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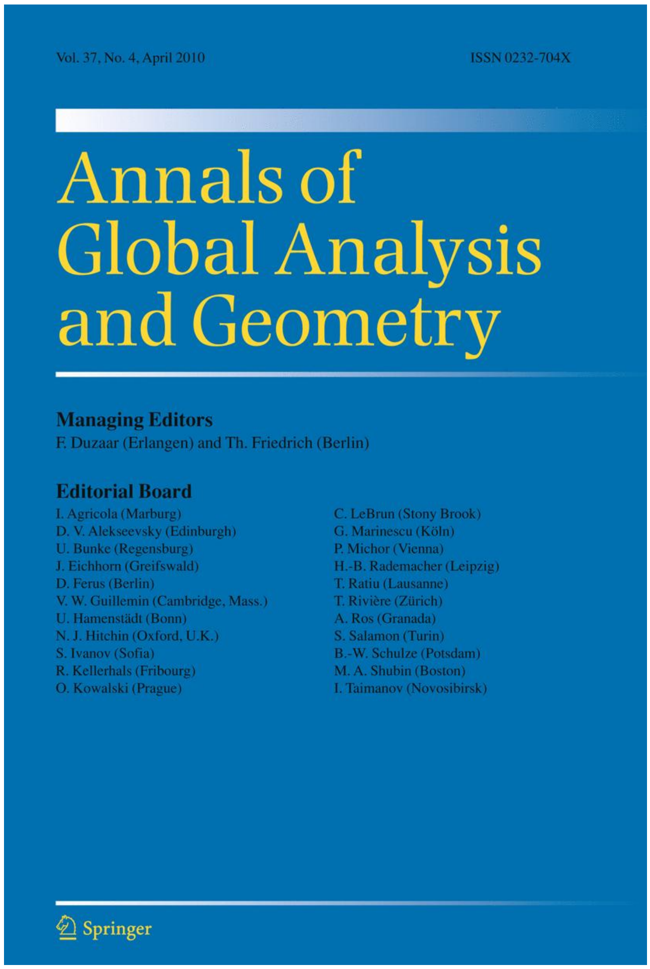
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ORIGINAL PAPER

Construction of conjugate functions

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Abstract We find all pairs of real analytic functions f and g in $\mathbb{R}^n$ such that $|\nabla f| = |\nabla g|$ and $(\nabla f)(\nabla g) = 0$.

Keywords Conjugate functions · Spinors · Kerr theorem

Mathematics Subject Classification (2000) 53C28 · 53C43 · 58E20

1 Introduction

We consider the following problem: Let $n \geq 3$. Find all pairs of functions (f, g), $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$|\nabla f| = |\nabla g|, \quad (\nabla f)(\nabla g) = 0.$$

Functions f and g constituting such a pair are called *conjugate* [1].

The notion of conjugate functions can be traced back to Jacobi's paper [4]. It appears naturally in the context of *harmonic morphisms* [2], i.e. the maps F between two Riemannian manifolds (M, g) and (N, h), $F : M \to N$, which pullback harmonic functions locally defined on N to harmonic functions locally defined on M. Taking as $M = \mathbb{R}^n$ and $N = \mathbb{R}^2$, and as $F = (f, g)$, it follows [2] that such F is a harmonic morphism, if and only if both f and g are harmonic and conjugate to each other.

Finding a pair of conjugate functions (f, g) is equivalent to finding a complex-valued function $h : \mathbb{R}^n \to \mathbb{C}$ such that

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$$(\nabla h)(\nabla h) = 0.$$

Having h, one gets f and g as its real and imaginary parts, respectively.

While solving the above-stated problem at the algebraic level, one needs to find a nice representation for *null* vectors in $\mathbb{C}^n$. In dimension $n = 3$ this may be done by means of spinors.

2 General analytic solution in dimension three

The solution of the problem in dimension $n = 3$ is as follows.

We consider $\mathbb{R}^3$ with coordinates (x, y, z). The most general form of ∇h is given in terms of two complex-valued functions (a spinor) (ϕ_1, ϕ_2)

$$h_x = \phi_1^2 - \phi_2^2, \quad h_y = i(\phi_1^2 + \phi_2^2), \quad h_z = 2\phi_1\phi_2.$$

Now, the integrability condition for existence of h is

$$\mathrm{d}\,[(\phi_1^2 - \phi_2^2)\mathrm{d}x + i(\phi_1^2 + \phi_2^2)\mathrm{d}y + 2\phi_1\phi_2\mathrm{d}z] = 0, \tag{2.1}$$

which is equivalent to

$$[\phi_1(\mathrm{d}x + i\mathrm{d}y) + \phi_2\mathrm{d}z] \wedge \mathrm{d}\phi_1 + [-\phi_2(\mathrm{d}x - i\mathrm{d}y) + \phi_1\mathrm{d}z] \wedge \mathrm{d}\phi_2 = 0.$$

This motivates introduction of two functions

$$X_1 = \phi_1(x + iy) + \phi_2 z, \quad X_2 = -\phi_2(x - iy) + \phi_1 z.$$

Having them the integrability condition is

$$\mathrm{d}(X_1\mathrm{d}\phi_1 + X_2\mathrm{d}\phi_2) = 0.$$

Its general solution analytic in (x, y, z) is

$$X_1 = F_1(\phi_1, \phi_2) \quad X_2 = F_2(\phi_1, \phi_2),$$

where $F = F(\phi_1, \phi_2)$ is a complex-valued function analytic in both variables, and $F_1 = \frac{\partial F}{\partial \phi_1}$, $F_2 = \frac{\partial F}{\partial \phi_2}$.

Explicitly, one finds ϕ_1 and ϕ_2 specifying F arbitrarily and solving the algebraic equations

$$\begin{array}{l} \phi_1(x + iy) + \phi_2 z = F_1(\phi_1, \phi_2) \\ -\phi_2(x - iy) + \phi_1 z = F_2(\phi_1, \phi_2) \end{array} \quad \text{for} \quad (\phi_1, \phi_2). \tag{2.2}$$

Once (ϕ_1, ϕ_2) is found then h is given by a simple integration of, e.g. equation $h_x = \phi_1^2 - \phi_2^2$. We then have

$$h = \int (\phi_1^2 - \phi_2^2)\mathrm{d}x.$$

This determines h modulo an addition of an arbitrary smooth function $s = s(y, z)$. Finally, to fix this function, we use $h_y = i(\phi_1^2 + \phi_2^2)$ and $h_z = 2\phi_1\phi_2$. These equations have a solution for s because Eq. 2.2, which we used to define (ϕ_1, ϕ_2), are equivalent, in an analytic category, to the integrability conditions (2.1). This solves the problem.

So, in dimension $n = 3$ the conclusion is that the general analytic solution is generated by one complex analytic function F of two variables.

Example As a simple example, we take a quadratic function F:

$$F = \alpha\phi_1^2 + 2\beta\phi_1\phi_2 + \gamma\phi_2^2 + \mu\phi_1 + \nu\phi_2,$$

where $\alpha, \beta, \gamma, \mu, \nu$ are complex constants. Then, Eq. 2.2 are

$$\begin{aligned} \phi_1(x+iy) + \phi_2 z &= 2\alpha\phi_1 + 2\beta\phi_2 + 2\mu \\ -\phi_2(x-iy) + \phi_1 z &= 2\beta\phi_1 + 2\gamma\phi_2 + 2\nu. \end{aligned}$$

.

Their solutions are given by

$$\begin{aligned} \tfrac{1}{2}\phi_1 &= \frac{\mu(2\gamma + x - iy) + \nu(z - 2\beta)}{(z-2\beta)^2 - (2\alpha - x - iy)(2\gamma + x - iy)} \\ \tfrac{1}{2}\phi_2 &= \frac{\nu(2\alpha - x - iy) + \mu(z - 2\beta)}{(z-2\beta)^2 - (2\alpha - x - iy)(2\gamma + x - iy)}. \end{aligned}$$

Now, according to the above procedure, the function h is given by

$$\begin{aligned} \tfrac{1}{4}h = \tfrac{1}{4}\int(\phi_1^2 - \phi_2^2)\mathrm{d}x = &\frac{\mu^2(2\gamma + x - iy) + \nu^2(2\alpha - x - iy) + 2\mu\nu(z-2\beta)}{(2\alpha - x - iy)(2\gamma + x - iy) - (z - 2b)^2} \\ &+ s(y,z), \end{aligned} \tag{2.3}$$

where $s = s(y,z)$ is an arbitrary (sufficiently smooth) complex-valued function of variables y and z. Imposing conditions $h_y = i(\phi_1^2 + \phi_2^2)$ and $h_z = 2\phi_1\phi_2$ we find that $s = s_0 =$ const $\in \mathbb{C}$. Taking $f = \mathrm{Re}(h)$ and $g = \mathrm{Im}(h)$ with h as in (2.3) and $s = s_0$, one can easily check[1] that $|\nabla f| = |\nabla g|$ and $(\nabla f)(\nabla g) = 0$.

3 General analytic solution in arbitrary dimension

Although, for arbitrary n, we cannot use spinors anymore, a similar procedure can be used to get a general analytic solution for any $n \geq 3$. This uses a bilinear parametrization of null vectors in $\mathbb{C}^n$, known in the theory od minimal surfaces [3].

In $\mathbb{R}^n$ we introduce coordinates $(x_1, x_2, \ldots, x_n)$ and parametrize a general null vector $(h_{x_1}, h_{x_2}, \ldots, h_{x_n})$ by means of $n-1$ complex functions $(\phi_1, \phi_2, \ldots, \phi_{n-1})$ via

$$\begin{aligned} h_{x_1} &= \phi_1^2 + \phi_2^2 + \cdots + \phi_{n-2}^2 - \phi_{n-1}^2 \\ h_{x_2} &= i(\phi_1^2 + \phi_2^2 + \cdots + \phi_{n-2}^2 + \phi_{n-1}^2) \\ h_{x_3} &= 2\phi_1\phi_{n-1} \\ &\ldots \\ h_{x_k} &= 2\phi_{k-2}\phi_{n-1} \\ &\ldots \\ h_{x_n} &= 2\phi_{n-2}\phi_{n-1}. \end{aligned} \tag{3.1}$$

One easily checks[2] that $(\nabla h)(\nabla h) = 0$.

[1] It is a straightforward but *lengthy* algebra. Using version 7.0 of the Mathematica symbolic calculation program, and Dell XPS computer with Intel Core Duo processor, this check lasted 333 s of the CPU time!

[2] I thank J.C. Wood (private communication) for pointing out this.

Now the integrability condition

$$\mathrm{d}\,[h_{x_1}\mathrm{d}x_1 + h_{x_2}\mathrm{d}x + \cdots + h_{x_n}\mathrm{d}x_n] = 0$$

for existence of $h : \mathbb{R}^n \to \mathbb{C}$ satisfying (3.1) is

$$\mathrm{d}[X_1\mathrm{d}\phi_1 + X_2\mathrm{d}\phi_2 + \cdots + X_{n-1}\mathrm{d}\phi_{n-1}] = 0$$

with $n-1$ functions $(X_1, X_2, \ldots, X_{n-1})$ given by

$$\begin{aligned}
X_1 &= \phi_1(x_1 + ix_2) + \phi_{n-1}x_3\\
X_2 &= \phi_2(x_1 + ix_2) + \phi_{n-1}x_4\\
&\ldots\\
X_{n-2} &= \phi_{n-2}(x_1 + ix_2) + \phi_{n-1}x_n\\
X_{n-1} &= -\phi_{n-1}(x_1 - ix_2) + \phi_1x_3 + \phi_2x_4 + \cdots + \phi_{n-2}x_n.
\end{aligned}$$

Thus, similarly to the $n = 3$ case, we first choose an arbitrary holomorphic function $F = F(\phi_1, \phi_2, \ldots, \phi_{n-1})$ of $n-1$ complex variables and, denoting its derivatives by $F_i = \frac{\partial F}{\partial \phi_i}$, $i = 1, 2, \ldots, n-1$, solve algebraic equations

$$\begin{aligned}
F_1 &= \phi_1(x_1 + ix_2) + \phi_{n-1}x_3\\
F_2 &= \phi_2(x_1 + ix_2) + \phi_{n-1}x_4\\
&\ldots\\
F_{n-2} &= \phi_{n-2}(x_1 + ix_2) + \phi_{n-1}x_n\\
F_{n-1} &= -\phi_{n-1}(x_1 - ix_2) + \phi_1x_3 + \phi_2x_4 + \cdots + \phi_{n-2}x_n,
\end{aligned} \tag{3.2}$$

for $(\phi_1, \phi_2, \ldots, \phi_{n-1})$ as functions of $(x_1, x_2, \ldots, x_n)$. Then, h is found by simple integration to be, e.g. $h = \int 2\phi_{n-2}\phi_{n-1}\mathrm{d}x_n$. This defines h modulo an addition of an arbitrary function $s = s(x_1, x_2, \ldots, x_{n-1})$ of the remaining variables $(x_1, x_2, \ldots, x_{n-1})$, which is fixed by imposing the first $(n-1)$ conditions (3.1) on such an h. This solves the problem for any $n \geq 3$.

Remark 3.1 We note that, under some regularity assumptions on the function F, Eq. 3.2 associate with any solution h an n-dimensional real submanifold M_n embedded in $\mathbb{C}^{n-1}$. One gets the equations for this submanifold in coordinates $(\phi_1, \phi_2, \ldots, \phi_{n-1}) \in \mathbb{C}^{n-1}$ by eliminating the real parameters $(x_1, x_2, \ldots, x_n)$ from Eq. 3.2. For example if $n = 3$, given $F = F(\phi_1, \phi_2)$, we have a 3-dimensional real hypersurface M_3 in $\mathbb{C}^2$ defined by

$$M_3 = \{(\phi_1, \phi_2) \in \mathbb{C}^2 \;:\; \mathrm{Im}(F_1\bar{\phi}_2 - F_2\bar{\phi}_1) = 0\}.$$

In the case of $n = 4$, given $F = F(\phi_1, \phi_2, \phi_3)$ we have

$$\begin{aligned}
M_4 = \{&(\phi_1, \phi_2, \phi_3) \in \mathbb{C}^3 :\\
&0 = \mathrm{Im}\big(\,\bar{\phi}_3\,[F_1(\phi_2^2\bar{\phi}_1^2 + |\phi_2|^4 - |\phi_1\phi_3|^2 - |\phi_3|^4)\\
&\qquad -F_2(|\phi_2|^2\phi_1\bar{\phi}_2 + (|\phi_1|^2 + |\phi_3|^2)\phi_2\bar{\phi}_1) + F_3\phi_3(\phi_1\bar{\phi}_2^2 + (|\phi_1|^2 + |\phi_3|^2)\bar{\phi}_1)]\,\big)\\
&0 = \mathrm{Im}\big(\,\bar{\phi}_3\,[F_1(|\phi_1|^2\phi_2\bar{\phi}_1 + (|\phi_2|^2 + |\phi_3|^2)\phi_1\bar{\phi}_2)\\
&\qquad -F_2(\phi_1^2\bar{\phi}_2^2 + |\phi_1|^4 - |\phi_2\phi_3|^2 - |\phi_3|^4) - F_3\phi_3(\phi_2\bar{\phi}_1^2 + (|\phi_2|^2 + |\phi_3|^2)\bar{\phi}_2)]\,\big)\}.
\end{aligned}$$

We also note that submanifolds M_n are foliated by $(n-2)$-dimensional leaves which are the images under the map $(x_1, x_2, \ldots, x_n) \mapsto (\phi_1, \phi_2, \ldots, \phi_{n-1})$ of the intersections in $\mathbb{R}^n$ of the level surfaces $f = c_1$ and $g = c_2$ corresponding to the conjugate functions (f, g)

associated with h. Thus, in particular, M_3 has a distinguished foliation by real curves and M_4 has a distinguished foliation by real surfaces.

4 Solution in terms of trilinear parametrization of null vectors in dimension five

It is interesting to note that in dimension $n = 5$, a slightly different procedure, based on a *tri*linear parametrization of complex null vectors, can be used to find solutions to the problem.

In $\mathbb{R}^5$ we use coordinates (x, y, z, t, u). Now we write a null vector $(h_x, h_y, h_z, h_t, h_u)$ in terms of *six* functions $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ as follows:

$$\begin{aligned} h_x &= i(\phi_1\phi_2 + \phi_3\phi_4)\phi_5 - \tfrac{1}{2}(\phi_1^2 + \phi_2^2 - \phi_3^2 - \phi_4^2)\phi_6, \\ h_y &= i(-\phi_1\phi_3 + \phi_2\phi_4)\phi_5 + (\phi_2\phi_3 - \phi_1\phi_4)\phi_6 \\ h_z &= (-\phi_1\phi_2 + \phi_3\phi_4)\phi_5 - \tfrac{i}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2)\phi_6 \\ h_t &= (\phi_1\phi_3 + \phi_2\phi_4)\phi_5 \\ h_u &= (\phi_1\phi_3 + \phi_2\phi_4)\phi_6. \end{aligned} \tag{4.1}$$

Check that $h_x^2 + h_y^2 + h_z^2 + h_t^2 + h_u^2 = 0$. Remarkably again, the integrability conditions

$$\mathrm{d}\,[h_x \mathrm{d}x + h_y \mathrm{d}y + h_z \mathrm{d}z + h_t \mathrm{d}t + h_u \mathrm{d}u] = 0$$

for the existence of $h : \mathbb{R}^5 \to \mathbb{C}$ satisfying (4.2), are equivalent to

$$\mathrm{d}(X\mathrm{d}\phi_1 + Y\mathrm{d}\phi_2 + Z\mathrm{d}\phi_3 + T\mathrm{d}\phi_4 + U\mathrm{d}\phi_5 + W\mathrm{d}\phi_6) = 0,$$

where

$$\begin{aligned} X &= \phi_3\phi_5(t - iy) - (\phi_1\phi_6 - i\phi_2\phi_5)(x + iz) - \phi_4\phi_6 y + \phi_3\phi_6 u \\ Y &= \phi_4\phi_5(t + iy) - (\phi_2\phi_6 - i\phi_1\phi_5)(x + iz) + \phi_3\phi_6 y + \phi_4\phi_6 u \\ Z &= \phi_1\phi_5(t - iy) + (\phi_3\phi_6 + i\phi_4\phi_5)(x - iz) + \phi_2\phi_6 y + \phi_1\phi_6 u \\ T &= \phi_2\phi_5(t + iy) + (\phi_4\phi_6 + i\phi_3\phi_5)(x - iz) - \phi_1\phi_6 y + \phi_2\phi_6 u \\ U &= \phi_1\phi_3(t - iy) + \phi_2\phi_4(t + iy) + i\phi_1\phi_2(x + iz) + i\phi_3\phi_4(x - iz) \\ W &= -\tfrac{1}{2}(\phi_1^2 + \phi_2^2)(x + iz) + \tfrac{1}{2}(\phi_3^2 + \phi_4^2)(x - iz) + (\phi_2\phi_3 - \phi_1\phi_4)y + (\phi_1\phi_3 + \phi_2\phi_4)u. \end{aligned}$$

Thus, taking an analytic function $F = F(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ of six variables, and solving algebraic equations

$$\begin{aligned} F_1 &= \phi_3\phi_5(t - iy) - (\phi_1\phi_6 - i\phi_2\phi_5)(x + iz) - \phi_4\phi_6 y + \phi_3\phi_6 u \\ F_2 &= \phi_4\phi_5(t + iy) - (\phi_2\phi_6 - i\phi_1\phi_5)(x + iz) + \phi_3\phi_6 y + \phi_4\phi_6 u) \\ F_3 &= \phi_1\phi_5(t - iy) + (\phi_3\phi_6 + i\phi_4\phi_5)(x - iz) + \phi_2\phi_6 y + \phi_1\phi_6 u \\ F_4 &= \phi_2\phi_5(t + iy) + (\phi_4\phi_6 + i\phi_3\phi_5)(x - iz) - \phi_1\phi_6 y + \phi_2\phi_6 u \\ F_5 &= \phi_1\phi_3(t - iy) + \phi_2\phi_4(t + iy) + i\phi_1\phi_2(x + iz) + i\phi_3\phi_4(x - iz) \\ F_6 &= -\tfrac{1}{2}(\phi_1^2 + \phi_2^2)(x + iz) + \tfrac{1}{2}(\phi_3^2 + \phi_4^2)(x - iz) + (\phi_2\phi_3 - \phi_1\phi_4)y + (\phi_1\phi_3 + \phi_2\phi_4)u \end{aligned}$$

for $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)$ as functions of (x, y, z, t, u), we generate a solution to $(\nabla h)(\nabla h) = 0$ via, for example, an integral $h = \int [(\phi_1\phi_3 + \phi_2\phi_4)\phi_6]\mathrm{d}u$.

Acknowledgements This note is motivated by a paper [1]. I thank Michael Eastwood and James C. Wood for clarifying discussions. The presented solution of the problem reminds very much the way in which the Kerr theorem is proved (see, e.g. [5,6]).

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