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LETTER TO THE EDITOR

A four-dimensional example of a Ricci flat metric admitting almost-Kähler non-Kähler structure*

Paweł Nurowski†§ and Maciej Przanowski‡

† Dipartimento di Scienze Matematiche, Universita degli Studi di Trieste, Trieste, Italy
‡ Instytut Fizyki, Politechnika Łódzka, Wólczańska 219, 93-005 Łódź, Poland

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Abstract. We construct an example of Ricci-flat almost-Kähler non-Kähler structure in four dimensions.

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1. Let $\mathcal{M}$ be a 4-manifold equipped with a metric g of signature (++++). The pair $(\mathcal{M}, g)$ is called a Riemannian 4-manifold.

An almost-Hermitian structure on $(\mathcal{M}, g)$ is a tensor field $J : T\mathcal{M} \to T\mathcal{M}$ such that $J^2 = -\mathrm{i}d$ and $g(JX, JY) = g(X, Y)$. An almost-Hermitian structure $(\mathcal{M}, g, J)$ is called Hermitian if J is integrable. Due to the Newlander–Nirenberg theorem this is equivalent to the vanishing of the Nijenhuis tensor $N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$ for J.

Given an almost-Hermitian structure $(\mathcal{M}, g, J)$ one defines the fundamental 2-form ω by $\omega(X, Y) = g(X, JY)$. An almost-Hermitian structure $(\mathcal{M}, g, J)$ is called almost-Kähler if its fundamental 2-form is closed. If, in addition, J is integrable then such structure is called Kähler.

This letter is motivated by the following conjecture [6].

Goldberg's conjecture

The almost Kähler structure of a compact Einstein manifold is necessarily Kähler.

The conjecture was proven in the case of non-negative scalar curvature of the Einstein manifold by Sekigawa in [11]. In recent work [12] he has additionally shown that Goldberg's conjecture holds in four dimensions. This result is relevant for gravitational instantons (see, e.g., [5]), since it implies that any compact Einstein gravitational instanton admitting an almost-Kähler structure is necessarily Kähler.

In this letter we show that the assumption about compactness of the Einstein manifold is essential for the Goldberg conjecture. In particular, we give an explicit example of a Ricci-flat

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§ Permanent address: Instytut Fizyki Teoretycznej, Wydział Fizyki, Uniwersytet Warszawski, ul. Hoża 69, Warszawa, Poland. E-mail address: `nurowski@fuw.edu.pl`

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almost-Kähler non-Kähler structure on a non-compact 4-manifold. This result is given by theorem 1 of point 4.

2. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^4$. Let $\theta^i = (M, \bar{M}, N, \bar{N})$ be four complex-valued 1-forms on $\mathcal{U}$ such that $M \wedge \bar{M} \wedge N \wedge \bar{N} \neq 0$. Using θ^i we define a metric g on $\mathcal{U}$ by

$$g = 2(M\bar{M} + N\bar{N}) := M \otimes \bar{M} + \bar{M} \otimes M + N \otimes \bar{N} + \bar{N} \otimes N.$$

Clearly $(\mathcal{U}, g)$ is a Riemannian 4-manifold.

The Weyl tensor W of the metric g splits into self-dual (W^+) and anti-self-dual (W^-) parts. $(\mathcal{U}, g)$ is said to be (anti-)self-dual iff $(W^+ \equiv 0)$ $W^- \equiv 0$. If $(W^+ \neq 0)$ $W^- \neq 0$ then in every point of $\mathcal{U}$ it defines at most two spinor directions $([\alpha^+, \beta^+])$ $[\alpha^-, \beta^-]$; see e.g. [7, 10]. (W^+) W^- is said to be of type D if (α^+) α^- coincides with (β^+) β^-.

Let $e_i = (m, \bar{m}, n, \bar{n})$ be a basis dual to $\theta^i = (M, \bar{M}, N, \bar{N})$. For any $\xi \in \mathbb{C}$ it is convenient to consider 1-forms

$$M_\xi = \frac{M - \bar{\xi}\bar{N}}{\sqrt{1 + \xi\bar{\xi}}} \qquad N_\xi = \frac{N + \bar{\xi}\bar{M}}{\sqrt{1 + \xi\bar{\xi}}}$$

and vector fields

$$m_\xi = \frac{m - \xi\bar{n}}{\sqrt{1 + \xi\bar{\xi}}} \qquad n_\xi = \frac{n + \xi\bar{m}}{\sqrt{1 + \xi\bar{\xi}}}.$$

The following lemma is well known (see for example [7, 10]).

Lemma 1.

(a) For any value of the complex parameter $\xi \in \mathbb{C} \cup \{\infty\}$ the expressions

$$J_\xi^+ = \mathrm{i}(\overline{M_\xi} \otimes \overline{m_\xi} - M_\xi \otimes m_\xi + \overline{N_\xi} \otimes \overline{n_\xi} - N_\xi \otimes n_\xi)$$
$$J_\xi^- = \mathrm{i}(M_\xi \otimes m_\xi - \overline{M_\xi} \otimes \overline{m_\xi} + \overline{N_\xi} \otimes \overline{n_\xi} - N_\xi \otimes n_\xi)$$

define almost-Hermitian structures on $(\mathcal{U}, g)$.

(b) The fundamental 2-forms corresponding to J_ξ^+ and J_ξ^- are given by, respectively

$$\omega_\xi^+ = \mathrm{i}(M_\xi \wedge \overline{M_\xi} + N_\xi \wedge \overline{N_\xi})$$
$$\omega_\xi^- = \mathrm{i}(\overline{M_\xi} \wedge M_\xi + N_\xi \wedge \overline{N_\xi}).$$

(c) Any almost-Hermitian structure on $(\mathcal{U}, g)$ is given either by one of J_ξ^+ or by one of J_ξ^-. Structures J_ξ^+ are different from J_ξ^-; also, different ξs correspond to different structures.

(d) If the metric g is not self-dual then among J_ξ^+s only at most four structures, corresponding to specific four values of the parameter ξ, may be integrable. Analogously, if the metric g is not anti-self-dual then only at most four J_ξ^-s may be integrable.

3. Let (x^1, x^2, x^3, x^4) be Euclidean coordinates on $\mathcal{U}$. Define

$$z_1 = x^1 + \mathrm{i}x^2 \qquad z_2 = x^3 + \mathrm{i}x^4. \tag{1}$$

Let $\partial_k = \partial/\partial z_k$ and $\partial_{\bar{k}} = \partial/\partial\overline{z_k}$, $k = 1, 2$.

Consider two 1-forms M and N on $\mathcal{U}$ defined by

$$M = f(\mathrm{d}z_1 + h\,\mathrm{d}z_2) \qquad N = \frac{1}{f}\,\mathrm{d}z_2, \tag{2}$$

where $f \neq 0$ (real) and h (complex) are functions on $\mathcal{U}$.

Since $M \wedge \bar{M} \wedge N \wedge \bar{N} = \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2 \neq 0$ then the metric $g = 2(M\bar{M} + N\bar{N})$ equips $\mathcal{U}$ with the Riemannian structure. Consider almost-Hermitian structures J_ξ^+ for such

$(\mathcal{U}, g)$. It is interesting to note that if $\xi = e^{i\phi} =$ constant then the corresponding fundamental 2-form $\omega^+_{e^{i\phi}}$ reads

$$\omega^+_{e^{i\phi}} = i(e^{i\phi}\, dz_2 \wedge dz_1 - e^{-i\phi}\, d\bar{z}_2 \wedge d\bar{z}_1)$$

and is closed. Thus, for any $e^{i\phi} \in \mathbb{S}^1$ we constructed an almost-Kähler structure $(\mathcal{U}, g, J^+_{e^{i\phi}})$. If the functions f and h are general enough, then the metric g has no chance to be self-dual. Moreover, since in such a case there are a finite number of Hermitian structures among J^+_ξ, then most of our structures must be non-Kähler. Summing up we have the following lemma.

Lemma 2. *Let $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ be coordinates on $\mathcal{U}$ as in (1). Then for each value of the real constant $\phi \in [0, 2\pi[$ the metric*

$$g = 2f^2(dz_1 + h\, dz_2)(d\bar{z}_1 + \bar{h}\, d\bar{z}_2) + 2\frac{1}{f^2}\, dz_2\, d\bar{z}_2 \tag{3}$$

and the almost-complex structure

$$J^+_{e^{i\phi}} = 2\mathrm{Re}\left\{ i\, e^{i\phi}\left[f^2(dz_1 + h\, dz_2) \otimes (\partial_{\bar{2}} - \bar{h}\partial_{\bar{1}}) - \frac{1}{f^2}\, dz_2 \otimes \partial_{\bar{1}}\right]\right\} \tag{4}$$

defines an almost-Kähler structure on $\mathcal{U}$.

If the functions f and h are general enough to prevent the metric from being self-dual then these structures are non-Kähler for almost all values of ϕ.

4. We look for not-self-dual Ricci-flat metrics among the metrics of lemma 2. For this purpose it is convenient to restrict to the metrics (3) whose anti-self-dual part of the Weyl tensor is strictly of type D. Such a restriction guarantees that all structures (4) are non-Kähler [7, 10].

We recall a useful lemma [8].

Lemma 3. *Let g be a Ricci-flat Riemannian metric in four dimensions. Assume that the anti-self-dual part of the Weyl tensor for g is strictly of type D. Then, locally there always exist complex coordinates (z_1, z_2) and a real function $K = K(v, z_2, \bar{z}_2)$, $v = z_1 + \bar{z}_1$ such that the metric can be written as*

$$g = \frac{\varepsilon K_{vv}}{(K_v)^{3/2}}\left(dz_1 + \frac{K_{v2}}{K_{vv}}\, dz_2\right)\left(d\bar{z}_1 + \frac{K_{v\bar{2}}}{K_{vv}}\, d\bar{z}_2\right) + 4\, e^{-K}\frac{(K_v)^{1/2}}{\varepsilon K_{vv}}\, dz_2\, d\bar{z}_2, \tag{5}$$

where $K_{v\bar{2}} = \partial^2 K/(\partial v\, \partial\bar{z}_2)$ etc. The function K satisfies

$$K_{vv}K_{2\bar{2}} - K_{v\bar{2}}K_{v2} - 2\, e^{-K}\left(K_{vv} + 2(K_v)^2\right) = 0, \tag{6}$$

$$K_v > 0, \qquad \varepsilon K_{vv} > 0 \tag{7}$$

where ε is either plus or minus one.

Also, every function $K = K(v, z_2, \bar{z}_2)$ satisfying (6) and (7) defines, via (5), a Ricci-flat metric. This metric has the anti-self-dual part of the Weyl tensor of strictly type D.

We ask when the metric (3) can be written in the form (5). Identifying coordinates (z_1, z_2) in both metrics we see that it is possible if

$$2f^2 = \frac{\varepsilon K_{vv}}{(K_v)^{3/2}} \qquad \text{and} \qquad \frac{2}{f^2} = 4\, e^{-K}\frac{(K_v)^{1/2}}{\varepsilon K_{vv}}.$$

These two equations are compatible only if $K_v\, e^K = 1$. It is a matter of straightforward integration that, modulo the coordinate transformations, the general solution of this equation which simultaneously satisfies equation (6) is† $K = \log(v - 2z_2\bar{z}_2)$. Using such K we easily find that in the region

$$\mathcal{U}' = \{\mathcal{U} \ni (z_1, z_2) \text{ such that } v - 2z_2\bar{z}_2 > 0\}$$

the metric (3) with

$$f = \frac{1}{\sqrt{2}(v - 2z_2\bar{z}_2)^{1/4}}, \qquad h = -2\bar{z}_2,$$

is Ricci-flat and strictly of type D on the anti-self-dual side of its Weyl tensor. The explicit expression for such g reads

$$g = \frac{1}{(v - 2z_2\bar{z}_2)^{1/2}}(\mathrm{d}z_1 - 2\bar{z}_2\,\mathrm{d}z_2)(\mathrm{d}\bar{z}_1 - 2z_2\,\mathrm{d}\bar{z}_2) + 4(v - 2z_2\bar{z}_2)^{1/2}\,\mathrm{d}z_2\,\mathrm{d}\bar{z}_2. \tag{8}$$

To gain a better insight into this metric we choose new coordinates

$$x = (v - 2z_2\bar{z}_2)^{1/2}, \qquad y = z_2 + \bar{z}_2, \qquad z = \mathrm{i}(\bar{z}_2 - z_2), \qquad q = \frac{z_1 - \bar{z}_1}{2\mathrm{i}}$$

on $\mathcal{U}'$. These coordinates are real. The metric (8) in these coordinates reads

$$g = x\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) + \frac{1}{x}\left(\tfrac{1}{2}z\,\mathrm{d}y - \tfrac{1}{2}y\,\mathrm{d}z + \mathrm{d}q\right)^2.$$

This shows that it belongs to the Gibbons–Hawking class [4] and that its self-dual part of the Weyl tensor vanishes.

We also recall [9] that a suitable Lie–Backlund transformation brings equation (6) to the Boyer–Finley–Plebański [2, 3] equation‡

$$F_{yy} + F_{zz} + \left(\mathrm{e}^F\right)_{xx} = 0$$

for one real function $F = F(x, y, z)$ of three real variables. It is interesting to note that the metric (8) corresponds to the simplest solution $F = 0$ of this equation.

Summing up we have the following theorem.

Theorem 1. *Let $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ be coordinates on $\mathcal{U} \subset \mathbb{R}^4 \cong \mathbb{C}^2$. The Riemannian manifold $(\mathcal{U}', g)$, where*

$$\mathcal{U}' = \{\mathcal{U} \ni (z_1, z_2) \text{ such that } v - 2z_2\bar{z}_2 > 0,\ v = z_1 + \bar{z}_1\}$$

and

$$g = \frac{1}{(v - 2z_2\bar{z}_2)^{1/2}}(\mathrm{d}z_1 - 2\bar{z}_2\,\mathrm{d}z_2)(\mathrm{d}\bar{z}_1 - 2z_2\,\mathrm{d}\bar{z}_2) + 4(v - 2z_2\bar{z}_2)^{1/2}\,\mathrm{d}z_2\,\mathrm{d}\bar{z}_2,$$

is Ricci-flat, anti-self-dual and has the anti-self-dual part of the Weyl tensor of type D. Moreover, $(\mathcal{U}', g)$ admits a circle of almost-Kähler non-Kähler structures

$$J^+_{\mathrm{e}^{\mathrm{i}\phi}} = 2\mathrm{Re}\left\{\mathrm{i}\,\mathrm{e}^{\mathrm{i}\phi}\left[\frac{1}{2(v - 2z_2\bar{z}_2)^{1/2}}(\mathrm{d}z_1 - 2\bar{z}_2\,\mathrm{d}z_2) \otimes (\partial_{\bar{2}} + 2z_2\partial_{\bar{1}}) - 2(v - 2z_2\bar{z}_2)^{1/2}\,\mathrm{d}z_2 \otimes \partial_{\bar{1}}\right]\right\}.$$

These structures are parametrized by the real constant $\phi \in [0, 2\pi[$. Their fundamental 2-forms are given by

$$\omega^+_{\mathrm{e}^{\mathrm{i}\phi}} = \mathrm{i}\left(\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}z_2 \wedge \mathrm{d}z_1 - \mathrm{e}^{-\mathrm{i}\phi}\,\mathrm{d}\bar{z}_2 \wedge \mathrm{d}\bar{z}_1\right).$$

† This solution was already known to Sławomir Białecki in 1984 [1].
‡ Also known to describe the $\boldsymbol{SU}(\infty)$ Toda lattice.

5. Interestingly, our examples can be globalized.

Indeed, the transformation

$$t = \frac{1}{2}\log(v - 2z_2\bar{z}_2), \qquad y = z_2 + \bar{z}_2, \qquad z = \mathrm{i}(\bar{z}_2 - z_2), \qquad q = \frac{z_1 - \bar{z}_1}{2\mathrm{i}}$$

brings the structures $(g, J^+_{\mathrm{e}^{\mathrm{i}\phi}}, \omega^+_{\mathrm{e}^{\mathrm{i}\phi}})$ of theorem 1 to a form which is regular for all the values of the real parameters $(t, y, z, q) \in \mathbb{R}^4$.

6. Finally, we observe that the metric (8), as being anti-self-dual, possesses a strictly Kähler structure. This is given by

$$J = \mathrm{i}[(\mathrm{d}z_1 - 2\bar{z}_2\,\mathrm{d}z_2) \otimes \partial_1 - (\mathrm{d}\bar{z}_1 - 2z_2\,\mathrm{d}\bar{z}_2) \otimes \partial_{\bar{1}} + \mathrm{d}\bar{z}_2 \otimes (\partial_{\bar{2}} + 2z_2\partial_{\bar{1}}) \\ -\mathrm{d}z_2 \otimes (\partial_2 + 2\bar{z}_2\partial_1)]$$

and belongs to structures of opposite orientation to $J^+_{\mathrm{e}^{\mathrm{i}\phi}}$. It is interesting whether there exist Ricci-flat metrics that admit almost-Kähler non-Kähler structures but do not admit any strictly Kähler structure.

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