M}$ adapted to $\mathcal{N}$. This means that $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$.

It is then very easy to see that the vanishing of the Ricci tensor on $\mathcal{N}$ is, due to our conventions, equivalent to the conditions

$$\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0.$$

Instead of proving Theorem 5.9, we prove a theorem that implies it. This is the complex version of the Goldberg–Sachs theorem, which generalizes the Lorentzian version due to Przanowski and Plebanski [23]. When stated in the Newman–Penrose language, this reads as follows:

Theorem 5.10 (1) *Suppose that a 4-dimensional metric g satisfies $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$ and $\kappa \equiv \sigma \equiv 0$. Then $\Psi_0 \equiv \Psi_1 \equiv 0$.*

(2) *If g is Einstein, $Ric(g) = \Lambda g$, and has a nowhere vanishing self-dual part of the Weyl tensor, then $\Psi_0 \equiv \Psi_1 \equiv 0$ implies $\kappa \equiv \sigma \equiv 0$.*

Before the proof, we make the following remarks:

Remark 5.11 *It is easy to see that part (1) of the above Theorem is equivalent to Theorem 5.9.*

Remark 5.12 *Note that $Ric = 0$ and more generally $Ric = \Lambda g$ are special cases of our condition $Ric_{|\mathcal{N}} = 0$.*

Proof (of Theorem 5.10). First we assume that κ and σ vanish everywhere on $\mathcal{M}$. To conclude that $\Psi_0 \equiv 0$ is very easy: Actually this conclusion is an immediate consequence of the Newman–Penrose equation (74). For if κ and σ are identically vanishing, then Eq. (74) gives $\Psi_0 \equiv 0$. Note that this conclusion holds even without *any* assumption about the components of the Schouten tensor P (or the Ricci tensor).

Now we prove the following

Lemma 5.13 *Suppose that a 4-dimensional metric g satisfies $\kappa \equiv \sigma \equiv 0$ and*

$$\delta\Psi_1 \equiv 2(\beta + 2\tau)\Psi_1, \tag{21}$$

$$D\Psi_1 \equiv 2(\varepsilon - 2\rho)\Psi_1. \tag{22}$$

Then it also satisfies

$$\Psi_1 \equiv 0.$$

Proof We use the commutator (13), and the Newman–Penrose equations (75–77) to obtain the compatibility conditions for (21) and (22). This is a pure calculation. We give its main steps below:

- applying $[\delta, D]$ to (21) and (22) we get:

$$[\delta, D]\Psi_1 \equiv 2\delta\left((\varepsilon - 2\rho)\Psi_1\right) - 2D\left((\beta + 2\tau)\Psi_1\right);$$

- next, using (13), and again (21) and (22), we transform this identity into:

$$\begin{aligned} 2(\rho' + \varepsilon' - \varepsilon)(\beta + 2\tau)\Psi_1 + 2(\alpha' + \beta + \pi')(\varepsilon - 2\rho)\Psi_1 \\ \equiv 2\delta\left((\varepsilon - 2\rho)\Psi_1\right) - 2D\left((\beta + 2\tau)\Psi_1\right); \end{aligned} \tag{23}$$

- now, the Leibniz rule, and a third use of (21) and (22), enables us to eliminate of the derivatives of Ψ_1 in (23);

- actually, simplifying (23), and using (21), (22) we get:

$$\big(2\delta(\varepsilon-2\rho)-2D(\beta+2\tau)+2(\varepsilon-\varepsilon'-\rho')(\beta+2\tau)-2(\alpha'+\beta+\pi')(\varepsilon-2\rho)\big)\,\Psi_1 \equiv 0; \tag{24}$$

- the last step in the proof of the lemma is to use the Newman–Penrose equations (75–77);
- these equations eliminate $\delta\varepsilon - D\beta$, (look at 75), $\delta\rho$, (look at 76), and $D\tau$, (look at 77), from the identity (24);
- this makes the identity (24) derivative-free;
- actually it transforms (24) to a remarkable identity:

$$(10\Psi_1)\Psi_1 \equiv 0; \tag{25}$$

- the identity (25) obviously implies $\Psi_1 \equiv 0$;

This proves Lemma 5.13. □

To conclude the proof of the part one of Theorem 5.10, we use our assumptions $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$, $\kappa \equiv \sigma \equiv 0$, and their consequence $\Psi_0 \equiv 0$, and insert them in the Bianchi identities (83) and (84). This trivially gives the relations (21) and (22), respectively. Then an obvious use of Lemma 5.13 finishes the proof of part one of Theorem 5.10.

We now pass to the proof of part two of Theorem 5.10.
When going from ($\Psi_0 \equiv \Psi_1 \equiv 0$) to ($\kappa \equiv \sigma \equiv 0$) we do as follows:

- Initially we only assume that $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$.
- Then the Bianchi identities (83) and (84) give:

$$2\mathsf{P}_{13}\kappa + (3\Psi_2 + \mathsf{P}_{12} - \mathsf{P}_{34})\sigma \equiv 0 \tag{26}$$

  and

$$(3\Psi_2 - \mathsf{P}_{12} + \mathsf{P}_{34})\kappa + 2\mathsf{P}_{24}\sigma \equiv 0, \tag{27}$$

  respectively.

At this stage, the following remark is in order:

Remark 5.14 *If we were able to conclude that the rank of the matrix*

$$m = \begin{pmatrix} 2\mathsf{P}_{13} & 3\Psi_2 + \mathsf{P}_{12} - \mathsf{P}_{34} \\ 3\Psi_2 - \mathsf{P}_{12} + \mathsf{P}_{34} & 2\mathsf{P}_{24} \end{pmatrix} \tag{28}$$

was identically *equal to* two*, this would immediately yield $\kappa \equiv \sigma \equiv 0$, which would conclude the proof. On the other extreme, if we were sure that the matrix m was* identically *equal to zero (i.e if it had rank identically equal to zero), we would argue as follows: The identically zero rank of m means that in addition to $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$ we have:* $\mathsf{P}_{13} \equiv \mathsf{P}_{24} \equiv \mathsf{P}_{12} - \mathsf{P}_{34} \equiv \Psi_2 \equiv 0$. *Then, combining the Bianchi identities (85) and (91), we get*

$$2\mathsf{P}_{33}\kappa + 2(\mathsf{P}_{23} - 3\Psi_3)\sigma \equiv 0.$$

Similarly, using the Bianchi identities (86) and (92) we get:

$$2(\mathsf{P}_{23} + 3\Psi_3)\kappa + 2\mathsf{P}_{22}\sigma \equiv 0.$$

Thus, in such case, the situation is similar to the previously considered case with the matrix m: Now we have

$$m_1 = \begin{pmatrix} \mathsf{P}_{33} & -3\Psi_3 + \mathsf{P}_{23} \\ 3\Psi_3 + \mathsf{P}_{23} & \mathsf{P}_{22} \end{pmatrix},$$

and if m_1 has rank identically *equal to* two*, we conclude that $\kappa \equiv \sigma \equiv 0$. If it has rank* identically *equal to* zero*, we in addition have $\mathsf{P}_{33} \equiv \mathsf{P}_{22} \equiv \mathsf{P}_{23} \equiv \Psi_3 \equiv 0$. This, due to the Bianchi identities, implies also that $\mathsf{P}_{12} \equiv \mathsf{P}_{34} \equiv$ const. Comparing this with (87) and (88) leads to*

$$\Psi_4\sigma \equiv \Psi_4\kappa \equiv 0,$$

which if we assume $\Psi_4 \neq 0$, yields $\kappa \equiv \sigma \equiv 0$.

This remark emphasizes that the local properties of the matrices m and m_1 are crucial for the behavior of κ and σ. Since we have *no guarantee* that rank of e.g. m is locally constant, returning to our proof, we must strengthen our assumptions on g by requiring that it satisfies more curvature conditions than $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$.

- The additional conditions which enable us to get $\kappa \equiv \sigma \equiv 0$ are:

  $$\mathsf{P}_{13} \equiv \mathsf{P}_{22} \equiv \mathsf{P}_{23} \equiv \mathsf{P}_{24} \equiv \mathsf{P}_{33} \equiv \mathsf{P}_{12} - \mathsf{P}_{34} \equiv 0.$$

  These, with the already assumed $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$, constitute the *full set of Einstein conditions $Ric(g) = \Lambda g$*, for the metric g.
- Under the Einstein assumption $Ric(g) = \Lambda g$ and the requirement that the self-dual part of the Weyl tensor is non-vanishing, we get $\kappa \equiv \sigma \equiv 0$ in a very easy way, by a successive inspection of the Bianchi identities (83–88).
- Indeed, the assumed Einstein equations $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv \mathsf{P}_{22} \equiv \mathsf{P}_{24} \equiv \mathsf{P}_{13} \equiv \mathsf{P}_{23} \equiv \mathsf{P}_{33} \equiv \mathsf{P}_{12} - \mathsf{P}_{34} \equiv 0$, the algebraical speciality conditions $\Psi_0 \equiv \Psi_1 \equiv 0$, and the Bianchi identities (83), (84), give $\sigma\Psi_2 \equiv 0$ and $\kappa\Psi_2 \equiv 0$. This means that whenever $\Psi_2 \neq 0$ we have $\kappa \equiv \sigma \equiv 0$. By continuity the points in which κ or σ are non-zero form *open* sets in $\mathcal{M}$. On these sets $\Psi_2 \equiv 0$ everywhere. Thus the discussed situation has only two possible outcomes: either $\kappa \equiv \sigma \equiv 0$ (which finishes the proof), or we have $\Psi_2 \equiv 0$ in an open set, in addition to the assumed $\Psi_0 \equiv \Psi_1 \equiv 0$.
- In this latter case we look at the Bianchi identities (85) and (86), obtaining: $\sigma\Psi_3 \equiv 0$ and $\kappa\Psi_3 \equiv 0$. This again leads to either $\sigma \equiv \kappa \equiv 0$ or to $\Psi_3 \equiv 0$ in addition to $\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0$.
- If $\Psi_3 \equiv 0$ the Bianchi identities (87) and (88) give: $\sigma\Psi_4 \equiv 0$ and $\kappa\Psi_4 \equiv 0$. Thus if we want to have non-vanishing self-dual part of the Weyl tensor, we are forced to have $\kappa \equiv \sigma \equiv 0$.
- This finishes the proof in this direction.

Thus in going from ($\Psi_0 \equiv \Psi_1 \equiv 0$) to ($\kappa \equiv \sigma \equiv 0$), we are only able to prove the theorem in the classical (although with a possibly non-zero cosmological constant) Goldberg–Sachs version, namely Theorem 5.10, (2). □

Whether it is possible to weaken the Einstein assumption above to $Ric_{|\mathcal{N}} \equiv 0$ is an open question.

5.2 Generalizing the Kundt–Thompson and the Robinson–Schild version

As noted by Kundt and Thompson [9] and Robinson and Schild [24], to achieve the algebraic speciality of the metric, when $\kappa \equiv \sigma \equiv 0$ has been assumed, it is sufficient to use weaker conditions than $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$. There are various approaches to obtain these conditions in the General Relativity literature (see e.g. [18]). In this section we present our approach, which is signature independent.

We first assume that $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$ holds *only conformally*. Thus, we merely assume that there exists a scale $\Upsilon : \mathcal{M} \to \mathbb{R}$ such that the rescaled metric $\hat{g} = e^{2\Upsilon} g$ satisfies

$$\hat{Ric}_{|\mathcal{N}} \equiv 0,$$

where $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$. This means that choosing a null coframe (M, P, N, K) for g, and the corresponding rescaled null coframe $\hat{M} = e^{\Upsilon} M$, $\hat{P} = e^{\Upsilon} P$, $\hat{N} = e^{\Upsilon} N$ and $\hat{K} = e^{\Upsilon} K$ for $\hat{g}$ we have

$$\hat{\mathsf{P}}_{11} \equiv \hat{\mathsf{P}}_{14} \equiv \hat{\mathsf{P}}_{44} \equiv 0. \tag{29}$$

Note that for this to be satisfied, we do not need to assume $\mathsf{P}_{11} \equiv \mathsf{P}_{14} \equiv \mathsf{P}_{44} \equiv 0$. Our aim now is to deduce what restrictions on g are imposed by Eq. (29).

As it is well known (see e.g. [5]) the rescaled Schouten tensor $\hat{\mathsf{P}}$ is related to P via:

$$\hat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2} \Upsilon_c \Upsilon^c g_{ab},$$

with $\Upsilon_a = \nabla_a \Upsilon$. Now, applying the covariant derivative ∇_c on both sides of this equation, antisymmetrizing over the indices $\{ca\}$ and using again this equation to eliminate the covariant derivatives of Υ_a we get

$$\nabla_{[c} \hat{\mathsf{P}}_{a]b} + \Upsilon_{[a} \hat{\mathsf{P}}_{c]b} + \Upsilon^d \hat{\mathsf{P}}_{d[a} g_{c]b} \equiv \tfrac{1}{2} (A_{bca} + C_{acb}{}^{d} \Upsilon_d). \tag{30}$$

Here A_{abc} is the Cotton tensor

$$A_{abc} = 2 \nabla_{[b} \mathsf{P}_{c]a},$$

and C_{abcd} is the Weyl tensor. Note that in addition to

$$A_{abc} = -A_{acb},$$

as a consequence of the first and the second Bianchi identities, we also have:

$$A_{abc} + A_{cab} + A_{bca} = 0, \tag{31}$$

and

$$A_{abc} = \nabla_d C^d_{abc}, \qquad A^a_{ab} = 0, \tag{32}$$

respectively.

The obtained identity (30) is a generalization of the identity known in the theory of conformally Einstein spaces (see e.g. [5]). It is interesting on its own, but it is particularly useful in our situation of Eq. (29).

Let as assume that in addition to (29), the distribution of totally null planes $\mathcal{N}$ is integrable. This means that in the frame (m, p, n, k) we have

$$\kappa \equiv \sigma \equiv 0,$$

which is the same as assuming that the respective connection coefficients satisfy

$$\Gamma_{414} \equiv \Gamma_{411} \equiv 0. \tag{33}$$

As we proved in the previous section this implies that the Weyl tensor coefficient

$$\Psi_0 \equiv 0.$$

Now, using the frame (m, p, n, k) and our assumptions (29) and (33) on the l.h.s of the identity (30), we directly check that the following proposition is true:

Proposition 5.15 *Suppose that a distribution of totally null 2-planes $\mathcal{N}$ on $(\mathcal{M}, g)$ be integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, and that the Schouten tensor $\hat{P}$ of the rescaled metric $\hat{g} = \mathrm{e}^{2\Upsilon} g$ is degenerate on $\mathcal{N}$, $\hat{P}_{|\mathcal{N}} \equiv 0$. Then for every three vector fields $X, Y, Z \in \mathcal{N}$ we have:*

$$X^a Y^b Z^c (\nabla_{[c} \hat{\mathsf{P}}_{a]b} + \Upsilon_{[a} \hat{\mathsf{P}}_{c]b} + \Upsilon^d \hat{\mathsf{P}}_{d[a} g_{c]b}) \equiv 0.$$

Since, in addition, in the coframe (m, p, n, k) the Weyl tensor coefficient $\Psi_0 \equiv 0$, the r.h.s. of (30), after being contracted with vectors X, Y, Z from $\mathcal{N}$, includes only the Weyl tensor coefficient Ψ_1. Thus the considered identity, when restricted to $\mathcal{N}$, reduces to two complex equations:

$$A_{141} - \Psi_1 \delta \Upsilon \equiv 0, \tag{34}$$

and

$$A_{441} - \Psi_1 D \Upsilon \equiv 0. \tag{35}$$

This relates the components $\{141\}$ and $\{441\}$ of the Cotton tensor *algebraically* to the Weyl tensor coefficient Ψ_1, and proves the following

Proposition 5.16 *A metric g with an integrable field of self-dual totally null 2-planes $\mathcal{N}$ on a 4-dimensional manifold $\mathcal{M}$ admits a conformal scale $\Upsilon : \mathcal{M} \to \mathbb{R}$ such that the rescaled metric $\hat{g}$ has Ricci tensor $\hat{Ric}$ degenerate on $\mathcal{N}$,*

$$\hat{Ric}_{|\mathcal{N}} \equiv 0,$$

only if the Cotton tensor A of the original metric satisfies Eq. (34), (35) in a null coframe in which $\kappa \equiv \sigma \equiv 0$.

It is interesting that the expressions (34) and (35) appear also in the following

Proposition 5.17 *Suppose that a metric g admits an integrable maximal totally null field of 2-planes. Then the Cotton tensor components A_{141} and A_{441} in the null coframe (M, P, N, K) in which $\kappa \equiv \sigma \equiv 0$ are related to the Cotton tensor components $\hat{A}_{\hat{1}\hat{4}\hat{1}}$ and $\hat{A}_{\hat{4}\hat{4}\hat{1}}$ in the null coframe $(\mathrm{e}^{\Upsilon} M, \mathrm{e}^{\Upsilon} P, \mathrm{e}^{\Upsilon} N, \mathrm{e}^{\Upsilon} K)$ of the rescaled metric $\hat{g} = \mathrm{e}^{2\Upsilon} g$ via*

$$\hat{A}_{\hat{1}\hat{4}\hat{1}} = \mathrm{e}^{-3\Upsilon} (A_{141} - \Psi_1 \delta \Upsilon), \tag{36}$$

$$\hat{A}_{\hat{4}\hat{4}\hat{1}} = \mathrm{e}^{-3\Upsilon} (A_{441} - \Psi_1 D \Upsilon). \tag{37}$$

The proof of this fact is straightforward. For example it can be checked in the Newman–Penrose formalism with $\kappa = \sigma = 0$, in which the relevant components of the Cotton tensor read:

$$A_{141} = D\mathsf{P}_{11} - \delta \mathsf{P}_{14} + (2\epsilon' - 2\epsilon + \rho')\mathsf{P}_{11} + (2\beta + 2\pi')\mathsf{P}_{14} - \lambda' \mathsf{P}_{44}, \tag{38}$$

$$A_{441} = D\mathsf{P}_{14} - \delta \mathsf{P}_{44} - \kappa' \mathsf{P}_{11} + (2\rho' - 2\epsilon)\mathsf{P}_{14} + (2\alpha' + 2\beta + \pi')\mathsf{P}_{44}. \tag{39}$$

Now, treating the Cotton tensor A as a linear map $\mathrm{T}\mathcal{M} \times \mathrm{T}\mathcal{M} \times \mathrm{T}\mathcal{M} \to \mathbb{R}$, we recall that A is *degenerate* on a vector distribution $\mathcal{Z}$, $A_{|\mathcal{Z}} = 0$, iff $A(Z_1, Z_2, Z_3) = 0$ for all $Z_1, Z_2, Z_3 \in \mathcal{Z}$. Then, if we take $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$, where (m, p, n, k) is a null frame, we see that $A_{441} = A_{141} = 0$ if and only if $A_{|\mathcal{N}} = 0$. This together with Propositions 5.16 and 5.17 imply the following

Corollary 5.18 *Suppose that a metric g admits an* integrable *maximal totally null field $\mathcal{N}$ of 2-planes. If the metric can be conformally rescaled to $\hat{g}$ so that the rescaled Ricci tensor $\hat{Ric}$ is degenerate on $\mathcal{N}$, $\hat{Ric}_{|\mathcal{N}} \equiv 0$, then in this scale the rescaled Cotton tensor $\hat{A}$ is degenerate on $\mathcal{N}$, $\hat{A}_{|\mathcal{N}} \equiv 0$.*

Remark 5.19 *We note that given an integrable totally null field of 2-planes $\mathcal{N}$ the condition $\hat{A}_{|\mathcal{N}} \equiv 0$ is* weaker *than $\hat{Ric}_{|\mathcal{N}} \equiv 0$. We saw that $\hat{Ric}_{|\mathcal{N}} \equiv 0$ implies $\hat{A}_{|\mathcal{N}} \equiv 0$, but the converse is not guaranteed.*

Now we use the Bianchi identities (93) and (94), which we display here as the following

Lemma 5.20 *On any 4-dimensional manifold with a metric g as in (1) we have*

$$A_{141} \equiv \triangle\Psi_0 + (\mu - 4\gamma)\Psi_0 - \delta\Psi_1 + 2(2\tau + \beta)\Psi_1 - 3\sigma\Psi_2 \tag{40}$$

$$A_{414} \equiv \partial\Psi_0 - (\pi + 4\alpha)\Psi_0 + D\Psi_1 + 2(2\rho - \varepsilon)\Psi_1 + 3\kappa\Psi_2. \tag{41}$$

Proof This is proved in the "Appendix", but we can also see this by observing that subtracting (40) from (38) and, respectively (41) from (39) we obtain the respective Bianchi identities (83) and (84). □

This Lemma is crucial for the rest of our arguments in this section. It has various consequences, the first being the following much sharper version of part one of Theorem 5.10:

Theorem 5.21 *Let $\mathcal{N} \subset \mathrm{T}^{\mathbb{C}}\mathcal{M}$ be a field of totally null 2-planes on a 4-dimensional manifold $(\mathcal{M}, g)$ equipped with metric g. Assume that the Cotton tensor A of the metric g, considered as a threelinear form on $\mathrm{T}^{\mathbb{C}}\mathcal{M}$, is degenerate on $\mathcal{N}$,*

$$A_{|\mathcal{N}} \equiv 0.$$

If in addition the field $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, everywhere on $\mathcal{M}$, then $(\mathcal{M}, g)$ is algebraically special at every point, with a field of multiple principal totally null 2-planes tangent to $\mathcal{N}$.

Proof In an adapted null coframe (M, P, N, K) our integrability assumption is $\kappa \equiv \sigma \equiv 0$, which as we know, implies $\Psi_0 \equiv 0$. The assumption about the degeneracy of the Cotton tensor means $A_{141} \equiv A_{441} \equiv 0$, which together with $\Psi_0 \equiv 0$ and Lemma 5.20 gives the identities: $\delta\Psi_1 \equiv 2(\beta + 2\tau)\Psi_1$ and $D\Psi_1 \equiv 2(\varepsilon - 2\rho)\Psi_1$. This implies $\Psi_1 \equiv 0$ by Lemma 5.13. Thus the field of (principal) totally null 2-planes $\mathcal{N}$ is multiple. □

Remark 5.22 *Note that as a result of this theorem, the assumption $A_{|\mathcal{N}} \equiv 0$ is* conformal. *Without knowing that $\kappa \equiv \sigma \equiv 0$ and $A_{141} \equiv A_{441} \equiv 0$ imply $\Psi_1 \equiv 0$, the assumption $A_{141} \equiv A_{441} \equiv 0$ seemed to be* not *conformal, because of the inhomogeneous terms in the transformations (36, 37). But since under the assumptions $\kappa \equiv \sigma \equiv 0$ and $A_{141} \equiv A_{441} \equiv 0$ we were able to discover that actually $\Psi_1 \equiv 0$, then A_{141} and A_{441} transform homogeneously under the conformal rescaling. Thus, in such case the condition $A_{|\mathcal{N}} \equiv 0$* is *conformal.*

The second application of Lemma 5.20 is included in the following

Remark 5.23 *Suppose that we would like to have a still sharper (than in Theorem 5.21) version of part one of Theorem 5.10. Thus instead of assuming* $Ric_{|\mathcal{N}} \equiv 0$*, or the weaker condition* $A_{|\mathcal{N}} \equiv 0$*, we would like to have an assumption about vanishing of still higher order derivatives of the curvature, that together with* $\kappa \equiv \sigma \equiv 0$ *would imply* $\Psi_1 \equiv 0$*. Then Lemma (5.20) assures that it is impossible, and the condition* $A_{|\mathcal{N}} \equiv 0$ *can not be weakened. Indeed, denoting such hypothetical condition by* $S \equiv 0$*, we would have* $(\kappa \equiv \sigma \equiv 0 \,\&\, S \equiv 0) \Rightarrow (\Psi_1 \equiv 0)$*. But since* $\kappa \equiv \sigma \equiv 0$*, in addition, implies that* $\Psi_0 \equiv 0$*, then Lemma 5.20 implies* $A_{141} \equiv A_{441} \equiv 0$*. Thus the hypothetically weaker than* $A_{|\mathcal{N}} \equiv 0$ *condition* $S \equiv 0$*, in turn, implies* $A_{|\mathcal{N}} \equiv 0$*. Since this alone, according to Theorem 5.21, is already sufficient to imply* $\Psi_1 \equiv 0$*, we do not need condition* $S \equiv 0$ *to obtain the desired result. This proves the following*

Theorem 5.24 *The weakest curvature condition which together with the integrability condition,* $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$*, implies that the field of totally null 2-planes* $\mathcal{N}$ *is principal and multiple is the degeneracy of the Cotton tensor on* $\mathcal{N}$*,* $A_{|\mathcal{N}} \equiv 0$*.*

Example 5.25 An example of a condition $S \equiv 0$ which is *a priori* weaker than $A_{|\mathcal{N}} \equiv 0$ may be obtained as follows. The procedure used in the proof of Lemma 5.13 may be equally applied to the situation in which the conditions (21, 22) are replaced by the Bianchi identities (40) and (41). Then, under the assumption that $\kappa \equiv \sigma \equiv 0$, and hence $\Psi_0 \equiv 0$, we literally repeat all the steps from the proof of Lemma 5.13. Indeed, starting with the application of δ on both sides of $A_{141} \equiv -\delta\Psi_1 + 2(2\tau + \beta)\Psi_1$ and D on both sides of $A_{414} \equiv D\Psi_1 + 2(2\rho - \varepsilon)\Psi_1$, after subtraction and use of the commutator (13), we obtain the following *identity*:

$$-10\Psi_1^2 \equiv DA_{141} - \delta A_{441} - (3\epsilon - \rho' - \epsilon' - 4\rho)A_{141} + (3\beta + \alpha' + \pi' + 4\tau)A_{441}. \tag{42}$$

This, is satisfied *always* when $\kappa \equiv \sigma \equiv 0$. Thus *the vanishing of the r.h.s of* (42) implies $\Psi_1 \equiv 0$. Moreover, since when $\kappa \equiv \sigma \equiv 0$ the vanishing of Ψ_1 is a conformal property, then the vanishing of the r.h.s. of (42) is a *conformal* property. In fact a direct calculation shows that if in a null coframe (M, P, N, K) we have $\kappa \equiv \sigma \equiv 0$ and

$$S = DA_{141} - \delta A_{441} - (3\epsilon - \rho' - \epsilon' - 4\rho)A_{141} + (3\beta + \alpha' + \pi' + 4\tau)A_{441} \tag{43}$$

then in the conformally rescaled metric $\hat{g} = e^{2\Upsilon} g$ and in the corresponding rescaled null coframe $(e^{\Upsilon}M, e^{\Upsilon}P, e^{\Upsilon}N, e^{\Upsilon}K)$ we have $\hat{\kappa} \equiv \hat{\sigma} \equiv 0$ and

$$\hat{S} = e^{-4\Upsilon} S.$$

Now using the explicit formulae for the covariant derivatives of the Cotton tensor components A_{141} and A_{441}:

$$\nabla_4 A_{141} = DA_{141} - (3\epsilon - \epsilon')A_{141} + \pi' A_{441}$$
$$\nabla_1 A_{441} = \delta A_{141} - (3\beta + \alpha')A_{441} - \rho' A_{141},$$

solving this for DA_{141} and δA_{141} and inserting in (43), we get

$$S = \nabla_4 A_{141} - \nabla_1 A_{441} + 4\rho A_{141} + 4\tau A_{441}. \tag{44}$$

We thus have a condition $S \equiv 0$, which together with $\kappa \equiv \sigma \equiv 0$ is *conformal* and *implies* that $\Psi_1 \equiv 0$. It is always satisfied when $A_{441} \equiv A_{141} \equiv 0$, i.e. we have $(A_{441} \equiv A_{141} \equiv 0) \Rightarrow S \equiv 0$, and at the first glance there is no reason for the implication $(S \equiv 0) \Rightarrow (A_{441} \equiv$

$A_{141} \equiv 0$) However, this implication is true, on the ground of the discussion in Remark 5.23. As a consequence we have

Proposition 5.26 *Under the assumption that the distribution of self-dual totally null 2-planes $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, the following two, conformally invariant, conditions are equivalent*

- *the Cotton tensor of the metric g is degenerate on $\mathcal{N}$, $A_{|\mathcal{N}} \equiv 0$*
- *the scalar S of the metric g, as defined in (44), identically vanishes, $S \equiv 0$.*

To discuss the next application of Lemma 5.20, we introduce

Definition 5.27 A metric g on a 4-dimensional manifold $\mathcal{M}$ is called *II-generic* if and only if the points in which its self-dual part of the Weyl tensor degenerates to Petrov types *III*, *N* or 0 are rare, in the sense that they belong to closed sets without interior in $\mathcal{M}$.

In particular every metric with self-dual part of the Weyl tensor being at each point of $\mathcal{M}$ algebraically general, or of mixed type: algebraically general on some subsets and type II or type D on their complements, is II-generic; a metric which is e.g. of type III in an open set of $\mathcal{M}$ is not II-generic.

Now we are ready to discuss a slight generalization of the known *conformal* versions of the Goldberg–Sachs theorem. In the Lorentzian case, such versions were given by Kundt and Thompson [9] and Robinson and Schild [24]. Penrose and Rindler [18] gave a complex (spinorial) version of the Kundt–Thompson/Robinson–Schild theorem. Here, we quote *our* complex version, which is a slight generalization:

Theorem 5.28 *Let $\mathcal{M}$ be a 4-dimensional manifold with a II-generic metric g. Let $\mathcal{N}$ be a field of self-dual totally null 2-planes on $\mathcal{M}$. Then any two of the following imply the third:*

(0) *The Cotton tensor of g is degenerate on $\mathcal{N}$, $A_{|\mathcal{N}} \equiv 0$.*
(i) *$\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$.*
(ii) *The self-dual part of the Weyl tensor is algebraically special on $\mathcal{M}$ with $\mathcal{N}$ being a multiple principal field of self-dual totally null 2-planes.*

Proof First, we observe that the implication ((0) & (i)) $\Rightarrow$ (ii) is true, as a simple application of Theorem 5.21. Note that for this we do not need the genericity assumption about the Weyl tensor.

To prove the other two implications we choose a null coframe on $(\mathcal{M}, g)$ so that $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$ and $g = 2(MP + NK)$ as in (1). Then

- the condition (0) is: $A_{141} \equiv A_{441} \equiv 0$,
- the condition (i) is: $\kappa \equiv \sigma \equiv 0$,
- the condition (ii) is: $\Psi_0 \equiv \Psi_1 \equiv 0$.

Now, the proof of ((i) & (ii)) $\Rightarrow$ (0) is an immediate consequence of Lemma 5.20, since the assumptions (i) & (ii) imply the identical vanishing of the r.h.s. of identities (40, 41), which means that also their l.h.s. identically vanish, $A_{141} \equiv A_{441} \equiv 0$. Note that also in the proof of this statement the genericity assumption about the Weyl tensor was not needed.

This assumption is however needed to get the last implication ((0) & (ii)) $\Rightarrow$ (i). Indeed assuming (i) & (ii), the identities (40, 41) from Lemma 5.20 reduce to the identities $-3\sigma\Psi_2 \equiv 0$ and $3\kappa\Psi_2 \equiv 0$. Now, similarly as in the proof of part two of the Theorem 5.10, to conclude that $\kappa \equiv \sigma \equiv 0$ in a neighborhood $\mathcal{U} \subset \mathcal{M}$, it is enough to assume that $\Psi_2 \neq 0$ on the complement of the closed sets without interior in $\mathcal{U}$. Since in our coframe in $\mathcal{U}$, according to (ii), we have $\Psi_0 \equiv \Psi_1 \equiv 0$, Proposition 5.7 assures that the coefficient Ψ_2 of the Weyl tensor is non-vanishing on the complement of the closed sets without interior in $\mathcal{U}$ if and only if the metric is II-generic in $\mathcal{U}$. Since this is the main assumption of Theorem 5.28 we see that $3\sigma\Psi_2 \equiv 0$ and $3\kappa\Psi_2 \equiv 0$ imply $\kappa \equiv \sigma \equiv 0$ in $\mathcal{U}$. This proves the part ((0) & (ii)) $\Rightarrow$ (i) of the theorem. □

As a consequence of this proof, we also have the following

Corollary 5.29 *Let $\mathcal{M}$ be a 4-dimensional manifold with a metric g and let $\mathcal{N}$ be a field of self-dual totally null 2-planes on $\mathcal{M}$. Assume that $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, and that the self-dual part of the Weyl tensor is algebraically special on $\mathcal{M}$, with $\mathcal{N}$ being a multiple principal field of self-dual totally 2-planes. Then the Cotton tensor of g is degenerate on $\mathcal{N}$, $A_{|\mathcal{N}} \equiv 0$.*

To discuss the sharpening of the Theorem 5.28 with respect to the implication ((0) & (ii)) $\Rightarrow$ (i) we introduce two more notions analogous to the II-genericity.

Definition 5.30 A metric g on a 4-dimensional manifold $\mathcal{M}$ is called *III-generic* if and only if the points in which its self-dual part of the Weyl tensor degenerates to Petrov types N or 0 belong to closed sets without interior in $\mathcal{M}$. Similarly, a metric g on a 4-dimensional manifold $\mathcal{M}$ is called *N-generic* if and only if the points in which its self-dual part of the Weyl tensor vanishes belong to closed sets without interior in $\mathcal{M}$.

For the III-generic metrics, we have the following

Theorem 5.31 *Let $\mathcal{M}$ be a 4-dimensional manifold with a III-generic metric g, whose self-dual part of the Weyl tensor is in addition algebraically special at all points of $\mathcal{M}$. Let $\mathcal{N}$ be the corresponding field of multiple principal totally null 2-planes on $\mathcal{M}$. If the Cotton tensor A of the metric g satisfies*

$$A(\,\cdot\,, Z_1, Z_2) \equiv 0, \qquad \forall Z_1, Z_2 \in \mathcal{N}$$

then the field $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, on $\mathcal{M}$.

Similarly for the N-generic metrics, we have

Theorem 5.32 *Let $\mathcal{M}$ be a 4-dimensional manifold with an N-generic metric g, whose self-dual part of the Weyl tensor is in addition algebraically special at all points of $\mathcal{M}$. Let $\mathcal{N}$ be the corresponding field of multiple principal totally null 2-planes on $\mathcal{M}$. Consider the 2-forms $A_Z = A(Z, \,\cdot\,, \,\cdot\,)$, where A is the Cotton tensor of the metric g and Z is a complex-valued vector field Z on $\mathcal{M}$. If for every vector field $Z \in \mathcal{N}$ the two form A_Z is anti-self-dual at each point of $\mathcal{M}$, then the field $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, on $\mathcal{M}$.*

We first prove Theorem 5.31.

Proof Again we choose a null coframe on $(\mathcal{M}, g)$ so that $\mathcal{N} = \mathrm{Span}_{\mathbb{C}}(m, k)$ and $g = 2(MP + NK)$ as in (1). Since $\mathcal{N}$ consists of *multiple* principal null 2-planes, according to Proposition 5.7, we have $\Psi_0 \equiv \Psi_1 \equiv 0$ in this coframe. Moreover, in this coframe, the condition $A(\,\cdot\,, Z_1, Z_2) \equiv 0\, \forall Z_1, Z_2 \in \mathcal{N}$ means that the coframe components $A_{i41}, i = 1, 2, 3, 4$ satisfy

$$A_{141} \equiv A_{214} \equiv A_{341} \equiv A_{414} \equiv 0. \tag{45}$$

Now we again use the Bianchi identities (93, 94) which reduce to

$$-3\sigma\Psi_2 \equiv 0, \qquad \text{and} \qquad 3\kappa\Psi_2 \equiv 0.$$

Similarly as in the proof of the second part of the Theoerm 5.10 this yields $\kappa \equiv \sigma \equiv 0$, with the exception when $\Psi_2 \equiv 0$. In such a case we have

$$\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv 0, \tag{46}$$

and these two Bianchi identities are tautologies. Thus to conclude something about κ and σ we need to use another pair of Bianchi identities. These are given by (95, 96) and refer to the respective components A_{341} and ${}_{214}$ of the Cotton tensor. Now, with the assumed (45) and (46) these identities reduce to

$$2\sigma\Psi_3 \equiv 0, \qquad \text{and} \qquad 2\kappa\Psi_3 \equiv 0.$$

This does not yield $\kappa \equiv \sigma \equiv 0$ only if $\Psi_3 \equiv 0$ in the neighbourhood. But this is forbidden by our assumption that the metric is III-generic in the considered neighbourhood.

Thus if the metric is III-generic in the neighbourhood we proved that $\kappa \equiv \sigma \equiv 0$ in a frame adapted to $\mathcal{N}$, which according to Proposition 5.1, means that $\mathcal{N}$ is integrable. □

Proof of Theorem 5.32. Choosing the null frame as in the above proof we first interpret the condition about the Cotton tensor 2-forms A_Z being all anti-self-dual. Since $\mathcal{N}$ is spanned by m and k we only need to consider the 2-forms $A_m = A(m, \cdot, \cdot)$ and $A_k = A(k, \cdot, \cdot)$. We have:

$$\begin{aligned} A_m = & A_{123}\theta^2 \wedge \theta^3 + \tfrac{1}{2}(A_{112} - A_{134})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + A_{114}\theta^1 \wedge \theta^4 \\ & + A_{113}\theta^1 \wedge \theta^3 + \tfrac{1}{2}(A_{112} + A_{134})(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) + A_{124}\theta^2 \wedge \theta^4 \end{aligned}$$

and

$$\begin{aligned} A_k = & A_{423}\theta^2 \wedge \theta^3 + \tfrac{1}{2}(A_{412} - A_{434})(\theta^1 \wedge \theta^2 - \theta^3 \wedge \theta^4) + A_{414}\theta^1 \wedge \theta^4 \\ & + A_{413}\theta^1 \wedge \theta^3 + \tfrac{1}{2}(A_{412} + A_{434})(\theta^1 \wedge \theta^2 + \theta^3 \wedge \theta^4) + A_{424}\theta^2 \wedge \theta^4. \end{aligned}$$

So looking at the bases (11) and (12) of the self-dual and anti-self-dual 2-forms Σ and Σ', we conclude that these 2-forms are anti-self-dual iff the following six conditions for the coframe components of the Cotton tensor are satisfied:

$$A_{114} \equiv A_{414} \equiv 0 \quad \& \tag{47}$$
$$A_{112} - A_{134} \equiv 0 \quad \& \tag{48}$$
$$A_{412} - A_{434} \equiv 0 \quad \& \tag{49}$$
$$A_{123} \equiv A_{423} \equiv 0. \tag{50}$$

Now we use the symmetries of the Cotton tensor to give equivalent forms of the conditions (48, 49). Using (32) we get $A_{112} \equiv A_{341} - A_{413}$ and using (31) we get $A_{134} \equiv -A_{413} - A_{341}$. Subtracting the latter from the former, we get the identity

$$A_{112} - A_{134} \equiv 2A_{341}.$$

In the similar way, we prove the identity

$$A_{412} - A_{434} \equiv 2A_{214}.$$

Comparing these two identities with (47–50) we conclude that the condition that A_Z is anti-self-dual for all $Z \in \mathcal{N}$, in our coframe, is equivalent to the six conditions

$$A_{114} \equiv A_{414} \equiv 0 \quad \&$$
$$A_{341} \equiv A_{214} \equiv 0 \quad \&$$
$$A_{123} \equiv A_{423} \equiv 0.$$

Since the first four conditions are precisely $A(\,\cdot\,, Z_1, Z_2) \equiv 0$ for $Z_1, Z_2 \in \mathcal{N}$, we now use Theorem 5.31 to conclude $\kappa \equiv \sigma \equiv 0$, provided that we are not in the situation when

$$\Psi_0 \equiv \Psi_1 \equiv \Psi_2 \equiv \Psi_3 \equiv 0 \tag{51}$$

in the neighbourhod. If this is the case, to show that we still have $\kappa \equiv \sigma \equiv 0$ we need the additional assumption (50). With this and (51) being assumed, using the Bianchi identities (97, 98), we easily obtain

$$-\sigma\Psi_4 \equiv 0 \qquad \text{and} \qquad \kappa\Psi_4 \equiv 0.$$

This implies that $\kappa \equiv \sigma \equiv 0$ in the neighborhood, on the ground of the N-genericity of the metric. This finishes the proof. □

As a counterpart to Corollary 5.29, we have

Corollary 5.33 *Let $\mathcal{M}$ be a 4-dimensional manifold with a metric g and let $\mathcal{N}$ be a field of self-dual totally null 2-planes on $\mathcal{M}$. Assume that $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, and that the self-dual part of the Weyl tensor is algebraically special on $\mathcal{M}$ with $\mathcal{N}$ being a multiple principal field of self-dual totally null 2-planes. Then if $\mathcal{N}$ has multiplicity equal to* three *the Cotton tensor of g satisfies $A(\,\cdot\,, Z_1, Z_2) \equiv 0$ for all $Z_1, Z_2 \in \mathcal{N}$. If $\mathcal{N}$ has multiplicity equal to* four *the 2-form A_Z of the Cotton tensor A of g is anti-self-dual.*

Proof The proof is an immediate application of the Bianci identities (93, 98). □

6 Interpretation in terms of a characteristic connection

The terms $4\rho A_{141} + 4\tau A_{441}$ that appear in formula (44) defining S in Example 5.25 suggests that to describe the geometry of manifolds with $\kappa \equiv \sigma \equiv 0$ it would be useful to have a *vectorial* object, say B_a, with components B_a being roughly

$$B_a = (B_1, B_2, B_3, B_4) = (4s^{-1}\tau, B_2, B_3, -4s^{-1}\rho), \tag{52}$$

where s is a complex constant. If we were able to find a geometric way of distinguishing such B_a, then the formula for S would be $S = (\nabla_4 - sB_4)A_{141} - (\nabla_1 - sB_1)A_{441}$ and would have an explicit geometric meaning. Note that the values of components B_2 and B_3 are totally irrelevant here!. In this section, we show how to geometrically distinguish such (partially determined) B_a.

6.1 Characteristic connection of a totally null 2-plane

Let us chose an arbitrary 1-form $B = B_a\theta^a$ on $(\mathcal{M}, g = g_{ab}\theta^a\theta^b)$. Given a choice of B one defines a new connection $\overset{W}{\nabla}$ on $\mathcal{M}$, which is related to the Levi-Civita connection as follows.

Let $\Gamma_{ab} = \Gamma_{abc}\theta^c$, be the Levi-Civita connection 1-forms as given in (4). Define

$$\overset{W}{\Gamma}_{abc} = \Gamma_{abc} + \tfrac{1}{2}(g_{ca}B_b - g_{cb}B_a + g_{ab}B_c). \tag{53}$$

Then the new connection $\overset{W}{\nabla}$ is defined on $\mathcal{M}$ by

$$\overset{W}{\nabla}_X e_b = X^c\overset{W}{\nabla}_c e_b = X^c\overset{W}{\Gamma}{}^a_{bc}e_a, \qquad \overset{W}{\Gamma}{}^a_{bc} = g^{ad}\overset{W}{\Gamma}_{dbc}, \tag{54}$$

where (e_a) is a frame dual to the coframe (θ^a), $e_a \lrcorner\, \theta^b = \delta^b_a$.

The connection $\overset{W}{\nabla}$ is called the *Weyl connection*. It is the unique *torsionless* connection satisfying

$$\overset{W}{\nabla} g = -Bg. \tag{55}$$

It has the nice property of being *conformal* in the sense that if the metric g undergoes a transformation $g \to \hat{g} = \mathrm{e}^{2\phi} g$, then Eq. (55) is preserved,

$$\overset{W}{\nabla} \hat{g} = -\hat{B}\hat{g},$$

with a mere change $B \to \hat{B} = B - 2\mathrm{d}\phi$.

The conformal properties of Weyl connections would be very interesting for our purpose of describing conformal conditions for the Goldberg–Sachs theorem, provided that, we were able to associate a unique Weyl form B with the main object of this theorem namely a field of totally null 2-planes $\mathcal{N}$. The following theorem shows that although such a natural way of chosing B is possible only *partially*, it nevertheless enables us to define a canonical connection on $\mathcal{N}$, which encodes its conformal properties.

Theorem 6.1 *Let $\mathcal{N}$ be a field of totally null 2-planes on $(\mathcal{M}, g)$, where g is a 4-dimensional metric of* any *(including complex) signature. Let us assume that $\mathcal{N}$ is integrable $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$. Then there exists a* unique *connection $\check{\nabla}$ on $\mathcal{N}$, which encodes the conformal properties of this field of totally null 2-planes.*

Proof We define the connection $\check{\nabla}$ in two steps.

Step One. We first look for a Weyl connection $\overset{W}{\nabla}$ on $\mathcal{M}$, as in (53, 54), which has the property that it preserves $\mathcal{N}$. This means that we ask if there exists a Weyl connection $\overset{W}{\nabla}$ on $\mathcal{N}$, such that

$$\overset{W}{\nabla}_Y X \in \mathcal{N} \qquad \forall\, X \in \mathcal{N} \quad \& \quad \forall\, Y \in \mathrm{T}\mathcal{M}\ ? \tag{56}$$

To answer this question, we work in the adapted null frame $(e_1, e_2, e_3, e_4) = (m, p, n, k)$, with the usual dual coframe $(\theta^1, \theta^2, \theta^3, \theta^4) = (M, P, N, K)$, so that the field of totally null 2-planes $\mathcal{N}$ is $\mathcal{N} = \mathrm{Span}(e_1, e_4) = \mathrm{Span}(m, k)$. Then the question (56) is equivalent to the question of existence of $\overset{W}{\nabla}$ such that

$$(\overset{W}{\nabla}_c e_1) \wedge e_1 \wedge e_4 = 0, \quad \& \quad (\overset{W}{\nabla}_c e_4) \wedge e_1 \wedge e_4 = 0, \quad \forall\, c = 1, 2, 3, 4,$$

where we abbreviated $\overset{W}{\nabla}_{e_c}$ to $\overset{W}{\nabla}_{e_c} = \overset{W}{\nabla}_c$. It is very easy to see that, since in the chosen frame the coefficients of the metric g_{ab} are all zero, except $g_{12} = g_{21} = g_{34} = g_{43} = 1$, then these conditions are equivalent to:

$$\left.\begin{aligned} (\overset{W}{\nabla}_c e_1) \wedge e_1 \wedge e_4 &= \overset{W}{\Gamma}{}_{11c} e_2 \wedge e_1 \wedge e_4 + \overset{W}{\Gamma}{}_{41c} e_3 \wedge e_1 \wedge e_4 = 0 \\ (\overset{W}{\nabla}_c e_4) \wedge e_1 \wedge e_4 &= \overset{W}{\Gamma}{}_{14c} e_2 \wedge e_1 \wedge e_4 + \overset{W}{\Gamma}{}_{44c} e_3 \wedge e_1 \wedge e_4 = 0 \end{aligned}\right\} \quad \forall\, c = 1, 2, 3, 4$$

or, what is the same,

$$\overset{W}{\Gamma}{}_{11c} = \overset{W}{\Gamma}{}_{14c} = \overset{W}{\Gamma}{}_{44c} = 0, \quad \forall c = 1, 2, 3, 4.$$

Comparing these last equations with (53), we easily see that $\overset{W}{\Gamma}{}_{11c} = \overset{W}{\Gamma}{}_{44c} = 0$ is *automatically* satisfied for all $c = 1, 2, 3, 4$, and then, by considering the remaining conditions

$\overset{W}{\Gamma}_{14c} = \overset{W}{\Gamma}_{41c} = 0$, we see that (56) is equivalent to:

$$\Gamma_{14c} + \tfrac{1}{2}(g_{c1}B_4 - g_{c4}B_1) = 0 \quad \forall\, c = 1, 2, 3, 4. \tag{57}$$

Now examining these equations for $c = 1$ and $c = 4$, we get the conditions that the Levi-Civita connection coefficients Γ_{141} and Γ_{144} must satisfy

$$\Gamma_{141} = \Gamma_{144} = 0. \tag{58}$$

Examining the Eq. (57) for $c = 2$ and $c = 3$, we get the relations between the components B_1 and B_4 of the 1-form B and the Levi-Civita connection coefficients Γ_{143} and Γ_{142}. These are as follows:

$$B_1 = 2\Gamma_{143}, \qquad B_4 = -2\Gamma_{142}. \tag{59}$$

Thus, the requirement that there is a Weyl connection preserving $\mathcal{N}$ is equivalent to the fact that in a coframe adapted to $\mathcal{N}$, we have (58) and (59). Since Γ_{141} and Γ_{144}, in the coframe adapted to $\mathcal{N}$, are $\Gamma_{141} = \sigma$ and $\Gamma_{144} = \kappa$, then we see that the connection $\overset{W}{\nabla}$ exists only if the field of totally null 2-planes $\mathcal{N}$ is *integrable*. When $\mathcal{N}$ is integrable then, in the adapted coframe (θ^i), the two of the components of the Weyl 1-form B, namely B_1 and B_4, are totally determined. They are equal to

$$B_1 = 2\tau, \qquad B_4 = -2\rho,$$

as desired in (52), with $s = 2$.

Concluding this part of the proof, we say that the condition (56) that the Weyl connection preserves $\mathcal{N}$ determines this connection only up to the terms B_2 and B_3 in the Weyl 1-form. In step two of the proof we restrict this connection to $\mathcal{N}$.

Step two. Since $\overset{W}{\nabla}$ preserves $\mathcal{N}$ in *any* direction then, in particular, it preserves it along $\mathcal{N}$. Thus $\overset{W}{\nabla}$, with *any* choice of B_2 and B_3, restricts naturally to $\mathcal{N}$. But *apriori* this restriction may depend on the choice of B_2 and B_3. That this is *not* the case follows from the following.

First observe that because of (58), we have

$$\begin{aligned}
&\overset{W}{\Gamma}_{211} = \Gamma_{211} + B_1, \quad \overset{W}{\Gamma}_{111} = 0, \quad \overset{W}{\Gamma}_{411} = 0, \quad \overset{W}{\Gamma}_{311} = \Gamma_{311}\\
&\overset{W}{\Gamma}_{214} = \Gamma_{214} + \tfrac{1}{2}B_4, \quad \overset{W}{\Gamma}_{114} = 0, \quad \overset{W}{\Gamma}_{414} = 0, \quad \overset{W}{\Gamma}_{314} = \Gamma_{314} + \tfrac{1}{2}B_1\\
&\overset{W}{\Gamma}_{241} = \Gamma_{241} + \tfrac{1}{2}B_4, \quad \overset{W}{\Gamma}_{141} = 0, \quad \overset{W}{\Gamma}_{441} = 0, \quad \overset{W}{\Gamma}_{341} = \Gamma_{341} + \tfrac{1}{2}B_1\\
&\overset{W}{\Gamma}_{244} = \Gamma_{244}, \quad \overset{W}{\Gamma}_{144} = 0, \quad \overset{W}{\Gamma}_{444} = 0, \quad \overset{W}{\Gamma}_{344} = \Gamma_{344} + B_4.
\end{aligned}$$

Thus the covariant derivatives

$$\begin{aligned}
\overset{W}{\nabla}_1 e_1 &= \overset{W}{\Gamma}{}^c_{11} e_c = \overset{W}{\Gamma}_{211} e_1 + \overset{W}{\Gamma}_{111} e_2 + \overset{W}{\Gamma}_{411} e_3 + \overset{W}{\Gamma}_{311} e_4\\
\overset{W}{\nabla}_4 e_1 &= \overset{W}{\Gamma}{}^c_{14} e_c = \overset{W}{\Gamma}_{214} e_1 + \overset{W}{\Gamma}_{114} e_2 + \overset{W}{\Gamma}_{414} e_3 + \overset{W}{\Gamma}_{314} e_4\\
\overset{W}{\nabla}_1 e_4 &= \overset{W}{\Gamma}{}^c_{41} e_c = \overset{W}{\Gamma}_{241} e_1 + \overset{W}{\Gamma}_{141} e_2 + \overset{W}{\Gamma}_{441} e_3 + \overset{W}{\Gamma}_{341} e_4\\
\overset{W}{\nabla}_4 e_4 &= \overset{W}{\Gamma}{}^c_{44} e_c = \overset{W}{\Gamma}_{244} e_1 + \overset{W}{\Gamma}_{144} e_2 + \overset{W}{\Gamma}_{444} e_3 + \overset{W}{\Gamma}_{344} e_4
\end{aligned}$$

of vectors (e_1, e_4) in the directions e_1 and e_4 spanning $\mathcal{N}$, are expressible purely in terms of the Levi-Civita connection coefficients Γ_{abc} and the totally determined part of B. In these relations, the unknown coefficients of B, namely B_2 and B_3, do not appear!

Thus $\overset{W}{\nabla}$ restricts to a *unique* and totally determined connection on $\mathcal{N}$. We define

$$\check{\nabla} = \overset{W}{\nabla}_{|\mathcal{N}} \quad \text{on} \quad \mathcal{N}.$$

Since this connection is constructed with only conformal objects, it is manifestly conformal.

The formulae for this connection in the Newman–Penrose formalism are as follows:

$$\begin{aligned} \check{\nabla}_m m &= (\beta - \alpha' + 2\tau)m - \lambda' k \\ \check{\nabla}_k m &= (\varepsilon - \varepsilon' - \rho)m + (\tau - \pi')k \\ \check{\nabla}_m k &= (\rho' - \rho)m + (\alpha' + \beta + \tau)k \\ \check{\nabla}_k k &= \kappa' m + (\varepsilon + \varepsilon' - 2\rho)k. \end{aligned} \tag{60}$$

□

The connection $\check{\nabla}$ defined in Theorem 6.1 is called the *characteristic connection* of an integrable totally null 2-plane $\mathcal{N}$ field.

Now, having any three (complex-valued) vector fields $X, Y, Z \in \mathcal{N}$, we define the torsion $\check{T}$ and the curvature $\check{R}$ of $\check{\nabla}$ via the usual:

$$\check{T}(X, Y) = \check{\nabla}_X Y - \check{\nabla}_Y X - [X, Y], \tag{61}$$

$$\check{R}(X, Y)Z = [\check{\nabla}_X, \check{\nabla}_Y]Z - \check{\nabla}_{[X,Y]}Z. \tag{62}$$

By construction these are *conformal* tensors defined on $\mathcal{N}$. Since both $\check{T}$ and $\check{R}$ are antisymmetric in X, Y they may have at most two, respectively four, independent components. Actually we have the following

Theorem 6.2 *The characteristic connection $\check{\nabla}$ of an integrable $\mathcal{N}$ is* torsionless*,*

$$\check{T} \equiv 0.$$

Its curvature, $\check{R}$, is given by

$$\check{R}(m, k)m = 4\Psi_1 m, \tag{63}$$

$$\check{R}(m, k)k = 4\Psi_1 k, \tag{64}$$

where Ψ_1 is the Weyl tensor coefficient of the Levi-Civita connection as defined in (10)*.*

Proof The torsionless property of the connection and the formulae (63, 64) for the curvature can be checked by a direct calculation. Indeed, for the torsionless we only have to show that $\check{T}(m, k) = 0$. One checks that this is a direct consequence of the definitions (61), (60) and the commutation relation $[\delta, D]$ from (13). To check (63) one uses the definition (62), the commutator $[\delta, D]$ and the Newman–Penrose equations (75), (77–79) and (82). Similarly, to check (64) one uses (62), (13) and the Newman–Penrose equations (75), (76), (79), (80) and (81). In all of these expressions one has to put the integrability conditions $\kappa \equiv \sigma \equiv 0$. The rest of the proof is easy pure algebra. □

Thus, we see that the curvature of $\check{\nabla}$ has only *one* independent component, which is a constant multiple of Ψ_1. Moreover, the entire curvature, which may be identified with the *curvature operator* $\check{R}(m, k) : \mathcal{N} \to \mathcal{N}$, satisfies

$$\check{R}(m, k) = (4\Psi_1)\mathrm{Id}_{\mathcal{N}}.$$

Recalling that Ψ_1 is that part of the self-dual part of the Weyl tensor, which if vanishes, makes it algebraically special, we have the following

Corollary 6.3 *A 4-dimensional manifold $\mathcal{M}$ with a metric g and an integrable field of totally null 2-planes $\mathcal{N}$ is algebraically special if and only if the characteristic connection $\check{\nabla}$ of $\mathcal{N}$ is flat, i.e. iff its curvature $\check{R} \equiv 0$.*

This proves the following Proposition.

Proposition 6.4 *A 4-dimensional manifold $(\mathcal{M}, g)$ is algebraically special iff it posesses an integrable field of totally null 2-planes whose characteristic connection is flat.*

6.2 Characteristic connection and the sharpest Goldberg–Sachs theorem

Given an integrable field of totally null 2-planes $\mathcal{N}$, we have the corresponding characteristic connection $\check{\nabla}$. Let $(f_A) = (f_1, f_2)$ be a frame in $\mathcal{N}$. In the previous section, we found that the curvature of $\check{\nabla}$ in the basis $(f_A) = (m, k)$ is

$$\check{R}^A_{\ BCD} = 4\Psi_1 \delta^A_B \epsilon_{CD},$$

where $A, B, C, D = 1, 2$, $(\delta^A_B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $(\epsilon_{CD}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, in particular, the 'Ricci tensor' $\check{R}_{AB} = \check{R}^C_{\ ACB}$ of this connection is *antisymmetric* and equal to

$$\check{R}_{AB} = 4\Psi_1 \epsilon_{AB}.$$

Since the curvature has only one component, it is obvious that the other possible contraction, namely $\check{R}^C_{\ CAB}$, is proportional to $\check{R}_{AB}$: $\check{R}^C_{\ CAB} = 2\check{R}_{AB}$. Using this Ricci tensor, we are able to formulate the following strengthening of the generalization of the Goldberg–Sachs theorem given in Theorem 5.21.

Theorem 6.5 *Let $\mathcal{N} \subset \mathrm{T}^{\mathbb{C}}\mathcal{M}$ be an integrable field of totally null 2-planes on a 4-dimensional manifold $(\mathcal{M}, g)$ equipped with metric g. Assume that the tensor $\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB}$ vanishes everywhere on $\mathcal{M}$,*

$$\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB} \equiv 0. \tag{65}$$

Then $(\mathcal{M}, g)$ is algebraically special at every point of $\mathcal{M}$, with a multiple field of principal totally null 2-planes tangent to $\mathcal{N}$.

Proof For every connection ∇_A, the action of the operator $\nabla_{[C}\nabla_{D]}$ on any tensor is a suitable linear action of the curvature of ∇_A on this tensor. Since for $\check{\nabla}_A$ the curvature has only one component Ψ_1, the quantity $\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB}$ only involves a constant coefficient sum of terms of the form $\Psi_1 \check{R}_{AB}$. Since $\check{R}_{AB}$ itself is proportional to Ψ_1, because of the symmetry, we conclude that

$$\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB} = c\ \Psi_1^2 \epsilon_{AB}\epsilon_{CD}, \qquad c = \text{const}.$$

The constant c may be calculated in a particular basis, e.g. in the basis $(f_A) = (m, k)$. Using this basis, the definitions (60) and the Newman–Penrose equations from the "Appendix", it is a matter of algebra to check that $c = -16$.

Now, if $\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB} \equiv 0$, then also $\Psi_1^2 \equiv 0$, and hence $\Psi_1 \equiv 0$. Since $\mathcal{N}$ is integrable, then we also have $\Psi_0 \equiv 0$, which means that $\mathcal{N}$ is a multiple totally null 2-plane. This finishes the proof. □

Remark 6.6 *Since S as in (44) is equal to $-10\Psi_1^2$, and this is turn is 8/5 of the only component of the conformal tensor $\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB}$, it is now clear why an 'adhoc' defined object S in (43) is a* weighted scalar.

Remark 6.7 *According to the discussion in Example 5.25, the assumption about the conformal tensor $\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB} \equiv 0$, replacing the Ricci flatness condition from the original Goldberg–Sachs theorem, can not be weakened if one wants to get the implication ($\kappa \equiv \sigma \equiv 0) \Rightarrow (\Psi_0 \equiv \Psi_1 \equiv 0)$. Thus, although the connection $\check{\nabla}$ provides plenty of a priori "weaker" conditions, such as for example $\check{\nabla}_{[E}\check{\nabla}_{F]}\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB} \equiv 0$, or conditions with more iterations of the curvature operator $\check{\nabla}_{[C}\check{\nabla}_{D]}$, they all are equivalent to the simplest condition $\check{\nabla}_{[C}\check{\nabla}_{D]}\check{R}_{AB} \equiv 0$.*

7 Generalizations of the Goldberg–Sachs theorem for real metrics

Theorems 5.10, 5.21, 5.24, 5.28, 5.31 and 5.32 were proved assuming that the metric g is *complex*. The proofs *also work* when g is *real*. To see this it is enough to look at the proofs assuming one of the reality conditions (L), (E), (S_c) or (S_r) of Remarks 4.1, 4.2, 4.3 and 4.4. They impose relations between the components of the Weyl tensor Ψ_μ and Ψ'_ν, between the Schouten tensor components P_{ab} and between the Cotton tensor components A_{abc}. These relations are harmless for the arguments in the proofs. They, however, may be used to *shorten* the proofs and may cause that some assumptions appearing in the complex versions can be dropped off.

We first discuss the Euclidean case.

7.1 Euclidean case

In this case, in every null coframe (M, P, N, K), as in (1), the reality conditions (E) imply that in particular:

$$\Psi_4 = \bar{\Psi}_0, \quad \Psi_3 = \bar{\Psi}_1, \quad \Psi_2 = \bar{\Psi}_2, \quad \Psi'_4 = \bar{\Psi}'_0, \quad \Psi'_3 = \bar{\Psi}'_1, \quad \Psi'_2 = \bar{\Psi}'_2. \tag{66}$$

In the rest of this section, we consider the self-dual part of the Weyl tensor and principal null 2-planes associated with it. The analysis of the anti-self-dual case is analogous.

Relations (66), when compared with the Eq. (19) defining the principal 2-planes, imply the following:

Proposition 7.1 *If $z = z_1$ is a solution of $\Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0$ then is so $z_2 = -\frac{1}{\bar{z}_1}$.*

Proof Inserting (66) and $z = z_1$ in the equation defining the principal null 2-planes (19), we get

$$\bar{\Psi}_0 z_1^4 - 4\bar{\Psi}_1 z_1^3 + 6\Psi_2 z_1^2 + 4\Psi_1 z_1 + \Psi_0 = 0.$$

Now dividing this by z_1^{-4} and taking the complex conjugation of the result, we get

$$\bar{\Psi}_0 z_2^4 - 4\bar{\Psi}_1 z_2^3 + 6\Psi_2 z_2^2 + 4\Psi_1 z_2 + \Psi_0 = 0,$$

which finishes the proof. □

Comparing this with Proposition 3.1, we have

Corollary 7.2 *Principal null 2-planes always appear in pairs corresponding to pairs of solutions* $(z_1, z_2) = (z_1, \frac{-1}{\bar{z}_1})$ *of Eq. (19).*
A pair of solutions $(z_1, z_2) = (z_1, \frac{-1}{\bar{z}_1})$ *of* Eq. (19) *at a point* x *distinguishes a pair* $(J(z_1), J(\frac{-1}{\bar{z}_1}))$ *of principal hermitian structures* $J(z_1)$ *and* $J(\frac{-1}{\bar{z}_1})$ *at* x*, which are conjugate to each other,* $\overline{J(\frac{-1}{\bar{z}_1})} = -J(z_1)$.

Proof The only thing to be proven is $\overline{J(\frac{-1}{\bar{z}_1})} = -J(z_1)$. By definition of these two structures we have $J(z)(m + zn) = i(m + zn)$, $J(z)(k - zp) = i(k - zp)$ and $J(\frac{-1}{\bar{z}})(m - \frac{1}{\bar{z}}n) = i(m - \frac{1}{\bar{z}}n)$, $J(\frac{-1}{\bar{z}})(k + \frac{1}{\bar{z}}p) = i(k + \frac{1}{\bar{z}}p)$. The second set of equations is equivalent to $J(\frac{-1}{\bar{z}})(\bar{z}m - n) = i(\bar{z}m - n)$ and $J(\frac{-1}{\bar{z}})(\bar{z}k + p) = i(\bar{z}k + p)$, which after conjugation and the use of the reality conditions (E) gives:

$$\overline{J(\tfrac{-1}{\bar{z}})}(k - zp) = -i(k - zp) = -J(z)(k - zp),$$
$$\overline{J(\tfrac{-1}{\bar{z}})}(m + zn) = -i(m + zn) = -J(z)(m + zn).$$

□

This corollary implies that at each point x of $\mathcal{M}$ the self-dual part of the Weyl tensor may be in one of the following Petrov types:

- type G: the generic type, in which the self-dual part of the Weyl tensor does not vanish at x, and in which we have two distinct pairs $(z_1, z_2) = (z_1, \frac{-1}{\bar{z}_1})$ and $(z_3, z_4) = (z_3, \frac{-1}{\bar{z}_3})$, $z_1 \neq z_3$, of solutions of Eq. (19). In such case the pairs (z_1, z_2) and (z_3, z_4) correspond to two pairs of *different* mutually conjugate principal hermitian structures $(J(z_1), J(z_2))$ and $(J(z_3), J(z_4))$ at x.
- type D: this is the degeneracy of type G. It occurs when z_1 is a double root of (19), i.e. when $z_3 = z_1$. In such case we have only *one* pair of double principal hermitian structures $(J(z_1), J(z_2))$ at x.
- type 0: this is the anti-self-dual type in which the self-dual part of the Weyl tensor *vanishes* at x. In this case the sphere of self-dual 2-planes has no distinguished points.

Note that always we may choose a Newman–Penrose frame in which $\Psi_0 = 0$ at x. In types G or D it is achieved by choosing the Newman–Penrose vectors m and k such that they span the principal null 2-plane corresponding to z_1. Then, in such a frame, the algebraically special type D is characterized by $\Psi_1 = 0$ and $\Psi_2 \neq 0$ at x. If in such a frame $\Psi_1 \neq 0$, then the self-dual part of the Weyl tensor is algebraically general (of type G) at x.

This proves the following

Theorem 7.3 *At every point of a 4-dimensional manifold* $\mathcal{M}$ *equipped with a real Euclidean-signature metric* g *the self-dual part of the Weyl tensor may be of one of the types G, D, and 0, with the analogous types for the anti-self-dual part of the Weyl tensor. Thus, at eavery point of a 4-manifold* $\mathcal{M}$ *equipped with a* Euclidean signature *metric* g *we have* $3 \times 3 = 9$ *'Petrov' types.*

Thus, the Euclidean reality conditions (E) imply that the number of possible Petrov types in the Euclidean case is much smaller than in the complex case. This implies that the complex theorems of the previous section have much stronger Euclidean versions. In particular, the proof of Theorem 5.28, when the reality conditions (E) are assumed, goes through as in the complex version, with the only exception, that the II-genericity property of g may now be weakened to the assumption that the self-dual part of the Weyl tensor is nowhere vanishing (or even to a still weaker assumption that the points at which the self-dual part of the Weyl tensor vanishes form closed sets without interior). Indeed, in the Euclidean case, the assumption $\Psi_0 \equiv \Psi_1 \equiv 0$ and $\Psi_2 \neq 0$, which is needed for the conclusion that $\kappa \equiv \sigma \equiv 0$, means only that the self-dual part of the Weyl tensor is non-vanishing, since now $\Psi_0 \equiv \Psi_1 \equiv 0$ implies that $\Psi_4 \equiv \Psi_3 \equiv 0$. This proves the Riemannian version of the Goldberg–Sachs Theorem 2.1.

One of the corollaries from the complex Theorem 5.28 is also the following

Corollary 7.4 *If the self-dual part of the Weyl tensor of a real metric g of Riemannian signature does not vanish on a 4-dimensional manifold $\mathcal{M}$, then modulo complex conjugation, such a metric admits at most two hermitian structures that agree with the orientation. If such hermitian structures exist their spaces of* (1,0) *vectors coincide with the self-dual principal totally null 2-planes. In particular, in type D we may have only one hermitian structure, which exists if and only if the Cotton tensor for g vanishes on its space of* (1,0) *vectors.*

The Euclidean version of Theorem 5.10 is also worth quoting. We have

Corollary 7.5 *Assume that a 4-dimensional manifold $\mathcal{M}$ equipped with a real metric of Riemannian signature g has a non-vanishing self-dual part of the Weyl tensor C^+. Suppose that it admits a hermitian structure J which agrees with the orientation, and that its Ricci tensor vanishes on the space $\mathcal{N}$ of* (1,0) *vectors of J. Then C^+ is of type D, with $\mathcal{N}$ being the only principal self-dual null 2-plane.*

7.2 Split signature case

To spell out all the possible Petrov types and their interpretations in this case, we first consider the Newman–Penrose coframe (M, P, N, K) with the reality conditions (S_c) from Remarks 4.1 and 4.2. In this coframe, the sphere of self-dual totally null 2-planes $\mathcal{N}_z$ is spanned by $m + zn$ and $k - zp$ as in (16). Now, having the reality conditions S_c, we ask which values of $z \in \mathbb{C}$ correspond to the *non-generic* self-dual totally null 2-planes which have real index equal to *two*. We have the following

Proposition 7.6 *A self-dual 2-plane $\mathcal{N}_z$ has real index equal to two if and only if the complex parameter $z \in \mathbb{C}$ lies on the unit circle $z\bar{z} = 1$.*

Proof Due to the reality conditions (S_c) a *real* non-vanishing vector $v = a(m + zn) + b(k - zp)$ from $\mathcal{N}_z$ must satisfy

$$a(m + zn) + b(k - zp) = \bar{a}(p - \bar{z}k) + \bar{b}(-n - \bar{z}m).$$

Equating to zero the respective coefficients at m, p, n, k we easily get that this is possible if and only if $z\bar{z} = 1$. Thus $\mathcal{N}_z$ includes real nozero vectors if and only if $z\bar{z} = 1$. We further observe that if $z\bar{z} = 1$ then v is real if and only if $b = -\bar{a}\bar{z}$. Thus, when z is fixed, we have a 1-complex-parameter-family $v = v(a)$ of real vectors in $\mathcal{N}_z$. Choosing two different values of a we get

$$v(a) \wedge v(a') = (a\bar{a}' - a'\bar{a})(m \wedge p - \bar{z}m \wedge k - zp \wedge n - n \wedge k).$$

This shows that $\mathcal{N}_z$ with $z\bar{z} = 1$ includes *independent* real vectors (take e.g. $a = 1$ and $a' = i$), thus it has real inedex two. This finishes the proof. □

Let us now choose a Newman–Penrose coframe as in (1). Then the reality conditions (S_c) imply that we have

$$\Psi_4 = \bar{\Psi}_0, \quad \Psi_3 = -\bar{\Psi}_1, \quad \Psi_2 = \bar{\Psi}_2, \quad \Psi'_4 = \bar{\Psi}'_0, \quad \Psi'_3 = -\bar{\Psi}'_1, \quad \Psi'_2 = \bar{\Psi}'_2, \tag{67}$$

and the reality conditions (S_r) mean that all Weyl tensor coefficients Ψ and Ψ' are real:

$$\Psi_0 = \bar{\Psi}_0, \quad \Psi_1 = \bar{\Psi}_1, \quad \Psi_2 = \bar{\Psi}_2, \quad \Psi_3 = \bar{\Psi}_3, \quad \Psi_4 = \bar{\Psi}_4, \tag{68}$$

(we also have analogous relations for Ψ').

We pass to the split signature version of the Petrov classification. We perform the analysis for the self-dual part of the Weyl tensor; the classification for the anti-self-dual case is analogous.

Let us fix a point $x \in \mathcal{M}$. Let (M, P, N, K) be a Newman–Penrose coframe around x satisfying the reality conditions (S_c), and as a consequence (67). We have the following

Proposition 7.7 *If $z = z_1$ is a solution of $\Psi_4 z^4 - 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0$ then is so $z_2 = \frac{1}{\bar{z}_1}$.*

Proof Inserting (67) and $z = z_1$ in the equation defining the principal null 2-planes (19) we get

$$\bar{\Psi}_0 z_1^4 + 4\bar{\Psi}_1 z_1^3 + 6\Psi_2 z_1^2 + 4\Psi_1 z_1 + \Psi_0 = 0.$$

Now dividing this by z_1^{-4} and taking the complex conjugation of the result, we get

$$\bar{\Psi}_0 z_2^4 + 4\bar{\Psi}_1 z_2^3 + 6\Psi_2 z_2^2 + 4\Psi_1 z_2 + \Psi_0 = 0,$$

which finishes the proof. □

Comparing this Proposition with Proposition 7.6 we get

Corollary 7.8 *Self-dual principal null 2-planes always appear in pairs corresponding to pairs of solutions $(z_1, z_2) = (z_1, \frac{1}{\bar{z}_1})$ of* Eq. (19). *The situation in which $z_1 = z_2$ happens only if the principal self-dual null 2-plane has real index two.*

Using Proposition 3.1 we may also reinterpret this corollary as follows

Corollary 7.9 *If* Eq. (19) *at a point x admits a principal self-dual null 2-plane of real index zero, then at this point we have two distinguished hermitian structures $J(z_1)$ and $J(\frac{1}{\bar{z}_1})$ associated with the solution z_1 of (19). Moreover these two structures are conjugate to each other.*

Proof The only thing to be proven is $\overline{J(\frac{1}{\bar{z}_1})} = -J(z_1)$. By definition of these two structures we have $J(z)(m+zn) = i(m+zn)$, $J(z)(k-zp) = i(k-zp)$ and $J(\frac{1}{\bar{z}})(m+\frac{1}{\bar{z}}n) = i(m+\frac{1}{\bar{z}}n)$, $J(\frac{1}{\bar{z}})(k - \frac{1}{\bar{z}}p) = i(k - \frac{1}{\bar{z}}p)$. The second set of equations is equivalent to $J(\frac{1}{\bar{z}})(\bar{z}m + n) = i(\bar{z}m + n)$ and $J(\frac{1}{\bar{z}})(\bar{z}k - p) = i(\bar{z}k - p)$, which after conjugation and the use of the reality conditions (S_c) gives:

$$\overline{J(\tfrac{1}{\bar{z}})}(k - zp) = -i(k - zp) = -J(z)(k - zp),$$
$$\overline{J(\tfrac{1}{\bar{z}})}(m + zn) = -i(m + zn) = -J(z)(m + zn).$$

Because of quite different reality conditions (67) and (68) at each point $x \in \mathcal{M}$ we need to consider separately two different cases: the generic one a) in which the self-dual part of the Weyl tensor admits *at least one* principal totally null 2-plane of real index *zero* at x, and the less generic one b) in which *all* principal null planes have real index *two* at x.

In the case a) we chose a Newman–Penrose coframe (M, P, N, K) around x such that it satisfies the reality conditions (S_c) and that the principal totally null 2-plane of real index zero corresponds to the solution $z = 0$ of (19). Then, in such a coframe $\Psi_0 = 0$, and the equation defining the principal null 2-planes becomes $4\bar{\Psi}_1 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z = 0$, or

$$2\bar{\Psi}_1 z^2 + 3\Psi_2 z + 2\Psi_1 = 0. \tag{69}$$

Thus in this coframe, we have two solutions $(z_1, z_2) = (0, \infty)$ corresponding to the mutually conjugate principal (almost) hermitian structures associated with two fields of principal 2-planes of index zero, and the rest of the principal 2-planes has to be determined as solutions to the quadratic Eq. (69). The roots of this equations are obviously

$$z_{3,4} = \frac{-3\Psi_2 \pm \sqrt{9\Psi_2^2 - 16\Psi_1\bar{\Psi}_1}}{4\bar{\Psi}_1}.$$

The interpretation depends on the sign of $9\Psi_2^2 - 16\Psi_1\bar{\Psi}_1$ and on whether Ψ_1 vanishes or not. It follows that at each point $x \in \mathcal{M}$ we have now *four* cases:

type G: the generic case in which $z_3 \neq z_4 = \frac{1}{\bar{z}_3}$, $z_3\bar{z}_3 \neq 1$, $z_3 \neq 0$ and $z_3 \neq \infty$. In such case, we have two pairs of *different* mutually conjugate principal hermitian structures at x corresponding to $(J(0), J(\infty))$ and $\left(J(z_3), J\left(\frac{1}{\bar{z}_3}\right)\right)$. This case happens when $9\Psi_2^2 > 16\Psi_1\bar{\Psi}_1$ and $\Psi_1 \neq 0$ at x.

type SG: in this case $z_3 \neq z_4 = \frac{1}{\bar{z}_3}$, $z_3\bar{z}_3 = 1$. Here, in addition to the pair of mutually conjugate principal hermitian structures $(J(0), J(\infty))$ at x, we have *two different* principal totally null 2-planes of real index *two* at x. These real 2-planes are associated with the solultions z_3 and z_4, which lie on the circle $z\bar{z} = 1$. This case happens when $9\Psi_2^2 < 16\Psi_1\bar{\Psi}_1$ at x.

type II: this is the degenerate case of the type SG. It happens when $9\Psi_2^2 = 16\Psi_1\bar{\Psi}_1$ and $\Psi_1 \neq 0$ at x, and the Eq. (69) has double root $z_3 = z_4$ at x. We necessarily have $z_3\bar{z}_3 = 1$ in this case, and thus, in addition to the pair of mutually conjugate principal hermitian structures $(J(0), (J(\infty))$ we have also *one* double principal null 2-plane of real index *two* at x.

type D: this is another degeneration of the type G. Now $\Psi_1 = 0$ at x and we have $z_3 = 0$ and $z_4 = \infty$ as solutions of (69). Thus in this case the points $z = 0$ and $z = \infty$ have multiplicity *two*, and we have only *one* pair of double principal hermitian structures $(J(0), J(\infty))$ at x.

We now pass to the cases in which we do *not* have a single principal null 2-plane which has a real index zero at x. The analysis here could still be performed in the Newman–Penrose coframe satisfying the reality conditions S_c, but since now all the solution of Eq. (19) would have to satisfy $z\bar{z} = 1$, we would not be able to choose the frame in such a way that Ψ_0 would be zero at x. This would lead to the analysis of the roots of the *quartic* Eq. (19), and it is why it is now much easier to reason in the coframe that satisfies the reality conditions (S_r). So now, we choose a Newman–Penrose coframe (M, P, N, K) around x, which satisfies the reality conditions (S_r) and, since now we have at least one principal null 2-plane of real index two at x, we may assume that we have $\Psi_0 = 0$ at x. In this coframe, our principal totally null 2-plane of real index two corresponds to $z_1 = 0$ and the other principal 2-planes are determined by

$$\Psi_4 z^3 - 4\Psi_3 z^2 + 6\Psi_2 z^2 + 4\Psi_1 = 0.$$

Here all the Ψ_1, Ψ_2, Ψ_3 and Ψ_4 are *real* and we admit only *real solutions* for z. (If the solution is complex, it corresponds to a 2-plane with real index zero, and corresponds to one of the cases G, SG, II, or D, considered earlier.)

Now, a fifteenth century substitution $z \to z - \frac{4\Psi_3}{3\Psi_4}$, brings this equation into the form $z^3 + pz + q = 0$, which has *three real* roots for z iff $27p^4 + 4q^3 \geq 0$. This inequality gives the restriction on the Weyl tensor, which determines the situation we are talking about here. If the self-dual part of the Weyl tensor satisfies this restriction, the Eq. (19) has *four real roots*. This, in addition to G, SG, II and D, defines the five new Petrov types:

type G_r: Equation (19), written in the coframe with reality conditions (S_r), has *four different real* roots, meaning that we have *four* different principal null 2-planes of real index two at x,

type II_r: Equation (19), written in the coframe with reality conditions (S_r), has *one double and two different real* roots, meaning that we have *three* different principal null 2-planes of real index two at x, one of them with multiplicity two,

type III_r: Equation (19), written in the coframe with reality conditions (S_r), has *one triple and one distinct real* roots, meaning that we have *two* different principal null 2-planes of real index two at x, one of them with multiplicity three,

type N_r: Equation (19), written in the coframe with reality conditions (S_r), has *one quadruple* root, meaning that we have a *single quadruple* principal null 2-planes of real index two at x,

type D_r: Equation (19), written in the coframe with reality conditions (S_r), has *two distinct double real* roots, meaning that we have *two* different principal null 2-planes of real index two at x, each of them having multiplicity two.

Finally we have the Petrov type corresponding to the situation when the *self-dual part of the Weyl tensor vanishes* at x (the metric is *anti-self-dual at* x).

This proves the following

Theorem 7.10 *At every point of a 4-dimensional manifold $\mathcal{M}$ equipped withh a real split-signature metric g the self-dual part of the Weyl tensor may be of one of the types G, SG, II, D, G_r, II_r, III_r, N_r, D_r, 0, with the analogous types for the anti-self-dual part of the Weyl tensor. Thus, at every point of a 4-manifold $\mathcal{M}$ equipped with a* split signature *metric g we have* $10 \times 10 = 100$ *'Petrov' types.*

The above analysis also suggest the following terminology: the name *algebraically special* for the self-dual part of the Weyl tensor in the split signature case is reserved to the types II, D, II_r, III_r, N_r, D_r and 0 only. Although the types SG and G_r are algebraically (and geometrically!) distinguished from the most general case G, we also call them *algebraically general*. With this terminology, Theorems 1.3 and 1.4 follow from our Theorem 5.10.

Because of the huge number of the algebraically special cases to be considered, we skip the discussion of the split signature versions of further theorems from Sect. 5 here. Such a discussion deserves a separate paper. This should also answer several interesting questions, such as for example, the following: 'are there split-signature Einstein metrics of type II?', 'is it possible to have a split signature Einstein 4-manifold on which an integrable totally null 2-planes can change its real index from 0 to 2?', etc.

We close this section by mentioning the recent paper [10]. It is entirely devoted to the Newman–Penrose formalism adapted to the split signature situation, and it provides a version of the split-signature Goldberg–Sachs theorem.

7.3 Lorentzian case

Here the Petrov types are precisely the same as in the complex case described by the Definition 5.4, i.e. we have types G, II, D, III, N and 0 here. The Lorentzian reality conditions (L) *do not make any restriction* on the Weyl tensor coefficients Ψ_μ. What they do is, they give a simple ralation between the self-dual part of the Weyl tensor and the anti-self-dual one. We have $\Psi'_\mu = \bar{\Psi}_\mu$, so here the anti-self-dual part of the Weyl tensor is totally determined by the self-dual one. Since in the proofs in Sect. 5 the coefficients Ψ'_μ never appear, and only Ψ_μs matter, all the proofs, and the theorems presented in Sect. 5 restrict naturally to the Lorentzian case without any alteration.

However, since in the Lorentzian signature the fields of totally null 2-planes have always real index one, it is customary to formulate the Lorentzian theorems in terms of the real vector field k such that $\mathrm{Span}_{\mathbb{C}}(k) = \mathcal{N} \cap \bar{\mathcal{N}}$. In particular, such a null real vector field is said to be *geodesic and shear-free* [25] if it satisfies

$$\mathcal{L}_k g = ag + g(k)\omega, \tag{70}$$

with a function a and a 1-form ω on $\mathcal{M}$. Here $g(k)$ is a 1-form on $\mathcal{M}$ such that $X \lrcorner\, g(k) = g(k, X)$ for any vector field $X \in \mathrm{T}\mathcal{M}$. When written in terms of the field $\mathcal{N}$ of the associated totally null 2-planes, condition (70) is equivalent to

$$[\mathcal{N}, \mathcal{N}] \subset \mathcal{N},$$

i.e. to the formal integrability condition for $\mathcal{N}$.

Suppose now the Weyl tensor C_{abcd} of $(\mathcal{M}, g)$ is *non-vanishing*. It is well known [2] that the algebraic equation

$$k_{[e}C_{a]bc[d}k_{f]}k^b k^c = 0, \tag{71}$$

for a null vector k has at most *four* solutions at every point $x \in \mathcal{M}$. The solutions k of Eq. (71) at $x \in \mathcal{M}$ are called the *principal null directions* (PNDs) at x. If Eq. (71) admits *exactly* four PNDs at $x \in \mathcal{M}$ then $(\mathcal{M}, g)$ is said to be *algebraically general* at x. If the number q of solutions to (71) at $x \in \mathcal{M}$ is $1 \leq q \leq 3$ then $(\mathcal{M}, g)$ is called *algebraically special* at x. In such case the *quartic* Eq. (71) has at least one *multiple root*, and the solution k corresponding to it is called a *multiple* PND. This notion of the algebraical speciality coincides with the one in terms of the principal null 2-planes, since on a Lorentzian oriented and time oriented 4-manifold $\mathcal{M}$, there is one to one correspondence between fields of totally null 2-planes in the complexification and real null vector fields, defined by the intersection of the 2-planes with their complex conjugations.

Having said this, we present the Lorentzian version of our complex Theorem 5.9.

Theorem 7.11 *Let $\mathcal{N} \subset \mathrm{T}^{\mathbb{C}}\mathcal{M}$ be a field of totally null 2-planes on a Lorentzian 4-dimensional manifold $(\mathcal{M}, g)$. Assume that the Ricci tensor Ric of $(\mathcal{M}, g)$, considered as a symmetric bilinear form on $\mathrm{T}^{\mathbb{C}}\mathcal{M}$, is degenerate on $\mathcal{N}$,*

$$Ric_{|\mathcal{N}} = 0.$$

If in addition the field $\mathcal{N}$ is integrable, $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$, everywhere on $\mathcal{M}$, or what is the same, if k such that $\mathrm{Span}_{\mathbb{C}}(k) = \mathcal{N} \cap \bar{\mathcal{N}}$, is geodesic and shear-free, then $(\mathcal{M}, g)$ is algebraically special at every point, with a multiple PND tangent to k.

Remark 7.12 *In [6] we used Theorem 7.11 without proof, since it would have made an already long paper even longer. Actually some statements equivalent to Theorem 7.11 are known to a few general relativists, see e.g. Lemma 2.2 on p. 577 of [23]. Since this equivalence is not easy to decipher, we decided to present this theorem here, as a corollary from the complex Theorem 5.9.*

7.4 Counterexample to Trautman's conjecture

Trautman in [26] asked if there exists an example of a 4-dimensional Bach flat metric with non-vanishing self-dual part of the Weyl tensor C^+, for which an integrable field of self-dual totally null 2-planes would not be principal for C^+. He *conjectured* that the answer to this question is 'no'. Although the question was formulated in the Lorentzian setting, it makes sense in any signature. It is also very closely related to the Goldberg–Sachs theorem.

Our analysis of this theorem from Sect. 5.2, especially the discussion in Example 5.25, suggests that the examples Trautman asks about, should be possible. This is because, the conditions needed for 'if and only if' between conditions (i) and (ii) in Theorem 5.28 are related to those derivatives of the Cotton tensor that are not present in the Bach tensor. This is clear from Example 5.25: the integrability conditions for $A_{|\mathcal{N}} \equiv 0$, give $S \equiv 0$, where S is given by (44). And although the Bach tensor components may be obtained by differentiating some components of the Cotton tensor, the derivatives of the Cotton tensor appearing in S are not (at least algebraically) expressible in terms of the components of the Bach tensor.

Below in this section we present a simple example of a metric with *Euclidean* signature which is Bach-flat, admits an *integrable hermitian structure* which *agrees with the orientation*, and whose self-dual part of the Weyl tensor is of general type G.

On $\mathbb{R}^4$, with local coordinates (x^1, x^2, x^3, x^4), consider $z = x^1 + ix^2$ and $w = x^3 + ix^4$, and a complex-valued function $f = f(w, z)$ *holomorphic* in *both* arguments w and z. Given f define a Riemannian metric

$$g = 2\left(\mathrm{d}w\mathrm{d}\bar{w} + \exp\left(f(w, z) + \bar{f}(\bar{w}, \bar{z})\right)\mathrm{d}z\mathrm{d}\bar{z}\right).$$

Now introduce the Newman–Penrose coframe by setting

$$M = \mathrm{d}\bar{w}, \quad P = \mathrm{d}w, \quad N = \mathrm{e}^f\mathrm{d}z, \quad K = \mathrm{e}^{\bar{f}}\mathrm{d}\bar{z}.$$

They obviously satisfy the Euclidean reality conditions (E). A short calculation shows, that modulo the complex conjugation, the only non-vanishing Newman–Penrose coefficients are:

$$\alpha = -\tfrac{1}{2}\pi = \beta' = -\tfrac{1}{2}\tau' = \tfrac{1}{4}f_w.$$

In particular $\kappa = \sigma = 0$, which is obvious since the field of self-dual totally null 2-planes $\mathcal{N}$ spanned by $m = \partial_{\bar{w}}$ and $k = \mathrm{e}^{-\bar{f}}\partial_{\bar{z}}$ is integrable. Now our main point is that the only non-vanishing components of the Weyl tensor are as follows:

$$\Psi_3 = \bar{\Psi}_1 = \tfrac{1}{4}\mathrm{e}^{-f}f_{wz}.$$

This in particular means that the field $\mathcal{N}$ is principal (since $\Psi_0 \equiv 0$), but when $f_{wz} \neq 0$ it is *not* multiple ($\Psi_3 \neq 0 \neq \Psi_1$). Moreover, since $\Psi'_0 \equiv \Psi'_1 \equiv \Psi'_2 \equiv \Psi'_3 \equiv \Psi'_4 \equiv 0$, i.e. the full anti-self-dual part of the Weyl tensor identically vanishes, the metric is *Bach flat*. This answers in positive the question of Trautman we mentioned at the beginning of this section. Moreover, if $f_{wz} \neq 0$, due to the Corollary 7.5, this self-dual metric can not have Ricci tensor vanishing on $\mathcal{N}$, and as such is never conformal to an Einstein metric.

7.5 Characteristic connection in real signatures

We now reexamine the arguments from Sect. 6 from the point of view of the reality conditions.

From Step one of the proof of Theorem 6.1 we know that the Weyl form B of the Weyl connection which preserves an integrable $\mathcal{N}$, in an adapted to $\mathcal{N}$ coframe is given by $B = 2\tau M + B_2 P + B_3 N - 2\rho K$. Thus in the complex case (or in the real cases in which we do not insist on B to be real) the Weyl 1-form is not totally determined by $\mathcal{N}$.

The situation is quite different in the Riemannian (E) and the split signature (S_c). In these two cases, the requirements that B is real determines it completely. Indeed, it is easy to see that the reality conditions (E) or (S_c) together with the requirement that B be real implies that B is equal to

$$B = 2\tau M + 2\pi P - 2\mu N - 2\rho K \tag{72}$$

or, what is the same,

$$\tfrac{1}{2}B = \Gamma_{143}\theta^1 + \Gamma_{234}\theta^2 + \Gamma_{321}\theta^3 + \Gamma_{412}\theta^4.$$

This proves the following theorem

Theorem 7.13 *Let $\mathcal{N}$ be a field of totally null 2-planes on $(\mathcal{M}, g)$, where g is a 4-dimensional metric of Riemannian or split signature. Let us assume that $\mathcal{N}$ is integrable $[\mathcal{N}, \mathcal{N}] \subset \mathcal{N}$ and that it has a real index 0 everywhere on $\mathcal{M}$. Then there exists a* canonical *Weyl connection $\overset{W}{\nabla}$ on $\mathcal{M}$, which encodes the conformal properties of the structure $(\mathcal{M}, g, \mathcal{N})$.*

The connection $\overset{W}{\nabla}$ is uniquely determined by the requirements that

- *it is real,*
- *it is torsionless,*
- *it satisfies: $\overset{W}{\nabla} g = -Bg$,*
- *it satisfies: $\overset{W}{\nabla}_X \mathcal{N} \subset \mathcal{N}$ for all $X \in T\mathcal{M}$.*

In terms of a coframe (θ^a) adapted to $\mathcal{N}$ and the connection 1-forms $\overset{W}{\Gamma}{}^a_{\ b} = g^{ad}\overset{W}{\Gamma}_{dbc}\theta^c$ the connection $\overset{W}{\nabla}$ is given by

$$\overset{W}{\Gamma}_{abc} = \Gamma_{abc} + \tfrac{1}{2}(g_{ca}B_b - g_{cb}B_a + g_{ab}B_c)$$

with

$$\tfrac{1}{2}B = \Gamma_{143}\theta^1 + \Gamma_{234}\theta^2 + \Gamma_{321}\theta^3 + \Gamma_{412}\theta^4.$$

Here Γ_{abc} are the Levi-Civita connection coefficients in the adapted coframe.

Definition 7.14 Let J be a hermitian (or pseudohermitian) structure on an $2n$-dimensional manifold $(\mathcal{M}, g)$ with a metric of Riemannian (or split) signature. A torsionless conection $\overset{HW}{\nabla}$ on $(\mathcal{M}, g, J)$ is called (pseudo)hermitian-Weyl iff

– $\overset{HW}{\nabla} J = 0$,
– and $\overset{HW}{\nabla} g = -Bg$ for some real 1-form B on $\mathcal{M}$.

According to our discussion in Sect. 3, integrable totally null 2-planes of real index 0 on a 4-dimensional manifold $(\mathcal{M}, g)$ are in one-to-one correspondence with (pseudo)hermitian structures J on $(\mathcal{M}, g)$, thus Theorem 7.13 can be reformulated as:

Theorem 7.15 *Every 4-dimensional (pseudo)hermitian manifold $(\mathcal{M}, g, J)$ defines a canonical (pseudo)hermitian-Weyl connection $\overset{HW}{\nabla}$. This connection encodes the conformal properties of the structure $(\mathcal{M}, g, J)$. It is given by $\overset{HW}{\nabla} = \overset{W}{\nabla}$, where $\overset{W}{\nabla}$ is as in Theorem 7.13.*

Thus in the (pseudo)hermitian case there is a better connection, namely $\overset{W}{\nabla}$, than the characteristic connection $\check{\nabla}$. It is better, since it enables to differentiate *any* vector from the tangent space of $\mathcal{M}$ along *any* other vector from $\mathrm{T}\mathcal{M}$. The connection $\check{\nabla}$ enables for the differentiation along $\mathcal{N} = \mathrm{T}^{(1,0)}\mathcal{M}$ only. And, $\overset{W}{\nabla}$ is better, because it contains *much more information* than $\check{\nabla}$. In particluar, $\check{\nabla}$ is simply the restriction of $\overset{W}{\nabla}$ to $\mathcal{N}$.

We now pass to the (pseudo)hermitian part of the Goldberg–Sachs Theorem 6.5.

We need some preparations:

Given the (pseudo)hermitian-Weyl connection $\overset{W}{\nabla}$, as in Theorem 7.15, we use the formula (53) to pass to the connection 1-forms $\overset{W}{\Gamma}_{ab} = \overset{W}{\Gamma}_{abc}\theta^c$. Here (θ^c) is a coframe *adapted* to J. The word 'adapted' (in accordance with the discussion in Sect. 3) means that the considered coframe is adapted to $\mathcal{N} = \mathrm{T}^{(1,0)}\mathcal{M}$ as in the definition of this notion at the begining of Sect. 5. Now, there is a sequence of definitions, which closely mimics the situation in Riemannian geometry:

Having the connection 1-forms $\overset{W}{\Gamma}_{ab}$, the metric g and its inverse, represented by g^{ab}, we also have the 1-forms $\overset{W}{\Gamma}{}^a_b = g^{ac}\overset{W}{\Gamma}_{cb}$. Using them, we define the curvature of the connection $\overset{W}{\nabla}$. We do it, in terms of the curvature 2-forms $\overset{W}{\Omega}{}^a_b$, analogous to those given in the formula (7), by:

$$\tfrac{1}{2}\overset{W}{R}{}^a_{bcd}\theta^c \wedge \theta^d = \mathrm{d}\overset{W}{\Gamma}{}^a_b + \overset{W}{\Gamma}{}^a_c \wedge \overset{W}{\Gamma}{}^c_b.$$

Here $\overset{W}{R}{}^a_{bcd}$ are the curvature coefficents in the coframe (θ^a). Then we define the Ricci tensor

$$\overset{W}{R}_{ab} = \overset{W}{R}{}^c_{acb},$$

and its scalar

$$\overset{W}{R} = g^{ab}\overset{W}{R}_{ab}.$$

The next step is to define the Schouten tensor

$$\overset{W}{\mathsf{P}}_{ab} = \tfrac{1}{2}\overset{W}{R}_{ab} - \tfrac{1}{12}\overset{W}{R}g_{ab}$$

and the Cotton tensor

$$\overset{W}{A}_{abc} = 2\overset{W}{\nabla}_{[b}\overset{W}{\mathsf{P}}_{c]a}.$$

This defines a linear map

$$\overset{W}{A} : \mathrm{T}\mathcal{M} \times \mathrm{T}\mathcal{M} \times \mathrm{T}\mathcal{M} \to \mathbb{R}$$

given by

$$\overset{W}{A} = \tfrac{1}{2}\overset{W}{A}_{abc}\theta^a \otimes (\theta^b \wedge \theta^c).$$

Then the (pseudo)hermitian part of Theorem 6.5 is:

Theorem 7.16 *Let $(\mathcal{M}, g, J)$ be a 4-dimensional (pseudo)hermitian manifold and let $\overset{W}{\nabla}$ be its canonical (pseudo)hermitian-Weyl connection $\overset{HW}{\nabla}$. Assume that*

$$\overset{W}{\nabla}_X \overset{W}{A}(Y, X, Y) \equiv \overset{W}{\nabla}_Y \overset{W}{A}(X, X, Y) \quad \text{for all vectors} \quad X, Y \in \mathcal{N} = \mathrm{T}^{(1,0)}\mathcal{M}. \tag{73}$$

Then the self-dual part of the Weyl tensor for $(\mathcal{M}, g)$ is algebraically special at every point of $\mathcal{M}$, with J being the multiple principal hermitian structure on $\mathcal{N}$.

Proof The proof of this Theorem consists of straightforward calculations using the above definitions. The key point in these calculations is that $\overset{W}{\nabla}_X \overset{W}{A}(Y, X, Y) - \overset{W}{\nabla}_Y \overset{W}{A}(X, X, Y)$, when X, Y run through all the vectors from $\mathcal{N}$, is always proportional to $\overset{W}{\nabla}_4 \overset{W}{A}_{141} - \overset{W}{\nabla}_1 \overset{W}{A}_{441}$. Here the indices 1 an 4 are the components from the coframe adapted to J, in which $e_1 = m$ and $e_4 = k$. By a direct calculation one can check that $\overset{W}{\nabla}_4 \overset{W}{A}_{141} - \overset{W}{\nabla}_1 \overset{W}{A}_{441} = 16\Psi_1^2$. Thus, when $\overset{W}{\nabla}_X \overset{W}{A}(Y, X, Y) \equiv \overset{W}{\nabla}_Y \overset{W}{A}(X, X, Y)$, as assumed, $\Psi_1 \equiv 0$, which proves the theorem.

Remark 7.17 *When calculating $\overset{W}{\nabla}_4 \overset{W}{A}_{141} - \overset{W}{\nabla}_1 \overset{W}{A}_{441}$, during the proof of the above theorem, we observed that the relation $\overset{W}{\nabla}_4 \overset{W}{A}_{141} - \overset{W}{\nabla}_1 \overset{W}{A}_{441} = 16\Psi_1^2$ is true even without the (pseudo) hermitian reality conditions (E) or (S_c). For this crucial relation to be true, we need to take B as in (72) and to assume the integrability of $\mathcal{N}$, i.e. to assume $\kappa \equiv \sigma \equiv 0$. If these two assumptions are satisfied then $\overset{W}{\nabla}_4 \overset{W}{A}_{141} - \overset{W}{\nabla}_1 \overset{W}{A}_{441} = 16\Psi_1^2$ irrespective of the signature of the metric. It is even true when the metric is complex! Thus the Weyl connection $\overset{W}{\nabla}$ with B as in (72) seems to be meaningful in case of g being complex, or having any signature. The only trouble with such a connection is that in the Lorentzian case it is complex. If one can live with this, one can replace the condition (65) in Theorem 6.5 by (73) and Theorem 6.5 will be true for complex metrics, as well for metrics of all the other real signatures.*

8 Appendix

The 36 signature independent Newman–Penrose equations, which include 16 first Bianchi identities, are:

$$\delta\kappa = D\sigma + \alpha'\kappa + 3\beta\kappa + \kappa\pi' - 3\varepsilon\sigma + \varepsilon'\sigma + \rho\sigma + \rho'\sigma + \kappa\tau + \Psi_0 \tag{74}$$
$$\partial\kappa' = D\sigma' + \alpha\kappa' + 3\beta'\kappa' + \kappa'\pi - 3\varepsilon'\sigma' + \varepsilon\sigma' + \rho'\sigma' + \rho\sigma' + \kappa'\tau' + \Psi_0'$$
$$D\beta = \delta\varepsilon - \alpha'\varepsilon - \beta\varepsilon' - \gamma\kappa - \kappa\mu - \varepsilon\pi' - \beta\rho' - \alpha\sigma + \pi\sigma - \Psi_1 \tag{75}$$
$$D\beta' = \partial\varepsilon' - \alpha\varepsilon' - \beta'\varepsilon - \gamma'\kappa' - \kappa'\mu' - \varepsilon'\pi - \beta'\rho - \alpha'\sigma' + \pi'\sigma' - \Psi_1'$$
$$\delta\rho = \partial\sigma + \kappa\mu' - \kappa\mu + \alpha'\rho + \beta\rho - 3\alpha\sigma + \beta'\sigma - \rho'\tau + \rho\tau - \Psi_1 - \mathsf{P}_{14} \tag{76}$$
$$\partial\rho' = \delta\sigma' + \kappa'\mu - \kappa'\mu' + \alpha\rho' + \beta'\rho' - 3\alpha'\sigma' + \beta\sigma' - \rho\tau' + \rho'\tau' - \Psi_1' - \mathsf{P}_{24}$$
$$D\tau = \triangle\kappa - \gamma'\kappa - 3\gamma\kappa + \pi'\rho + \pi\sigma - \sigma\tau' - \varepsilon'\tau + \varepsilon\tau - \rho\tau - \Psi_1 + \mathsf{P}_{14} \tag{77}$$
$$D\tau' = \triangle\kappa' - \gamma\kappa' - 3\gamma'\kappa' + \pi\rho' + \pi'\sigma' - \sigma'\tau - \varepsilon\tau' + \varepsilon'\tau' - \rho'\tau' - \Psi_1' + \mathsf{P}_{24}$$

$$\triangle\rho = \partial\tau - \kappa\nu + \gamma\rho + \gamma'\rho - \mu'\rho - \lambda\sigma - \alpha\tau + \beta'\tau - \tau\tau' - \Psi_2 - \mathsf{P}_{12} - \mathsf{P}_{34}$$
$$\triangle\rho' = \delta\tau' - \kappa'\nu' + \gamma'\rho' + \gamma\rho' - \mu\rho' - \lambda'\sigma' - \alpha'\tau' + \beta\tau' - \tau\tau' - \Psi_2' - \mathsf{P}_{12} - \mathsf{P}_{34}$$
$$\triangle\alpha = \partial\gamma + \beta'\gamma + \alpha\gamma' - \beta\lambda - \alpha\mu' - \varepsilon\nu + \nu\rho - \lambda\tau - \gamma\tau' + \Psi_3$$
$$\triangle\alpha' = \delta\gamma' + \beta\gamma' + \alpha'\gamma - \beta'\lambda' - \alpha'\mu - \varepsilon'\nu' + \nu'\rho' - \lambda'\tau' - \gamma'\tau + \Psi_3'$$
$$\triangle\lambda = \partial\nu - 3\gamma\lambda + \gamma'\lambda - \lambda\mu - \lambda\mu' + 3\alpha\nu + \beta'\nu - \nu\pi - \nu\tau' - \Psi_4$$
$$\triangle\lambda' = \delta\nu' - 3\gamma'\lambda' + \gamma\lambda' - \lambda'\mu' - \lambda'\mu + 3\alpha'\nu' + \beta\nu' - \nu'\pi' - \nu'\tau - \Psi_4'$$
$$D\lambda = \partial\pi - 3\varepsilon\lambda + \varepsilon'\lambda - \kappa'\nu + \alpha\pi - \beta'\pi - \pi^2 - \lambda\rho - \mu\sigma' - \mathsf{P}_{22}$$
$$D\lambda' = \delta\pi' - 3\varepsilon'\lambda' + \varepsilon\lambda' - \kappa\nu' + \alpha'\pi' - \beta\pi' - \pi'^2 - \lambda'\rho' - \mu'\sigma - \mathsf{P}_{11} \tag{78}$$
$$D\mu = \delta\pi - \varepsilon\mu - \varepsilon'\mu - \kappa\nu - \alpha'\pi + \beta\pi - \pi\pi' - \mu\rho' - \lambda\sigma - \Psi_2 - \mathsf{P}_{12} - \mathsf{P}_{34}$$
$$D\mu' = \partial\pi' - \varepsilon'\mu' - \varepsilon\mu' - \kappa'\nu' - \alpha\pi' + \beta'\pi' - \pi\pi' - \mu'\rho - \lambda'\sigma' - \Psi_2' - \mathsf{P}_{12} - \mathsf{P}_{34}$$
$$D\alpha = \partial\varepsilon + \alpha\varepsilon' - 2\alpha\varepsilon - \beta'\varepsilon - \gamma\kappa' - \kappa\lambda - \varepsilon\pi - \alpha\rho + \pi\rho - \beta\sigma' + \mathsf{P}_{24}$$
$$D\alpha' = \delta\varepsilon' + \alpha'\varepsilon - 2\alpha'\varepsilon' - \beta\varepsilon' - \gamma'\kappa - \kappa'\lambda' - \varepsilon'\pi' - \alpha'\rho' + \pi'\rho' - \beta'\sigma + \mathsf{P}_{14} \tag{79}$$
$$\triangle\beta = \delta\gamma + \alpha'\gamma + 2\beta\gamma - \beta\gamma' - \alpha\lambda' - \beta\mu - \varepsilon\nu' + \nu\sigma - \gamma\tau - \mu\tau - \mathsf{P}_{13}$$
$$\triangle\beta' = \partial\gamma' + \alpha\gamma' + 2\beta'\gamma' - \beta'\gamma - \alpha'\lambda - \beta'\mu' - \varepsilon'\nu + \nu'\sigma' - \gamma'\tau' - \mu'\tau' - \mathsf{P}_{23}$$
$$D\rho = \partial\kappa - 3\alpha\kappa - \beta'\kappa - \kappa\pi + \varepsilon\rho + \varepsilon'\rho - \rho^2 - \sigma\sigma' - \kappa'\tau - \mathsf{P}_{44} \tag{80}$$
$$D\rho' = \delta\kappa' - 3\alpha'\kappa' - \beta\kappa' - \kappa'\pi' + \varepsilon'\rho' + \varepsilon\rho' - \rho'^2 - \sigma\sigma' - \kappa\tau' - \mathsf{P}_{44} \tag{81}$$
$$\triangle\mu = \delta\nu - \lambda\lambda' - \gamma\mu - \gamma'\mu - \mu^2 + \alpha'\nu + 3\beta\nu - \nu'\pi - \nu\tau - \mathsf{P}_{33}$$
$$\triangle\mu' = \partial\nu' - \lambda\lambda' - \gamma'\mu' - \gamma\mu' - \mu'^2 + \alpha\nu' + 3\beta'\nu' - \nu\pi' - \nu'\tau' - \mathsf{P}_{33}$$
$$D\nu = \triangle\pi - \varepsilon'\nu - 3\varepsilon\nu + \lambda\pi' - \gamma'\pi + \gamma\pi + \mu\pi - \mu\tau' - \lambda\tau + \Psi_3 - \mathsf{P}_{23}$$
$$D\nu' = \triangle\pi' - \varepsilon\nu' - 3\varepsilon'\nu' + \lambda'\pi - \gamma\pi' + \gamma'\pi' + \mu'\pi' - \mu'\tau - \lambda'\tau' + \Psi_3' - \mathsf{P}_{13}$$
$$D\gamma = \triangle\varepsilon - 2\varepsilon\gamma - \varepsilon'\gamma - \varepsilon\gamma' - \kappa\nu + \beta\pi + \alpha\pi' - \alpha\tau + \pi\tau - \beta\tau' - \Psi_2 + \mathsf{P}_{34}$$
$$D\gamma' = \triangle\varepsilon' - 2\varepsilon'\gamma' - \varepsilon\gamma' - \varepsilon'\gamma - \kappa'\nu' + \beta'\pi' + \alpha'\pi - \alpha'\tau' + \pi'\tau' - \beta'\tau - \Psi_2' + \mathsf{P}_{34}$$
$$\partial\mu = \delta\lambda - \alpha'\lambda + 3\beta\lambda - \alpha\mu - \beta'\mu + \mu\pi - \mu'\pi - \nu\rho + \nu\rho' - \Psi_3 - \mathsf{P}_{23}$$
$$\delta\mu' = \partial\lambda' - \alpha\lambda' + 3\beta'\lambda' - \alpha'\mu' - \beta\mu' + \mu'\pi' - \mu\pi' - \nu'\rho' + \nu'\rho - \Psi_3' - \mathsf{P}_{13}$$
$$\delta\tau = \triangle\sigma + \kappa\nu' + \lambda'\rho - 3\gamma\sigma + \gamma'\sigma + \mu\sigma - \alpha'\tau + \beta\tau + \tau^2 + \mathsf{P}_{11} \tag{82}$$
$$\partial\tau' = \triangle\sigma' + \kappa'\nu + \lambda\rho' - 3\gamma'\sigma' + \gamma\sigma' + \mu'\sigma' - \alpha\tau' + \beta'\tau' + \tau'^2 + \mathsf{P}_{22}$$
$$\delta\alpha = \partial\beta + \alpha\alpha' - 2\alpha\beta + \beta\beta' - \varepsilon\mu + \varepsilon\mu' + \gamma\rho + \mu\rho - \gamma\rho' - \lambda\sigma - \Psi_2 + \mathsf{P}_{12}$$
$$\partial\alpha' = \delta\beta' + \alpha\alpha' - 2\alpha'\beta' + \beta\beta' - \varepsilon'\mu' + \varepsilon'\mu + \gamma'\rho' + \mu'\rho' - \gamma'\rho - \lambda'\sigma' - \Psi_2' + \mathsf{P}_{12}$$

The 20 second Bianchi identities are:

$$\begin{aligned}\delta\Psi_1 =\ & \triangle\Psi_0 - D\mathsf{P}_{11} + \delta\mathsf{P}_{14} - 4\gamma\Psi_0 + \mu\Psi_0 + 2\beta\Psi_1 - 3\sigma\Psi_2 + 4\tau\Psi_1\\ & -2\kappa\mathsf{P}_{13} + 2\varepsilon\mathsf{P}_{11} - 2\varepsilon'\mathsf{P}_{11} - 2\beta\mathsf{P}_{14} - 2\pi'\mathsf{P}_{14} + \lambda'\mathsf{P}_{44} - \rho'\mathsf{P}_{11} - \sigma\mathsf{P}_{12} + \sigma\mathsf{P}_{34}\end{aligned} \tag{83}$$
$$\begin{aligned}\partial\Psi_1' =\ & \triangle\Psi_0' - D\mathsf{P}_{22} + \partial\mathsf{P}_{24} - 4\gamma'\Psi_0' + \mu'\Psi_0' + 2\beta'\Psi_1' - 3\sigma'\Psi_2' + 4\tau'\Psi_1'\\ & -2\kappa'\mathsf{P}_{23} + 2\varepsilon'\mathsf{P}_{22} - 2\varepsilon\mathsf{P}_{22} - 2\beta'\mathsf{P}_{24} - 2\pi\mathsf{P}_{24} + \lambda\mathsf{P}_{44} - \rho\mathsf{P}_{22} - \sigma'\mathsf{P}_{12} + \sigma'\mathsf{P}_{34}\end{aligned}$$
$$\begin{aligned}D\Psi_1 =\ & -\partial\Psi_0 - D\mathsf{P}_{14} + \delta\mathsf{P}_{44} + 4\alpha\Psi_0 + \pi\Psi_0 + 2\varepsilon\Psi_1 - 3\kappa\Psi_2 - 4\Psi_1\rho + \kappa'\mathsf{P}_{11}\\ & + \kappa\mathsf{P}_{12} + 2\varepsilon\mathsf{P}_{14} - \kappa\mathsf{P}_{34} - 2\alpha'\mathsf{P}_{44} - 2\beta\mathsf{P}_{44} - \pi'\mathsf{P}_{44} - 2\rho'\mathsf{P}_{14} - 2\sigma\mathsf{P}_{24}\end{aligned} \tag{84}$$
$$\begin{aligned}D\Psi_1' =\ & -\delta\Psi_0' - D\mathsf{P}_{24} + \partial\mathsf{P}_{44} + 4\alpha'\Psi_0' + \pi'\Psi_0' + 2\varepsilon'\Psi_1' - 3\kappa'\Psi_2' - 4\Psi_1'\rho' + \kappa\mathsf{P}_{22}\\ & + \kappa'\mathsf{P}_{12} + 2\varepsilon'\mathsf{P}_{24} - \kappa'\mathsf{P}_{34} - 2\alpha\mathsf{P}_{44} - 2\beta'\mathsf{P}_{44} - \pi\mathsf{P}_{44} - 2\rho\mathsf{P}_{24} - 2\sigma'\mathsf{P}_{14}\end{aligned}$$
$$\begin{aligned}\triangle\Psi_1 =\ & \delta\Psi_2 + D\mathsf{P}_{13} - \delta\mathsf{P}_{34} + \nu\Psi_0 + 2\gamma\Psi_1 - 2\mu\Psi_1 - 2\sigma\Psi_3 - 3\tau\Psi_2\\ & -\pi\mathsf{P}_{11} - \pi'\mathsf{P}_{12} + 2\varepsilon'\mathsf{P}_{13} + \mu\mathsf{P}_{14} + \lambda'\mathsf{P}_{24} + \kappa\mathsf{P}_{33} + \pi'\mathsf{P}_{34} + \rho'\mathsf{P}_{13} + \sigma\mathsf{P}_{23}\end{aligned} \tag{85}$$

$$\begin{aligned}
\triangle\Psi_1' &= \partial\Psi_2' + D\mathsf{P}_{23} - \partial\mathsf{P}_{34} + \nu'\Psi_0' + 2\gamma'\Psi_1' - 2\mu'\Psi_1' - 2\sigma'\Psi_3' - 3\tau'\Psi_2' \\
&\quad -\pi'\mathsf{P}_{22} - \pi\mathsf{P}_{12} + 2\varepsilon\mathsf{P}_{23} + \mu'\mathsf{P}_{24} + \lambda\mathsf{P}_{14} + \kappa'\mathsf{P}_{33} + \pi\mathsf{P}_{34} + \rho\mathsf{P}_{23} + \sigma'\mathsf{P}_{13} \\
\partial\Psi_1 &= -D\Psi_2 + D\mathsf{P}_{12} - \delta\mathsf{P}_{24} + \lambda\Psi_0 + 2\alpha\Psi_1 + 2\pi\Psi_1 + 2\kappa\Psi_3 - 3\rho\Psi_2 \\
&\quad + \kappa'\mathsf{P}_{13} + \pi\mathsf{P}_{14} + \kappa\mathsf{P}_{23} + 2\alpha'\mathsf{P}_{24} + \pi'\mathsf{P}_{24} - \mu\mathsf{P}_{44} + \rho'\mathsf{P}_{12} - \rho'\mathsf{P}_{34} + \sigma\mathsf{P}_{22} \qquad (86) \\
\delta\Psi_1' &= -D\Psi_2' + D\mathsf{P}_{12} - \partial\mathsf{P}_{14} + \lambda'\Psi_0' + 2\alpha'\Psi_1' + 2\pi'\Psi_1' + 2\kappa'\Psi_3' - 3\rho'\Psi_2' \\
&\quad + \kappa\mathsf{P}_{23} + \pi'\mathsf{P}_{24} + \kappa'\mathsf{P}_{13} + 2\alpha\mathsf{P}_{14} + \pi\mathsf{P}_{14} - \mu'\mathsf{P}_{44} + \rho\mathsf{P}_{12} - \rho\mathsf{P}_{34} + \sigma'\mathsf{P}_{11} \\
\triangle\Psi_2 &= -\delta\Psi_3 + \triangle\mathsf{P}_{12} - \partial\mathsf{P}_{13} + 2\nu\Psi_1 - 3\mu\Psi_2 - 2\beta\Psi_3 + 2\tau\Psi_3 + \sigma\Psi_4 \\
&\quad + \lambda\mathsf{P}_{11} + \mu'\mathsf{P}_{12} - 2\beta'\mathsf{P}_{13} + \nu\mathsf{P}_{14} + \nu'\mathsf{P}_{24} - \rho\mathsf{P}_{33} - \mu'\mathsf{P}_{34} + \tau\mathsf{P}_{23} + \tau'\mathsf{P}_{13} \qquad (87) \\
\triangle\Psi_2' &= -\partial\Psi_3' + \triangle\mathsf{P}_{12} - \delta\mathsf{P}_{23} + 2\nu'\Psi_1' - 3\mu'\Psi_2' - 2\beta'\Psi_3' + 2\tau'\Psi_3' + \sigma'\Psi_4' \\
&\quad + \lambda'\mathsf{P}_{22} + \mu\mathsf{P}_{12} - 2\beta\mathsf{P}_{23} + \nu'\mathsf{P}_{24} + \nu\mathsf{P}_{14} - \rho'\mathsf{P}_{33} - \mu\mathsf{P}_{34} + \tau'\mathsf{P}_{13} + \tau\mathsf{P}_{23} \\
D\Psi_3 &= \partial\Psi_2 + \triangle\mathsf{P}_{24} - \partial\mathsf{P}_{34} - 2\lambda\Psi_1 - 3\pi\Psi_2 - 2\varepsilon\Psi_3 + \kappa\Psi_4 - 2\rho\Psi_3 + \lambda\mathsf{P}_{14} \\
&\quad + \rho\mathsf{P}_{23} - 2\gamma'\mathsf{P}_{24} + \mu'\mathsf{P}_{24} + \nu\mathsf{P}_{44} + \sigma'\mathsf{P}_{13} - \tau\mathsf{P}_{22} - \tau'\mathsf{P}_{12} + \tau'\mathsf{P}_{34} \qquad (88) \\
D\Psi_3' &= \delta\Psi_2' + \triangle\mathsf{P}_{14} - \delta\mathsf{P}_{34} - 2\lambda'\Psi_1' - 3\pi'\Psi_2' - 2\varepsilon'\Psi_3' + \kappa'\Psi_4' - 2\rho'\Psi_3' + \lambda'\mathsf{P}_{24} \\
&\quad + \rho'\mathsf{P}_{13} - 2\gamma\mathsf{P}_{14} + \mu\mathsf{P}_{14} + \nu'\mathsf{P}_{44} + \sigma\mathsf{P}_{23} - \tau'\mathsf{P}_{11} - \tau\mathsf{P}_{12} + \tau\mathsf{P}_{34} \\
\triangle\Psi_3 &= -\delta\Psi_4 - \triangle\mathsf{P}_{23} + \partial\mathsf{P}_{33} - 3\nu\Psi_2 - 2\gamma\Psi_3 - 4\mu\Psi_3 - 4\beta\Psi_4 + \tau\Psi_4 + \nu\mathsf{P}_{12} \\
&\quad - 2\lambda\mathsf{P}_{13} + \nu'\mathsf{P}_{22} - 2\gamma\mathsf{P}_{23} - 2\mu'\mathsf{P}_{23} + 2\alpha\mathsf{P}_{33} + 2\beta'\mathsf{P}_{33} - \nu\mathsf{P}_{34} - \tau'\mathsf{P}_{33} \qquad (89) \\
\triangle\Psi_3' &= -\partial\Psi_4' - \triangle\mathsf{P}_{13} + \delta\mathsf{P}_{33} - 3\nu'\Psi_2' - 2\gamma'\Psi_3' - 4\mu'\Psi_3' - 4\beta'\Psi_4' + \tau'\Psi_4' + \nu'\mathsf{P}_{12} \\
&\quad - 2\lambda'\mathsf{P}_{23} + \nu\mathsf{P}_{11} - 2\gamma'\mathsf{P}_{13} - 2\mu\mathsf{P}_{13} + 2\alpha'\mathsf{P}_{33} + 2\beta\mathsf{P}_{33} - \nu'\mathsf{P}_{34} - \tau\mathsf{P}_{33} \\
\partial\Psi_3 &= D\Psi_4 - \triangle\mathsf{P}_{22} + \partial\mathsf{P}_{23} - 3\lambda\Psi_2 - 2\alpha\Psi_3 + 4\pi\Psi_3 + 4\varepsilon\Psi_4 + \rho\Psi_4 \\
&\quad + 2\gamma'\mathsf{P}_{22} - 2\gamma\mathsf{P}_{22} - \mu'\mathsf{P}_{22} - \lambda\mathsf{P}_{12} + 2\alpha\mathsf{P}_{23} - 2\nu\mathsf{P}_{24} + \lambda\mathsf{P}_{34} + \sigma'\mathsf{P}_{33} - 2\tau'\mathsf{P}_{23} \qquad (90) \\
\delta\Psi_3' &= D\Psi_4' - \triangle\mathsf{P}_{11} + \delta\mathsf{P}_{13} - 3\lambda'\Psi_2' - 2\alpha'\Psi_3' + 4\pi'\Psi_3' + 4\varepsilon'\Psi_4' + \rho'\Psi_4' \\
&\quad + 2\gamma\mathsf{P}_{11} - 2\gamma'\mathsf{P}_{11} - \mu\mathsf{P}_{11} - \lambda'\mathsf{P}_{12} + 2\alpha'\mathsf{P}_{13} - 2\nu'\mathsf{P}_{14} + \lambda'\mathsf{P}_{34} + \sigma\mathsf{P}_{33} - 2\tau\mathsf{P}_{13} \\
\delta\mathsf{P}_{12} &= D\mathsf{P}_{13} + \triangle\mathsf{P}_{14} + \partial\mathsf{P}_{11} - 2\delta\mathsf{P}_{34} - 2\alpha\mathsf{P}_{11} + 2\beta'\mathsf{P}_{11} - \pi\mathsf{P}_{11} - \pi'\mathsf{P}_{12} + 2\varepsilon'\mathsf{P}_{13} \\
&\quad + 2\rho\mathsf{P}_{13} - 2\gamma\mathsf{P}_{14} + \mu\mathsf{P}_{14} + 2\mu'\mathsf{P}_{14} + \lambda'\mathsf{P}_{24} + \kappa\mathsf{P}_{33} + \pi'\mathsf{P}_{34} + \nu'\mathsf{P}_{44} + \rho'\mathsf{P}_{13} \qquad (91) \\
&\quad + \sigma\mathsf{P}_{23} - \tau\mathsf{P}_{12} + \tau\mathsf{P}_{34} - \tau'\mathsf{P}_{11} \\
\partial\mathsf{P}_{12} &= D\mathsf{P}_{23} + \triangle\mathsf{P}_{24} + \delta\mathsf{P}_{22} - 2\partial\mathsf{P}_{34} - 2\alpha'\mathsf{P}_{22} + 2\beta\mathsf{P}_{22} - \pi'\mathsf{P}_{22} - \pi\mathsf{P}_{12} + 2\varepsilon\mathsf{P}_{23} \\
&\quad + 2\rho'\mathsf{P}_{23} - 2\gamma'\mathsf{P}_{24} + \mu'\mathsf{P}_{24} + 2\mu\mathsf{P}_{24} + \lambda\mathsf{P}_{14} + \kappa'\mathsf{P}_{33} + \pi\mathsf{P}_{34} + \nu\mathsf{P}_{44} + \rho\mathsf{P}_{23} \\
&\quad + \sigma'\mathsf{P}_{13} - \tau'\mathsf{P}_{12} + \tau'\mathsf{P}_{34} - \tau\mathsf{P}_{22} \\
D\mathsf{P}_{34} &= -2D\mathsf{P}_{12} + \triangle\mathsf{P}_{44} + \partial\mathsf{P}_{14} + \delta\mathsf{P}_{24} - \rho\mathsf{P}_{12} - \kappa'\mathsf{P}_{13} - 2\alpha\mathsf{P}_{14} - \pi\mathsf{P}_{14} - \kappa\mathsf{P}_{23} \\
&\quad -2\alpha'\mathsf{P}_{24} - \pi'\mathsf{P}_{24} + \rho\mathsf{P}_{34} - 2\gamma\mathsf{P}_{44} - 2\gamma'\mathsf{P}_{44} + \mu\mathsf{P}_{44} + \mu'\mathsf{P}_{44} - \rho'\mathsf{P}_{12} + \rho'\mathsf{P}_{34} \qquad (92) \\
&\quad -\sigma\mathsf{P}_{22} - \sigma'\mathsf{P}_{11} - 2\tau\mathsf{P}_{24} - 2\tau'\mathsf{P}_{14} \\
\triangle\mathsf{P}_{34} &= D\mathsf{P}_{33} - 2\triangle\mathsf{P}_{12} + \partial\mathsf{P}_{13} + \delta\mathsf{P}_{23} - \lambda\mathsf{P}_{11} - \mu\mathsf{P}_{12} - \mu'\mathsf{P}_{12} + 2\beta'\mathsf{P}_{13} - 2\pi\mathsf{P}_{13} \\
&\quad -\nu\mathsf{P}_{14} - \lambda'\mathsf{P}_{22} + 2\beta\mathsf{P}_{23} - 2\pi'\mathsf{P}_{23} - \nu'\mathsf{P}_{24} + 2\varepsilon\mathsf{P}_{33} + 2\varepsilon'\mathsf{P}_{33} + \rho\mathsf{P}_{33} + \mu\mathsf{P}_{34} \\
&\quad +\mu'\mathsf{P}_{34} + \rho'\mathsf{P}_{33} - \tau\mathsf{P}_{23} - \tau'\mathsf{P}_{13}
\end{aligned}$$

Using relations (31, 32) we can reexpress identities (83–90) in terms of the components of the Cotton tensor. After this the Cotton tensor components 'hide' the terms with the Schouten tensor components P_{ij}, and the respective identities assume a more compact form as follows:

$$A_{141} = \triangle\Psi_0 + (\mu - 4\gamma)\Psi_0 - \delta\Psi_1 + 2(2\tau + \beta)\Psi_1 - 3\sigma\Psi_2 \qquad (93)$$

$$A_{414} = \partial\Psi_0 - (\pi + 4\alpha)\Psi_0 + D\Psi_1 + 2(2\rho - \varepsilon)\Psi_1 + 3\kappa\Psi_2 \qquad (94)$$

$$A_{341} = \triangle\Psi_1 + 2(\mu - \gamma)\Psi_1 - \delta\Psi_2 + 3\tau\Psi_2 - \nu\Psi_0 + 2\sigma\Psi_3 \qquad (95)$$

$$A_{214} = \partial\Psi_1 - 2(\alpha + \pi)\Psi_1 + D\Psi_2 + 3\rho\Psi_2 - \lambda\Psi_0 - 2\kappa\Psi_3 \qquad (96)$$

$$A_{132} = \triangle\Psi_2 + 3\mu\Psi_2 + \delta\Psi_3 + 2(\beta - \tau)\Psi_3 - 2\nu\Psi_1 - \sigma\Psi_4 \tag{97}$$
$$A_{423} = \partial\Psi_2 - 3\pi\Psi_2 - D\Psi_3 - 2(\varepsilon + \rho)\Psi_3 - 2\lambda\Psi_1 + \kappa\Psi_4 \tag{98}$$
$$A_{323} = \triangle\Psi_3 + 2(\gamma + 2\mu)\Psi_3 + \delta\Psi_4 + (4\beta - \tau)\Psi_4 + 3\nu\Psi_2 \tag{99}$$
$$A_{223} = \partial\Psi_3 + 2(\alpha - 2\pi)\Psi_3 - D\Psi_4 - (\rho + 4\varepsilon)\Psi_4 + 3\lambda\Psi_2, \tag{100}$$

with the analogous identities for the primed quantities.

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