# *Einstein's Equations and The Embedding of 3-dimensional CR Manifolds*

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ABSTRACT. We prove several theorems concerning the connection between the local CR embeddability of 3-dimensional CR manifolds, and the existence of algebraically special Maxwell and gravitational fields. We reduce the Einstein equations for spacetimes associated with such fields to a system of CR invariant equations on a 3-dimensional CR manifold defined by the fields. Using the reduced Einstein equations we construct two independent CR functions for the corresponding CR manifold. We also point out that the Einstein equations, imposed on spacetimes associated with a 3-dimensional CR manifold, imply that the spacetime metric, after an appropriate rescaling, becomes well defined on a circle bundle over the CR manifold. The circle bundle itself emerges as a consequence of Einstein's equations.

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## 1. INTRODUCTION

***1.1. 3-dimensional CR structures.*** Let $M$ be an open set in $\mathbb{R}^3$, with *real* coordinates $(x, y, z)$. We consider a complex vector field

$$Z = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}. \tag{1.1}$$

Here the coefficients

$$a = a(x, y, z), \qquad b = b(x, y, z), \qquad c = c(x, y, z)$$

are *complex-valued* functions on $M$. We assume that the vector field $Z$ and its complex conjugate $\bar{Z}$ are *linearly independent* at each point of $M$. The vector field $Z$ spans a 1-dimensional complex distribution $\mathrm{T}^{1,0}$ in $\mathbb{C}\mathrm{T}M$. By definition the pair $(M, \mathrm{T}^{1,0})$ is an abstract *3-dimensional CR manifold*. The CR structure $(M, \mathrm{T}^{1,0})$ on $M$ is said to be of class $C^k$ iff the coefficients $a = a(x, y, z)$, $b = b(x, y, z)$ and $c = c(x, y, z)$ are of differentiability class $C^k$.

If the vector fields $Z$, $\bar{Z}$ and $[Z, \bar{Z}]$ are linearly independent at each point of $M$ then the CR structure $(M, \mathrm{T}^{1,0})$ is called *strictly pseudoconvex*. When $[Z, \bar{Z}]$ is linearly dependent on $Z$ and $\bar{Z}$ at a point $p$, the *Levi form* of the structure $(M, \mathrm{T}^{1,0})$ vanishes at $p$.

Natural examples of 3-dimensional CR manifolds are hypersurfaces $M$ of one real codimension embedded in $\mathbb{C}^2$. They acquire a CR structure $\mathrm{T}^{1,0}M$ from the ambient space by $\mathrm{T}^{1,0}M = \{X - iJX \mid X \in \mathrm{T}M \cap J(\mathrm{T}M)\}$, where $J$ is the standard complex structure on $\mathbb{C}^2$.

Given an abstract CR manifold $(M, \mathrm{T}^{1,0})$ one asks if there exists a local diffeomorphism

$$\iota : M \to \iota(M) \subset \mathbb{C}^2$$

such that

$$\iota_* \mathrm{T}^{1,0} = \mathrm{T}^{1,0}\iota(M).$$

Such a diffeomorphism is called a *local CR embedding* of $(M, \mathrm{T}^{1,0})$ into $\mathbb{C}^2$.

The question of whether or not a given 3-dimensional CR manifold $(M, \mathrm{T}^{1,0})$ can be locally CR embedded as a hypersurface in $\mathbb{C}^2$ is related to the problem of the local existence of solutions to the *linear* partial differential equation

$$\bar{Z}\zeta = 0, \tag{1.2}$$

where

$$\bar{Z} = \bar{a}\frac{\partial}{\partial x} + \bar{b}\frac{\partial}{\partial y} + \bar{c}\frac{\partial}{\partial z}$$

is a section of the bundle $\bar{\mathrm{T}}^{1,0} = \mathrm{T}^{0,1}$. The solution $\zeta = \zeta(x, y, z)$ is called a *CR function*. Of course any holomorphic function $h = h(\zeta)$ of a CR function is a CR function. When speaking about different CR functions, we will mean only those which are not functionally related to each other by a holomorphic function in this sense.

If the CR structure $(M, \mathrm{T}^{1,0})$ on $M$ is *real analytic*, then by the Cauchy-Kowalewska theorem, equation (1.2) locally admits *two* functionally independent CR functions

$$\zeta = \zeta(x, y, z), \quad \eta = \eta(x, y, z)$$

such that

$$\mathrm{d}\zeta \wedge \mathrm{d}\eta \neq 0.$$

Then the CR manifold is locally CR embeddable with the local CR embedding being real analytic, and given by [1, 24]

$$\iota : M \ni (x, y, z) \mapsto (\zeta, \eta) = (\zeta(x, y, z), \eta(x, y, z)) \in \mathbb{C}^2.$$

The situation in which (1.2) has two functionally independent solutions may also happen when the CR structure on $M$ is sufficiently smooth but *not* real analytic. The important thing is that the requirement that (1.2) locally admits two functionally independent CR functions is *equivalent* to the local embeddability of $M$. The real analyticity is not needed for this equivalence to hold.

It turns out that if one abandons the assumption about the real analyticity of $(M, \mathrm{T}^{1,0})$, e.g. if one assumes that $(M, \mathrm{T}^{1,0})$ is only of class $C^\infty$, then the equation (1.2) *may have no other local solutions than the trivial ones* $\zeta = const$. This remarkable result is due to Louis Nirenberg [26, 27], where he gave the first example of a $Z$ having $C^\infty$ coefficients such that (1.2) has only constant local solutions. As a consequence such abstract CR manifolds are *not* locally CR embeddable as hypersurfaces in $\mathbb{C}^2$.

There are a few situations in which a $C^\infty$ 3-dimensional abstract CR manifold *is* locally embeddable. Among them are the *Levi flat* structures (those whose Levi form vanishes identically in a neighborhood), as well as those which admit a local *symmetry* (i.e. a local real vector field $X$ such that $[X, Z] \wedge Z = 0$, in a neighborhood). It is also possible to find $C^\infty$ 3-dimensional abstract CR structures on $M$

which have one CR function $\zeta$, with $d\zeta \neq 0$, but such that any other local CR function is functionally dependent on $\zeta$; hence such structures are also not locally CR embeddable.

However the surprising and remarkable thing about $C^\infty$ 3-dimensional CR structures is that *the generic situation is like the example of Nirenberg*: The complex vector fields $Z$, having only locally constant solutions, *are dense* (in the $C^\infty$ topology). This is a consequence of a Baire category argument.

Given a strictly pseudoconvex structure $(M, \mathrm{T}^{1,0})$ on $M$ which is locally CR embeddable, an arbitrary small randomly chosen $C^\infty$ perturbation of it will no longer be locally CR embeddable, see [14].

Consequently, for smooth 3-dimensional CR structures, those which are locally embeddable form a *thin set* (in the sense of Baire category) in the space of all such structures. So we have the question: What sort of conditions (aside from real analyticity) can be imposed on a sufficiently smooth 3-dimensional CR structure in order to guarantee local CR embedding? In this paper, among other things, we show how imposing the vacuum Einstein equations on an associated space time can single out elements of this thin set.

***1.2. The dual formulation.*** Since we have $Z \wedge \bar{Z} \neq 0$ we can supplement these two complex vector fields by one *real* vector field $Z_0$ on $M$ so that the triple $(Z_0, Z, \bar{Z})$ constitutes a basis for complex vector fields on $M$. Associated to the basis $(Z_0, Z, \bar{Z})$ there is its dual basis of 1-forms $(\lambda, \mu, \bar{\mu})$ on $M$ satisfying

$$Z_0 \lrcorner \lambda = 1, \quad Z \lrcorner \mu = 1, \quad \bar{Z} \lrcorner \bar{\mu} = 1,$$

all other contractions being identically equal to zero. Note that the form $\lambda$ is real valued.

The forms $(\lambda, \mu, \bar{\mu})$ are determined by the CR structure $(M, \mathrm{T}^{1,0})$ up to the following transformations:

$$\lambda \mapsto \lambda' = f\lambda, \quad \mu \mapsto \mu' = h\mu + p\lambda, \quad \bar{\mu} \mapsto \bar{\mu}' = \bar{h}\bar{\mu} + \bar{p}\lambda, \tag{1.3}$$

where $f \neq 0$ (real) and $h \neq 0, p$ (complex) are functions on $M$.

It is obvious that the differential equation

$$d\zeta \wedge \lambda \wedge \mu = 0$$

for a complex valued function $\zeta$ on $M$ is invariant under the transformations (1.3). It is the dual version of the tangential CR equation (1.2) and its solutions are just CR functions.

The Levi form of a 3-dimensional CR structure $(M, \mathrm{T}^{1,0})$ at a point is a nonvanishing multiple of the value of the real valued function $\omega$ defined by

$$\lambda' \wedge d\lambda' = i\omega\lambda' \wedge \mu' \wedge \bar{\mu}'. \tag{1.4}$$

Thus the CR structure is strictly pseudoconvex iff

$$\lambda \wedge \mathrm{d}\lambda \neq 0$$

on $M$. (Note that this statement is independent of the choice of the representative $\lambda'$ in (1.3).)

This enables us to formulate an equivalent definition of a 3-dimensional CR structure, the original one, that was actually used by Elie Cartan [4]:

***Definition 1.1.*** A 3-dimensional CR structure is a 3-dimensional real manifold $M$ equipped with an equivalence class of pairs of 1-forms $(\lambda, \mu)$ such that:

- $\lambda$ is real-valued, $\mu$ is complex-valued;
- $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ at each point of $M$;
- two pairs $(\lambda, \mu)$ and $(\lambda', \mu')$ are in the same class iff there exists functions $f$, $h$, $p$ such that (1.3) holds.

We will use this definition in the following and, rather than $(M, \mathrm{T}^{(1,0)})$, we will write $(M, (\lambda, \mu))$ to stress that a CR structure is associated with a class $[(\lambda, \mu)]$.

***1.3. Lifting CR manifolds to Lorentzian spacetimes.*** Three dimensional CR structures are very closely related to the so-called congruences[1] of null geodesics without shear in a spacetime [36, 45, 52]. These are well known in general relativity theory and proved to be very useful in the process of constructing nontrivial solutions to the vacuum Einstein equations in 4-dimensional manifolds equipped with Lorentzian metrics.

Given a 3-dimensional CR manifold $M$, we consider a representative $(\lambda, \mu)$ of the class $[(\lambda, \mu)]$ defining the CR structure on it. On the Cartesian product $\mathcal{M} = M \times \mathbb{R}$ we have a distinguished field of directions $k$, which is tangent to the $\mathbb{R}$ factor in $\mathcal{M}$. The four manifold $\mathcal{M} = M \times \mathbb{R}$ naturally projects onto $M$ with a projection $\pi : \mathcal{M} \to M$ and $\pi_*(k) = 0$. We choose a coordinate $r$ along the $\mathbb{R}$ factor, so that that $k$ may be represented by $k = \partial_r$. Omitting the pullbacks $\pi^*$ when expressing the forms on $\mathcal{M}$, i.e. for example, denoting by $\mu$ the pullback $\pi^*(\mu)$, we equip $\mathcal{M}$ with a class of metrics [45, 46, 56]

$$g = 2P^2 \, [\, \mu\bar{\mu} + \lambda(\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda) \,], \tag{1.5}$$

where $P \neq 0, H$ (real) and $W$ (complex) are arbitrary functions on $\mathcal{M}$; the expressions like e.g. $\mu\bar{\mu}$ denote the symmetrized tensor product: $\mu\bar{\mu} = \frac{1}{2}(\mu \otimes \bar{\mu} + \bar{\mu} \otimes \mu)$. We note that that the coordinate $r$ has no geometrical meaning; it can be replaced by any other function $r' = r'(r, x, y, z)$ such that $\partial r'/\partial r \neq 0$.

Now we consider the entire *class* of metrics (1.5), which depends on arbitrary functions $P$, $W$, $H$ and the class of coordinates $r'$. We claim that this *class* is

[1] In mathematical language the physicists' term 'congruence' means: 'foliation of a manifold by curves'; in this particular situation it means: 'foliation by means of a three parameter family of null and shearfree geodesics'.

*naturally* attached to the CR structure $(M,(\lambda,\mu))$. To see this, start with another representative $(\lambda',\mu')$ of the class $[(\lambda,\mu)])$. The new forms $(\lambda',\mu')$ are related to the previous choice $(\lambda,\mu)$ via (1.3). Now, maintaining the same $P$, $W$, $H$ and $r$, write a metric $g'$ using formula (1.5) with $(\lambda,\mu,\bar\mu)$ replaced by $(\lambda',\mu',\bar\mu')$. Using definitions (1.3) the metric $g'$ can be reexpressed in terms of the original forms $(\lambda,\mu,\bar\mu)$. A short calculation shows that after this, $g'$ has again form (1.5) with merely the functions $P,W,H$ and the coordinate $r$ being changed. Thus with each 3-dimensional CR structure $(M,(\lambda,\mu))$ there is an associated 4-dimensional manifold $\mathcal{M}=M\times\mathbb{R}$ with a class of Lorentzian metrics $g$ as in (1.5). Now we can compare $g$ and $g'$. It follows that there exists a nonvanishing real function $\alpha$ and a 1-form $\varphi$ on $\mathcal{M}$ such that

$$g'=\alpha^2 g+2g(k)\varphi. \tag{1.6}$$

Here $g(k)$ is the 1-form on $\mathcal{M}$ such that $X\lrcorner\, g(k)=g(k,X)$. The class of Lorentzian 4-dimensional metrics $[g]$ with the equivalence relation $g\sim g'$ iff $g$ and $g'$ are related by (1.5), (1.6) is called the class of metrics *adapted* to the CR structure $(M,(\lambda,\mu))$.

In each of the metrics from this class the lines tangent to the integral curves of the vector field $k=\partial_r$, are *null*. They have the further property of satisfying

$$\mathcal{L}_k g=\Theta g+2g(k)\vartheta, \tag{1.7}$$

with a real function $\Theta$, the *expansion*, and a real 1-form $\vartheta$ on $\mathcal{M}$. This equation in particular means that the integral curves of $k$ are *geodesics*. It also implies that the curves are *shearfree*, meaning that they preserve the natural conformal metric defined by the class $[g]$ in the quotient space $k^\perp/k$.

In the traditional language of physicsts $\mathcal{M}=M\times\mathbb{R}$ is equipped with a *congruence* of null and shearfree geodesics tangent to $k$. Physicists say that this congruence is *diverging* at a point of $\mathcal{M}$ iff the expansion $\Theta$ in (1.7) is nonvanishing at this point.

The leaf space of integral curves of the congruence generated by $k$ can be identified with $M$. The property of the congruence of being null geodesic and shearfree means precisely that the 3-dimensional leaf space $M$ of its integral curves has an abstract CR structure.

The above described procedure of associating a metric $g$ from $[g]$, to a 3-dimensional CR structure $(M,(\lambda,\mu))$, will be called a *lift of the CR structure to a spacetime* [44, 45, 56].

Given a lift of a CR structure $(M,(\lambda,\mu))$ to a spacetime, we now briefly define two concepts, which are important for the formulation of our main results. More detailed definitions are to be found in Section 3.1 and Section 3.3.1, respectively.

The first notion (see the end of Section 3.1) is defined as follows:

Having chosen a representative $(\lambda,\mu)$ of $[(\lambda,\mu)]$ we pull it back to $\mathcal{M}=\mathbb{R}\times M$ by $\pi^*$. Then we observe that the 2-dimensional complex distribution $N$, consisting of complex vector fields $Y$ on $\mathcal{M}$ satisfying $Y \lrcorner \pi^*(\lambda \wedge \mu) = 0$ is well defined. This is because the 2-forms $\pi^*(\lambda \wedge \mu)$ and $\pi^*(\lambda' \wedge \mu')$ corresponding to different representatives of the class $[(\lambda, \mu)]$ merely differ by the scale of a nonvanishing complex function. The distribution $N$ is called the *distribution of $\alpha$ planes* associated with the CR structure $(M, (\lambda, \mu))$ in the lifted spacetime $\mathcal{M}$.

The second notion we define here (again skipping the details to Section 3.3.1, which includes a beatiful example due to Ivor Robinson) is a *null Maxwell field aligned with the congruence*. Sometimes we will use the term 'a *null aligned Maxwell field*', for short. Suppose that in the class of forms $[(\lambda, \mu)]$ there is a pair $(\lambda', \mu')$ with the property that $\mathrm{d}(\pi^*(\lambda' \wedge \mu')) = 0$. Then we say that the CR structure $(M, (\lambda, \mu))$ admits a null aligned Maxwell field $\mathcal{F} = \pi^*(\lambda' \wedge \mu')$.

***1.4. From congruences of shearfree and null geodesics to CR manifolds.*** The fact that any 3-dimensional CR manifold $(M, (\lambda, \mu))$ locally defines a class of Lorentzian metrics (1.5) on $M \times \mathbb{R}$ in which the lines tangent to the $\mathbb{R}$ factor are null and shearfree geodesics has also its converse. This is given by the following theorem [36, 44, 45, 49, 52, 56].

***Theorem 1.2.*** *Let $(\mathcal{M}, g)$ be a 4-dimensional manifold equipped with a Lorentzian metric. Suppose that $\mathcal{M}$ is foliated by a 3-parameter family of curves which are null geodesics without shear. Then $\mathcal{M}$ is locally a cartesian product $\mathcal{M} = M \times \mathbb{R}$ with $M$ being a 3-dimensional CR manifold. The CR structure $(M, (\lambda, \mu))$ on $M$ is uniquely determined by $(\mathcal{M}, g)$ and the shearfree congruence on $\mathcal{M}$. If $r$ is a real coordinate such that $k = \partial_r$ is tangent to the congruence, then the Lorentzian metric $g$ on $\mathcal{M}$ can be locally represented by (1.5) with some specific functions $P, W, H$ depending on the choice of the representatives $(\lambda, \mu)$ of the corresponding CR structure.*

It should be noticed that 4-dimensional Lorentzian metrics of the form (1.5) have been studied by physicists since the late 1950s. In particular physicists found a lot of examples of such metrics which satisfy Einstein's equations $\mathrm{Ric}(g) = \Lambda g$. Among them is the Schwarzschild solution (the corresponding CR manifold is Levi flat), the Taub-NUT solution (the corresponding CR manifold is almost everywhere CR equivalent to the Heisenberg group CR structure), the Kerr rotating black hole solution (the corresponding CR manifold is almost everywhere strictly pseudoconvex and has only a 2-dimensional group of local CR symmetries; so it is not CR equivalent to the Heisenberg group CR structure), and many others (see [15], sections devoted to algebraically special solutions, for huge families of Einstein examples with various CR structures). Physicists were looking for the Einstein solutions among the metrics (1.5) because such metrics were believed to correspond to gravitational radiation. Although the understanding of the mathematical fact that there is a CR structure behind the scenes came much later, physicists were from the very begining aware that radiative Maxwell or Einstein fields impose a sort of a complex structure in the underlying spacetimes. The notion of a CR structure was implicit in such papers as [6, 9, 17, 28, 34, 35, 40–43, 47–50, 54, 55], but physicists did not manage to abstract the concept for about twenty years

[50, 52, 56]. Ironically, at about the same time when the systematic work on gravitational radiation started, the notion of a CR structure was being revived in mathematics, due to the discovery of the nonsolvability of equations of Hans Lewy type [25]. Mathematicians were however unaware of the developement in general relativity theory and also did not make the connection. We hope that this paper fills the gap between these two areas of mathematics and physics.

Our main motivation is the paper [22] and the research on relations between 3-dimensional CR structures and algebraically special gravitational fields undertaken by the Warsaw Relativity Group in the 1980s [18–21, 23, 29, 30, 32, 44–46, 52, 53, 56, 57]. We are also inspired and impressed by the works of relativists on gravitational radiation; in particular by the contributions of Andrzej Trautman, Ivor Robinson, Roger Penrose, Ray Sachs, Roy Kerr, Ted Newman and Jacek Tafel.

## 2. LOCAL CR EMBEDDABILITY THEOREMS

In the sequel we assume our CR structures have a sufficiently high *finite* order of differentiability, in particular they need *not* be real analytic. All considerations are *local* on $M$.

***Theorem 2.1.*** *Let $M$ be a sufficiently smooth 3-dimensional CR manifold. If it has a lift to a spacetime whose congruence of null and shearfree geodesics is diverging over the points where the Levi form vanishes, and whose complexified Ricci tensor vanishes on the distribution of $\alpha$ planes associated with the congruence, then $M$ admits a CR function $\zeta$ such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$ is nowhere zero.*

In particular Theorem 2.1 applies to the strictly pseudoconvex case. Thus it rules out the generic situation, like the example of Nirenberg, in which all CR functions are locally constant.

Actually, in the strictly pseudoconvex case, we have a stronger theorem.

***Theorem 2.2.*** *Let $M$ be a sufficiently smooth strictly pseudoconvex 3-dimensional CR manifold. Then the following two conditions are equivalent.*

(i) *$M$ admits a lift to a spacetime whose complexified Ricci tensor vanishes on the corresponding distribution of $\alpha$ planes.*
(ii) *$M$ admits one CR function $\zeta$ such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}$ is nowhere zero.*

Our next result gives an if and only if criterion for the *local embeddability* of a sufficiently smooth, *not* necessarily real analytic, strictly pseudoconvex 3-dimensional CR manifold:

***Theorem 2.3.*** *Let $M$ be a sufficiently smooth strictly pseudoconvex 3-dimensional CR manifold. It is locally CR embeddable as a hypersurface in $\mathbb{C}^2$ if and only if:*

(i) *it admits a lift to a spacetime whose complexified Ricci tensor vanishes on the corresponding distribution of $\alpha$ planes, and*
(ii) *it admits a nontrivial null Maxwell field aligned with the null congruence of shearfree geodesics corresponding to the CR structure on $M$.*

Next we abandon the requirement about the existence of a null Maxwell field and replace it by further assumptions about the curvature of the lifted spacetime. This leads to a remarkable theorem.

***Theorem 2.4.*** *Let $M$ be a sufficiently smooth strictly pseudoconvex 3-dimensional CR manifold. If $M$ admits a lift to a Ricci flat spacetime which has locally constant Petrov type, then it is locally CR embeddable as a hypersurface in $\mathbb{C}^2$.*

The hypothesis of *locally constant Petrov type* is a technical assumption, which will be explained in detail in Section 3.1. Here we only mention that according to the theory of exact solutions of Einstein equations, a spacetime metric at a point can have one of the six Petrov types [3, 34, 38]. These are: Petrov types $I$, $II$, $D$, $III$, $N$ or 0, and they may change from point to point in the spacetime. With this information we may formulate the following theorem.

***Theorem 2.5.*** *Let $(M, (\lambda, \mu))$ be a real analytic strictly pseudoconvex 3-dimensional CR manifold. Then it always has a lift to a spacetime satsifying the Einstein equations* $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$ *whose Petrov type is $II$ or $D$.*

Here $\Lambda$ is the *cosmological constant* and the function $\Phi$ corresponds to the energy momentum tensor of *pure radiation*. We believe that this theorem is also true when we replace the term 'real analytic' with 'sufficiently smooth embeddable' (see Remark 3.25).

The proofs of the above theorems will emerge in the discussion which follows. Actually the theorems stated above, constitute only a selection of the results we prove in the paper. In special cases, which are systematically studied in the main body of the paper, we obtain sharper results than stated here.

We close this section with a remark about the nontriviality of Theorems 2.1 and 2.4. As was already mentioned after Theorem 1.2 both theorems are far from being empty. There is an abundance of Ricci flat Lorentzian 4-metrics which admit a congruence of null and shearfree geodesics. The encyclopedia book [15] gives an up to date catalog of such metrics in the sections devoted to algebraically special vacuum solutions (Sections 26 through 30 in the second edition). Every Ricci flat metric in these sections of the book has a corresponding 3-dimensional CR structure. This may be Levi flat everywhere (as is the case for the Schwarzschild metric) or strictly pseudoconvex in 3-dimensional regions and Levi flat on some lower dimmensional sets as in the following example:

We consider the metric

$$g = 2\Big(\mathcal{P}^2\mu\bar{\mu} + \lambda(\mathrm{d}r' + \mathcal{W}\mu + \bar{\mathcal{W}}\bar{\mu} + \mathcal{H}\lambda)\Big),$$

where

$$\lambda = \mathrm{d}u + \frac{i(2b + (a+b)\zeta\bar{\zeta})}{\zeta\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^2}\mathrm{d}\zeta - \frac{i(2b + (a+b)\zeta\bar{\zeta})}{\bar{\zeta}\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^2}\mathrm{d}\bar{\zeta}, \quad \mu = \mathrm{d}\zeta,$$

$$\mathcal{P}^2 = \frac{r'^2}{\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^2} + \frac{\left(b - a + (b+a)\frac{\zeta\bar{\zeta}}{2}\right)^2}{\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^4},$$

$$\mathcal{W} = \frac{ia\bar{\zeta}}{\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^2}, \quad \mathcal{H} = -\frac{1}{2} + \frac{mr' + b^2 - ab\dfrac{1 - \frac{\zeta\bar{\zeta}}{2}}{1 + \frac{\zeta\bar{\zeta}}{2}}}{r'^2 + \dfrac{\left(b - a + (b+a)\frac{\zeta\bar{\zeta}}{2}\right)^2}{\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^2}},$$

and $m$, $a$, $b$ are *real* constants.

Clearly this metric is in the form (1.5), and as such may be considered as the lift of a CR structure $(\lambda, \mu)$ which is defined on the 3-dimensional manifold $M$ parametrized by $(u, \mathrm{Re}(\zeta), \mathrm{Im}(\zeta))$.

The interesting feature of this scarry-looking 3-parameter family of metrics is that it is *Ricci flat* for all values of the coordinates $(u, \mathrm{Re}(\zeta), \mathrm{Im}(\zeta), r')$ in which $g$ is not singular [15]. Actually if $b = 0$ the above metric is just the Kerr rotating black hole with mass $m$ and the angular momentum parameter $a$; if $a = b = 0$ the metric $g$ describes the Schwarzsschild black hole with mass $m$. If $m = a = 0$ the corresponding metric is the Taub-NUT vacuum metric.

Calculating $\lambda \wedge \mathrm{d}\lambda$, we get

$$\lambda \wedge \mathrm{d}\lambda = i\frac{(a+b)\zeta\bar{\zeta} - 2(a-b)}{\left(1 + \frac{\zeta\bar{\zeta}}{2}\right)^3}\mathrm{d}u \wedge \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}.$$

This means that for each value of the three parameters $(m, a, b)$ the corresponding CR structure $(M, (\lambda, \mu))$ is strictly pseudoconvex everywhere except the points where

$$(a+b)\zeta\bar{\zeta} - 2(a-b) = 0.$$

Note that if $a > b$ and $a \neq -b$, there is an entire cylinder

$$\zeta\bar{\zeta} = \frac{2(a-b)}{a+b}$$

in $M$ on which $\lambda \wedge d\lambda = 0$. In such case the corresponding CR structure is Levi flat on this cylinder and strictly pseudoconvex outside it. A short calculation shows that on this cylinder the shearfree congruence of null geodesics tangent to $k = \partial_{r'}$ is diverging everywhere. So this case is a nontrivial example of a metric which appears in Theorem 2.1. Many more such examples can be found in the appropriate sections of [15].

## 3. The Einstein equations and CR functions

***3.1. The first CR function.*** Here we prove Theorem 2.1 by adapting the argument presented in [15].

We consider a general 4-dimensional spacetime $\mathcal{M}$, i.e. a 4-manifold equipped with a metric $g$ of Lorentzian signature $(+,+,+,-)$. We assume that the spacetime $\mathcal{M}$ admits a null congruence of shearfree and null geodesics. This may be described as follows:

The congruence is tangent to a nonvanishing vector field $k$ which is *null*, $g(k,k) = 0$. Having $k$, we introduce a coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ on $\mathcal{M}$ such that

$$g = 2(\theta^1\theta^2 + \theta^3\theta^4), \quad \theta^3 = g(k,\cdot), \quad \text{and} \quad k \lrcorner\, \theta^1 = k \lrcorner\, \theta^2 = 0. \tag{3.1}$$

Note that, due to the signature of the metric $g$, this definition implies that the 1-forms $\theta^3$, $\theta^4$ are *real* valued, whereas the 1-forms $\theta^1$, $\theta^2$ are *complex* valued with $\bar{\theta}^2 = \theta^1$. Note also that the coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ is *not* uniquely defined by (3.1). It is given up to a linear transformation associated with a 4-dimensional parabolic subgroup of the Lorentz group preserving the null direction $k$. Explicitly:

$$\begin{aligned}
\theta^{1\prime} &= e^{i\varphi}(\theta^1 + \bar{B}\theta^3)\\
\theta^{2\prime} &= e^{-i\varphi}(\theta^2 + B\theta^3)\\
\theta^{3\prime} &= A\theta^3\\
\theta^{4\prime} &= A^{-1}(\theta^4 - B\theta^1 - \bar{B}\theta^2 - B\bar{B}\theta^3),
\end{aligned} \tag{3.2}$$

where $A \neq 0$, $\varphi$ are real functions, and $B$ is a complex function. The coframes (3.2) are said to be *adapted* to $k$.

Imagine now that we have $k$, which in some adapted coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ satisfies:

$$\begin{aligned}
d\theta^3 \wedge \theta^1 \wedge \theta^3 &= -\bar{\kappa}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4\\
d\theta^1 \wedge \theta^1 \wedge \theta^3 &= -\bar{\sigma}\, \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4,
\end{aligned} \tag{3.3}$$

with some complex functions $\kappa$ and $\sigma$. Then, it follows that in any other adapted coframe (3.2) we have

$$\begin{aligned} d\theta^{3'} \wedge \theta^{1'} \wedge \theta^{3'} &= -A^2 e^{i\varphi} \bar{\kappa}\, \theta^{1'} \wedge \theta^{2'} \wedge \theta^{3'} \wedge \theta^{4'} \\ d\theta^{1'} \wedge \theta^{1'} \wedge \theta^{3'} &= -A e^{2i\varphi} (\bar{B}\bar{\kappa} + \bar{\sigma})\, \theta^{1'} \wedge \theta^{2'} \wedge \theta^{3'} \wedge \theta^{4'}, \end{aligned}$$

as can be easily checked. This means that the *simultaneous vanishing* or *not* of both coefficients $\kappa$, $\sigma$ at a point, is independent of the choice of the adapted coframe, and thus is a property of a null congruence associated with $k$. If

$$\kappa = \sigma = 0 \tag{3.4}$$

everywhere, the null congruence associated with $k$ is called a *shearfree* congruence of null *geodesics*. We mention in addition that the vanishing of $\kappa$ alone at a point, is also an invariant property of the congruence. If we assume nothing about $\sigma$ but require that $\kappa = 0$ everywhere, such a congruence turns out to consist of null *geodesics*, which may or may not be shearfree.

It is also worthwhile to look at the transformations of $d\theta^3 \wedge \theta^3$. To be consistent with (3.3) we write it as:

$$d\theta^3 \wedge \theta^3 = i\Omega\theta^1 \wedge \theta^2 \wedge \theta^3 - (\kappa\theta^1 + \bar{\kappa}\theta^2) \wedge \theta^3 \wedge \theta^4, \tag{3.5}$$

with a real function $\Omega$ on $\mathcal{M}$. Changing the adapted coframe to (3.2) we get:

$$d\theta^{3'} \wedge \theta^{3'} = A(\bar{B}\kappa - B\bar{\kappa} + i\Omega)\theta^{1'} \wedge \theta^{2'} \wedge \theta^{3'} - A^2(e^{-i\varphi}\kappa\theta^{1'} + e^{i\varphi}\bar{\kappa}\theta^{2'}) \wedge \theta^{3'} \wedge \theta^{4'}.$$

In the case of a geodesic null congruence, $\kappa \equiv 0$, these equations reduce respectively to:

$$d\theta^3 \wedge \theta^3 = i\Omega\theta^1 \wedge \theta^2 \wedge \theta^3 \tag{3.6}$$

and

$$d\theta^{3'} \wedge \theta^{3'} = iA\Omega\theta^{1'} \wedge \theta^{2'} \wedge \theta^{3'}.$$

This proves that in such case the vanishing or not of $\Omega$ at a point is also an invariant property of $k$, and thus, of the congruence. Consider a *null* and *geodesic* congruence at a point $x \in \mathcal{M}$. If

$$\Omega \neq 0$$

at $x$, we say that the congruence is *twisting* there. If

$$\Omega = 0$$

at $x$, we say that the congruence is *not twisting* at $x$.

From now on we assume that we have $k$ with $\kappa = \sigma = 0$ everywhere, i.e. that we have a null congruence of shearfree geodesics in $\mathcal{M}$. Our next step is to give the geometric interpretation of (3.4).

Choosing an adapted coframe (3.1), using Cartan's formula $\mathcal{L}_k\theta = k \lrcorner \mathrm{d}\theta + \mathrm{d}(k \lrcorner \theta)$ for the Lie derivative of a 1-form $\theta$, and the respective equations (3.6), $(3.3)_2$ with $\sigma = 0$, we easily get

$$(\mathcal{L}_k\theta^3) \wedge \theta^3 = 0 \quad \text{and} \quad (\mathcal{L}_k\theta^1) \wedge \theta^1 \wedge \theta^3 = 0,$$

everywhere in $\mathcal{M}$. The meaning of these two equations is obvious: the real 1-form $\theta^3$, when Lie transported by the flow $\varphi_t$ generated by the congruence, transforms as $\varphi_t^*(\theta^3) = f\theta^3$, where $f$ is a real function on $I \times \mathcal{M}$, $t \in I$; the complex 1-form $\theta^1$ transforms as $\varphi_t^*(\theta^1) = h\theta^1 + p\theta^3$, where $h$, $p$ are complex functions on $I \times \mathcal{M}$. Since $\varphi_t$ is a local diffeomorphism, the functions $f$ and $h$ are locally nonvanishing.

Now, taking any hypersurface $M$ in $\mathcal{M}$ transversal to $k$, we equip it with a CR structure $(\lambda', \mu', \bar{\mu}')$ as in (1.3) by setting

$$\lambda' = \iota^*(\theta^3), \quad \mu' = \iota^*(\theta^1), \quad \bar{\mu}' = \iota^*(\theta^2).$$

Here $\iota : M \to \mathcal{M}$ is the natural inclusion of $M$ in $\mathcal{M}$, so that $\iota^*\theta$ is just the restriction of $\theta$ to $M$. The above discussed changes of $\theta^3$ and $\theta^1$ along $k$ imply that the CR structures on any two transversal hypersurfaces are CR equivalent. The pseudoconvexity property of these CR structures is easily described by means of equation (3.6). Indeed, pulling back this equation by means of $\iota^*$, from the spacetime to $M$, we get

$$\mathrm{d}\lambda' \wedge \lambda' = i\iota^*(\Omega)\mu' \wedge \bar{\mu}' \wedge \lambda'.$$

This means that the Levi form $\omega$, as defined by (1.4), is $\omega = \iota^*(\Omega)$. Thus the CR structure $(\lambda', \mu', \bar{\mu}')$ has a nonvanishing Levi form $\omega \neq 0$ at $p \in M$ iff the unique congruence curve passing through $p$ is twisting at $\iota(p)$.

Summarizing we have the following lemma [45].

***Lemma 3.1.*** *The 3-dimensional leaf space of a null congruence of shearfree geodesics in spacetime is locally a CR manifold. This CR manifold is strictly pseudoconvex at a point iff the congruence curve is twisting at the corresponding point in spacetime.*

It is convenient to choose a real function $r$ on $\mathcal{M}$, so that $k = P^{-1}\partial_r$, where $P \neq 0$ is a real function on $\mathcal{M}$. Then our Lemma guarantees that the adapted coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ can be chosen in such a way that

$$\begin{aligned} \theta^1 &= P\mu \\ \theta^2 &= P\bar{\mu} \\ \theta^3 &= P\lambda \\ \theta^4 &= P(\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda), \end{aligned} \tag{3.7}$$

where locally $\mathcal{M} = \mathbb{R} \times M$, $\lambda, \mu$ are the 1-forms on $M$ which define the CR structure there, and the functions $H$ (real) and $W$ (complex) are functions on $\mathcal{M}$. This in particular means that in addition to $k \lrcorner \lambda = k \lrcorner \mu = 0$ we also have $k \lrcorner \mathrm{d}\lambda = k \lrcorner \mathrm{d}\mu = 0$. Thus we just demonstrated how a shearfree congruence of null geodesics in spacetime restricts the spacetime metric to the form (1.5). As we already mentioned in Section 1.3, we also have the statement in the opposite direction: given a 3-dimensional CR structure represented on $M$ via forms $(\lambda, \mu, \bar{\mu})$ we lift it to a spacetime $\mathcal{M} = \mathbb{R} \times M$ with metric (1.5) and with a null congruence of shearfree geodesics represented by $k = P^{-1}\partial_r$.

Whether we start with a 3-dimensional CR structure and then define the spacetime with a null congruence of shearfree geodesics, or immediately start with a spacetime with such congruence, we may try to impose some curvature conditions on $g$. The question arises if these conditions say something about the underlying CR geometry.

To study this question we consider Cartan's structure equations for the metric (1.5), written in an adapted coframe (3.7). These are:

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^k = 0 \tag{3.8}$$

$$\mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \tfrac{1}{2} R^i{}_{jk\ell}\theta^k \wedge \theta^\ell. \tag{3.9}$$

Here the 1-forms $\Gamma^i{}_j$ are the Levi-Civita connection 1-forms. They define $\Gamma_{ij} = g_{ik}\Gamma^k{}_j$, which satisfy $\Gamma_{ij} = -\Gamma_{ji}$. Modulo the symmetry the only nonzero components of the metric are $g_{12} = g_{34} = 1$. Its inverse $g^{ij}$, again modulo symmetry, has $g^{12} = g^{34} = 1$ as the only nonvanishing components. The coeffcients $R^i{}_{jk\ell}$ are the Riemann tensor coeffcients. The Ricci tensor is defined as $R_{ij} = R^k{}_{ikj}$. Using the metric $g_{ij}$ we also define $R_{ijk\ell} = g_{im}R^m{}_{jk\ell}$. This can be used to define the covariant components of the Weyl tensor $C^i{}_{jk\ell}$ via

$$\begin{aligned} C_{ijk\ell} = R_{ijk\ell} &+ \tfrac{1}{6}R(g_{ik}g_{lj} - g_{il}g_{kj}) \\ &+ \tfrac{1}{2}(g_{il}R_{kj} - g_{ik}R_{lj} + g_{jk}R_{li} - g_{j\ell}R_{ki}). \end{aligned} \tag{3.10}$$

Here $R = R_{ij}g^{ij}$ is the Ricci scalar. The Weyl tensor $C^i_{jk\ell} = g^{im}C_{mjk\ell}$ carries the conformal information about the spacetime.

***Remark 3.2.*** Note that, due to the reality conditions: $\bar\theta^1 = \theta^2$, $\bar\theta^3 = \theta^3$, $\bar\theta^4 = \theta^4$ we have $\bar R_{13} = R_{23}$, $\bar R_{33} = R_{33}$, $\bar\Gamma_{14} = \Gamma_{24}$, etc. This means that complex conjugation interchanges the indices $1 \leftrightarrow 2$ and leaves the indices 3 and 4 unchanged.

Now, since $\kappa = \sigma = 0$, the connection 1-form $\Gamma_{24}$ satisfying Cartan's first structure equation (3.8) and, being compatible with our conventions (3.3) and (3.5), must be a linear combination of $\theta^1$ and $\theta^3$ only:

$$\Gamma_{24} = -\bar\rho\theta^1 - \bar\tau\theta^3. \tag{3.11}$$

This equation defines complex functions $\rho$ and $\tau$. Our conventions do not give any restrictions on $\tau$. On the other hand, to be compatible with (1.7) and (3.6) the function $\rho$ has to assume the form

$$\rho = \tfrac{1}{2}(-\Theta + i\Omega),$$

with real $\Omega$ being the twist, and real $\Theta$ being the expansion of the congruence.

Now we pass to the curvature analysis. The first option is to impose the Einstein equations

$$R_{ij} = \Lambda g_{ij}, \quad i,j = 1,2,3,4 \tag{3.12}$$

on $g$. These equations conveniently split into three types of equations [15]:

(a) $R_{22} = R_{24} = R_{44} = 0$,
(b) $R_{12} = R_{34} (= \Lambda)$,
(c) $R_{33} = R_{23} = 0$.

According to Remark 3.2 the set (a) consists of five real equations ($R_{44}$ is real!), the set (b) consists of two real equations and the set (c) consists of three real equations.

To proceed with the proof of Theorem 2.1 we now focus our attention on equations (a).

First, using the definition of the Ricci tensor, we write equations (a) in terms of the Riemann tensor coefficients. A short calculation shows that, modulo the symmetries of the Riemann tensor, they are equivalent to:

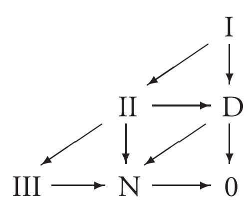
$$\begin{aligned} R_{44} = 0 \quad &\Leftrightarrow \quad R_{2414} = 0\\ R_{24} = 0 \quad &\Leftrightarrow \quad R_{2412} - R_{2434} = 0\\ R_{22} = 0 \quad &\Leftrightarrow \quad R_{2423} = 0. \end{aligned}$$

Second, we invoke the celebrated theorem due to Goldberg and Sachs [9]. We will need this theorem in a very technical version. Before giving its formulation suitable for our purposes it is worthwhile to present its original meaning.

As noted by Cartan [3], every 4-dimensional spacetime $\mathcal{M}$ distinguishes at each point at most *four* null directions - the *principal null directions*, as they are nowadays called. If at a point $p \in \mathcal{M}$ the Weyl tensor of the metric is not zero, then the number $s$ of distinct directions is $1 \leq s \leq 4$. The number $s$ at a point depends on the Weyl tensor in an algebraic fashion: it is the number of distinct roots of a certain fourth order complex polynomial associated with the Weyl tensor. If $s = 4$ at $p \in \mathcal{M}$ the spacetime is called *algebraically general* at $p$. If $1 \leq s \leq 3$, the spacetime is called *algebraically special* at $p$. If $1 \leq s \leq 3$, then at least two of Cartan's principal null directions coincide. The coinciding principal null directions are called *multiple* principal null directions.

All the possible degeneracies of the principal null directions at a point can be enumerated by the possible partitions of the number *four*. Thus we have cases [1111], [112], [22], [13], [4]. The algebraically general case corresponds to [1111]. The case where there is only one doubly degenerate principal null direction corresponds to [112]. The case with two doubly degenerate principal null direction coresponds to [22], and so on. This classification of points in a spacetime, is known as the *Petrov types* [38], although it was known to Elie Cartan [3], and was brought to its contemporary form by Roger Penrose [34]. In Penrose's formulation it reads as follows:

(i) Petrov type I ('non-degenerate') [1111],
(ii) Petrov type II [112],
(iii) Petrov type III [13],
(iv) Petrov type D ('degenerate') [22],
(v) Petrov type N ('null') [4].




The 0 in the diagram above, the *Penrose diagram* as it is called, represents a vanishing Weyl tensor at a point. The arrows point towards more special cases.

A convenient way to determine the Petrov type is to calculate the *Weyl scalars* $\Psi_0$, $\Psi_1$, $\Psi_2$, $\Psi_3$, $\Psi_4$ at a point. These quantities are complex numbers at a point, which fully determine the 10 independent components of the Weyl tensor at this point. In a frame $(e_1, e_2, e_3, e_4) = (m, \bar{m}, \ell, k)$, dual to the coframe adapted to a null vector field $k$, they are defined by

$$\begin{aligned}
\Psi_0 &= C_{ijk\ell} k^i m^j k^k m^\ell = C_{4141} = R_{4141} \\
\Psi_1 &= C_{ijk\ell} k^i \ell^j k^k m^\ell = C_{4341} = \tfrac{1}{2}(R_{4341} + R_{1421}) \\
\Psi_2 &= C_{ijk\ell} k^i m^j \ell^k \bar{m}^\ell = C_{4132} \\
\Psi_3 &= C_{ijk\ell} \ell^i k^j \ell^k \bar{m}^\ell = C_{3432} = \tfrac{1}{2}(R_{3432} + R_{2312}) \\
\Psi_4 &= C_{ijk\ell} \ell^i \bar{m}^j \ell^k \bar{m}^\ell = C_{3232} = R_{3232}.
\end{aligned}$$

For the convenience of the reader, we we have used here formula (3.10), to reexpress $\Psi_0$, $\Psi_1$, $\Psi_3$ and $\Psi_4$ in terms of the Riemann tensor components.

The importance of the Weyl scalars for the Petrov classification consists in the following observations:

- $\Psi_0 = 0 \iff k$ is a principal null direction;
- $\Psi_0 = \Psi_1 = 0, \quad \Psi_2 \neq 0 \iff k$ has degeneracy [112] or [22];
- $\Psi_0 = \Psi_1 = \Psi_2 = 0, \quad \Psi_3 \neq 0 \iff k$ has degeneracy [13];
- $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \quad \Psi_4 \neq 0 \iff k$ has degeneracy [4];
- $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \iff$ Weyl tensor is zero.

We will use them in Sections 3.2 and 3.3.

In turns out, and this has been known for years by physicists, that if the spacetime admits a congruence of shearfree and null geodesics, then tangent vectors to the congruence at a point should be aligned with one of the principal null directions. The Goldberg-Sachs theorem says more [9]:

***Theorem 3.3*** (Goldberg-Sachs)**.** *Suppose that a 4-dimensional spacetime satisfies Einstein's equations* $\mathrm{Ric}(g) = \Lambda g$*. Then the following conditions are equivalent:*

(i) *The spacetime admits a null congruence of shearfree geodesics tangent to a vector field* $k$*.*

(ii) *The spacetime is algebraically special with a multiple principal null direction tangent to* $k$*.*

Algebraically special fields are important in physics because they are connected with what *gravitational radiation* could be. By this we mean the following.

Roughly speaking, if one observes a *general* gravitational field *far from the sources* and measures the 'distance' from the sources by means of $r > 0$, then studying the $r$-dependence of the Weyl tensor $C$ of the metric (which in the *empty* spacetime describes the gravitational field strength), he will discover a

$$C = \frac{N}{r} + \frac{III}{r^2} + \frac{II + D}{r^3} + \frac{I}{r^4} + O\left(\frac{1}{r^5}\right)$$

behavior, as $r \to \infty$.

Here $N$, $III$, $II$, $D$ and $I$ are tensorial quantities with all the symmetries of the Weyl tensor. They have the respective algebraic Petrov type denoted by the corresponding symbols. Thus, a *general* gravitational field far from the sources is of Petrov type $N$, as first observed in [54]. Approaching the sources, the field becomes less and less algebraically special. This is the so called *peeling property* of the gravitational field [47, 48]. It gives an *algebraic* criterion for a gravitational field to be 'radiative'.

Since the original paper of Goldberg and Sachs [9] the theorem was strengthend in various ways. In particular, it is known that to achieve the implication (i) $\Rightarrow$ (ii) in Theorem 3.3 the full set of the Einstein equations is not needed. One can weaken $\mathrm{Ric}(g) = \Lambda g$ to our equations (a) and the implication (i) $\Rightarrow$ (ii) in Theorem 3.3 is still true. Also the geometric condition about algebraic speciality can be reformulated directly in terms of the vanishing of certain components of the Weyl tensor, and in turn, of the Riemann tensor. Such a form of the theorem is needed for proving our Theorem 2.1. We quote it below [10] (see also Lemma 2.2 on p. 577 in [39]):

***Theorem 3.4.*** *Suppose that a 4-dimensional manifold $\mathcal{M}$ is equipped with a Lorentzian metric which in a null coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ has the form*

$$g = 2(\theta^1\theta^2 + \theta^3\theta^4),$$

*with $\bar\theta^1 = \theta^2$, $\bar\theta^3 = \theta^3$, $\bar\theta^4 = \theta^4$. Let $\kappa$ and $\sigma$ be given by (3.3).*

*Assume that the Ricci tensor of $g$, in the coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$, satisfies*

$$R_{22} = R_{24} = R_{44} = 0$$

*everywhere on $\mathcal{M}$. Then:*

(i) $\kappa = \sigma = 0$ *everywhere on $\mathcal{M}$*

*implies*

(ii) $\Psi_0 = \Psi_1 = 0$ *everywhere on $\mathcal{M}$.*

Third, using this theorem, and the definitions of $\Psi_0$ and $\Psi_1$, we conclude that if our metric (3.1), (3.7) satisfies (a), then, in our adapted coframe it has:

$$R_{2412} = R_{2424} = R_{2414} = R_{2423} = R_{2434} = 0, \tag{3.13}$$

everywhere on $\mathcal{M}$.

Fourth, we write down explicitly the second Cartan's structure equations for indices $\{24\}$:

$$\mathrm{d}\Gamma_{24} + (-\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{24} = \tfrac{1}{2} R_{24kl}\theta^k \wedge \theta^\ell.$$

Due to (3.13) the r.h.s. of the above equality has only one nonvanishing term:

$$\tfrac{1}{2}R_{24kl}\theta^k \wedge \theta^\ell = R_{2413}\theta^1 \wedge \theta^3.$$

Thus, assuming equations (a) we have an identity:

$$\mathrm{d}\Gamma_{24} + (-\Gamma_{12} + \Gamma_{34}) \wedge \Gamma_{24} - R_{2413}\theta^1 \wedge \theta^3 = 0.$$

Since $\Gamma_{24}$ as given by (3.11) is a linear combination of $\theta^1$ and $\theta^3$, after wedging this identity with $\Gamma_{24}$, we conclude that:

$$\mathrm{d}\Gamma_{24} \wedge \Gamma_{24} = 0 \tag{3.14}$$

everywhere on $\mathcal{M}$. A short calculation shows that

$$\Gamma_{24} \wedge \bar\Gamma_{24} = |\rho|^2\theta^1 \wedge \theta^2 + \bar\rho\tau\theta^1 \wedge \theta^3 - \rho\bar\tau\theta^2 \wedge \theta^3. \tag{3.15}$$

From now on we assume that

$$\rho \neq 0$$

at every point on $\mathcal{M}$. This is equivalent to saying that the congruence of null and shearfree geodesics is diverging at points where the associated CR structure has vanishing Levi form.

Our assumption about nonvanishing $\rho$, when compared with (3.15), implies that

$$\Gamma_{24} \wedge \bar{\Gamma}_{24} \neq 0 \tag{3.16}$$

at every point on $\mathcal{M}$.

Now we need the following lemma.

***Lemma 3.5.*** *Let $\varphi$ be a smooth complex valued 1-form defined locally in $\mathbb{R}^n$, $n \geq 3$, such that $\varphi \wedge \bar{\varphi} \neq 0$. Then*

$$\mathrm{d}\varphi \wedge \varphi \equiv 0 \quad \text{if and only if } \varphi = h\,\mathrm{d}\zeta\,,$$

*where $\zeta$ is a smooth complex function such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$, and $h$ is a smooth nonvanishing complex function.*

*Proof.* Consider an open set $U \in \mathbb{R}^n$ in which we have $\varphi$ such that $\mathrm{d}\varphi \wedge \varphi = 0$ and $\varphi \wedge \bar{\varphi} \neq 0$. We define *real* 1-forms $\varphi^1 = \mathrm{Re}(\varphi)$ and $\varphi^2 = \mathrm{Im}(\varphi)$. They satisfy $\varphi^1 \wedge \varphi^2 \neq 0$ in $U$. Our assumption $\mathrm{d}\varphi \wedge \varphi = 0$, when written in terms of the real 1-forms $\varphi^1$, $\varphi^2$ is:

$$\mathrm{d}\varphi^1 \wedge \varphi^1 - \mathrm{d}\varphi^2 \wedge \varphi^2 + i(\mathrm{d}\varphi^2 \wedge \varphi^1 + \mathrm{d}\varphi^1 \wedge \varphi^2) = 0.$$

Taking the real and imaginary parts we have

$$\begin{aligned} \mathrm{d}\varphi^1 \wedge \varphi^1 - \mathrm{d}\varphi^2 \wedge \varphi^2 &= 0 \\ \mathrm{d}\varphi^2 \wedge \varphi^1 + \mathrm{d}\varphi^1 \wedge \varphi^2 &= 0. \end{aligned}$$

Now the argument splits into two cases. For dimension $n \geq 4$ we wedge the first equality, and then the second equality with $\varphi^2$, and get

$$\mathrm{d}\varphi^1 \wedge \varphi^1 \wedge \varphi^2 = 0, \quad \mathrm{d}\varphi^2 \wedge \varphi^1 \wedge \varphi^2 = 0. \tag{3.17}$$

If $n = 3$ these equations are trivially satisfied. In whatever dimension $n \geq 3$ we are in, once we have noticed that (3.17) is true, we see that the *real* forms $\varphi^1$, $\varphi^2$ form a closed differential ideal. Thus we can use the *real* Fröbenius theorem, which implies that there exists a coordinate chart $(x, y, u^\nu)$, $\nu = 3, 4, \ldots n$, in $U$ such that $\varphi^1 = t_{11}\mathrm{d}x + t_{12}\mathrm{d}y$ and $\varphi^2 = t_{21}\mathrm{d}x + t_{22}\mathrm{d}y$, with some *real* functions $t_{ij}$ in $U$ such that $t_{11}t_{22} - t_{12}t_{21} \neq 0$. Thus in the coordinates $(x, y, u^\nu)$ the form $\varphi = \varphi^1 + i\varphi^2$ can be written as $\varphi = c_1 \mathrm{d}x + c_2 \mathrm{d}y$, where now $c_1$, $c_2$ are *complex* functions such that $c_1\bar{c}_2 - \bar{c}_1 c_2 \neq 0$ on $U$, so neither $c_1$ nor $c_2$ can be zero. The $\mathrm{d}\varphi \wedge \varphi \equiv 0$ condition for $\varphi$ written in this representation is simply $c_2^2 \mathrm{d}(c_1/c_2) \wedge \mathrm{d}x \wedge \mathrm{d}y \equiv 0$. Thus all the partial derivatives

$$\frac{\partial\left(\frac{c_1}{c_2}\right)}{\partial u^\nu} \equiv 0, \qquad \forall \nu = 3, 4, \dots, n.$$

This means that the ratio $c_1/c_2$ does not depend on $u^\nu$. This ratio defines a nonvanishing *complex* function $F(x,y) = c_1/c_2$ of only *two* real variables $x$ and $y$. Returning to $\varphi$ we see that it is of the form $\varphi = c_2(\mathrm{d}y + F(x,y)\mathrm{d}x)$. Consider the real bilinear symmetric form

$$G = 2\varphi\bar{\varphi} = |c_2|^2\Big(\mathrm{d}y^2 + 2\big(F(x,y) + \bar{F}(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y + \big|F(x,y)\big|^2\,\mathrm{d}x^2\Big).$$

Invoking the classical theorem on the existence of isothermal coordinates we are able to find an open set $U' \subset U$ with new coordinates $(\xi, \eta, u^\nu)$ in which $G = h^2(\mathrm{d}\xi^2 + \mathrm{d}\eta^2)$, where $h = h(\xi, \eta, u^\nu)$ is a real function in $U'$. This means that in these coordinates $\varphi = h\mathrm{d}(\xi + i\eta) = h\,\mathrm{d}\zeta$, and because of $\varphi \wedge \bar{\varphi} \neq 0$ we have $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$. The proof in the other direction is obvious. □

Since the connection 1-form $\Gamma_{24}$ satisfies (3.14) and (3.16), we can apply the above Lemma for $\varphi = \Gamma_{24}$. Now, $n = 4$ and we have

$$\Gamma_{24} = h\,\mathrm{d}\zeta \quad \text{with } \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0 \text{ on } U' \subset \mathcal{M}.$$

Using (3.11), which relates $\Gamma_{24}$ to the coframe 1-forms $\theta^1$ and $\theta^3$, and expressing these two in terms of the 1-forms $(\lambda, \mu, \bar{\mu})$ by (3.7), we get:

$$h\mathrm{d}\zeta = \Gamma_{24} = -P(\bar{\rho}\mu + \bar{\tau}\lambda).$$

Since the function $P$ is nowhere vanishing we can write this last equation as $-P^{-1}h\,\mathrm{d}\zeta = \bar{\rho}\mu + \bar{\tau}\lambda$. Wedging this with $\mathrm{d}\zeta \wedge \lambda$, we get $\bar{\rho}\ \mathrm{d}\zeta \wedge \lambda \wedge \mu = 0$ on $U'$. Because of our assumption that $\rho$ is nowhere vanishing we finally obtain

$$\mathrm{d}\zeta \wedge \lambda \wedge \mu = 0 \quad \text{with } \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0 \text{ on } U' \subset \mathcal{M}. \tag{3.18}$$

The last equation pullsback to the CR manifold $M$ providing a CR function there.

Our construction of $\zeta$ obviously works if the CR structure is of class $C^3$. Actually we think that class $C^{2,1}$ suffices, see [13].

The last part of the proof consists in giving a geometric interpretation to the Einstein conditions (a).

To discuss this we go back to a general spacetime $\mathcal{M}$ equipped with a null congruence associated with a vector field $k$. We do not require that this congruence is geodesic and shearfree here. Such a congruence defines a class of adapted coframes (3.1), (3.2). It follows from equations (3.2) that although $k$ does not specify the coframe 1-forms $\theta^1$, $\theta^3$ uniquely, we have

$$\theta^{1'} \wedge \theta^{3'} = A\mathrm{e}^{i\varphi}\theta^1 \wedge \theta^3.$$

This may be used to define complex valued vector fields $Y$ on $\mathcal{M}$ such that

$$Y \lrcorner (\theta^1 \wedge \theta^3) = 0.$$

The complex valued distribution $N$ consisting of all such vector fields is uniquely defined by the null congruence on $\mathcal{M}$. It follows that

$$N = \{ae_2 + be_4\}$$

where $a, b$ are arbitrary complex valued functions on $\mathcal{M}$ and $e_2$, $e_4$ is a part of the null frame $(e_1, e_2, e_3, e_4)$ dual to $(\theta^1, \theta^2, \theta^3, \theta^4)$, i.e. $e_i \lrcorner \theta^j = \delta^j{}_i$ and, in particular, $e_4 \wedge k = 0$.

Thus a null congruence defines at each point $x \in M$ a 2-complex-dimensional plane $N_x$. These planes are called $\alpha$ planes. They have the property of being *totally null*:

$$g(Y_1, Y_2) = 0, \quad \forall\, Y_1, Y_2 \in N.$$

Thus all vectors in $N_x$ are null and orthogonal to each other.

The Ricci tensor of a spacetime may be considered as a symmetric bilinear form. We extend it to the complexification by linearity.

Now combining these two facts we may require that we have a spacetime in which the Ricci tensor, extended to the complexification, has a similar property with respect to $N$ as the metric. We say that the complexified Ricci tensor $\mathrm{Ric}(g)$ *vanishes on the distribution* $N$ iff

$$\mathrm{Ric}(g)(Y_1, Y_2) = 0, \quad \forall\, Y_1, Y_2 \in N.$$

We will denote this condition by $\mathrm{Ric}(g)_{|N} = 0$. Obviously it is weaker than the Ricci flatness condition. Actually if $N$ is the distribution of $\alpha$ planes associated with the congruence, then in an adapted coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$

$$\mathrm{Ric}(g)_{|N} = 0 \;\Leftrightarrow\; R_{22} = R_{24} = R_{44} = 0.$$

Hence the vanishing of Ricci on $N$ is precisely equivalent to our conditions (a).

This completes the proof of Theorem 2.1. □

***3.2. Reduction of the Einstein equations to the CR manifold.*** In this section we derive a maximally reduced system corresponding to the Einstein equations for a spacetime admitting a congruence of null and shearfree geodesics. We will assume that the congruence has nonvanishing twist at every point of the spacetime. This is the same as to assume that the underlying CR structure is strictly pseudoconvex.

As we know, the Einstein equations for such spacetimes split conveniently into three types of equations, which in Section 3.1 were denoted by (a), (b) and (c). Our reduction procedure will follow this split: we first impose equations (a), then (b) and finally (c).

We start with equations (a). They will be reduced according to the following scheme. As we have proven in Section 3.1 equations (a) imply that the corresponding CR structure admits at least one CR-function. This result enables us to start our reduction procedure with a 4-manifold $\mathcal{M} = \mathbb{R} \times M$ by taking the metric in the form

$$g = 2(\theta^1\theta^2 + \theta^3\theta^4), \tag{3.19}$$

where

$$\theta^1 = P\mu, \quad \theta^2 = P\bar{\mu}, \quad \theta^3 = P\lambda, \quad \theta^4 = P[\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda],$$

with the 1-forms $(\lambda, \mu, \bar{\mu})$ satisfying

$$\mathrm{d}\mu = 0, \qquad \mathrm{d}\bar{\mu} = 0, \tag{3.20}$$

$$\mathrm{d}\lambda = i\mu \wedge \bar{\mu} + (c\mu + \bar{c}\bar{\mu}) \wedge \lambda. \tag{3.21}$$

The forms $(\lambda, \mu, \bar{\mu})$ define the CR structure on $M$. Note that formula (3.20) follows from our result on the existence of one CR function. This result enables us to put

$$\mu = \mathrm{d}\zeta,$$

where $\zeta$ is the first CR function obtained in (3.18). We remain with this choice for $\mu$ in the rest of this section. Formula (3.21), which says that there is a choice of $\lambda$ such that the coefficient of the $i\mu \wedge \bar{\mu}$ term is equal to one, is equivalent to our assumption about strict pseudoconvexity.

At this stage we introduce a basis

$$(\partial_0, \partial, \bar{\partial})$$

of vector fields on $M$, which is dual to the coframe $(\lambda, \mu, \bar{\mu})$ on $M$. The complex vector field $\bar{\partial}$ is the *tangential CR operator on* $M$.

We also note that the closure of the system (3.20)-(3.21) implies that:

$$\partial\bar{c} = \bar{\partial}c, \tag{3.22}$$

so that $\partial\bar{c}$ is *real*.

Thus from now on we assume that we have a CR manifold $M$ with the defining forms $(\lambda, \mu, \bar{\mu})$ satisfying (3.20)-(3.21). Our goal is to lift this CR structure to an Einstein spacetime.

***Remark 3.6.*** We stress that although we did not impose the full equations (a) on our spacetime, we already used in (3.20)-(3.21) a consequence of these equations, which enabled us to assume that $(\lambda, \mu, \bar{\mu})$ are in the form (3.21). It is a justified procedure: since ultimately we are interested in the maximal reduction of the full system (a), we may freely use its consequence at any stage of the reduction procedure.

The reduction of equations (a) goes as follows:

- We first use the Goldberg-Sachs theorem, which says that if equations (a) are satisfied then $R_{2412} + R_{2434} = 0$. Modulo complex conjugation this is equivalent to the requirement that the function $W$ satisfies:

$$W_r - iW_{rr} = 0.$$

  This equation may be easily integrated, proving that the most general form of a $W$ satisfying (a) is given by

$$W = i\mathrm{e}^{-ir}x + y, \tag{3.23}$$

  where the complex functions $x$ and $y$ are $r$-independent, $x_r = y_r = 0$. Thus the requirement that our spacetime is algebraically special (the requirement that is implied by (a)) is equivalent to the form (3.23) of the function $W$.
- The first of equations (a), namely $R_{44} = 0$, is equivalent to the differential equation on $P$:

$$-4PP_{rr} + P_r^2 + P^2 = 0.$$

  This again can be easily solved to get:

$$P = \frac{p}{\cos\left(\frac{r+s}{2}\right)}, \tag{3.24}$$

  with real functions $p \neq 0$ and $s$ satisfying $p_r = s_r = 0$.
- Equation $R_{24} = 0$ is equivalent to

$$y = -ic - 2i\partial\log p + \partial s + 2i\mathrm{e}^{is}x,$$

  i.e., to an equation which expresses the function $y$ in terms of the unknowns $p$, $s$, $x$ and the CR quantities $c$ and $\partial$.

- Now it is convenient to introduce a *new* unknown $t$, a complex valued function on $M$, which replaces the unknown $x$. The quantity $x$ is related to $t$ via:

$$e^{is}x = c + 2\partial \log p - t. \tag{3.25}$$

  This enables us to write $y$ as

$$y = ic + 2i\partial \log p + \partial s - 2it. \tag{3.26}$$

- In terms of $t$ the Einstein equation $R_{22} = 0$ is equivalent to

$$\partial t + (c - t)t = 0. \tag{3.27}$$

We summarize this in the following proposition.

***Proposition 3.7.*** *The metric* (3.19)-(3.21) *satisfies the Einstein equations* $R_{44} = R_{24} = R_{22} = 0$ *if and only if the function* $W$ *is given by* (3.23)*, the function* $P$ *is given by* (3.24) *with* $p$, $s$ (real), $x$, $y$, $t$ (complex) *satisfying* $p_r = s_r = x_r = y_r = t_r = 0$*, definitions* (3.25)*,* (3.26) *and the differential equation (*(3.27)*).*

Observe that equation (3.27) always has the solution $t \equiv 0$. Using this observation we prove the converse of Theorem 2.1 in the strictly pseudoconvex case. Indeed if we have a strictly pseudoconvex 3-dimensional CR manifold with one CR function $\zeta$ such that $d\zeta \wedge d\bar{\zeta} \neq 0$, we define $\mu = d\zeta$. We next choose $\lambda$ so that (3.21) is satisfied. Then we take $t \equiv 0$ and choose sufficiently smooth arbitrary real functions $p$ and $s$. Using them we define a lift to a spacetime having the metric $g$ as in (3.19), with $P$ given by (3.24), and $W$ given by (3.23), (3.25), (3.26). Then it follows that such a metric satisfies Einstein's equations (a), independently of the choice of a real function $H$ on $\mathcal{M}$. Combining this with Theorem 2.1, we have proved Theorem 2.2. ◻

We now pass to the reduction of equations (b). The steps here are:

- Assuming equations (a) to be satisfied, so that the metric is given by Proposition 3.7, we first impose a consequence of (b), namely $R_{12} + R_{34} = 2\Lambda =$ const, which is just the condition that the Ricci scalar is constant and equal to $4\Lambda$. This equation determines the $r$ dependence of the function $H$. It reads:

$$H = \frac{m}{p^4}e^{2i(r+s)} + \frac{\bar{m}}{p^4}e^{-2i(r+s)} + Qe^{i(r+s)} + \bar{Q}e^{-i(r+s)} + h, \tag{3.28}$$

with

$$Q = \frac{3m + \bar{m}}{p^4} + \frac{2}{3}\Lambda p^2 + \frac{2\partial p \bar{\partial} p - p(\partial\bar{\partial} p + \bar{\partial}\partial p)}{2p^2}$$
$$- \frac{i}{2}\partial_0 \log p - 2t\bar{\partial} p - \bar{t}\partial \log p + \frac{3}{2}\bar{\partial} t - \bar{c}t - \frac{1}{2}c\bar{t} + \frac{5}{2}t\bar{t} + \partial\bar{t} - \bar{\partial} c,$$

$$h = 3\frac{m + \bar{m}}{p^4} + 2\Lambda p^2 + \frac{2\partial p \bar{\partial} p - p(\partial\bar{\partial} p + \bar{\partial}\partial p)}{p^2}$$
$$- 3(t\bar{\partial}\log p + \bar{t}\partial \log p) + \frac{5}{2}(\partial\bar{t} + \bar{\partial} t) + 6t\bar{t} - \frac{3}{2}(c\bar{t} + \bar{c}t) - 2\bar{\partial} c + \partial_0 s.$$

Note that $h$ is *real* due to (3.22). The unknown $m$ is complex and satisfies $m_r = 0$.

- At this stage we have found the explicit $r$ dependence of the entire metric (3.19)-(3.21). For the full determination of this dependence we only needed equations (a) and the subset of equations (b) given by $R_{12} + R_{34} = 2\Lambda =$ const.
- We now impose another consequence of (b), namely $R_{12} - R_{34} = 0$. This, together with $R_{12}+R_{34} = 2\Lambda =$ const, is equivalent to (b). The reduction of this equation gives the following differential equation connecting $p$, $t$ and $m$:

$$\left[\partial\bar{\partial} + \bar{\partial}\partial + \bar{c}\partial + c\bar{\partial} + \frac{1}{2}c\bar{c} + \frac{3}{4}(\partial\bar{c} + \bar{\partial} c) - \frac{3}{2}(\partial\bar{t} + \bar{\partial} t + t\bar{t})\right] p = \frac{m + \bar{m}}{p^3} + \tfrac{2}{3}\Lambda p^3. \tag{3.29}$$

This completes the reduction of equations (b).

We summarize this in the following theorem.

***Theorem 3.8.*** *A strictly pseudoconvex CR structure* $(M, (\lambda, \mu))$ *lifts to a space-time satisfying Einstein equations* (a) *and* (b) *if and only if it admits at least one CR function* $\zeta$ *with* $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$ *and, in addition, it admits a solution to equation* (3.29) *for a real function* $p$ *on* $M$*, with* $c$*,* $t$ *obeying respectively* (3.21) *and* (3.27).

If we do not insist on the full system (a) and (b), we conclude with the following remarkable theorem.

***Theorem 3.9.*** *A strictly pseudoconvex CR structure* $(M, (\lambda, \mu))$ *lifts to a space-time having constant Ricci scalar and satisfying equations* (a) *if and only if it admits at least one CR function* $\zeta$ *with* $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$*. In such case the spacetime metric satisfying* (a) *and having constant Ricci scalar equal to* $4\Lambda$ *is* (3.19)-(3.21) *with* $W$ *given by* (3.23)*,* $P$ *given by* (3.24)*,* $x$*,* $y$ *given by* (3.25)*,* (3.26)*, and* $H$*,* $Q$*,* $h$ *given by*

(3.28). *The functions* $m$ (complex), $p$ (real) *are arbitrary, and the complex function* $t$ *satisfies the partial differential equation* (3.27).

This theorem is remarkable for the reasons highlighted in the following remarks.

***Remark 3.10.*** Given a CR structure with one CR function, to determine the most general lift to a spacetime with a metric satisfying the Ricci conditions of Theorem (3.9), we need to have a general solution for only *one* complex equation (3.27) for the complex function $t$ on the CR manifold. This equation has always one solution, namely $t = 0$. Surprisingly the question if this equation has other solutions is equivalent to the question if the CR structure admits more CR functions, and hence is locally embeddable. To see this take the CR manifold $(M, (\lambda, \mu))$ satisfying (3.20)-(3.21) and consider the complex 1-form $\varphi$ defined by:

$$\varphi = \mu + i\bar{t}\lambda. \tag{3.30}$$

Then due to (3.20)-(3.21) we have

$$\mathrm{d}\varphi \wedge \varphi = i \, [\, \bar{\partial}\bar{t} + (\bar{c} - \bar{t})\bar{t} \,] \, \mu \wedge \bar{\mu} \wedge \lambda.$$

Thus $\mathrm{d}\varphi \wedge \varphi = 0$ is equivalent to the Einstein equation (3.27). Since obviously $\varphi \wedge \bar{\varphi} \neq 0$, then according to Lemma 3.5, $\varphi$ defines a CR function $\eta$ such that $h\,\mathrm{d}\eta = \varphi$. Thus the equation

$$h\,\mathrm{d}\eta = \mu + i\bar{t}\lambda \tag{3.31}$$

relates the CR functions $\eta$ and solutions $t$ of the Einstein equation (3.27). As an example take the trivial solution $t = 0$ of (3.27). Since $\mu = \mathrm{d}\zeta$ and since for $t = 0$ equation (3.31) gives $h\,\mathrm{d}\eta = \mu$, the CR function $\eta$ is dependent on the CR function $\zeta$ of (3.18). To get a $\zeta$-independent CR function $\eta$ we need a nonzero solution of (3.27). That this requirement is also sufficient follows from the relation $h\,\mathrm{d}\eta \wedge \mathrm{d}\zeta = i\bar{t}\lambda \wedge \mu$ implied by (3.31). This proves the following corollary.

***Corollary 3.11.*** *Every nonzero solution* $t$ *of the Einstein equation* (3.27) *provides a second CR function* $\eta$ *such that* $\mathrm{d}\eta \wedge \mathrm{d}\zeta \neq 0$. *Also the converse is true: every CR function* $\eta$ *defines a complex function* $t$ *satisfying the Einstein equation* (3.27). *The transformation between* $\eta$ *and* $t$ *is given by* $h\,\mathrm{d}\eta = \mathrm{d}\zeta + i\bar{t}\lambda$.

Thus, in particular, if the starting CR structure is locally embedded, which means that the general solution to the tangential CR equation is explicitly known, we may write the general solution for $t$ satisfying (3.27) and obtain the most general solution for the metric satisfying the highly nonlinear system of equations (a) and $R_{12} + R_{34} = 2\Lambda = \text{const}$.

***Remark 3.12.*** Another remarkable feature of Theorem 3.9 is that the metrics (3.19)-(3.21) satisfying (a) and $R_{12} + R_{34} = 2\Lambda = \text{const}$ have an explicit $r$ dependence which is very particular. Note that the functions $W$ of (3.23) and $H$ of (3.28) are *periodic* in $r$ with period $2\pi$. If we forget about the conformal factor we may write the metrics in the form [30]

$$\hat{g} = 2p\left[\mu\bar{\mu} + \lambda\left[\mathrm{d}r + (i\mathrm{e}^{-ir}x + y)\mu + (-i\mathrm{e}^{ir}\bar{x} + \bar{y})\bar{\mu} + \left(\frac{m}{p^4}\mathrm{e}^{2i(r+s)} + \frac{\bar{m}}{p^4}\mathrm{e}^{-2i(r+s)} + Q\mathrm{e}^{i(r+s)} + \bar{Q}\mathrm{e}^{-i(r+s)} + h\right)\right]\right], \tag{3.32}$$

which is periodic and *regular* in $r$. Thus, what the Einstein equations (a) and $R_{12} + R_{34} = 2\Lambda = \text{const}$ impose on the spacetime $\mathcal{M} = \mathbb{R} \times M$ is a *circle bundle structure* $\mathbb{S}^1 \to \hat{\mathcal{M}} \to M$ on the Lorentzian manifold $(\hat{\mathcal{M}}, \hat{g})$ which has $\mathcal{M}$ as its universal cover. The Lorentzian manifold $(\hat{\mathcal{M}}, \hat{g})$ is called by physicists a *conformal (fiberwise) compactification* of $(\mathcal{M}, g)$. It is used by them to study the asymptotic behavior of a gravitational field. We summarize in the following corollary.

***Corollary 3.13.*** *The Einstein equations* (a) *and the constancy of the Ricci scalar imposed on the metrics* (3.19)-(3.21) *imply that all the metric functions are periodic in the $r$ coordinate, so that there is a natural circle bundle over the strictly pseudoconvex CR manifold onto which all the Einstein metrics* (3.19)-(3.21) *descend.*

***Remark 3.14.*** In 1976 Fefferman [7] introduced a natural *conformal* Lorentzian metric $\hat{g}_F$ on a circle bundle $\mathbb{S}^1 \to \hat{\mathcal{M}}_F \to M$ over any strictly pseudoconvex 3-dimensional CR manifold $(M, (\lambda, \mu))$ embedded in $\mathbb{C}^2$. A natural question is how his circle bundle and his Lorentzian metrics are related to our $(\hat{\mathcal{M}}, \hat{g})$ above. The answer is the following:

- Fefferman metrics constitute a simple subclass of our metrics (3.32).
- They happen to be conformally Einstein *only* in the case when the corresponding CR manifold is locally CR equivalent to the Heisenberg group CR structure [18]; in such case $\hat{g}_F$ is conformally flat.
- Given a CR structure as in (3.20)-(3.21) the Fefferman metric $\hat{g}_F$ is obtained from our $\hat{g}$ by putting $x = m = Q = 0$, $y = -\frac{i}{3}c$, $h = -\frac{1}{12}(\partial\bar{c} + \bar{\partial}c)$.
- Thus, in our setting, the Fefferman metrics (or, strictly speaking, their generalizations to strictly pseudoconvex CR manifolds which admit one CR function) are represented by

$$\hat{g}_F = 2\left[\mu\bar{\mu} + \lambda\left[\mathrm{d}r - \frac{i}{3}c\mu + \frac{i}{3}\bar{c}\bar{\mu} - \frac{1}{12}(\partial\bar{c} + \bar{\partial}c))\lambda\right]\right]. \tag{3.33}$$

Note that the Fefferman metrics are $r$-independent. This reflects the well-known fact [18, 30, 31, 51] that the null congruence of shearfree geodesics associated with the $k = \partial_r$ direction is a *conformal Killing* vector for each Fefferman metric. Actually, the above formula for the Fefferman metric for CR manifolds having one CR function is obtained by

(i) imposing the requirement that the metric (3.32) has a conformal Killing vector alligned with $k = \partial_r$ (this forces $x$, $m$ and $Q$ to vanish), and
(ii) imposing another requirement that the metrics (3.32) be of Petrov type $N$ (this specifies that $y$ and $h$ must be expressed in terms of $c$ as above).

That the requirements (i) and (ii) are neccessary and sufficient to distinguish the Fefferman metrics among metrics (3.32) is a well-known fact [18, 30, 31, 51].

Looking at the Fefferman metrics (3.33) one may say that the circle bundle structure of the spacetime is not visible, since the metrics are *constant* along the $r$-direction. Only if a more general class of metrics (3.19)-(3.21) is taken into account does the circle bundle structure emerge. And it emerges in a natural way, as a consequence of the Einstein equations (a) and $R_{12} + R_{34} = 2\Lambda = \text{const}$.

The fact that the Fefferman metrics are of Petrov type $N$ everywhere essentially means that the Weyl tensor has only one nonvanishing complex component. This is proportional to

$$\Psi_4 = \partial\bar{\partial}\partial c + 3c\bar{\partial}\partial c - 7ic\partial_0 c - 3i\partial\partial_0 c + (\partial c + 2c^2)\bar{\partial}c.$$

The vanishing or not of $\Psi_4$ is a CR invariant property. It is also a conformal invariant property. In the CR context, vanishing of $\Psi_4$ is an if and only if condition for the CR structure (3.20)-(3.21) to be CR equivalent to the Heisenberg group CR structure [2, 18]. In the conformal context, the vanishing of $\Psi_4$ means that $\hat{g}$ is conformally flat. Thus the $\Psi_4$ component of the Weyl tensor for $\hat{g}_F$ is proportional to Cartan's lowest order invariant [4] of a strictly pseudoconvex CR structure.

We thus have the following corollary.

***Corollary 3.15.*** *The CR structure* (3.20)-(3.21) *is locally CR equivalent to the Heisenberg group CR structure if and only if the function* $c$ *in* (3.21) *satisfies*

$$\partial\bar{\partial}\partial c + 3c\bar{\partial}\partial c - 7ic\partial_0 c - 3i\partial\partial_0 c + (\partial c + 2c^2)\bar{\partial}c = 0.$$

***Remark 3.16.*** We are now prepared to give the geometric interpretation of the leading terms in the partial differential equation (3.29). Having chosen a strictly pseudoconvex CR structure $(M, (\lambda, \mu))$ and functions $t$ and $m$ on $M$ this is a partial differential equation for a real function $p$ on $M$. Below we give the interpretation of the *linear* operator on its left hand side.

Since the Fefferman metrics $\hat{g}_F$ are defined up to a conformal scale it is reasonable to consider the *conformally invariant* wave operator $(-*\mathrm{d}*\mathrm{d} - \frac{1}{6}R)$, with $-*\mathrm{d}*\mathrm{d}$ being the D'Alembert operator and $R$ being the Ricci scalar; both being calculated in the Fefferman metric $\hat{g}_F$. Let us apply this operator to a real function $f$ on the Fefferman bundle $\hat{\mathcal{M}}_F$, which is constant along the fibres. Remarkably we get [30]:

$$(- * \mathrm{d} * \mathrm{d} - \tfrac{1}{6}R)f = \left[\partial\bar\partial + \bar\partial\partial + c\bar\partial + \bar c\partial + \frac{1}{2}c\bar c + \frac{3}{8}(\partial\bar c + \bar\partial c)\right]f. \tag{3.34}$$

It is worthwhile to add that $R = -3c\bar c - \frac{9}{4}(\partial\bar c + \bar\partial c)$ and that $*\mathrm{d} * \mathrm{d}f = -(\partial\bar\partial + \bar\partial\partial + c\bar\partial + \bar c\partial)f$, but neither $R$ nor the 3-dimensional operator $\partial\bar\partial + \bar\partial\partial + c\bar\partial + \bar c\partial$ has a CR geometrical meaning. Only their sum

$$\triangle_{CR} = \partial\bar\partial + \bar\partial\partial + c\bar\partial + \bar c\partial + \frac{1}{2}c\bar c + \frac{3}{8}(\partial\bar c + \bar\partial c)$$

is CR meaningful.

Using (3.34), we rewrite the Einstein equation (3.29) in the following equivalent form:

$$\left[\triangle_{CR} + \frac{3}{8}(\partial\bar c + \bar\partial c) - \frac{3}{2}(\partial\bar t + \bar\partial t + t\bar t)\right]p = \frac{m + \bar m}{p^3} + \frac{2}{3}\Lambda p^3.$$

The appearence of the $\frac{3}{8}(\partial\bar c + \bar\partial c)$ term here in the potential is unpleasent, but we can not avoid it.

Finally we pass to equations (c):

- Assuming (a) and (b) are satisfied, we first reduce the complex equation $R_{13} = 0$. This is equivalent to one complex equation

$$\partial m + 3(c - t)m = 0, \tag{3.35}$$

  for the complex function $m$ of (3.28).
- Now assuming that the metric (3.19)-(3.21) satisfies the Einstein equations (3.23), (3.24), (3.25), (3.26), (3.27), (3.28), (3.29) and (3.35) we calculate the Weyl tensor. Since, via Goldberg-Sachs, the metric is algebraically special, only three of the five Weyl scalars $\Psi_0$, $\Psi_1$, $\Psi_2$, $\Psi_3$, $\Psi_4$ are in principle nonvanishing. These are: $\Psi_2$, $\Psi_3$, $\Psi_4$. The Weyl spinor $\Psi_2$, whose vanishing means that the metric is of Petrov type $III$, $N$ or 0, has a particularly neat form:

$$\Psi_2 = \frac{(1 + \mathrm{e}^{i(r+s)})^3}{2p^6}m. \tag{3.36}$$

- Although we succeded in reducing the last equation $R_{33} = 0$, the explicit reduced form of it is too complicated to be presented here.

The following remark is in order.

***Remark 3.17.*** If a lift to the spacetime $(\mathcal{M}, g)$ of *any* CR structure $(M, (\lambda, \mu))$ is considered, one can try to write down curvature conditions that are compatible with the underlying CR geometry of the associated congruence of shearfree and null geodesics. We already met the curvature conditions that respect the underlying CR geometry. These are the Einstein equations (a), or in more geometric terms, conditions forcing the complexified Ricci tensor to be identically zero on the associated distribution of $\alpha$ planes. It turns out that the Einstein conditions (a), (b) and $R_{13} = 0$ also have geometric meaning. They are equivalent to

$$\text{Ric}(g) = \Lambda g + \Phi \lambda \otimes \lambda, \tag{3.37}$$

where $\Phi$ is an arbitrary real function on $\mathcal{M}$. Physicists call these equations the *cosmological constant Einstein's equations with a pure radiation field*, since they describe gravitational fields with the energy momentum tensor in which all the energy is propagated with the speed of light along the direction determined by the congruence of shearfree geodesics defined by $M$.

***3.3. The second CR function.*** We start this section by considering a 3-dimensional CR manifold which lifts to a spacetime with a twisting congruence of null and shearfree geodesics and which satisfies Einstein equations (a). Theorem 2.1 assures that such a CR manifold has at least one CR function, say $\zeta$, with $d\zeta \wedge d\bar{\zeta} \neq 0$. Our approach to the problem of obtaining an independent CR function, say $\eta$, is by finding a complex equation, let us call it ($2^{\text{nd}}CR$), which when assumed to be satisfied, guarantees that $\eta$ exists. The main idea here is to find ($2^{\text{nd}}CR$) among the full system of Einstein equations (a), (b) and (c), especially among the reduced Einstein's equations of Section 3.2.

It turns out that, depending on some additional assumptions about the lifted spacetime, various choices of ($2^{\text{nd}}CR$) are possible. Among all of these choices the simplest is to consider equation (3.27) as the ($2^{\text{nd}}CR$). Indeed, if our CR manifold lifts to a spacetime whose metric (3.19)-(3.21) has nonvanishing $t$ in (3.27) then, as we already noticed in Corollary (3.11), it is locally embeddable, with an embedding given by means of the CR functions $\zeta$ of (3.18) and $\eta$ of (3.31). The trouble with equation (3.27) is that the Einstein equations (a) do not guarantee that (3.27) has any other solution than $t \equiv 0$. Nevertheless equation (3.27) may be used as a *criterion for embeddability*. Suppose one knows that

(i) a 3-dimensional strictly pseudoconvex CR manifold admits a CR function $\zeta$ such that $d\zeta \wedge d\bar{\zeta} \neq 0$ and, in addition, one knows that

(ii) this manifold lifts to a spacetime with a metric $g$ which satisfies Einstein's equations (a).

Then he can write the metric $g$ in the form (3.19)-(3.21) with $\mu = d\zeta$ and simply *calculate* the function $t$. If he finds that $t \neq 0$, then he concludes that the CR structure is locally embeddable. This is due to the fact that the calculated $t$ automatically satisfies (3.27) since $g$ satisfies (a).

***Remark 3.18.*** At this stage we remark that equation (3.27) is interesting on its own, without any reference to the fact that it originates from the Einstein equation for the lifted spacetime. Indeed, it follows from our discusion above, that one can use this equation to get a sharp criterion for the embeddability of a strictly pseudoconvex CR structure that admits one CR function. Here the procedure is as follows:

- Suppose we are given a strictly pseudoconvex 3-dimensional CR manifold $M$ which has one CR function $\zeta$ such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$.
- Given $\zeta$, we write $\mu = \mathrm{d}\zeta$, and choose $\lambda$ so that equation (3.21) is satisfied. This, in particular, defines the function $c$ on $M$.
- Define $(\partial, \bar{\partial}, \partial_0)$ as dual to $(\mu, \bar{\mu}, \lambda)$.
- Consider the equation $\partial t + (c - t)t = 0$ for a complex function $t$ on $M$.
- Then we have the following theorem.

***Theorem 3.19.*** *The CR structure $(M, (\lambda, \mu = \mathrm{d}\zeta))$ is locally embeddable if and only if the equation $\partial t + (c - t)t = 0$ has a solution $t$ such that $t \neq 0$.*

We may combine this result with a result of Hanges [11], who found another criterion for the existence of the second CR function for a 3-dimensional strictly pseudoconvex CR manifold. It is well known, that if a 3-dimensional strictly pseudoconvex CR manifold $M$ admits one CR function $\zeta$ as above, then one can supplement $\zeta$ and $\bar{\zeta}$ by a real function $u$ on $M$ so that $(\mathrm{Re}\zeta, \mathrm{Im}\zeta, u)$ constitute a coordinate system on $M$, in which

$$\mu = \mathrm{d}\zeta, \quad \text{and} \quad \lambda = \frac{\mathrm{d}u + L\mathrm{d}\zeta + \bar{L}\mathrm{d}\bar{\zeta}}{i(\bar{\partial}L - \partial\bar{L})},$$

and in which the complex valued function $L = L(\zeta, \bar{\zeta}, u)$ vanishes at the origin, $L(0, 0, 0) = 0$. Hanges' result is that the CR structure $(M, (\lambda, \mu))$ is locally embeddable near the origin if and only if the function $L = L(\zeta, \bar{\zeta}, u)$ is the *boundary value* of a function $\tilde{L} = \tilde{L}(\zeta, \bar{\zeta}, w)$ which is *holomorphic* in the complex variable $w = u + iv$. Using this result and writing the differential equation $\partial t + (c - t)t = 0$ in the local coordinates $(\zeta, \bar{\zeta}, u)$ we get the following remarkable corollary.

***Corollary 3.20.*** *The nonlinear partial differential equation*

$$(\partial\bar{L} - \bar{\partial}L)\partial t - [\partial(\partial\bar{L} - \bar{\partial}L) + (\partial\bar{L} - \bar{\partial}L)(L_u + t)]t = 0,$$

*with $\partial = \partial_\zeta - L\partial_u$, and with the complex valued function $L = L(\zeta, \bar{\zeta}, u)$ vanishing at the origin, is locally solvable near the origin for a complex function $t \neq 0$ if and only if $L$ is the boundary value of a function $\tilde{L} = \tilde{L}(\zeta, \bar{\zeta}, w)$ which is holomorphic in the variable $w$. If this is the case the CR structure $(M, (\lambda, \mu))$ is locally embeddable.*

Returning to our discussion of the relations between the second CR function and the Einstein equations of the lifted spacetime, we are now in a position to say that, if we have a solution $t \neq 0$ of equation (3.27), the problem of the local embeddability of a 3-dimensional manifold $M$ is solved. If we are in an unlucky situation which negates the existence of a solution $t \neq 0$, two things may happen:

- either the only solution to (3.27) is $t$ vanishing everywhere,
- or $t = 0$ at a point around which we want to embedd $M$ into $\mathbb{C}^2$.

In the first case, we can put $t \equiv 0$ in all the equations we have derived in Section 3.2. In the second case, some care is needed, and we need some preparations. In what follows, our considerations will be of a bit more general nature than is required to treat this case, but at a certain moment, they will lead us to the conclusion that the case $t = 0$ at a point is, actually, the same as the case $t \equiv 0$.

To get to this conclusion, consider a situation in which we have a 3-dimensional CR structure $(M, (\lambda, \mu))$ with $\mu$ and $\lambda$ satisfying (3.20)-(3.21). Assume, in addition, that the CR structure admits complex functions $h \neq 0$ and $t_0$ such that the complex valued 1-form

$$\mu' = h^{-1}(\mu + i\bar{t}_0\lambda) \quad \text{is closed,} \quad d\mu' = 0. \tag{3.38}$$

We *assume* it is the case. If we have such $h$ and $t_0$, we define

$$\lambda' = |h|^{-2}\lambda. \tag{3.39}$$

The forms $(\lambda', \mu', \bar{\mu}')$ define the same CR structure on $M$ as the forms $(\lambda, \mu, \bar{\mu})$. Moreover, because of our choice of $\lambda'$, we have $d\lambda' = i\mu' \wedge \bar{\mu}' + (c'\mu' + \bar{c}'\bar{\mu}') \wedge \lambda'$, with $c' = h(c - t_0 - \partial \log(h\bar{h}))$. Thus, if our CR structure admits $\mu'$ of (3.38) then $(\lambda', \mu', \bar{\mu}')$ satisfy, qualitatively, the same structural equations (3.20)-(3.21) as $(\lambda, \mu, \bar{\mu})$. Therefore, in such a situation, when lifting the CR manifold $M$ to a spacetime satisfying Einstein equations (a), we can use $(\lambda, \mu, \bar{\mu})$ and $(\lambda', \mu', \bar{\mu}')$ on the same footing. We know that if we start with $(\lambda, \mu, \bar{\mu})$, we get our conclusions (3.23)-(3.27). Similarly, using $(\lambda', \mu', \bar{\mu}')$ we get the same conclusions, with the mere change that all the variables in (3.23)-(3.27) have now *primes*. It is easy to get the relations between the 'primed' and the 'nonprimed' variables. For us the most important is the relation between $t$ calculated for $(\lambda, \mu, \bar{\mu})$ and $t'$ calculated for $(\lambda', \mu', \bar{\mu}')$. This, when calculated, is

$$t' = h(t - t_0). \tag{3.40}$$

The hypothetic situation, in which we have $\mu'$ as in (3.38), is realized in practice if we have a CR structure as in Section 3.2 which satisfies equations (a). For such a structure, chosing $\mu = d\zeta$, we get $t$ satisfying (3.27). Then, given such a $t$, we define $\varphi = \mu + i\bar{t}\lambda$ as in (3.30). Since, as we know, this $\varphi$ satisfies $d\varphi \wedge \varphi = 0$, $\varphi \wedge \bar{\varphi} \neq 0$, then by Lemma 3.5, we are guaranteed an existence of $h \neq 0$ such that $\mu' = h^{-1}(\mu + i\bar{t}\lambda)$ is closed, $d\mu' = 0$. Thus, passing from $(\lambda, \mu, \bar{\mu})$ to $(\lambda' = |h|^{-2}\lambda, \mu', \bar{\mu}')$, as in (3.39), we must use $t_0 = t$ (compare the present $\mu'$ with this of (3.38)). This means that, after transforming all the variables appearing in (3.23)-(3.27) to their primed counterparts, we get, in particular,

$$t' = h(t - t_0) = h(t - t) = 0, \quad \text{everywhere.}$$

This shows that even if $t$ is not identically zero, including the case when it is zero at a point and nonzero off this point, we can transform $t$ to *zero everywhere* by an appropriate choice of the adapted coframe.

***Remark 3.21.*** That $t$ may be gauged to zero everywhere was *subconciously* known to physicists, and used by them [15, 43], in their derivations of the maximally reduced system of equations for the *algebraically special Einstein metrics*. Actually they have never encountered our variable $t$, since at the very beginning of their considerations, they used a very specific choice of the adapted coframe, that forbidded $t$ to ever appear. Being aware of this 'physicists trick' [43], we were not gauging $t$ to zero here up to now, since nonzero $t$, if it exists, provides us with a second CR function. However, if $t$ does not give us a second CR function at the point around which we want to embedd our CR manifold (because, for example, it is *vanishing* at this point), we use the argument above to gauge $t$ to *zero everywhere*. In this way we proved that the two cases: $t \equiv 0$, and $t = 0$ at a point, differ by the choice of an adapted coframe.

We summarize this in the following corollary.

***Corollary 3.22.*** *If a strictly pseudoconvex CR structure can be lifted to a space-time which satisfies Einstein's equations* (a) *then, without loss of generality, we may assume that the variable* t *is identically equal to zero in all of the reduced Einstein equations (3.23),* (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.35) *and in the equation* $R_{33} = 0$.

In accordance with this corollary, we now put $t = 0$ everywhere and look for the second CR function in terms of other quantities than $t$. In the rest of the paper we will frequently use the following crucial lemma.

***Lemma 3.23.*** *Suppose that a strictly pseudoconvex CR manifold M is represented by forms* $\lambda$ *and* $\mu = \mathrm{d}\zeta$, *with* $\mu \wedge \bar{\mu} \neq 0$, *as in* (3.20)-(3.21). *Then, if in addition, M admits a solution to the equation*

$$\partial_0 \partial \bar{\eta}' = 0, \tag{3.41}$$

*for a complex valued function* $\eta'$ *on M such that*

$$\partial_0 \bar{\eta}' \neq 0, \tag{3.42}$$

*then* $(M, (\lambda, \mu))$ *is locally embeddable. Here, as always, the operators* $(\partial, \bar{\partial}, \partial_0)$ *are dual to* $(\mu, \bar{\mu}, \lambda)$.

*Proof.* Since $\partial_0$ is a *real* operator, we can always find a real function $s$ on $M$ such that locally $(\zeta, \bar{\zeta}, s)$ are coordinates on $M$ and $\partial_0 = \partial/\partial s$. Given a solution $\eta'$ to (3.41) we calculate $\bar{z} = -\partial\bar{\eta}'$ obtaining $z = z(\zeta, \bar{\zeta}, s)$. We restrict this function to the hypersurface $s = 0$ in $M$ getting $z_0 = z(\zeta, \bar{\zeta}, 0)$. We search for a function $\omega_0 = \omega_0(\zeta, \bar{\zeta})$ on $s = 0$ in $M$, such that

$$\partial\bar{\omega}_0 - \bar{z}_0 = 0.$$

This equation, as the conjugate of the inhomogeneous CR equation in the complex plane, can always be locally solved for $\omega_0$. Given such an $\omega_0$ on $s = 0$ we extend it to a complex valued function $\omega$ in $M$ by the requirement that it is constant along $s$,

$$\frac{\partial}{\partial s}\omega \equiv 0, \quad \omega_{|s=0} \equiv \omega_0.$$

Now we define

$$\eta = \eta' + \omega.$$

We have $\partial_0\partial\bar{\eta} = \partial_0\partial\bar{\eta}' + \partial_0\partial\bar{\omega}$, and since $\partial_0\partial\bar{\eta}' \equiv 0$ and the commutator

$$\partial_0\partial - \partial\partial_0 = c\partial_0, \tag{3.43}$$

we get $\partial_0\partial\bar{\eta} = \partial\partial_0\bar{\omega} + c\partial_0\bar{\omega} \equiv 0$. Thus our complex function $\eta$ satisfies

$$\frac{\partial}{\partial s}(\partial\bar{\eta}) = 0 \tag{3.44}$$

everywhere. Since we have chosen $\omega$ so that

$$(\partial\bar{\eta})_{|s=0} = (\partial\bar{\eta}' + \partial\bar{\omega})_{|s=0} = -\bar{z}_0 + \partial\bar{\omega}_0 = 0,$$

then equation (3.44), considered as a differential equation for the unknown $\partial\bar{\eta}$, satisfies the initial condition $(\partial\bar{\eta})_{|s=0} = 0$. Thus, $\partial\bar{\eta}$ must vanish everywhere, $\partial\bar{\eta} \equiv 0$. This proves that if $\eta'$ satisfies (3.41) we have a *CR function* $\eta$ associated with it. Moreover, because of our assumption (3.42) we have

$$\mathrm{d}\eta \wedge \mathrm{d}\zeta \wedge \bar{\mu} = \mathrm{d}\eta \wedge \mu \wedge \bar{\mu} = (\mathrm{d}\eta' + \mathrm{d}\omega) \wedge \mu \wedge \bar{\mu} = (\partial_0\eta')\lambda \wedge \mu \wedge \bar{\mu} \neq 0.$$

This in particular means that $\mathrm{d}\zeta \wedge \mathrm{d}\eta \neq 0$. Thus the CR functions $\zeta$ and $\eta$ are independent and as such they provide a local embedding of the CR manifold $M$ in $\mathbb{C}^2$. This finishes the proof of the lemma. □

The rest of the paper uses this lemma, under various further assumptions about the lifted spacetime, to produce a new CR function which, together with the $\zeta$ of (3.18), provides the embedding of the CR manifold.

*3.3.1. Existence of a null Maxwell field aligned with the congruence.* As a warm up we start with a CR manifold $M$ and assume it lifts to a spacetime that merely satisfies Einstein equations (a). As we know, in such a case, we automatically have the CR function $\zeta$ of (3.18) which can be used to choose the forms $\lambda$ and $\mu$ as in (3.20)-(3.21). Next we add the assumption about the corresponding spacetime metric (3.19). We will assume for a while that the lifted spacetime $(\mathcal{M}, g)$ admits a *null Maxwell field* which is *aligned* with the congruence of null geodesics corresponding to $(M, (\lambda, \mu))$. The terms in itallics mean the following:

- In any oriented spacetime $(\mathcal{M}, g)$ a *Maxwell field* is a real 2-form $F$ such that $\mathrm{d}F = \mathrm{d} * F = 0$, where $*$ is the Hodge star operator associated with the metric $g$.
- Every real 2-form $F$ in spacetime defines a complex 2-form $\mathcal{F} = F + i * F$. This is antiselfdual, i.e. by definition, it satisfies $*\mathcal{F} = -i\mathcal{F}$. Also the converse is true. Every complex antiselfdual 2-form $\mathcal{F}$ defines a real 2-form $F$, via $F = \mathrm{Re}\mathcal{F}$. The so defined $F$ has the property that $\mathrm{Im}\mathcal{F} = *F$.
- Thus Maxwell fields are in one to one correspondence with closed antiselfdual complex 2-forms $\mathcal{F}$ in $\mathcal{M}$. From now on we will identify Maxwell fields with such $\mathcal{F}$s.
- A nonzero Maxwell field $\mathcal{F}$ is called *null* iff $\mathcal{F} \wedge \mathcal{F} \equiv 0$. Thus a null Maxwell is algebraically special.
- An example of a null Maxwell field is given by a plane electromagnetic wave, in which the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are orthogonal to each other $\mathbf{E}\mathbf{B} = 0$ and have equal length $\mathbf{E}^2 - \mathbf{B}^2 = 0$. In this case $F = \mathrm{d}t \wedge (\mathbf{E}\mathrm{d}\mathbf{r}) + \frac{1}{2}\mathrm{d}\mathbf{r} \wedge (\mathbf{B} \times \mathrm{d}\mathbf{r})$ and $*F$ is obtained from $F$ by the replacement $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$.
- A nontrivial example is due to Ivor Robinson [58]. We present it here, due to its influence on the entire subject:

  In Minkowski spacetime $\mathcal{M} = \mathbb{R}^4$, with the metric $g = 2(\mathrm{d}u'\mathrm{d}r + \mathrm{d}\zeta\mathrm{d}\bar{\zeta})$, consider the following (complex) change of variables:

  $$u' = u - rz\bar{z}, \quad \zeta = (r + i)z, \quad \bar{\zeta} = (r - i)\bar{z}.$$

  After this transformation the metric is $g = 2(\lambda\mathrm{d}r + 2(r^2 + 1)\mu\bar{\mu})$, with $\lambda = \mathrm{d}u + i(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z})$ and $\mu = \mathrm{d}z$. Now consider an antiselfdual 2-form $\mathcal{F} = f\lambda \wedge \mu$ with a nonvanishing (sufficiently smooth) complex valued function $f$ in $\mathcal{M}$. It is obviously *null*, and it defines a *Maxwell* field, i.e. it satisfies $\mathrm{d}\mathcal{F} = 0$, if and only if
  - ◇ $f = f(u, r, z, \bar{z})$ is independent of the real coordinate $r$, $f_r = 0$, and
  - ◇ $f$ satisfies the *linear* PDE: $(\partial f/\partial\bar{z}) + iz(\partial f/\partial u) = 0$.

  The beauty of this example is that

  $$\frac{\partial}{\partial\bar{z}} + iz\frac{\partial}{\partial u}$$

  is the Hans Lewy operator [25, 58].

- Null Maxwell fields are radiative in a similar sense as the algebraically special gravitational fields. Far from the sources the leading term of the field strength $F$ behaves as

$$F = \frac{\text{Null}}{r} + O\left(\frac{1}{r^2}\right), \quad \text{as } r \to \infty.$$

- A null Maxwell field is always of the form $\mathcal{F} = \tilde{f}\tilde{\lambda} \wedge \tilde{\mu}$, with some real 1-form $\tilde{\lambda}$, some complex 1-form $\tilde{\mu}$ and a complex function $\tilde{f}$ on $\mathcal{M}$.
- One of the implications of the Robinson theorem [41] is that if the spacetime $\mathcal{M}$ admits a null Maxwell field $\mathcal{F} = \tilde{f}\tilde{\lambda} \wedge \tilde{\mu}$, then it is locally a product $\mathcal{M} = \mathbb{R} \times M$, with $M$ being a CR manifold. The CR structure on $M$ is induced by the forms $\lambda$, $\mu$ on $M$ such that $\tilde{\lambda} = \pi^*(\lambda)$ and $\tilde{\mu} = \pi^*(\mu)$, with $\pi : \mathcal{M} \to M$ being the projection which forgets about the $\mathbb{R}$ factor in $\mathcal{M}$.
- Since a null Maxwell field in the spacetime induces the CR structure as above, then the congruence in $\mathcal{M}$, being tangent to the $\mathbb{R}$ factor, is null geodesic and shearfree.
- Now the construction in the other direction can be attempted. Given a 3-dimensional CR structure $(M, (\lambda, \mu))$ one considers its lift to the spacetime $\mathcal{M} = \mathbb{R} \times M$, which is then naturally equipped with a null congruence of shearfree geodesics tangent to the $\mathbb{R}$ factor. Then the null Maxwell field $\mathcal{F} = f\lambda \wedge \mu$ is called *aligned* with this congruence.

Thus let us assume that in addition to the strict pseudoconvexity, and to the assumption that the lifted spacetime satisfies equations (a), we also have a nonvanishing null Maxwell field aligned with the congruence associated to CR structure $(M, (\lambda, \mu))$. This additional assumption will play the role of our equation $(2^{\text{nd}}CR)$. Assuming this, we are guaranteed the existence of a complex function $f$ on $M$ such that

$$\mathrm{d}(f\lambda \wedge \mu) = 0.$$

For $(M, (\lambda, \mu))$ as in (3.20)-(3.21) we easily check that this equation is equivalent to

$$(\bar{\partial} f + \bar{c} f)\lambda \wedge \mu \wedge \bar{\mu} = 0.$$

This in turn is equivalent to a single complex equation [52]

$$\partial \bar{f} + c\bar{f} = 0 \tag{3.45}$$

on $M$, which is our new $(2^{\text{nd}}CR)$. If we now introduce a function $\bar{\eta}'$ related to $f$ by

$$\partial_0 \eta' = f, \tag{3.46}$$

we have $\partial_0 \bar{\eta}' \neq 0$, since otherwise the Maxwell field would vanish. Moreover, inserting the definition (3.46) in the Maxwell equation (3.45), we see that if a nonvanishing $f$ satisfies (3.45), then the corresponding $\eta'$ satisfies

$$\partial\partial_0\bar{\eta}' + c\partial_0\bar{\eta}' = 0. \tag{3.47}$$

Using the commutator (3.43) we conclude that this equation is finally equivalent to $\partial_0\partial\bar{\eta}' = 0$. Since, as we have already noticed $\partial_0\bar{\eta}' \neq 0$, our present $\eta'$ satisfies all the assumptions of Lemma 3.23. Using it we define a CR function $\eta$ which is independent of $\zeta$. Thus we have proven Theorem 2.3 in the direction (i) and (ii) $\Rightarrow$ embeddability.

To get the converse we do as follows. Assuming embeddability, we have two independent CR functions. Let us choose one, say $\zeta$, such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$. Then, using $\zeta$, we construct a spacetime whose Ricci tensor satisfies equations (a), as in the proof of Theorem 2.2. After achieving this we, in particular, have embeddability $\Rightarrow$ (i). In addition, we have $\mu = \mathrm{d}\zeta$ and $\lambda$ of (3.21). Now we take an independent CR function, say $\eta$. Because of the independence condition $\mathrm{d}\zeta \wedge \mathrm{d}\eta \neq 0$, we have that $\partial_0\eta \neq 0$. Then we define $f = \partial_0\eta \neq 0$ and observe that $\mathcal{F} = f\lambda \wedge \mu$ satsifies $\mathrm{d}\mathcal{F} = 0$. This provides us with a nontrivial null aligned Maxwell field, proving that embeddability $\Rightarrow$ (ii). Thus Theorem 2.3 is proven.

This completes our discussion of the existence of a null Maxwell field in the spacetime. We mention however that Trautman [57] has conjectured that Theorem 2.3 remains valid without condition (i).

*3.3.2. Petrov type II or D.* We now return to the pure Einstein situation, in which we have a strictly pseudoconvex CR manifold $M$ whose lifted spacetime $(\mathcal{M}, g)$ satisfies Einstein equations (a). We work in the gauge $t \equiv 0$ and we impose further Einstein equations on the lifted spacetime. From now on we will always assume that the lifted spacetime satsifies Einstein's equation (a), (b) and one of the equations (c), namely $R_{13} = 0$. These, according to Remark 3.17 are equivalent to $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$. As a consequence we are guaranteed the existence of a complex function $m$ on the CR manifold $M$ such that

$$\partial m + 3cm = 0 \tag{3.48}$$

(compare with (3.35) assuming $t \equiv 0$).

Einstein's equations (a), (b) and $R_{13} = 0$ do not guarantee that $m$ is nonvanishing. This shall be *assumed*, and the equation (3.48) with $m \neq 0$ will be our new $(2^{\mathrm{nd}}CR)$.

The assumption about the existence of a *nonvanishing* $m$ has a clear spacetime meaning. This is due to equation (3.36). It says that $m \neq 0$ at a point if and only if the spacetime metric at this point is of no more special Petrov type than $II$ or $D$. So let us assume that a strictly pseudoconvex 3-dimensional CR manifold $M$ lifts to a spacetime of Petrov type $II$ or $D$, but no more algebraically special, which in addition satisfies Einstein's equations (a), (b) and $R_{13} = 0$. Having assumed this we replace equation (3.48) with an equation for the complex function $\eta'$ such that

[22]

$$m = (\partial_0 \bar{\eta}')^3. \tag{3.49}$$

By our assumption about the Petrov type we have

$$\partial_0 \bar{\eta}' \neq 0.$$

Moreover, inserting (3.49) into (3.48), after the trivial simplification which uses the assumed $\partial_0 \bar{\eta}' \neq 0$, we get $\partial \partial_0 \bar{\eta}' + c \partial_0 \bar{\eta}' = 0$. This is again equation (3.47) for $\eta'$ satisfying $\partial_0 \bar{\eta}' \neq 0$. This means that the argument from the previous subsection applies, and using Lemma 3.23, we can modify $\eta'$ to $\eta$ being the second CR function on $M$. This proves the following theorem.

***Theorem 3.24.*** *Assume that a strictly pseudoconvex CR manifold M*

(i) *admits a lift to a spacetime satisfying Einstein's equations* $\mathrm{Ric}(g) = \Lambda g + \Phi \lambda \otimes \lambda$, *and*

(ii) *has Petrov type II or D, but no more special.*

*Then such a CR structure is locally CR embeddable.*

At this stage it is worthwhile to note that if we have $m$ satisfying (3.35) we can use its associated $\eta'$ to define $f$ by formula (3.46). Then we can use this $f$ to define a Maxwell 2-form $\mathcal{F}$ by $\mathcal{F} = f \lambda \wedge \mu$. Due to (3.47) this $\mathcal{F}$ satisfies the Maxwell equations $\mathrm{d}\mathcal{F} = 0$. Thus the lifted spacetime of Theorem 3.24 admits an aligned null Maxwell field.

Also the converse to Theorem 3.24, namely Theorem 2.5, can now be proven, using a similar argument.

*Proof of Theorem 2.5.* Indeed, given an embeddable strictly pseudoconvex CR manifold $M$, we choose one of its CR functions $\zeta$ such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$ to define $\mu = \mathrm{d}\zeta$ and $\lambda$ satisfying (3.21). Then we take $t \equiv 0$ and $s \equiv 0$. To construct an Einstein spacetime satisfying $\mathrm{Ric}(g) = \Lambda g + \Phi \lambda \otimes \lambda$ we need first to find a complex function $m$ such that equation (3.35) with $t \equiv 0$ is satisfied. We can do it in two ways. Either we choose $m \equiv 0$, or we can prove that we can find $m \neq 0$ satisfying (3.35).

Let us first consider the second possibility. Since our real analytic CR manifold is locally embeddable [1] we are guaranteed that a second CR function $\eta$, independent of $\zeta$ exists on $M$. Thus we have $\partial_0 \eta \neq 0$. Then we define $m = (\partial_0 \eta)^3$, which obviously does not vanish. Because $\eta$ satisfies the tangential CR equation, we easily get that our $m$ satsifies $\partial m + 3cm = 0$. After determining $\eta$ we must solve the last of the reduced Einstein equations $\mathrm{Ric}(g) = \Lambda g + \Phi \lambda \otimes \lambda$, namely equation (3.29) for $p$. This is a *real* equation for a *real* function on the CR manifold $M$. Looking at (3.29) we see that if we specify a real constant $\Lambda$, the only unknown in this equation is $p$, since now $c$, $t \equiv 0$ and $m$, as well as $\partial$ and $\bar{\partial}$ are just specified. For every fixed constant $\Lambda$, this equation is a second order real

PDE in 3-dimensions, with a quite well behaved nonlinear part. We do not know for sure about its solvability unless we assume real analyticity in the variables appearing in it. To have this it is enough to assume that the CR manifold $M$ is *real analytic*. Then we can always find a local solution for $p$. Inserting this $p$, and $m$, $t \equiv 0$, $s \equiv 0$, $c$ into (3.19), with functions $P$, $W$ and $H$ as in (3.23), (3.24), (3.25), (3.26), (3.28) we define a metric $g$. This satisfies $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$, with our fixed $\Lambda$ and some real function $\Phi$, which is determined by all our choices. Due to the comitted choice of $\eta$, we have $m \neq 0$, therefore the lifted spacetime is of Petrov type $II$ or $D$. This proves Theorem 2.5. □

***Remark 3.25.*** We strongly believe that equation (3.29) with arbitrary sufficiently smooth functions $t$, $c$, $m$ and arbitrary real constant $\Lambda$, has a local nonvanishing solution for $p$ on any sufficiently smooth CR manifold $M$, and that it could be proved by standard methods. If this is true then we could replace Theorem 2.5 with a stronger one, in which the term 'real analytic' would be replaced by 'sufficiently smooth embeddable'.

The authors are unaware of a precise reference to the literature in which the existence of nonzero solutions to (3.29), without the real analyticity assumption, is proved. In the rest of this section we will *assume* that it is true.

Let us now discuss the first possibility mentioned above. Actually, instead of using the second CR function, we could have chosen $m \equiv 0$ in addition to $t \equiv 0$ and $s \equiv 0$. Then inserting these functions into the equation (3.29) for $p$ and fixing a constant $\Lambda$, we conclude that it admits a local solution. Thus defining the metric as before we again get a spacetime with $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$. There is however an important difference between this situation and the one considered before. The spacetime now has $m \equiv 0$, so that it is of Petrov $III$ or its specialization. Even more important is the fact that in constructing the metric now we *did not use the second CR function*. Thus, modulo our current assumption, we have the following corollary.

***Corollary 3.26.*** *Every sufficiently smooth strictly pseudoconvex 3-dimensional CR structure which admits one CR function $\zeta$ such that $\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} \neq 0$ has a lift to a spacetime which satisfies Einstein equations $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$ and which is of Petrov type $III$, or its specializations $N$ or $0$.*

This means that for the price of generality in the Petrov type, we may replace the embeddability assumption in Theorem 2.5, by a *weaker* assumption about the mere existence of one CR function, and still get the Einstein condition $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$ for the lift.

*3.3.3.* $\mathrm{Ric}(g) = 0$ *and Petrov type III.* If we assume that our strictly pseudoconvex CR structure $(M, (\lambda, \mu))$ has a lift to a spacetime $(\mathcal{M}, g)$ satisfying Einstein's equations $\mathrm{Ric}(g) = \Lambda g + \Phi\lambda \otimes \lambda$, which in addition, is of Petrov type $III$ or its specializations, then *without further assumptions* about $(\mathcal{M}, g)$ we are *unable* to produce the second CR function for $M$. Of course, to get the embeddability of

$M$, we may assume that our spacetime admits an aligned null Maxwell field and then use Theorem 2.3. But, if we lack a Maxwell field detector, we need to invent a new ($2^{\text{nd}}CR$) equation that guarantees the existence of a second CR function $\eta$. This can be done by imposing more special restrictions on $\mathrm{Ric}(g)$, as we will do in this section.

So now we assume that the lifted spacetime of our CR manifold $(M, (\lambda, \mu))$ is of Petrov type *III* or more special, and that it satsifies Einstein's equations (a), (b) and the first of equations (c), namely $R_{13} = 0$. We work in the gauge

$$t \equiv 0$$

and, due to our assumption about the Petrov type, we have

$$m \equiv 0.$$

Then, guided by the theory of exact solutions of Einstein equations we introduce a function

$$I = \partial(\partial \log p + c) + (\partial \log p + c)^2. \tag{3.50}$$

This enable us to significantly simplify the formulae for the last componenent of the Ricci tensor and the Weyl scalar coefficient $\Psi_3$. These are given in the following proposition.

***Proposition 3.27.*** *If the lifted spacetime satsifies the Einstein equations* (3.23), (3.24), (3.25), (3.26), (3.27), (3.28), (3.29), (3.35), *then the Ricci tensor component* $R_{33}$ *is given by*

$$R_{33} = 8\frac{\cos^4\left(\frac{r+s}{2}\right)}{p^4}(\partial + 2c)(p^2\partial\bar{I}) + O(\Lambda) \tag{3.51}$$

*and the Weyl scalar* $\Psi_3$ *is given by*

$$\Psi_3 = 2i\,\partial\bar{I}\,\frac{e^{(i(r+s))/2}}{p^2}\cos^3\left(\frac{r+s}{2}\right) + O(\Lambda), \tag{3.52}$$

*as* $\Lambda \to 0$.

***Remark 3.28.*** The omitted $O(\Lambda)$ term in $R_{33}$ reads

$$O(\Lambda) = -8\Lambda\cos^4\left(\frac{r+s}{2}\right)\left(\frac{4}{3}\Lambda p^2 + 6(\bar{c}\partial + c\bar{\partial})\log p + 12\partial\log p\,\bar{\partial}\log p \right.$$
$$\left. + 3c\bar{c} - \frac{1}{2}(\partial\bar{c} + \bar{\partial}c) - 2i\partial_0\log p\right),$$

and the omitted $\Lambda$ term in $\Psi_3$ is

$$O(\Lambda) = -4i\Lambda(2\bar{\partial}\log p + \bar{c})\, e^{(i(r+s))/2}\cos^3\left(\frac{r+s}{2}\right).$$

We note that the $O(\Lambda)$ term in $R_{33}$ is *complex*. It includes the purely imaginary $-2i\partial_0 \log p$. Thus the first term in $R_{33}$ is *also complex*, since $R_{33}$ is *real*. This means that the first term in $R_{33}$ includes a purely imaginary $\Lambda$ term which cancels $-2i\partial_0 \log p$. If $\Lambda = 0$ the first term in $R_{33}$ becomes real, and $R_{33}$ becomes real as it should be.

The appearance of the unwanted $O(\Lambda)$ terms in (3.51) and (3.52) forces us to assume that $\Lambda = 0$. So in our search for the second CR function we will assume from now on that the lifted spacetime has vanishing cosmological constant

$$\Lambda = 0.$$

Then, if we in addidtion assume that the lifted spacetime is Ricci flat, we may easily use the function $\partial\bar{I}$ to construct the second CR function.

Let us thus assume that the lifted spacetime has $\Lambda = 0$ and $R_{33} = 0$ everywhere, and that in addition it is of strictly Petrov type $III$. This last assumption means that $\partial\bar{I} \neq 0$. Moreover, since $R_{33} = 0$ and $\Lambda = 0$ guarantees that

$$(\partial + 2c)(p^2\partial\bar{I}) = 0, \tag{3.53}$$

we may use our standard trick of considering $\eta'$ related to $I$ via:

$$p^2\partial\bar{I} = (\partial_0\bar{\eta}')^2.$$

Inserting this into (3.53) and utilising the assumption $\partial_0\bar{\eta}' \neq 0$ about the Petrov type, we again obtain $\partial_0\partial\bar{\eta}' = 0$, which is enough to conclude that the following theorem is true:

***Theorem 3.29.*** *Assume that a strictly pseudoconvex CR manifold M*

(i) *admits a lift to the spacetime satisfying Einstein's equations* $\mathrm{Ric}(g) = 0$, *and*

(ii) *has Petrov type $III$, but no more special.*

*Then such a CR structure is locally embeddable.*

Of course, as in the end of the last subsection we can now use our second CR function, to construct an aligned null Maxwell field in our Ricci flat spacetime of type $III$.

*3.3.4. Petrov type N.* Staying in the gauge $t \equiv 0$, with the cosmological constant set to $\Lambda = 0$, an assumption that our spacetime is of type $N$ means that $m = 0$ and $\partial\bar{I} = 0$ everywhere (see (3.36) and (3.52)). In the context of our search for the second CR function this is a very fortunate Petrov type. Indeed, assuming type $N$ we have

$$\partial \bar{I} \equiv 0,$$

which not only *implies* that $R_{33} = 0$, but also implies that $I$ is a CR function! The only question is if this CR function is *independent* of $\zeta$. For this we need

$$\partial_0 \bar{I} \neq 0. \tag{3.54}$$

To conclude the independence we calculate the last Weyl scalar $\Psi_4$. Assuming that $t \equiv 0$, $\Lambda = 0$, $m \equiv 0$ and $\partial \bar{I} \equiv 0$ we get:

$$\Psi_4 = 2i\, \partial_0 \bar{I}\, \frac{\mathrm{e}^{-(i(r+s))/2}}{p^2} \cos^3\left(\frac{r+s}{2}\right).$$

Thus the condition (3.54) for $I$ to be an independent CR function is equivalent to the condition on Petrov type not to be degenerate to the conformally flat type 0. Thus we have the following theorem.

***Theorem 3.30.*** *Assume that a strictly pseudoconvex CR manifold M*

(i) *admits a lift to a spacetime satisfying Einstein equations* (a)*,* (b) *with* $\Lambda = 0$ *and* $R_{13} = 0$ *and which, in addition,*

(ii) *has Petrov type N and is nowhere degenerate to Petrov type 0.*

*Then such a CR structure is locally embeddable. Moreover, in such case the spacetime is Ricci flat.*

The remark about the existence of the aligned Maxwell field, as at the end of the previous subsections, applies here also.

*3.3.5. Conformally flat case.* If we only know that among the lifted spacetimes of a strictly pseudoconvex CR structure there is a Minkowski metric, we may proceed with our search for the second function in the same spirit as we were doing in the previous subsections. However, in such a case there is a simpler more elegant geometric way of achieving our goal. This comes from Penrose's *twistor theory*.

Let us now forget about all the results from the entire Section 3 and assume that we are given a 3-dimensional CR structure $(M, (\lambda, \mu))$, not neccessarily strictly pseudoconvex (!), which has a lift to a *conformally flat* spacetime $\mathcal{M}$. We do not need the Einstein equations for the rest of the argument. It is known (see e.g. [36]) that the space of all null geodesics in a neighbourhood in $\mathcal{M}$ is a 5-dimensional CR manifold $N$, which is naturally locally CR embedded in $\mathbb{C}^3$. Actually $N$ may be identified with an open set in the following *real projective quadric*

$$PN = \{\mathbb{CP}^3 \ni (W_1 : W_2 : W_3 : W_4) \mid |W_1|^2 + |W_2|^2 - |W_3|^3 - |W_4|^2 = 0\}$$

CR embedded in $\mathbb{CP}^3$. The manifold $PN$ is called the *space of projective twistors*.

Since points of $N$ are null geodesics in $\mathcal{M}$ then a congruence of null geodesics in $\mathcal{M}$ is just a 3-dimensional manifold $M_N$ in $N$. Crucial to our argument is the fact that if a congruence of null geodesics in $\mathcal{M}$ is *shearfree* then $M_N$ is a *CR submanifold* [33] of the 5-dimensional *embedded* CR manifold $N$. Thus having a congruence of null and shearfree geodesics in $\mathcal{M}$ we first are guaranteed that $M$ is locally CR embedded as a submanifold $M_N$ in $\mathbb{C}^3$. But this implies that $M$ also has a local CR embedding in a $\mathbb{C}^2$, see [1, 12]. The argument is very simple:

Take a point $p \in M_N$, and define $\mathbb{C}^2$ to be the smallest complex vector space which contains $\mathrm{T}_p M_N$. The local projection $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ is holomorphic, hence its restriction $\varphi = \pi_{|M_N}$ is a CR map, whose image in $\mathbb{C}^2$ is the desired CR embedding. This proves the following theorem.

***Theorem 3.31.*** *Every 3-dimensional CR manifold which has a lift to a conformally flat spacetime is locally embeddable.*

Using Theorems 3.24, 3.29, 3.30, and 3.31 we obtain Theorem 2.4.

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