Conformal transformations and the beginning of the Universe

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We consider two consecutive Universes (eons) with positive cosmological constants within the framework of Penrose's conformal cyclic cosmology. If we assume that both eons are filled with perfect fluids and that they both are conformal to the metric of Einstein's static universe by an analytic conformal transformation, then, using the Eisntein field equations, we prove that a) the fluids can only belong to five classes (cosmological constant, radiation, dust, and two other classes with negative pressures corresponding to a gas of strings and a gas of domain walls), b) the field equations on both sides of the future/eternity hypersurface exhibit certain duality, and c) both eons (one at the end and the other at the beginning) are almost critical so that the future eon is dominated by radiation and resembles the beginning of our Universe.

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In Penrose's conformal cyclic cosmology (CCC) [1], the metric $\check{g}$ of the Universe is conformally flat at the surface $t=0$ of the initial singularity [2]. Consider a conformal class $[\check{g}]$ of metrics conformal to $\check{g}$. Assume that the conformal class $[\check{g}]$ is regular in a strip $t \in]-\epsilon, \epsilon[$. In particular, this means that there exists a Lorentzian metric g in the class $[\check{g}]$ that is regular for all $t \in]-\epsilon, \epsilon[$. Penrose calls g the *intermediate metric* and relates it to two *physical* metrics:

(i) the metric $\check{g}$ describing the Universe close to the singularity, when $t \in]0, \epsilon[$, and

(ii) the metric $\hat{g}$, which is interpreted as the physical metric of the previous Universe (previous eon), when $t \in]-\epsilon, 0[$.

Formally, having chosen the intermediate metric g, one gets three metrics: $\hat{g}$, g, and $\check{g}$ in the entire bandage region $t \in]-\epsilon, \epsilon[$. From now on, we will only consider spatially homogeneous Universes. In such a case, according to Ref. [3], the three metrics are related via

$$\hat{g} = \frac{1}{f^2} g, \qquad \check{g} = f^2 g, \tag{1}$$

where $f = f(t)$ and $f(t)$ is chosen in such a way that $\check{g}$ coincides with the metric $\check{g}$ of the current Universe (current eon), when $t \in [0, \epsilon[$, and $\hat{g}$ coincides with the physical metric $\hat{g}$ of the previous eon, when $t \in]-\epsilon, 0]$.

In Penrose's proposal for the CCC, it is the conformal geometry $[g]$ of the metric g that is relevant for the cosmology of the Universe in the bandage region $t \in]-\epsilon, \epsilon[$. According to the paradigm of CCC, around the end of an old eon ($t \to 0^-$) and the beginning of the new eon ($t \to 0^+$), the Universe loses a part of the information about the (pseudo-)Riemannian physical metrics ($\hat{g}$ in $[-\epsilon, 0[$ and $\check{g}$ in $]0, \epsilon[$). The physical remnant of these (pseudo)-Riemannian geometries around the hypersurface $t=0$ is the *conformal* geometry $[g]$ of g. The question of what kind of dynamics this conformal geometry obeys is not stated by the CCC. Nevertheless, a passage from the (pseudo-)Riemannian to the conformal setting around the $t=0$ hypersurface eliminates the problem of "singularity" of the physical Universe there. In a conformal setting, the $t=0$ hypersurface is *regular*. Actually, the main invariant of the conformal geometry, namely, the *Weyl tensor*, is *vanishing*, so not only the *conformal geometry* $[g]$ is not singular, but it is even *flat* at $t=0$. The singularity of the (pseudo-)Reimannian metric $\check{g}$ at $t=0$ is merely a "wrong choice of coordinates" of the conformal geometry $[g]$, which at $t=0$ can be represented by a "good coordinate" g or by a "bad coordinate" $\check{g}$.

In this brief paper, we provide a simple model in which the Einstein equations for the spatially homogeneous metrics $\hat{g}$ and $\check{g}$ can be consistently solved, when one assumes that the two consecutive eons have positive cosmological constants and are filled with perfect fluids. A framework for the model was discussed in Refs. [4,5].

Before passing to the general discussion, for a motivation, we present the following simple example. We assume that $\hat{g}$ in the strip $t \in]-\epsilon, 0[$ is given by the de Sitter metric,

$$\hat{g} = \frac{1}{H^2 t^2}(-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2) = \frac{1}{H^2 t^2} g,$$

with g the flat metric in $]-\epsilon, \epsilon[$ and $H = \mathrm{const}$. Thus, comparing with (1), we get $f(t) = Ht$ in $]-\epsilon, 0[$. Extending $f(t)$ from $]-\epsilon, 0[$ to $f(t) = Ht$ in $]-\epsilon, \epsilon[$, we get

$$\check{g} = H^2 t^2 (-\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2), \tag{2}$$

and in $]0, \epsilon[$, we obtain the metric for the radiation-dominated Universe, for $t > 0$. Indeed, a simple time coordinate transformation $t = \sqrt{\frac{2\check{t}}{H}}$, brings (2) to the standard form

$$\check{g} = -\mathrm{d}\check{t}^2 + \frac{\check{t}}{t_0}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2),$$

with $t_0 = \frac{1}{2H}$. This transforms the de Sitter Universe from the times when t was before $t = 0$ to the radiation-dominated Universe at times $t > 0$ by a conformal transformation between the regions $]-\epsilon, 0[$ and $]0, \epsilon[$.

The rest of our paper is a generalization of this example to more general situations, when the current ($t > 0$) Universe has the energy-momentum tensor of *more general type* than the *pure radiation*.

In our generalization, the intermediate metric g is a Friedman-Lemaître-Robertson-Walker (FLRW) metric,

$$g = \left[-\mathrm{d}t^2 + \frac{h^2}{(1 + \frac{\kappa}{4}(x^2 + y^2 + z^2))^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\right], \tag{3}$$

with spatial curvature $\kappa = \mathrm{const}$ and with $h = h(t)$.

We assume that the metric g is regular at $t = 0$ so that the singularity of $\check{g}$ at the transition ($t \to 0$) is due to the behavior of the conformal factor $f(t) \to 0$ in $\check{g}$. We therefore have

$$h(t) \to 1 \quad \text{and} \quad f(t) \to Ct \quad \text{as} \quad t \to 0, \tag{4}$$

where the linear dependence of $f(t)$ on t for $t \to 0$ comes from the assumption of the dominance of the cosmological constant (de Sitter solution) at the end of the previous eon.

The dynamics of the model is governed by the Einstein equations, satisfied by each of the metrics $\hat{g}$ and $\check{g}$ separately. The Einstein equations, respectively for $\hat{g}$ and $\check{g}$, come with their specific cosmological constants $\hat{\lambda}$ and $\check{\lambda}$ and with their own energy momentum tensors. We make an assumption that both of them are the energy-momentum tensors of perfect fluids characterized by their respective energy densities $\hat{\mu}$ and $\check{\mu}$ and pressures $\hat{p}$ and $\check{p}$. The Einstein equations for $\hat{g}$ are

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\hat{g}_{\mu\nu} + \hat{\lambda}\hat{g}_{\mu\nu} = 8\pi G\hat{T}_{\mu\nu}, \tag{5}$$

with

$$\hat{T}_{\mu\nu} = (\hat{\mu} + \hat{p})\hat{u}_\mu\hat{u}_\nu + \hat{p}\hat{g}_{\mu\nu}. \tag{6}$$

We also have the identical-looking equations for $\check{g}$ with all the hats replaced by the checks. The respective velocities of the fluids are

$$u^\mu = (1, 0, 0, 0), \qquad \check{u}^\mu = \frac{1}{f}u^\mu \quad \hat{u}^\mu = fu^\mu. \tag{7}$$

We assume that the cosmological constant is non-negative for both $\hat{g}$ and $\check{g}$. Consequently, we write

$$\check{\lambda} = 3\check{H}_\lambda^2, \qquad \hat{\lambda} = 3\hat{H}_\lambda^2. \tag{8}$$

Now, our aim is to find solutions to the above-mentioned Einstein equations. We make an ansatz for $\hat{g}$ and $\check{g}$, in which $\hat{g}$ and $\check{g}$ are given by (1) and in which the intermediate metric g is the general FLRW metric (3). Thus, to specify a solution compatible with Penrose's CCC proposal, we have to determine f, h, $\hat{\mu}$, $\check{\mu}$, $\hat{p}$, and $\check{p}$, which are the unknown real functions of time.

To find solutions for f, h, $\hat{\mu}$, $\check{\mu}$, $\hat{p}$, and $\check{p}$, we recall that a *consequence* of the Einstein equations (5) and (6) is the conservation of the energy-momentum tensor, $\hat{\nabla}^\nu\hat{T}_{\mu\nu} = 0$, which reduces to

$$\hat{\nabla}^\nu\hat{T}_{\mu\nu} = \left(f\left(\hat{\mu}' + 3(\hat{\mu} + \hat{p})\left(\ln\frac{h}{f}\right)'\right), 0, 0, 0\right) = 0.$$

Thus, when solving the Einstein equations, we have to solve the equation

$$\hat{\mu}' + 3(\hat{\mu} + \hat{p})\left(\ln\frac{h}{f}\right)' = 0 \tag{9}$$

as well as the analogous equation

$$\check{\mu}' + 3(\check{\mu} + \check{p})(\ln(hf))' = 0 \tag{10}$$

for the checked quantities. The physical interpretation of Eqs. (9) and (10) is that these are the *continuity equations* for the fluids $(\hat{\mu}, \hat{p})$ and $(\check{\mu}, \check{p})$, respectively.

Now, we make the main assumptions to obtain special solutions to the "hatted" and "checked" Eqs. (5) and (6) with appealing physical properties.

First, we assume that both perfect fluids $(\hat{\mu}, \hat{p})$ and $(\check{\mu}, \check{p})$ are mixtures of a number of respective perfect fluids $(\hat{\mu}_i, \hat{p}_i)$ and $(\check{\mu}_i, \check{p}_i)$. This means that we have

$$\hat{\mu} = \sum_i \hat{\mu}_i, \qquad \check{\mu} = \sum_i \check{\mu}_i,$$
$$\hat{p} = \sum_i \hat{p}_i, \qquad \check{p} = \sum_i \check{p}_i.$$

Second, we assume that the equation of state for each of the perfect fluids in the mixtures is given by

$$\hat{p}_i = \hat{w}_i\hat{\mu}_i, \qquad \check{p}_i = \check{w}_i\check{\mu}_i, \tag{11}$$

with $\hat{w}_i$ and $\check{w}_i$ being *constants*.

In this way, the unknowns we try to determine by using the hatted and checked Einstein Eqs. (5) and (6) are f, h, $\hat{\mu}_i$, $\check{\mu}_i$, and the constants $\hat{w}_i$ and $\check{w}_i$.

Third, we assume that each perfect fluid $(\hat{\mu}_i, \hat{p}_i)$ considered separately satisfies its own continuity equation; i.e., we assume that

$$\hat{\mu}_i{}' + 3(\hat{\mu}_i + \hat{p}_i)\left(\ln\frac{h}{f}\right)' = 0 \tag{12}$$

and

$$\check{\mu}_i{}' + 3(\check{\mu}_i + \check{p}_i)(\ln(hf))' = 0. \tag{13}$$

Note that, since Eqs. (9) and (10) are linear in μs and ps, each solution of (12) and (13) also solves (9) and (10). However, the solutions of (12) and (13) constitute only a subset of all solutions to (9) and (10). Thus, our third assumption restricts the general solution of the hatted and checked Einstein Eqs. (5) and (6) to its subset in which, within a given eon, the mixtures of perfect fluids neither interact among themselves nor convert from one fluid into another.

One may worry that this assumption is incompatible with the rest of the Einstein Eqs. (5) and (6). It turns out, however, that this is *not* the case, as the rest of our analysis clearly shows.

To see this, we first solve (12) and (13). Here, the general solutions are

$$\hat{\mu} = \sum_i \frac{3\hat{\mu}_{0i}}{8\pi G}\left(\frac{h}{f}\right)^{-3-3\hat{w}_i}$$
$$\check{\mu} = \sum_i \frac{3\check{\mu}_{0i}}{8\pi G}(hf)^{-3-3\check{w}_i}, \tag{14}$$

with $\hat{\mu}_{0i}$ and $\check{\mu}_{0i}$ integration constants. This reduces the set of our unknowns from f, h, $\hat{\mu}_i$, $\check{\mu}_i$, and the constants $\hat{w}_i$ and $\check{w}_i$ to functions f, h, and constants $\hat{w}_i$ and $\check{w}_i$.

It follows that the rest of the hatted and checked Einstein Eqs. (5) and (6) reduces to only two ODEs. They are parametrized by the constants $\hat{\mu}_{0i}$, $\check{\mu}_{0i}$, $\hat{w}_i$, and $\check{w}_i$ and read

$$f^2\left(\frac{h}{f}\right)'^2 - \sum_i \hat{\mu}_{0i}\left(\frac{h}{f}\right)^{-1-3\hat{w}_i} - \hat{H}_\lambda^2\left(\frac{h}{f}\right)^2 = -\kappa \tag{15}$$

for the hats and

$$\frac{(fh)'^2}{f^2} - \sum_i \check{\mu}_{0i}(hf)^{-1-3\check{w}_i} - \check{H}_\lambda^2(fh)^2 = -\kappa \tag{16}$$

for the checks. Thus, for each choice of constants $\hat{\mu}_{oi}$, $\check{\mu}_{oi}$, $\hat{w}_i$, and $\check{w}_i$, we are left with two ODEs. In this way, we end up with two ODEs for two unknown functions—the system that definitely has solutions.

From these solutions, we pick up those that are mathematically elegant and desirable by Penrose's CCC proposal. In particular, we want the solutions to satisfy matching conditions (4) at the big bang/big crunch hypersurface $t = 0$.

We consider a solution "mathematically elegant" if the functions $f(t)$ and $h(t)$ are analytic in the variable t around $t = 0$. This means that such solutions for $f = f(t)$ and $h = h(t)$ of our two ODEs are expressible in terms of the Taylor series in t centered at $t = 0$. From now on, we consider only such solutions. Thus, assuming the Taylor series expansions for $f(t)$ and $h(t)$, and the matching conditions (4), we get

$$\left(\frac{h}{f}\right) = \frac{a(t)}{t}, \qquad f^2\left(\frac{h}{f}\right)'^2 = \frac{b(t)}{t^2},$$
$$fh = tc(t), \qquad \frac{(fh)'^2}{f^2} = \frac{q(t)}{t^2},$$

where $a = a(t)$, $b = b(t)$, $c = c(t)$, and $q = q(t)$ are functions analytic in an interval around $t = 0$, such that $a(0) \neq 0$, $b(0) \neq 0$, $c(0) \neq 0$, and $q(0) \neq 0$. Inserting these relations into the ODEs (15) and (16) gives

$$b(t) - \sum_i \hat{\mu}_{0i} t^{3+3\hat{w}_i} a^{-1-3\hat{w}_i}(t) - \hat{H}_\lambda^2 a^2(t) + t^2\kappa = 0$$

and

$$q(t) - \sum_i \check{\mu}_{0i} t^{1-3\check{w}_i} q^{-1-3\check{w}_i}(t) - \check{H}_\lambda^2 t^4 q^2(t) + t^2\kappa = 0.$$

These equations have consequences. First, they show that $-1 - 3\hat{w}_i$ and $-1 - 3\check{w}_i$ must be integers. In particular, each of the ws must be an integer multiple of $\frac{1}{3}$. Second, this compared with the usual dominant energy conditions for the fluids

$$-1 \leq \hat{w}_i \leq 1, \quad \text{and} \quad -1 \leq \check{w}_i \leq 1,$$

imply that the possible values of $\hat{w}_i$s are

$$\hat{w}_i = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1, \tag{17}$$

and that the possible values of $\check{w}_i$s are

$$\check{w}_i = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1.$$

Using these values for $\check{w}_i$s we get:

$$q(t) - \check{\mu}_{01}(tq)^4 - \check{\mu}_{02}(tq)^3 - \check{\mu}_{03}(tq)^2 - \check{\mu}_{04}tq - \check{\mu}_{05}$$
$$- \check{\mu}_{06}(tq)^{-1} - \check{\mu}_{07}(tq)^{-2} - \check{H}_\lambda^2 t^4 q^2 + t^2\kappa = 0.$$

Since $q(0) \neq 0$, and $q(t)$ is analytic in an interval around 0, we have $\lim_{t\to 0} q^{-1}(t) = \frac{1}{q(0)}$ and $\lim_{t\to 0} q^{-2}(t) = \frac{1}{q(0)^2}$. Thus multiplying the above equation first by t^2 and passing to the limit $t \to 0$, and then multiplying it by t and passing to the limit $t \to 0$ we obtain $\check{\mu}_{07} = \check{\mu}_{06} = 0$. This shows that out of seven possibilities for $\check{w}_i$s we are left with only five:

$$\check{w}_i = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}. \tag{18}$$

The perfect fluids corresponding to $\check{\mu}_{07}$ and $\check{\mu}_{06}$, i.e. those with $\check{w}_i = \frac{2}{3}$, 1, are excluded by our assumptions!

This is all what we can deduce about our solutions from the requirement of their analyticity, matching conditions (4), and the dominant energy conditions for the energy-momentum tensors.

The solutions must satisfy two ODEs (15) and (16) with $\hat{w}_i$s and $\check{w}_i$s as in (17) and (18). It is convenient to write down these two ODEs explicitly. We write them in terms of new unknowns $\hat{F} = \hat{F}(t)$ and $\check{F} = \check{F}(t)$, which are related to $f(t)$ and $h(t)$ via

$$\hat{F} = \frac{f}{h}, \qquad \check{F} = fh. \tag{19}$$

In these variables, Eq. (15) reads

$$\frac{\check{F}}{\hat{F}}\hat{F}'^2 = (\hat{H}_\lambda^2 + \hat{\mu}_{01}) + \hat{\mu}_{02}\hat{F} + (\hat{\mu}_{03} - \kappa)\hat{F}^2 + \hat{\mu}_{04}\hat{F}^3 + \hat{\mu}_{05}\hat{F}^4 + \hat{\mu}_{06}\hat{F}^5 + \hat{\mu}_{07}\hat{F}^6, \tag{20}$$

and Eq. (16) reads

$$\frac{\check{F}}{\hat{F}}\check{F}'^2 = \check{\mu}_{05} + \check{\mu}_{04}\check{F} + (\check{\mu}_{03} - \kappa)\check{F}^2 + \check{\mu}_{02}\check{F}^3 + (\check{H}_\lambda^2 + \check{\mu}_{01})\check{F}^4. \tag{21}$$

The system of ODEs (20) and (21) definitely has local solutions satisfying our matching conditions (4). However, we cannot say much about them. For this reason, we make a further (the last) assumption. Before introducing it, we give a motivation.

Note that the intermediate metric (3) when $h(t) = 1$ and $\kappa = 1$ is the metric of Einstein's static universe. In the history of General Relativity, this metric was proposed to describe cosmology as the first. After Hubble's discovery of the expansion of the Universe, the Einstein static universe was abandoned, and more general Friedman-Lemaître-Roberston-Walker cosmologies began their life to be in accordance with observations.

But what if, as almost always, Einstein was in a sense right? Note that if one assumes Penrose's CCC scenario there is a prominent place for the Einstein static universe metric in (CC) cosmology without being in major contradiction with astronomical observations. What if the Penrose intermediate metric g—the background for the transition from one eon to another—is just the Einstein static universe metric? We investigate this question, in a bit more general setting of general κ, in the rest of this paper.

In the context of our solutions, the Einstein static universe situation corresponds to the unknown $h(t) \equiv 1$ and $\kappa = 1$. This suggests looking for the analytic solutions to (15) and (16) with

$$h(t) \equiv 1.$$

This is equivalent to the assumption that the intermediate metric g describes one of the following three things: the Einstein static universe metric ($\kappa > 0$), the Minkowski metric ($\kappa = 0$), or the static universe with spatial sections as hyperboloids ($\kappa < 0$). We assume this from now on. In such a case,

$$\hat{F} = \check{F} = f,$$

and the two ODEs (20) and (21) to be solved become

$$f'^2 = (\hat{H}_\lambda^2 + \hat{\mu}_{01}) + \hat{\mu}_{02}f + (\hat{\mu}_{03} - \kappa)f^2 + \hat{\mu}_{04}f^3 + \hat{\mu}_{05}f^4 + \hat{\mu}_{06}f^5 + \hat{\mu}_{07}f^6$$

and

$$f'^2 = \check{\mu}_{05} + \check{\mu}_{04}f + (\check{\mu}_{03} - \kappa)f^2 + \check{\mu}_{02}f^3 + (\check{H}_\lambda^2 + \check{\mu}_{01})f^4. \tag{22}$$

Subtracting these two equations gives restrictions on the constants $\hat{\mu}_{0i}$. Indeed, since we have

$$(\hat{H}_\lambda^2 + \hat{\mu}_{01} - \check{\mu}_{05}) + (\hat{\mu}_{02} - \check{\mu}_{04})f + (\hat{\mu}_{03} - \check{\mu}_{03})f^2 + (\hat{\mu}_{04} - \check{\mu}_{02})f^3 + (\hat{\mu}_{05} - \check{H}_\lambda^2 - \check{\mu}_{01})f^4 + \hat{\mu}_{06}f^5 + \hat{\mu}_{07}f^6 = 0$$

and f is an analytic function of t, then we must have

$$\begin{aligned}\hat{\mu}_{07} &= 0\\ \hat{\mu}_{06} &= 0\\ \hat{\mu}_{05} &= \check{\mu}_{01} + \check{H}_\lambda^2\\ \hat{\mu}_{04} &= \check{\mu}_{02}\\ \hat{\mu}_{03} &= \check{\mu}_{03}\\ \hat{\mu}_{02} &= \check{\mu}_{04}\\ \hat{\mu}_{01} &= \check{\mu}_{05} - \hat{H}_\lambda^2.\end{aligned} \tag{23}$$

Note the first two equalities; they exclude the existence of fluids with $\hat{w} = \frac{2}{3}$ and $\hat{w} = 1$ in the previous eon. So, now both eons have the same number of fluids in the mixture.

In addition, we have duality; for each $i = 1, 2, 3, 4, 5$, if in the current eon there is a fluid with $\check{w}_i$, then it gets transformed from the fluid with $\hat{w}_{6-i}$ from the previous eon. We have

$$\check{w}_i = \hat{w}_i = \frac{i-4}{3}, \qquad i = 1, 2, 3, 4, 5.$$

In this sense, fluids with $i = 3$ corresponding to $w = -\frac{1}{3}$ in both eons are self-dual.

The fluids in both eons get "quantized," and in each eon, they can only appear in five types. The fluid in the current eon with $i = 1$ corresponds to the cosmological constant. The two fluids from the current eon with $i = 2$ and $i = 3$ have negative pressure and are usually not considered in cosmology. It is nevertheless worth noting that $w = -1/3$ corresponds to a *gas of strings* (Ref. [6], p. 228), while $w = -2/3$ corresponds to a *gas of domain walls* (Ref. [6], p. 219). The two remaining values of $\check{w}_i$, namely, $\check{w}_4 = 0$ and $\check{w}_5 = \frac{1}{3}$, are usually the only ones used in cosmology on physical grounds; $i = 5$ corresponds to *radiation*, and $i = 4$ corresponds to *matter*. We find it interesting that all *five* of these physically justified values of w come out naturally—in both eons—from the requirement of smoothness of conformal transformations relating the consecutive eons. It is also interesting that the radiation is dual to the cosmological constant, matter is dual to a gas of domain walls, and a gas of strings is self-dual on two sides of the transition.

We close this discussion with a remark that now the only equation to be solved is Eq. (22) for $f = f(t)$. It is solved in quadratures, and its most general solution gives $f = f(t)$ as an inverse of a general elliptic function.[1]

Let us summarize. The metrics

$$\hat{g} = \frac{1}{f^2} g \quad \text{and} \quad \check{g} = f^2 g, \quad \text{with}$$

$$g = -\mathrm{d}t^2 + \frac{(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}{(1 + \frac{\kappa}{4}(x^2 + y^2 + z^2))^2},$$

in which the function $f = f(t)$ is given implicitly by the elliptic integral [see Eq. (22)]

$$t = \int \frac{\mathrm{d}f}{\sqrt{\check{\mu}_{05} + \check{\mu}_{04} f + (\check{\mu}_{03} - \kappa) f^2 + \check{\mu}_{02} f^3 + (\check{H}_\lambda^2 + \check{\mu}_{01}) f^4}},$$

describe a mixture of perfect fluids $(\hat{\mu}_i, \hat{w}_i \hat{\mu}_i)$ and $(\check{\mu}_i, \check{w}_i \check{\mu}_i)$, in the bandage region of Penrose's CCC proposal. The metric $\check{g}$ defining the initial cosmology of the current eon contains the mixture of five fluids with $\check{w}_i = -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}$, similarly to the metric $\hat{g}$ defining the final stages of cosmology of the previous eon, in which each of the fluids with $\check{w}_{6-i}$ is replaced by a fluid with $\hat{w}_i$. The fluids in the mixtures have the following respective energy densities:

$$\hat{\mu}_i = \frac{3(-\hat{H}_\lambda^2 \delta_{i1} + \check{H}_\lambda^2 \delta_{i5} + \check{\mu}_{0,6-i})}{8\pi G} f^{3+3\check{w}_i} \quad \text{and}$$

$$\check{\mu}_i = \frac{3\check{\mu}_{0i}}{8\pi G} f^{-3-3\check{w}_i}.$$

In the previous eon, each of them is moving with the 4-velocity

$$\hat{u} = f \partial_t,$$

and in the current eon, each of them is moving with 4-velocity

$$\check{u} = \frac{1}{f} \partial_t.$$

The metrics $\hat{g}$ and $\check{g}$ satisfy Einstein's equations with the energy momentum of the fluids. They also satisfy matching conditions (4) at the wound hypersurface $t = 0$.

Finally, we have the following comment: assuming that $h(t) \equiv 1$, the Hubble parameter for $t \to 0$ is equal to

$$H(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{f}\right) \to \frac{1}{Ct^2}, \tag{24}$$

so the criticality, using (14) and (23), tends to 1 for very early times:

$$\Omega = \frac{8\pi G \hat{\mu}}{3H^2} \to 1. \tag{25}$$

Thus, in our model, the Universe is critical at the early times. Note that it is also critical at the end when the dominant contribution to the energy comes from the cosmological constant.

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[1]Equation (22) after a birational change of variables can be transformed to the Weierstrass differential equation equation for the famous Weierstrass $\wp$ function (Ref. [7], p. 91).

[1] R. Penrose, *Cycles of Time: An Extraordinary New View of the Universe* (Bodley Head, London, 2010).
[2] K. P. Tod, Classical Quantum Gravity **20**, 521 (2003).
[3] V. G. Gurzadyan and R. Penrose, Eur. Phys. J. Plus **128**, 22 (2013).
[4] K. P. Tod, Gen. Relativ. Gravit. **47**, 17 (2015).
[5] E. T. Newman, arXiv:1309.7271.
[6] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Reading, MA, 1990).
[7] D. Hausemöller, *Elliptic Curves*, Graduate Texts in Mathematics (Springer, New York, 2004), Vol. 111.