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# Non-rigid parabolic geometries of Monge type

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ABSTRACT

In this paper we study a novel class of parabolic geometries which we call parabolic geometries of Monge type. These parabolic geometries are defined by gradings such that their $-1$ component contains a nonzero co-dimension 1 abelian subspace whose bracket with its complement is non-degenerate. We completely classify the simple Lie algebras with such gradings in terms of elementary properties of the defining set of simple roots. In addition we characterize those parabolic geometries of Monge type which are non-rigid in the sense that they have nonzero harmonic curvatures in positive weights. Standard models of all non-rigid parabolic geometries of Monge type are described by under-determined ODE systems. The full symmetry algebras for these under-determined ODE systems are explicitly calculated; surprisingly, these symmetries are all just prolonged point symmetries.



## 1. Introduction

Early in the development of the structure theory for simple Lie algebras, W. Killing [9,10] conjectured that there exists a rank 2, 14-dimensional simple Lie algebra $\mathfrak{g}_2$ which

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admits a realization as a Lie algebra of vector fields on a 5-dimensional manifold. This realization was discovered independently by F. Engel and E. Cartan[1] and is given by the infinitesimal symmetries of the rank 2 distribution in 5 variables for the under-determined ordinary differential equation

$$\frac{dz}{dx} = \left[\frac{d^2y}{dx^2}\right]^2. \tag{1.1}$$

Recall that for any distribution $\mathcal{D}$ defined on a manifold $M$, the ***Lie algebra of infinitesimal symmetries*** $\mathfrak{X}(\mathcal{D})$ is the set of vector fields $X$ on $M$ such that $[X, \mathcal{D}] \subset \mathcal{D}$. Eq. (1.1) subsequently re-appeared as the flat model in Cartan's solution [4] to the equivalence problem for rank 2 distributions in 5 variables and in papers by Hilbert [11] and Cartan [6] on the problem of closed form integration of under-determined ODE systems.

It is therefore natural to ask if all simple Lie algebras admit such elegant realizations as the infinitesimal symmetries of under-determined systems of ordinary differential equations. We shall formulate this question within the context of parabolic geometry and give a complete answer in terms of the novel concept of a ***parabolic geometry of Monge type***. These geometries are defined intrinsically in terms of the $-1$ grading component and exist for all types of simple Lie algebras. In this paper we shall **[i]** completely classify all parabolic geometries of Monge type; **[ii]** identify those geometries which are non-rigid and describe the spaces of fundamental curvatures in terms of the second Lie algebra cohomology; **[iii]** give under-determined ODE realizations for the standard models; and **[iv]** explicitly calculate the infinitesimal symmetries for the standard models. For each classical simple Lie algebra, one particular parabolic Monge geometry of depth 3 stands out from all the others. We believe that these particular geometries, listed in Theorem A, merit further study similar to that for the well-known $|1|$-gradations and contact gradations.

To explain this work in more detail, we first recall a few basic definitions from the general theory of parabolic geometry. As presented in [1,17], the underlying structure for any parabolic geometry is a semi-simple Lie algebra $\mathfrak{g}$ and a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-k}. \tag{1.2}$$

Such a decomposition is called a $|k|$-***grading*** if: **[i]** $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$; **[ii]** the negative part of this grading

$$\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-k}$$

is generated by $\mathfrak{g}_{-1}$, that is, $[\mathfrak{g}_{-1}, \mathfrak{g}_\ell] = \mathfrak{g}_{-1+\ell}$ for $\ell < 0$; and **[iii]** $\mathfrak{g}_k \neq 0$ and $\mathfrak{g}_{-k} \neq 0$. The negatively graded part $\mathfrak{g}_-$ is a graded nilpotent Lie algebra while the non-negative part of this grading

[1] Their articles appear sequentially in 1893 in Comptes Rendu [2,7].

$$\mathfrak{p} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0$$

is always a parabolic subalgebra. We remark that for a fixed choice of simple roots $\Delta^0$ of $\mathfrak{g}$, there is a one-to-one correspondence between the subsets $\Sigma$ of $\Delta^0$ and the gradings of $\mathfrak{g}$ [1, pp. 292–293]. We will denote the corresponding parabolic geometry constructed this way by $(\mathfrak{g}, \Sigma)$.

For every $|k|$-grading of a simple Lie algebra $\mathfrak{g}$, there is unique element $E \in \mathfrak{g}_0$, called the ***grading element***, such that $[E, x] = jx$ for all $x \in \mathfrak{g}_j$ and $-k \leq j \leq k$. Let $\Lambda^q(\mathfrak{g}_-, \mathfrak{g})$ be the vector space of $q$-forms on $\mathfrak{g}_-$ with values in $\mathfrak{g}$ and set $\Lambda^q(\mathfrak{g}_-, \mathfrak{g})_p$ to be the subspace of $q$-forms which are homogeneous of weight $p$, that is,

$$\Lambda^q(\mathfrak{g}_-, \mathfrak{g})_p = \{\omega \in \Lambda^q(\mathfrak{g}_-, \mathfrak{g}) \mid \mathcal{L}_E(\omega) = p\,\omega\}.$$

The spaces $\Lambda^*(\mathfrak{g}_-, \mathfrak{g})_p$ define a co-chain complex with respect to the standard Lie algebra differential. The cohomology of this co-chain complex is denoted by[2] $H^q(\mathfrak{g}_-, \mathfrak{g})_p$. A parabolic geometry is called ***rigid*** if all the degree 2 cohomology spaces in positive weights vanish and ***non-rigid*** otherwise. The cohomology spaces $H^q(\mathfrak{g}_-, \mathfrak{g})_p$ can be calculated by the celebrated method of Kostant [13] (see also [1, §3.3] and [17, §5.1]).

With these preliminaries dispatched, fix a $|k|$-grading of $\mathfrak{g}$, let $N$ be the simply connected Lie group with Lie algebra $\mathfrak{g}_-$ and let $\mathcal{D}(\mathfrak{g}_{-1})$ be the distribution on $N$ generated by the left invariant vector fields corresponding to the $\mathfrak{g}_{-1}$ component of $\mathfrak{g}_-$. This distribution is called the ***standard differential system*** associated to the given parabolic geometry.

It is a fundamental result of N. Tanaka (see [17, Sections 2 and 5], especially pages 432 and 475) that if $H^1(\mathfrak{g}_-, \mathfrak{g})_p = 0$ for $p \geq 0$, then the Tanaka prolongation of $\mathfrak{g}_-$ coincides with $\mathfrak{g}$ and we have the following Lie algebra isomorphism

$$\mathfrak{X}(\mathcal{D}(\mathfrak{g}_{-1})) \cong \mathfrak{g}. \tag{1.3}$$

*In this way, one can construct many examples of distributions $\mathcal{D}$ whose symmetry algebra $\mathfrak{X}(\mathcal{D})$ is a given finite dimensional simple Lie algebra* $\mathfrak{g}$. Indeed, pick a subset $\Sigma \subset \Delta^0$ of the simple roots and construct the associated grading (1.2), which we require to satisfy $H^1(\mathfrak{g}_-, \mathfrak{g})_p = 0$ for $p \geq 0$. This cohomology condition is generally satisfied, with the few exceptions enumerated in [1, Proposition 4.3.1] or [17, Proposition 5.1]. Then calculate the left invariant vector fields on the nilpotent Lie group $N$. By (1.3) the Lie algebra of the infinitesimal symmetries of the standard differential system $\mathcal{D}(\mathfrak{g}_{-1})$ is the given simple Lie algebra $\mathfrak{g}$. Finally write down a system of ordinary or partial differential equations whose canonical differential system is $\mathcal{D}(\mathfrak{g}_{-1})$.

All of these calculations can be done with the Maple *DifferentialGeometry* package and this allowed the authors to generate many examples of differential equations with

[2] The notation in Yamaguchi [17] is $H^{p,q}(\mathfrak{g}_-, \mathfrak{g}) = H^q(\mathfrak{g}_-, \mathfrak{g})_{p+q-1}$.

prescribed simple Lie algebras of infinitesimal symmetries. For each classical simple Lie algebra *one* particular parabolic geometry immediately stood out from all the others. These are listed in the following theorem.

**Theorem A.** *The standard differential systems for the parabolic geometries* $A_\ell\{\alpha_1, \alpha_2, \alpha_3\}$, $C_\ell\{\alpha_{\ell-1}, \alpha_\ell\}$, $B_\ell\{\alpha_1, \alpha_2\}$ *and* $D_\ell\{\alpha_1, \alpha_2\}$, *are realized as the canonical differential systems for the under-determined ordinary differential equations*

$$\mathbf{I}: \ A_\ell\{\,\alpha_1, \alpha_2, \alpha_3\,\},\ \ell \geq 3, \qquad \dot{z}^i = \dot{y}^0 \dot{y}^i,\ 1 \leq i \leq \ell - 2. \tag{1.4}$$

$$\mathbf{II}: \ C_\ell\{\,\alpha_{\ell-1}, \alpha_\ell\,\},\ \ell \geq 3, \qquad \dot{z}^{ij} = \dot{y}^i \dot{y}^j,\ 1 \leq i \leq j \leq \ell - 1. \tag{1.5}$$

$$\mathbf{III}: \ B_\ell\{\,\alpha_1, \alpha_2\,\},\ \ell \geq 3, \qquad \dot{z} = \frac{1}{2}\sum_{i,j=1}^{2\ell-3} \kappa_{ij} \dot{y}^i \dot{y}^j. \tag{1.6}$$

$$\mathbf{IV}: \ D_\ell\{\,\alpha_1, \alpha_2\,\},\ \ell \geq 4,\ D_3\{\,\alpha_1, \alpha_2, \alpha_3\,\}, \quad \dot{z} = \frac{1}{2}\sum_{i,j=1}^{2\ell-4} \kappa_{ij} \dot{y}^i \dot{y}^j. \tag{1.7}$$

*Here* $(\kappa_{ij})$ *is a symmetric, non-degenerate constant matrix of an arbitrary signature* $(r, s)$, *where* $r + s = 2\ell - 3$ *for* $B_\ell$ *or* $r + s = 2\ell - 4$ *for* $D_\ell$. *The symmetry algebras of* **I** *through* **IV** *are isomorphic, as real Lie algebras, to* $\mathfrak{sl}(\ell + 1, \mathbb{R})$, $\mathfrak{sp}(\ell, \mathbb{R})$, $\mathfrak{so}(r + 2, s + 2)$, *and* $\mathfrak{so}(r + 2, s + 2)$, *respectively.*

We note that the only repetition in the above list is $A_3$ and $D_3$, where the matrix $(\kappa_{ij})$ has signature $(1, 1)$, corresponding to the isomorphism $\mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3)$.

Evidently, Eqs. (1.6) and (1.7) are the differential equations for a curve $\gamma(x) = (x, y^i(x), z(x))$ to lie on the null cone of the metric

$$g = dx\, dz - \frac{1}{2}\sum_{ij} \kappa_{ij} dy^i\, dy^j.$$

Similarly, the Monge equations (1.4) and (1.5) can be interpreted as the differential equations for curves to lie on the common null cones of families of (degenerate) quadratic forms

$$\{dx\, dz^i - dy^0\, dy^i\} \quad \text{and} \quad \{dx\, dz^{ij} - dy^i\, dy^j\}.$$

The geometric characterization of these families of quadratic forms and their roles as geometric structures associated to parabolic geometries are interesting problems in their own right, which we hope to address in a future publication.

*The main result of this paper is an intrinsic characterization of those parabolic geometries arising in Theorem A, as well as the* $\mathfrak{g}_2$ *parabolic geometries defining Eq.* (1.1). To motivate this result, two key observations are needed. First, under-determined systems of ordinary differential equations such as **I**–**IV** are often referred to, in the geometric differential equation literature, as ***Monge equations***. As distributions these Monge equations are all generated by vector fields $\{X, Y_1, Y_2, \ldots, Y_d\}$ such that $[Y_i, Y_j] = 0$ and such that the $2d + 1$ vector fields $\{X, Y_i, [X, Y_i]\}$ are all point-wise independent. This first observation suggests the following fundamental definition.

**Definition 1.1.** A parabolic geometry

$$\mathfrak{g} = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{-k}$$

is of ***Monge type*** if its $-1$ grading component $\mathfrak{g}_{-1}$ contains a co-dimension 1 non-zero abelian subalgebra $\mathfrak{y}$ and $\dim \mathfrak{g}_{-2} = \dim \mathfrak{y}$.

The second observation is that each of the parabolic geometries arising in Theorem A, as well as the Hilbert–Cartan equation (1.1), is non-rigid. These two observations motivate our second theorem.

**Theorem B.** *Let $\mathfrak{g}$ be a split simple Lie algebra of rank $\ell$ with simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. The following is a complete list of non-rigid parabolic geometries of Monge type.*

$$\begin{array}{lll}
\mathbf{Ia}: A_\ell\{\,\alpha_1, \alpha_2, \alpha_3\,\},\ \ell \geq 3 & \mathbf{Ib}: A_\ell\{\,\alpha_1, \alpha_2\,\},\ \ell \geq 2 & \\
\mathbf{IIa}: C_\ell\{\,\alpha_{\ell-1}, \alpha_\ell\,\},\ \ell \geq 3 & \mathbf{IIb}: C_3\{\,\alpha_1, \alpha_2, \alpha_3\,\} & \\
\mathbf{IIIa}: B_\ell\{\,\alpha_1, \alpha_2\,\},\ \ell \geq 2 & \mathbf{IIIb}: B_2\{\,\alpha_2\,\} & \mathbf{IIIc}: B_3\{\,\alpha_2, \alpha_3\,\} \\
\mathbf{IIId}: B_3\{\,\alpha_1, \alpha_2, \alpha_3\,\} & & \\
\mathbf{IVa}: D_\ell\{\,\alpha_1, \alpha_2\,\},\ \ell \geq 4 & & \\
\mathbf{Va}: G_2\{\,\alpha_1\,\} & \mathbf{Vb}: G_2\{\,\alpha_1, \alpha_2\,\}. &
\end{array}$$

A number of remarks concerning Theorem B are in order.

**1.** The standard differential systems for cases **Ia**, **IIa**, …, **Va** are precisely those given by Eqs. (1.4), (1.5), (1.6) and (1.7) (for $\kappa_{ij}$ with split signature), and (1.1). Cases **Ib**, **IIIb**, and **IIIa** with $\ell = 2$ are the only cases where $H^1(\mathfrak{g}_-, \mathfrak{g})_p \neq 0$ for some $p \geq 0$. The standard models for **Ib** and **IIIb** are easily seen to be the jet spaces $J^1(\mathbb{R}^1, \mathbb{R}^{\ell-1})$ and $J^1(\mathbb{R}^1, \mathbb{R}^1)$. The standard models for **IIb**, **IIIc**, and **IIId** are respectively

$$\dot{z}^1 = \dot{y}^1\dot{y}^2 \quad \dot{z}^2 = x\dot{y}^2 \quad \dot{z}^3 = (y^1 + \dot{y}^1x)\dot{y}^2 \quad \dot{z}^4 = y^1\dot{y}^1\dot{y}^2, \tag{1.8}$$

$$\dot{z} = \ddot{y}^1\dot{y}^2, \quad \text{and} \tag{1.9}$$

$$\dot{z}^1 = \dot{y}^1\dot{y}^2 \quad \dot{z}^2 = \frac{1}{2}(\dot{y}^2)^2 \quad \dot{z}^3 = \frac{1}{2}\dot{y}^1(\dot{y}^2)^2 \quad \dot{z}^4 = \frac{1}{2}\dot{y}^2(x\dot{y}^1\dot{y}^2 - y^1\dot{y}^2 - 2\dot{y}^1y^2). \tag{1.10}$$

Finally, the standard differential system in case **Vb** is simply a partial prolongation of the standard differential system for (1.1) (see also [17, §1.3]). We provide the details for these calculations in Section 4.

**2.** It is a relatively straightforward matter to extend this classification of non-rigid parabolic Monge geometries to *all real simple Lie algebras*. In the real case the $|k|$-gradings are defined by those subsets of simple roots which are disjoint from the compact roots and invariant under the Satake involution [1, Theorem 3.2.9]. This requirement, our classification of parabolic Monge gradations in Theorem 2.4 and the classification of real simple Lie algebras (see, for example, [1, Table Appendix B.4]), show that, in addition to the split real forms listed in Theorem B, one only has to include the real parabolic geometries listed in Theorem A **III** and **IV** for $\kappa_{ij}$ of general signature.

**3.** It is rather disappointing that none of the exceptional Lie algebras $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ appear in Theorem B but, simply stated, there are rather few non-rigid parabolic geometries for these algebras [17] and none of these satisfy the Monge criteria of Theorem 2.4. (See, however, Cartan [3] for the standard differential system for $\mathfrak{f}_4\{\alpha_4\}$ which is not of Monge type. In the same spirit, see [17, p. 480] for some other linear PDE systems with simple Lie algebras of symmetries.)

**4.** We remark also that just as the Hilbert–Cartan equation (1.1) arises as the reduction of the parabolic Goursat equation

$$32u_{xy}^3 - 12u_{yy}^2u_{xy}^2 + 9u_{xx}^2 - 36u_{xx}u_{xy}u_{yy} + 12u_{xx}u_{yy}^3 = 0$$

(see [5] and [15]), one also finds in [5, p. 414] that Eq. (1.6), with $\ell = 3$, appears as the reduction of a certain second order system of 3 non-linear partial differential equations for 1 unknown function in 3 independent variables. See the Ph.D. thesis of S. Sitton [14] for details.

**5.** With regards to the Cartan equivalence problem associated to each of these non-rigid parabolic geometries of Monge type, it hardly needs to be said that the $\mathfrak{g}_2$ parabolic geometry defined by $\{\alpha_1\}$ was solved in full detail by Cartan [4]. For the remaining interesting cases, that is, except cases **Ib**, **IIIb**, and **IIIa** with $\ell = 2$, we remark that, unlike the $\mathfrak{g}_2$ equivalence problem, *all fundamental invariants appear in the solution to the equivalence problem as torsion.*[3] The equivalence problems associated to the parabolic geometries **IIb** and **IIId** are quite remarkably simple – each admits only a scalar torsion invariant and no curvature invariants.

**6.** As pointed out by the referee, the papers [18] and [19] by Yamaguchi and Yatsui are closely related to the subject of this article. Let us recall that for any regular distribution $\mathcal{D}$ on a manifold $M$ one can associate a graded nilpotent Lie algebra

$$\sigma(\mathcal{D}) = \sigma_{-1} \oplus \sigma_{-2} \oplus \cdots \oplus \sigma_{-k}$$

called the symbol algebra. In accordance with Definition 1.1, we then say that $\mathcal{D}$ is a Monge distribution if $\sigma_{-1}$ admits a co-dimension 1 abelian subalgebra $\mathfrak{y}$. More generally,

[3] For the definition of the torsion of a Cartan connection, see [1, p. 85].

one says that $\mathcal{D}$ defines a pseudo-product structure on $M$ if $\sigma_{-1}$ is the vector space direct sum of two 2 abelian subalgebras $\zeta$ and $\mathfrak{y}$. A general method of constructing pseudo-product structures is given in [18] which produces, from a different viewpoint, the Monge gradations in cases **IIa**, **IIIa**, and **IVa** of Theorem B.

For pseudo-product structures there is a duality construction for the standard models – one can view either one of the subalgebras $\zeta$ or $\mathfrak{y}$ as defining the horizontal distribution of total vector fields for a system of differential equations. In this paper we have taken this horizontal distribution to be the 1-dimensional complement to the codimension-1 abelian sub-algebra, resulting in the realization of the standard models as *under-determined non-linear ODE* or Monge equations. In [19], the co-dimension 1 abelian subalgebra is taken as the horizontal space, leading to realization of the standard models as *over-determined linear PDE*. The dual systems to the Monge equations **III** and **IV** corresponding to $B_\ell$ and $D_\ell$ in Theorem A are

$$\frac{\partial^2 y}{\partial x_p \partial x_q} = \kappa_{pq}\frac{\partial^2 y}{\partial x_1^2}, \quad 1 \leq p, q \leq n,$$

where $y$ is the unknown function and the $x_p$ for $1 \leq p \leq n$ are the independent variables with $n = 2\ell - 3$ for $B_\ell$ and $n = 2\ell - 4$ for $D_\ell$. For the dual systems to Monge equations **II** corresponding to $C_\ell$, we refer the reader to Eqs. (3.6) and (3.7) in [19]. It is an interesting and instructive exercise to explicitly exhibit the transformations between these two realizations.

The paper is organized as follows. In Section 2 we give a complete classification of the grading subsets $\Sigma$ for parabolic geometries of Monge type. We show, in particular, that for simple Lie algebras of rank $\ell \geq 3$, there is a unique simple root $\zeta \in \Sigma$ which is connected in the Dynkin diagram to every other element of $\Sigma$. In Section 3, we adapt the arguments of Yamaguchi [17] to describe all the non-rigid parabolic geometries of Monge type, thereby proving Theorem B. We also describe the cohomology spaces $H^2(\mathfrak{g}_-, \mathfrak{g})_p$ with positive homogeneity weights (as irreducible representations of $\mathfrak{g}_0$) for each non-rigid parabolic geometry. This gives a characterization of the curvature for the normal Cartan connection which will play an important role in our subsequent study of the Cartan equivalence problem for non-rigid parabolic geometries of Monge type. In Section 4, we explicitly give the structure equations for the nilpotent Lie algebras $\mathfrak{g}_-$ for each non-rigid parabolic geometry of Monge type. In each case we integrate these structure equations to obtain the Monge equation realizations of the standard differential systems. This establishes Theorem A. Finally in Section 5 we use standard methods to explicitly calculate the infinitesimal symmetry generators for our standard models in Theorem A. Remarkably, these infinitesimal symmetries are all prolonged point transformations.

## 2. Parabolic geometries of Monge type

In the introduction we defined the notion of a parabolic geometry of Monge type (Definition 1.1) as one for which the $\mathfrak{g}_{-1}$ component contains a co-dimension 1 non-zero abelian subalgebra $\mathfrak{y}$ satisfying $\dim \mathfrak{g}_{-2} = \dim \mathfrak{y}$. In this section we obtain a remarkable intrinsic classification of these parabolic geometries in terms of the defining set of simple roots $\Sigma$. The key to this classification is the fact that the set $\Sigma$ must contain a distinguished root $\zeta$ which is adjacent to all the other roots of $\Sigma$ in the Dynkin diagram of $\mathfrak{g}$ (see Theorem 2.4).

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra of rank $\ell$ with Cartan subalgebra $\mathfrak{h}$ and roots $\Delta$, positive roots $\Delta^+$ and simple roots $\Delta^0$. The height of a root $\beta = \sum_{\alpha\in\Delta^0} n_\alpha \alpha$ with respect to $\Sigma$ is defined as $\mathrm{ht}_\Sigma(\beta) = \sum_{\alpha\in\Sigma} n_\alpha$, and the set of roots with height $j$ is denoted by $\Delta^j_\Sigma$. The $j$-th grading component in (1.2) is

$$\mathfrak{g}_j = \bigoplus_{\beta\in\Delta^j_\Sigma} \mathfrak{g}_\beta \quad \text{for } j \neq 0 \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\beta\in\Delta^0_\Sigma} \mathfrak{g}_\beta .$$

It is clear that $\dim \mathfrak{g}_j = \dim \mathfrak{g}_{-j}$.

While we shall primarily be concerned with the case $\dim \mathfrak{g}_{-1} > 2$, we shall, nevertheless, be required to carefully analyze the case $\dim \mathfrak{g}_{-1} = 2$ since this contains the exceptional Lie algebra $\mathfrak{g}_2$ for the Hilbert–Cartan equation (1.1). For this special case, we shall use the following.

**Lemma 2.1.** *Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra. If $\dim \mathfrak{g}_{-1} = 2$, then $\operatorname{rank} \mathfrak{g} = 2$.*

**Proof.** We show that if $\operatorname{rank} \mathfrak{g} > 2$ then $\dim \mathfrak{g}_1 > 2$. Let $\Sigma \subset \Delta^0$ be any non-empty subset of the simple roots $\Delta^0$ for $\mathfrak{g}$. If $\operatorname{rank} \mathfrak{g} > 2$, then the set $\Sigma$ must non-trivially intersect a set of 3 connected simple roots $\{\alpha, \beta, \gamma\}$. Then $\alpha$, $\alpha + \beta$, $\alpha + \beta + \gamma$, $\beta$, $\beta + \gamma$, and $\gamma$ are all roots. Regardless of which of these 3 simple roots $\alpha$, $\beta$, $\gamma$ are in $\Sigma$, there will always be at least 3 roots with height 1 relative to $\Sigma$ and therefore $\dim \mathfrak{g}_{-1} \geq 3$. For example, if the intersection with $\Sigma$ contains just $\beta$, then $\beta$, $\alpha + \beta$, and $\beta + \gamma$ have height 1 while if the intersection contains $\alpha$ and $\gamma$, then the roots $\alpha$ and $\alpha + \beta$, $\beta + \gamma$, and $\gamma$ have height 1. $\square$

**Theorem 2.2.** *Let $\mathfrak{g}$ be a $|k|$-graded simple Lie algebra of Monge type with $\dim \mathfrak{g}_{-1} = 2$. Then the possibilities are:*

1. $A_2\{\alpha_1, \alpha_2\}$
2. $B_2\{\alpha_2\}$ (*the short root*)
3. $B_2\{\alpha_1, \alpha_2\}$
4. $G_2\{\alpha_1\}$ (*the short root*)
5. $G_2\{\alpha_1, \alpha_2\}$

**Proof.** By the above lemma $\operatorname{rank} \mathfrak{g} = 2$ and hence $\mathfrak{g}$ is of type $A_2$, $B_2 = C_2$, or $G_2$. The gradations not in the above list are $A_2\{\alpha_1\}$, $A_2\{\alpha_2\}$, $B_2\{\alpha_1\}$ and $G_2\{\alpha_2\}$, and they are not of Monge type; specifically, the gradations $A_2\{\alpha_1\}$, $A_2\{\alpha_2\}$, and $B_2\{\alpha_1\}$ have depth $k = 1$ while for $G_2\{\alpha_2\}$ one easily checks that $\dim \mathfrak{g}_{-1} = 4$ and $\dim \mathfrak{g}_{-2} = 1$. $\square$

For the rest of this section we focus on the case $\dim \mathfrak{g}_{-1} > 2$.

**Proposition 2.3.** *Let $\mathfrak{g}$ be a $|k|$-graded semi-simple Lie algebra of Monge type with $\dim \mathfrak{g}_{-1} > 2$, and let $\Sigma$ be the subset of simple roots which defines the gradation of $\mathfrak{g}$.*
**[i]** *The abelian subalgebra $\mathfrak{y} \subset \mathfrak{g}_{-1}$ is $\mathfrak{g}_0$-invariant.*
**[ii]** *There is a 1-dimensional $\mathfrak{g}_0$-invariant subspace $\mathfrak{x}$ such that $\mathfrak{g}_{-1} = \mathfrak{x} \oplus \mathfrak{y}$.*
**[iii]** *There is a unique simple root $\zeta \in \Sigma$ and roots $\{\beta_1, \beta_2, \ldots, \beta_d\} \subset \Delta^1_\Sigma$ such that*

$$\mathfrak{x} = \mathfrak{g}_{-\zeta} \quad \text{and} \quad \mathfrak{y} = \mathfrak{g}_{-\beta_1} \oplus \mathfrak{g}_{-\beta_2} \oplus \cdots \oplus \mathfrak{g}_{-\beta_d}. \tag{2.1}$$

**[iv]** *The set $\Sigma$ consists precisely of the root $\zeta$ and all roots adjacent to $\zeta$ in the Dynkin diagram for $\mathfrak{g}$.*
**[v]** *If $\mathfrak{g}_0$ contains no simple ideal of $\mathfrak{g}$, then the Lie algebra $\mathfrak{g}$ is simple.*

**Proof.** **[i]** Let $\{y_1, y_2, \ldots, y_d\}$ be a basis for $\mathfrak{y}$ and let $\{x, y_1, y_2, \ldots, y_d\}$ be a basis for $\mathfrak{g}_{-1}$. The generating condition $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$ and the fact that $\dim \mathfrak{g}_{-2} = \dim \mathfrak{y}$ imply that

$$\mathrm{ad}_x : \mathfrak{y} \to \mathfrak{g}_{-2} \text{ is an isomorphism.} \tag{2.2}$$

This implies that the vectors $z_i = [x, y_i]$ form a basis for $\mathfrak{g}_{-2}$. Let $u \in \mathfrak{g}_0$. Since the action of $\mathfrak{g}_0$ on $\mathfrak{g}$ preserves the $|k|$-grading, it follows that

$$[u, y_i] = a_i x + b_i^j y_j.$$

Since the vectors $y_i$ commute, the Jacobi identity for the vectors $u$, $y_i$, $y_j$ yields

$$a_i z_j - a_j z_i = 0 \quad \text{for all } 1 \le i < j \le d.$$

Since $d > 1$ this implies that $a_i = 0$ and hence $[u, y_i] \in \mathfrak{y}$. This proves **[i]**.

**[ii]** Since $\mathfrak{g}$ is a complex semi-simple Lie algebra, $\mathfrak{g}_0$ is a reductive Lie algebra and the center $\mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}$ by [1, Theorem 3.2.1]. Hence the center acts on $\mathfrak{g}_{-1}$ by semi-simple endomorphisms. Therefore the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is completely reducible (see for example [1, p. 316]). Thus the $\mathfrak{g}_0$-invariant subspace $\mathfrak{y}$ admits a $\mathfrak{g}_0$-invariant complement $\mathfrak{x}$.

**[iii]** Since the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ used to define the root space decomposition of $\mathfrak{g}$ is, by definition, contained in $\mathfrak{g}_0$, the $\mathfrak{g}_0$-invariant subspaces $\mathfrak{x}$ and $\mathfrak{y}$ must be direct sums of the (1-dimensional) root spaces corresponding to roots in $\Delta^1_\Sigma$. This proves Eq. (2.1).

Put $\mathfrak{x} = \mathfrak{g}_{-\zeta}$. In order to complete the proof of **[iii]**, we must verify that $\zeta$ is a simple root. Suppose not. Since $\zeta$ is a positive root of height 1, we can therefore write $\zeta = \zeta' + \zeta''$, where $\zeta'$ is a positive root of height 0 and $\zeta''$ is a positive root of height 1. Then, on the one hand,

$$[\mathfrak{g}_{\zeta'}, \mathfrak{x}] = [\mathfrak{g}_{\zeta'}, \mathfrak{g}_{-\zeta}] = \mathfrak{g}_{-\zeta''}.$$

On the other hand, $\mathfrak{g}_{\zeta'} \subset \mathfrak{g}_0$ and so $[\mathfrak{g}_{\zeta'}, \mathfrak{x}] \subset \mathfrak{x}$ since $\mathfrak{x}$ is $\mathfrak{g}_0$-invariant. This contradicts the above equation and therefore $\zeta$ must be a simple root which belongs to $\Sigma$.

**[iv]** Let $\beta \in \Sigma \backslash \zeta$ and let $x \in \mathfrak{g}_{-\zeta}$ and $y \in \mathfrak{g}_{-\beta}$ be non-zero vectors. By (2.2), $[x, y] \in \mathfrak{g}_{-2}$ is non-zero, $\zeta + \beta$ must be a root, and therefore $\beta$ is adjacent to $\zeta$ in the Dynkin diagram for $\mathfrak{g}$. Conversely, let $\beta$ be any simple root adjacent to $\zeta$. Then $\beta + \zeta$ is a root and $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{-\zeta}] = \mathfrak{g}_{-\beta-\zeta}$. If $\beta \notin \Sigma$, then $\beta \in \Delta^0_\Sigma$ and therefore, by the $\mathfrak{g}_0$-invariance of $\mathfrak{g}_{-\zeta}$, $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{-\zeta}] \subset \mathfrak{g}_{-\zeta}$. This is a contradiction and hence $\beta \in \Sigma$.

**[v]** Suppose that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{k}$, where $\mathfrak{l}$ and $\mathfrak{k}$ are semi-simple. The condition that $\mathfrak{g}_0$ contains no simple ideal of $\mathfrak{g}$ implies that $\Sigma$ must contain simple roots of $\mathfrak{l}$ and $\mathfrak{k}$. Therefore $\Sigma$ is disconnected in the Dynkin diagram of $\mathfrak{g}$, which contradicts **[iv]**. □

In view of **[v]**, we henceforth assume that $(\mathfrak{g}, \Sigma)$ is a parabolic geometry of Monge type with $\mathfrak{g}$ simple. By Proposition 2.3, there is a simple root $\zeta$ such that all the other roots in $\Sigma$ are connected to $\zeta$ in the Dynkin diagram for $\mathfrak{g}$. We say that the root $\zeta$ is the ***leader*** of $\Sigma$. However not every simple root of $\mathfrak{g}$ can serve as a leader for a parabolic geometry of Monge type. To complete our characterization, we now turn our attention to the gradation of $\mathfrak{g}$ by the leader $\zeta$ itself, and in particular to the decomposition of the semi-simple part $\mathfrak{g}^{ss}_{\zeta,0}$ of its 0-grading component. By virtue of the connectivity of $\Sigma$, there is a one-to-one correspondence between the remaining roots $\Sigma \backslash \zeta$ and the connected components of graph obtained by removing the node $\zeta$ in the Dynkin diagram for $\mathfrak{g}$. Label these connected components by $\Upsilon_\alpha$ for $\alpha \in \Sigma \backslash \zeta$ so that

$$\Delta^0 = \{\zeta\} \cup \bigcup_{\alpha \in \Sigma \backslash \zeta} \Upsilon_\alpha.$$

Let $\mathfrak{g}(\Upsilon_\alpha)$ be the complex simple Lie algebra with Dynkin diagram $\Upsilon_\alpha$. Then by [1, Proposition 3.2.2] we have the following decomposition

$$\mathfrak{g}^{ss}_{\zeta,0} = \bigoplus_{\alpha \in \Sigma \backslash \zeta} \mathfrak{g}(\Upsilon_\alpha).$$

**Theorem 2.4.** *Let $\mathfrak{g}$ be a parabolic geometry of a complex simple Lie algebra as determined by the set of simple roots $\Sigma$. If $\dim \mathfrak{g}_{-1} > 2$, then $\mathfrak{g}$ is a parabolic geometry of Monge type if and only if*

**[i]** *there is root $\zeta \in \Sigma$ which is adjacent to every other root in $\Sigma$ in the Dynkin diagram of $\mathfrak{g}$; and*

**[ii]** *For each $\alpha \in \Sigma \backslash \zeta$, the parabolic geometry for the complex simple Lie algebra $\mathfrak{g}(\Upsilon_\alpha)$ defined by the root $\{\alpha\}$ is $|1|$-graded.*

Condition **[ii]** of Theorem 2.4 excludes just two possibilities. The first is $F_4$ with $\zeta = \alpha_4$, the short root, and the other exclusion is $C_m$ with $\zeta = \alpha_i$, for $1 \le i \le m - 2$, $m \ge 3$. (See Corollary 2.6 for details.) Therefore there are many Monge gradations, and the condition of non-rigidity is essential for arriving at the short list of Monge gradations in Theorem B.

In order to prove this theorem, we consider the set of roots $\Upsilon^1_\alpha$ of $\mathfrak{g}(\Upsilon_\alpha)$ with height 1 relative to the gradation by $\{\alpha\}$, that is,

$$\Upsilon^1_\alpha = \{\beta \in \Delta \mid \beta = \alpha + \sum_{i=1}^{m} n_i\beta_i \quad \text{where } \beta_i \in \Upsilon_\alpha\backslash\alpha,\ n_i > 0,\ \text{and } m \geq 0\}. \tag{2.3}$$

Furthermore, define subspaces of $\mathfrak{g}$ by

$$\mathfrak{y}_{-\alpha} = \bigoplus_{\beta\in\Upsilon^1_\alpha} \mathfrak{g}_{-\beta}. \tag{2.4}$$

These are the $-1$-grading components of $\mathfrak{g}(\Upsilon_\alpha)$ with respect to $\{\alpha\}$. The proof of Theorem 2.4 depends on the following lemma.

**Lemma 2.5.** *Let $\Sigma$ be a set of simple roots satisfying condition* [i] *of Theorem 2.4.*

[i] *Then $\Delta^1_\Sigma = \{\zeta\} \cup \bigcup_{\alpha\in\Sigma\backslash\zeta} \Upsilon^1_\alpha$, and hence we have the following decomposition*

$$\mathfrak{g}_{-1} = \mathfrak{g}_{-\zeta} \oplus \bigoplus_{\alpha\in\Sigma\backslash\zeta} \mathfrak{y}_{-\alpha}. \tag{2.5}$$

[ii] *If $\beta \in \Upsilon^1_\alpha$ and $\beta' \in \Upsilon^1_{\alpha'}$ with $\alpha \neq \alpha'$, then $\beta + \beta'$ is not a root, and hence*

$$[\mathfrak{y}_{-\alpha}, \mathfrak{y}_{-\alpha'}] = 0. \tag{2.6}$$

[iii] *If $\beta \in \Upsilon^1_\alpha$ then $\zeta + \beta \in \Delta$, and hence* $\dim[\mathfrak{g}_{-\zeta}, \mathfrak{y}_{-\alpha}] = \dim \mathfrak{y}_{-\alpha}$.

[iv] *If $\gamma \in \Delta^0_\Sigma$, $\beta \in \Upsilon^1_\alpha$ and $\gamma + \beta \in \Delta$, then $\gamma + \beta \in \Upsilon^1_\alpha$. Thus the $\mathfrak{y}_{-\alpha}$ in (2.5) are $\mathfrak{g}_0$-invariant subspaces of $\mathfrak{g}_{-1}$.*

**Proof.** [i] Clearly $\Upsilon^1_\alpha \subset \Delta^1_\Sigma$ and so it suffices to show that if $\beta \in \Delta^1_\Sigma\backslash\zeta$ then there is a root $\alpha \in \Sigma$ such that $\beta \in \Upsilon^1_\alpha$. Indeed, since $\beta$ has height 1 with respect to $\Sigma$, there is a root $\alpha \in \Sigma$ and simple roots $\beta_i \in \Delta^0\backslash\Sigma$ such that

$$\beta = \alpha + \sum_{i=1}^{m} n_i\beta_i \quad \text{where } n_i > 0 \text{ and } m \geq 0. \tag{2.7}$$

Since $\beta$ is a root, the set of simple roots $\{\alpha, \beta_1, \ldots, \beta_m\}$ must define a connected subgraph of the Dynkin diagram for $\mathfrak{g}$. Therefore $\{\alpha, \beta_1, \ldots, \beta_m\} \subset \Upsilon_\alpha$. This equation implies that $\beta_i \in \Upsilon_\alpha\backslash\alpha$ and $\beta \in \Upsilon^1_\alpha$.

**[ii]** In view of (2.3), the roots $\beta \in \Upsilon^1_\alpha$ and $\beta' \in \Upsilon^1_{\alpha'}$, with $\alpha \neq \alpha'$, are given by

$$\beta = \alpha + \sum_{i=1}^{m} n_i \beta_i \quad \text{and} \quad \beta' = \alpha' + \sum_{i=1}^{m'} n'_i \beta'_i. \tag{2.8}$$

Since $\Upsilon_\alpha$ and $\Upsilon_{\alpha'}$ are disjoint, the totality of roots $\{\alpha, \alpha', \beta_i, \beta'_i\}$ is not a connected subgraph in the Dynkin diagram of $\mathfrak{g}$ and therefore $\beta + \beta'$ cannot be a root. Consequently $[\mathfrak{g}_{-\beta}, \mathfrak{g}_{-\beta'}] = 0$ and (2.6) follows.

**[iii]** Let $(\cdot,\cdot)$ be the positive-definite inner product on the root space induced from the Killing form. Since $\zeta$ is adjacent to $\alpha$ but not any of the $\beta_i$, it follows that

$$(\beta, \zeta) = \big(\alpha + \sum_{i=1}^{m} n_i \beta_i, \zeta\big) = (\alpha, \zeta) < 0,$$

and therefore $\beta + \zeta$ is a root by [8, p. 324 (6)].

**[iv]** We note that $\gamma + \beta \in \Delta^1_\Sigma$, and then use **[i]** to conclude that $\gamma + \beta \in \Upsilon^1_\alpha$. The $\mathfrak{g}_0$-invariance of the summands $\mathfrak{y}_{-\alpha}$ immediately follows. $\square$

**Proof of Theorem 2.4.** Suppose that $\mathfrak{g}$ is a parabolic geometry of Monge type. Then condition **[i]** follows from Proposition 2.3. From (2.1) and (2.5), we know that

$$\mathfrak{y} = \bigoplus_{\alpha \in \Sigma \setminus \zeta} \mathfrak{y}_{-\alpha}. \tag{2.9}$$

Since $\mathfrak{y}$ is abelian, each of the summands $\mathfrak{y}_{-\alpha}$ in this decomposition must be abelian. Since $\mathfrak{y}_{-\alpha}$ is the $-1$-grading component for the gradation of $\mathfrak{g}(\Upsilon_\alpha)$ defined by $\alpha$, this must be a $|1|$-gradation and condition **[ii]** in Theorem 2.4 is established.

Conversely, given a $|k|$-grading defined by $\Sigma$ such that **[i]** and **[ii]** hold, define $\mathfrak{y}$ by (2.9). By (2.5), $\mathfrak{y}$ is a co-dimension 1 subspace of $\mathfrak{g}_{-1}$. We now check the conditions of Definition 1.1 for a parabolic Monge geometry. To prove that $\mathfrak{y}$ is abelian, we first note that each summand $\mathfrak{y}_{-\alpha}$ is abelian by hypothesis **[ii]**. Eq. (2.6) then proves that $\mathfrak{y}$ is abelian. That the dimension of $[\mathfrak{g}_{-\zeta}, \mathfrak{y}]$ equals the dimension of $\mathfrak{y}$ follows directly from part **[iii]** of Lemma 2.5. $\square$

An explicit list of parabolic geometries of Monge type can now be constructed from the classification of $|1|$-graded simple Lie algebras given in the table on page 297 of [1]. We see that condition **[ii]** of Theorem 2.4 holds if and only if the graded simple algebras $\mathfrak{g}(\Upsilon_\alpha)$ are $A$, $B$, $C$, $D$, $E_6$ and $E_7$ with the gradation given by a simple root at the end of its Dynkin diagram as specified in the table. In the case that $\mathfrak{g}(\Upsilon_\alpha)$ is of type $B_m$, $\alpha$ cannot equal $\alpha_m$ which means that the original $|k|$-graded algebra $\mathfrak{g}$ cannot be $F_4$ with $\zeta = \alpha_4$, the short root. Similarly, in the case that $\mathfrak{g}(\Upsilon_\alpha)$ is of type $C_m$, $\alpha$ cannot equal $\alpha_1$ which means that the original $|k|$-graded algebra $\mathfrak{g}$ cannot be $C_m$ with $\zeta = \alpha_i$, for

$1 \le i \le m-2$, $m \ge 3$. One can check that these two exceptions occur precisely when $\Sigma$ consists of just short roots. This proves the following.

**Corollary 2.6.** *Let $\mathfrak{g}$ be a parabolic geometry of a simple Lie algebra as determined by the set of simple roots $\Sigma \subset \Delta^0$. If $\dim \mathfrak{g}_{-1} > 2$, then $\mathfrak{g}$ is a parabolic geometry of Monge type if and only if condition* [**i**] *of Theorem 2.4 holds and $\Sigma$ contains a long root.*

## 3. Non-rigid parabolic geometries of Monge type

Let $\mathfrak{g}$ be a simple Lie algebra and let

$$\mathfrak{g} = \mathfrak{g}_k \oplus \dots \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-k} \tag{3.1}$$

be a $|k|$-grading of $\mathfrak{g}$ determined by the set of simple roots $\Sigma \subset \Delta^0$. We suppose that the parabolic geometry defined by this grading is of Monge type (see Definition 1.1) so that $\Sigma$ satisfies the conditions of Theorem 2.4. The purpose of this section is to determine which parabolic geometries of Monge type are non-rigid, that is, we will characterize the Monge subsets of simple roots $\Sigma$ with non-vanishing second degree Lie algebra cohomology in positive homogeneity weight

$$H^2(\mathfrak{g}_-, \mathfrak{g})_p \neq 0 \quad \text{for some } p > 0. \tag{3.2}$$

In a remarkable paper K. Yamaguchi [17] gives a complete list of all sets of simple roots for which the corresponding parabolic geometry satisfies (3.2). Initially, we simply determined which of the 40 or so cases in Yamaguchi's classification were of Monge type and in this way we arrived at Theorem B. It is a rather surprising fact that of all the possible sets of simple roots $\Sigma$ of Monge type, those which are non-rigid contain either the first or the last root and for the algebras $B$, $C$, and $D$ of rank $\ge 4$ all contain exactly 2 roots. Since these two facts alone effectively reduce the proof of Theorem B to the examination of just a few cases and since both facts can be directly established with relative ease, we have chosen to give the detailed proofs here.

We shall use Kostant's theorem [13] to calculate the Lie algebra cohomology.[4] To briefly describe how this calculation proceeds, we first establish some standard notation. Recall that we denote the set of all roots by $\Delta$ and the positive and negative roots by $\Delta^+$ and $\Delta^-$. For a subset of simple roots $\Sigma \subset \Delta^0$, we denote by

$$\Delta^+_\Sigma = \bigcup_{k>0} \Delta^k_\Sigma \tag{3.3}$$

the set of roots with positive heights with respect to $\Sigma$.

[4] Kostant's theorem applies more generally to the Lie algebra cohomology $H^q(\mathfrak{g}_-, V)$, where $V$ is any representation space of $\mathfrak{g}$, but we limit our discussion to just the case where $V = \mathfrak{g}$ is the adjoint representation of $\mathfrak{g}$.

For each simple root $\alpha_i \in \Delta^0$ the simple Weyl reflection $s_i$ on the root space is defined by $s_i(\beta) = \beta - \langle \beta, \alpha_i \rangle \, \alpha_i$, where $\beta \in \Delta$ and $\langle \beta, \alpha_i \rangle = \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}$. The finite group generated by all simple Weyl reflections is the Weyl group $W$ of $\mathfrak{g}$. For any element $\sigma \in W$, we define another set of roots by

$$\Delta_\sigma = \sigma(\Delta^-) \cap \Delta^+, \tag{3.4}$$

that is, $\Delta_\sigma$ is the set of positive roots that are images of negative roots under the action of $\sigma$. It is an important fact, established in many textbooks, that if $q = \mathrm{card}\Delta_\sigma$, then $\sigma$ can be written as a product of exactly $q$ simple Weyl reflections $s_i$, in other words, $\mathrm{length}(\sigma) = \mathrm{card}\Delta_\sigma$. Finally, define

$$W_\Sigma = \{\sigma \in W \,|\, \Delta_\sigma \subset \Delta^+_\Sigma\} \quad \text{and} \quad W^q_\Sigma = \{\sigma \in W_\Sigma \,|\, \mathrm{card}\Delta_\sigma = q\}. \tag{3.5}$$

Hasse diagrams provide an effective method for finding the sets $W^q_\Sigma$ (see [1, §§3.2.14–16]).

Kostant's method is based upon two key results. The first result states that the cohomology spaces $H^q(\mathfrak{g}_-, \mathfrak{g})$ are isomorphic to the kernel of a certain (algebraic) Laplacian $\Box : \Lambda^q(\mathfrak{g}_-, \mathfrak{g}) \to \Lambda^q(\mathfrak{g}_-, \mathfrak{g})$. The forms in $\ker \Box$ are said to be harmonic – they define distinguished cohomology representatives. Since $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$, the Lie algebra $\mathfrak{g}_0$ naturally acts on the forms $\Lambda^q(\mathfrak{g}_-, \mathfrak{g})$. The second key observation is that this action commutes with $\Box$.

The first assertion in Kostant's theorem is that $\ker \Box$ decomposes as a direct sum of irreducible representations of $\mathfrak{g}_0$, each occurring with multiplicity 1, and that there is a one-to-one correspondence between the irreducible summands in this decomposition and the Weyl group elements in $W^q_\Sigma$. For each $\sigma \in W^q_\Sigma$, we label the corresponding summand by $H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$ and write

$$H^q(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{\sigma \in W^q_\Sigma} H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g}). \tag{3.6}$$

Kostant's theorem also describes the lowest weight vector for $H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$ as an irreducible $\mathfrak{g}_0$-representation.[5] Fix a basis $e_\alpha$ for the root space $\mathfrak{g}_\alpha$, and let $\omega_\alpha$ be the 1-form dual to $e_\alpha$ under the Killing form: $\omega_\alpha(x) = B(e_\alpha, x)$. Let $\theta$ denote the ***highest root*** of $\mathfrak{g}$, which is also the highest weight for the adjoint representation of $\mathfrak{g}$. For $\sigma \in W^q_\Sigma$, let $\Delta_\sigma = \{\beta_1, \beta_2, \ldots, \beta_q\}$. Then

$$\omega_\sigma = e_{-\sigma(\theta)} \otimes \omega_{-\beta_1} \wedge \omega_{-\beta_2} \wedge \cdots \wedge \omega_{-\beta_q} \tag{3.7}$$

is the harmonic representative for the lowest weight vector in $H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$. The homogeneity weight $w_\Sigma(\omega_\sigma)$ of this form with respect to the grading is the homogeneity weight

[5] Actually Kostant [13] studied the cohomology $H^{q,\sigma}(\mathfrak{g}_+, \mathfrak{g})$ and gave the highest weight vector for this irreducible $\mathfrak{g}_0$-representation. Through the Killing form, we have $(H^{q,\sigma}(\mathfrak{g}_+, \mathfrak{g}))^* = H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$, and therefore the negative of the highest weight of the former becomes the lowest weight for the latter.

of all the forms in $H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$, since the orbit of the $\mathfrak{g}_0$-action on $\omega_\sigma$ is all of $H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$ and the grading element $E$ commutes with $\mathfrak{g}_0$.

To calculate the homogeneity weight $w_\Sigma(\omega_\sigma)$ is generally quite complicated but it is possible to obtain a compact formula in the case of immediate interest to us, namely when $q = 2$. Then the length of $\sigma$ is 2 and there are two simple Weyl reflections $s_i$ and $s_j$, $i \neq j$ such that $\sigma = \sigma_{ij} = s_i \circ s_j$.

**Lemma 3.1.** *If $\sigma = s_i \circ s_j \in W^2_\Sigma$, then $\Delta_\sigma = \{\, \alpha_i, s_i(\alpha_j) \,\}$, $\alpha_i \in \Sigma$, and*

$$w_\Sigma(\omega_\sigma) = -\mathrm{ht}_\Sigma(\theta) + \langle \theta, \alpha_i \rangle + 1 + (\langle \theta, \alpha_j \rangle + 1)\, \mathrm{ht}_\Sigma(s_i(\alpha_j)). \tag{3.8}$$

*Therefore the parabolic geometry defined by $\Sigma$ is non-rigid if and only if*

$$\langle \theta, \alpha_i \rangle + (\langle \theta, \alpha_j \rangle + 1)\, \mathrm{ht}_\Sigma(s_i(\alpha_j)) \geq \mathrm{ht}_\Sigma(\theta). \tag{3.9}$$

**Proof.** The formula (3.8) for the homogeneity weight of $H^{q,\sigma}(\mathfrak{g}_-, \mathfrak{g})$ is essentially the same as that given by Yamaguchi in Section 5.3 of [17] and we follow the arguments given there.

We first show that $\Delta_\sigma = \{\, \alpha_i, s_i(\alpha_j)\}$. Since $\sigma = s_i \circ s_j$, we have

$$\sigma^{-1}(\alpha_i) = -s_j(\alpha_i) \in \Delta^- \quad \text{and} \quad \sigma^{-1}(s_i(\alpha_j)) = s_j(\alpha_j) = -\alpha_j \in \Delta^-$$

and therefore $\alpha_i$ and $s_i(\alpha_j)$ are the two distinct elements of $\Delta_\sigma$. Set $\beta_1 = \alpha_i$ and $\beta_2 = s_i(\alpha_j)$. The requirement $\Delta_\sigma \in \Delta^+_\Sigma$ now implies that $\alpha_i \in \Sigma$ and therefore $\mathrm{ht}_\Sigma(\alpha_i) = 1$.

Since

$$\sigma(\theta) = s_i(\theta - \langle \theta, \alpha_j \rangle \alpha_j) = \theta - \langle \theta, \alpha_i \rangle \alpha_i - \langle \theta, \alpha_j \rangle s_i(\alpha_j),$$

we have that the weight of the harmonic representative (3.7) (with $q = 2$) is

$$w_\Sigma(\omega_\sigma) = -\mathrm{ht}_\Sigma(\theta) + \langle \theta, \alpha_i \rangle\, \mathrm{ht}_\Sigma(\alpha_i) + \langle \theta, \alpha_j \rangle \mathrm{ht}_\Sigma(s_i(\alpha_j)) + \mathrm{ht}_\Sigma(\beta_1) + \mathrm{ht}_\Sigma(\beta_2),$$

which reduces to (3.8). $\square$

To continue, we list the expressions for $\theta$ and the nonzero $\langle \theta, \alpha_i \rangle$ for the classical Lie algebras.

$$A_\ell : \theta = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell, \qquad \langle \theta, \alpha_1 \rangle = \langle \theta, \alpha_\ell \rangle = 1 \tag{3.10}$$

$$B_\ell : \theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell, \qquad \langle \theta, \alpha_2 \rangle = 1 \tag{3.11}$$

$$C_\ell : \theta = 2\alpha_1 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell, \qquad \langle \theta, \alpha_1 \rangle = 2 \tag{3.12}$$

$$D_\ell : \theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell, \qquad \langle \theta, \alpha_2 \rangle = 1 \tag{3.13}$$

With these formulas and Lemma 3.1 it is now a straightforward matter to determine all the non-rigid parabolic geometries of Monge type. *Simply stated, the reason that there are relatively few such geometries is because the Monge conditions in Theorem 2.4 lead to a large lower bound for the value of* $\mathrm{ht}_\Sigma(\theta)$.

**Proposition 3.2.** *Every Monge parabolic geometry of type $A_\ell$ with $\ell \geq 5$ whose simple roots $\Sigma$ are interior to the Dynkin diagram is rigid. Apart from the standard symmetry of the Dynkin diagram for $A_\ell$, the non-rigid Monge parabolic geometries of type $A_\ell$ are $A_\ell\{\alpha_1, \alpha_2\}$ for $\ell \geq 2$ and $A_\ell\{\alpha_1, \alpha_2, \alpha_3\}$ for $\ell \geq 3$.*

**Proof.** If $\Sigma$ is interior to the Dynkin diagram, then by Theorem 2.4 it is a connected set of 3 roots. From (3.10), we have $\mathrm{ht}_\Sigma(\theta) = 3$. Since $\alpha_1, \alpha_\ell \notin \Sigma$, (3.9) reduces to

$$(\langle \theta, \alpha_j \rangle + 1)(\mathrm{ht}_\Sigma(\alpha_j) - \langle \alpha_j, \alpha_i \rangle) \geq 3. \tag{3.14}$$

But $\langle \theta, \alpha_j \rangle \leq 1$, $\mathrm{ht}_\Sigma(\alpha_j) \leq 1$ and $-\langle \alpha_j, \alpha_i \rangle \leq 1$ (from the Cartan matrix for $A_\ell$), so (3.14) is satisfied only when

$$\langle \theta, \alpha_j \rangle = 1, \quad \mathrm{ht}_\Sigma(\alpha_j) = 1, \quad \text{and } \langle \alpha_j, \alpha_i \rangle = -1. \tag{3.15}$$

The second equation implies that $\alpha_j \in \Sigma$, which is interior to the Dynkin diagram. Then the first equation cannot be satisfied by (3.10). Therefore the first statement in the proposition is established, and hence the only non-rigid cases for $A_\ell$ with $\ell \geq 5$ are $A_\ell\{\, \alpha_1, \alpha_2 \,\}$ and $A_\ell\{\, \alpha_1, \alpha_2, \alpha_3 \,\}$.

For $\ell \leq 4$ the Monge systems are $A_2\{\alpha_1, \alpha_2\}$, $A_3\{\alpha_1, \alpha_2\}$, $A_3\{\alpha_1, \alpha_2, \alpha_3\}$, $A_4\{\alpha_1, \alpha_2\}$ and $A_4\{\alpha_1, \alpha_2, \alpha_3\}$ and hence, in summary, the only possible non-rigid parabolic geometries of type $A_\ell$ are those listed in the second statement of the proposition. To show that these possibilities are actually all non-rigid, one calculates the following table of Weyl reflections in $W^2_\Sigma$ from the Hasse diagrams and the associated weights from (3.8).

| Monge systems | $W^2_\Sigma$ | Weights of $\sigma_{ij}$ |
|---|---|---|
| $A_2\{\alpha_1, \alpha_2\}$ | $[\sigma_{12}, \sigma_{21}]$ | [4, 4] |
| $A_3\{\alpha_1, \alpha_2\}$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}]$ | [2, 3, 1] |
| $A_\ell\{\alpha_1, \alpha_2\}$, $\ell \geq 4$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}]$ | [2, 3, 0] |
| $A_3\{\alpha_1, \alpha_2, \alpha_3\}$ | $[\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{32}]$ | [1, 1, 2, 2, 1] |
| $A_4\{\alpha_1, \alpha_2, \alpha_3\}$ | $[\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{32}, \sigma_{34}]$ | [1, 0, 2, 0, 0, 0] |
| $A_\ell\{\alpha_1, \alpha_2, \alpha_3\}$, $\ell \geq 5$ | $[\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{32}, \sigma_{34}]$ | [1, 0, 2, 0, 0, −1] |

□

**Proposition 3.3.** *Every Monge parabolic geometry of type $C_\ell$ with $\ell \geq 4$ for a set $\Sigma$ containing 3 simple roots is rigid. The non-rigid Monge parabolic geometries of type $C_\ell$ are $C_3\{\alpha_1, \alpha_2, \alpha_3\}$ and $C_\ell\{\alpha_{\ell-1}, \alpha_\ell\}$ for $\ell \geq 3$.*

**Proof.** By Corollary 2.6, $\Sigma$ must contain the long simple root and therefore $\Sigma = \{\alpha_{\ell-2}, \alpha_{\ell-1}, \alpha_\ell\}$ if it contains 3 simple roots. Then from (3.12) we have $\mathrm{ht}_\Sigma(\theta) = 5$. Since $\ell \geq 4$, we have $\alpha_1 \notin \Sigma$. Then (3.12) shows that $\langle \theta, \alpha_i \rangle = 0$, and therefore (3.8) reduces to

$$(\langle \theta, \alpha_j \rangle + 1)(\mathrm{ht}_\Sigma(\alpha_j) - \langle \alpha_j, \alpha_i \rangle) \geq 5. \tag{3.16}$$

Now we have

$$\langle \theta, \alpha_j \rangle \leq 2, \quad \mathrm{ht}_\Sigma(\alpha_j) \leq 1, \quad -\langle \alpha_j, \alpha_i \rangle \leq 2. \tag{3.17}$$

If $\langle \theta, \alpha_j \rangle = 0$, then (3.16) is not possible. The only other possible value is $\langle \theta, \alpha_j \rangle = 2$ but then $\alpha_j = \alpha_1$ and we have $\mathrm{ht}_\Sigma(\alpha_j) = 0$ and $-\langle \alpha_j, \alpha_i \rangle \leq 1$ by the Dynkin diagram. Then (3.16) fails again. The first statement in the proposition is established and the list of possible non-rigid parabolic geometries of type $C_\ell$ are those listed in the second statement of the proposition. These are all non-rigid.

| Monge systems | $W^2_\Sigma$ | Weights of $\sigma_{ij}$ |
|---|---|---|
| $C_3\{\, \alpha_1,\, \alpha_2,\, \alpha_3 \,\}$ | $[\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{32}]$ | $[0,\, -1,\, 2,\, -1,\, -2]$ |
| $C_3\{\, \alpha_2,\, \alpha_3 \,\}$ | $[\sigma_{21}, \sigma_{23}, \sigma_{32}]$ | $[1,\, 1,\, 0]$ |
| $C_\ell\{\, \alpha_{\ell-1},\, \alpha_\ell \,\}$, $\ell \geq 4$ | $[\sigma_{\ell-1\,\ell-2}, \sigma_{\ell-1\,\ell}, \sigma_{\ell\,\ell-1}]$ | $[-1,\, 1,\, 0]$ |

□

**Proposition 3.4.** *Every Monge parabolic geometry of type $B_\ell$ with $\ell \geq 4$ for a set $\Sigma$ containing 3 simple roots is rigid. The non-rigid Monge parabolic geometries of type $B_\ell$ are $B_2\{\alpha_2\}$, $B_3\{\alpha_2, \alpha_3\}$, $B_3\{\alpha_1, \alpha_2, \alpha_3\}$ and $B_\ell\{\alpha_1, \alpha_2\}$ for $\ell \geq 2$.*

*Likewise, every Monge parabolic geometry of type $D_\ell$ with $\ell \geq 4$ for a set $\Sigma$ containing 3 or more simple roots is rigid. The non-rigid Monge parabolic geometries of type $D_\ell$ for $\ell \geq 4$ are $D_\ell\{\alpha_1, \alpha_2\}$.*

**Proof.** We note that for $D_\ell$, the Monge grading set $\Sigma$ can contain 4 simple roots. For either $B_\ell$ or $D_\ell$ with $\ell \geq 4$, if $\Sigma$ contains 3 or more simple roots then, from (3.11) and (3.13), we find that $\mathrm{ht}_\Sigma(\theta) = 5$ or 6. Since

$$\langle \theta, \alpha_i \rangle \leq 1, \quad \langle \theta, \alpha_j \rangle \leq 1, \quad \langle \theta, \alpha_i \rangle \neq \langle \theta, \alpha_j \rangle, \quad \mathrm{ht}_\Sigma(s_i(\alpha_j)) \leq 3,$$

(3.8) can only hold when $\langle \theta, \alpha_i \rangle = 0$ and $\langle \theta, \alpha_j \rangle = 1$. In this case $\alpha_j = \alpha_2$ by (3.11) and (3.13), and (3.8) becomes

$$2(\mathrm{ht}_\Sigma(\alpha_2) - \langle \alpha_2, \alpha_i \rangle) \geq 5.$$

For $D_\ell$, this is not possible because $-\langle\alpha_2, \alpha_i\rangle \leq 1$. For $B_\ell$, this inequality holds only if $\alpha_2 \in \Sigma$, $\alpha_i = \alpha_3$ and $\ell = 3$. The first statement in the proposition for each type $B_\ell$ or $D_\ell$ is therefore established.

In view of this result and Theorem 2.2 the possible non-rigid, Monge parabolic geometries of type $B_\ell$ are $B_2\{\,\alpha_2\,\}$, $B_2\{\,\alpha_1,\,\alpha_2\,\}$, $B_3\{\,\alpha_1,\,\alpha_2\,\}$, $B_3\{\,\alpha_2,\,\alpha_3\,\}$, $B_3\{\,\alpha_1,\,\alpha_2,\,\alpha_3\,\}$, $B_\ell\{\,\alpha_1,\,\alpha_2\,\}$ for $\ell \geq 4$ and $B_\ell\{\,\alpha_{\ell-1},\,\alpha_\ell\,\}$ for $\ell \geq 4$. The Monge parabolic geometries $B_\ell\{\,\alpha_{\ell-1},\,\alpha_\ell\,\}$ are rigid; all the others are non-rigid.

| Monge systems | $W^2_\Sigma$ | Weights of $\sigma_{ij}$ |
|---|---|---|
| $B_2\{\alpha_1, \alpha_2\}$ | $[\sigma_{12}, \sigma_{21}]$ | [4, 3] |
| $B_2\{\,\alpha_2\,\}$ | $[\sigma_{21}]$ | [3] |
| $B_3\{\,\alpha_1,\,\alpha_2\,\}$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}]$ | [2, 1, 0] |
| $B_3\{\,\alpha_2,\,\alpha_3\,\}$ | $[\sigma_{21}, \sigma_{23}, \sigma_{32}]$ | [−1, 0, 3] |
| $B_3\{\,\alpha_1, \alpha_2,\,\alpha_3\,\}$ | $[\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{32}]$ | [0, −3, −1, −1, 2] |
| $B_\ell\{\,\alpha_1,\,\alpha_2\,\}$, $\ell \geq 4$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}]$ | [2, 1, 0] |
| $B_4\{\,\alpha_3,\,\alpha_4\,\}$, $\ell \geq 4$ | $[\sigma_{32}, \sigma_{34}, \sigma_{43}]$ | [−1, −1, 0] |
| $B_\ell\{\,\alpha_{\ell-1},\,\alpha_\ell\,\}$, $\ell \geq 5$ | $[\sigma_{\ell-1\,\ell-2}, \sigma_{\ell-1\,\ell}, \sigma_{\ell\,\ell-1}]$ | [−2, −1, 0] |

The possible non-rigid, Monge parabolic geometries of type $D_\ell$ are $D_\ell\{\alpha_1, \alpha_2\}$, $D_\ell\{\alpha_{\ell-2}, \alpha_{\ell-1}\}$ and $D_\ell\{\alpha_{\ell-2}, \alpha_\ell\}$. Note that $D_4\{\alpha_2, \alpha_4\}$ is equivalent to $D_4\{\alpha_1, \alpha_2\}$ and $D_\ell\{\alpha_{\ell-2}, \alpha_{\ell-1}\}$ and $D_\ell\{\alpha_{\ell-2}, \alpha_\ell\}$ are equivalent for all $\ell \geq 4$. For $\ell \geq 5$ the geometries $D_\ell\{\alpha_{\ell-2}, \alpha_\ell\}$ are rigid.

| Monge systems | $W^2_\Sigma$ | Weights of $\sigma_{ij}$ |
|---|---|---|
| $D_4\{\,\alpha_1,\,\alpha_2\,\}$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}, \sigma_{24}]$ | [2, 1, 0, 0] |
| $D_\ell\{\,\alpha_1,\,\alpha_2\,\}$, $\ell \geq 5$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}]$ | [2, 1, 0] |
| $D_5\{\,\alpha_3,\,\alpha_5\,\}$ | $[\sigma_{32}, \sigma_{34}, \sigma_{35}, \sigma_{53}]$ | [0, −1, 0, 0] |
| $D_\ell\{\,\alpha_{\ell-2},\,\alpha_\ell\,\}$, $\ell \geq 6$ | $[\sigma_{\ell-2\,\ell-3}, \sigma_{\ell-2\,\ell-1}, \sigma_{\ell-2\,\ell}, \sigma_{\ell\,\ell-2}]$ | [−1, −1, 0, 0] |

□

For the exceptional Lie algebras the highest weights and non-zero $\langle\theta, \alpha_i\rangle$ are:

$$\begin{aligned}
&G_2\ : \theta = 3\alpha_1 + 2\alpha_2, && \langle\theta, \alpha_2\rangle = 1\\
&F_4\ : \theta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, && \langle\theta, \alpha_1\rangle = 1\\
&E_6\ : \theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, && \langle\theta, \alpha_2\rangle = 1\\
&E_7\ : \theta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, && \langle\theta, \alpha_1\rangle = 1\\
&E_8\ : \theta = 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8, && \langle\theta, \alpha_8\rangle = 1
\end{aligned}$$

Here the roots are labeled as in [17, p. 454] or [12, p. 58].

**Proposition 3.5.** *The only non-rigid Monge parabolic geometries for the exceptional simple Lie algebras are $G_2\{\alpha_1\}$ and $G_2\{\alpha_1, \alpha_2\}$.*

**Proof.** Consider first the case of $F_4$. If card$\Sigma \geq 3$, then $\mathrm{ht}_\Sigma(\theta) = 9$ and with $\langle \theta, \alpha_i \rangle \leq 1$, $\langle \theta, \alpha_j \rangle \leq 1$, and $\mathrm{ht}_\Sigma(s_i(\alpha_j)) \leq 3$, the inequality (3.8) cannot hold. For parabolic geometries of Monge type, $\Sigma$ must contain the long root and this leaves just $F_4\{\alpha_1, \alpha_2\}$ as the only possibility. But it is easy to check that this is rigid.

For $E_6$, $E_7$ and $E_8$ we have $\langle \theta, \alpha_i \rangle \leq 1$, $\langle \theta, \alpha_j \rangle \leq 1$, $\mathrm{ht}_\Sigma(s_i(\alpha_j)) \leq 2$ and $\langle \theta, \alpha_i \rangle \neq \langle \theta, \alpha_j \rangle$ so that the left-hand side of (3.8) does not exceed 4. If card$\Sigma \geq 3$, then by the connectivity of $\Sigma$ we have $\mathrm{ht}_\Sigma(\theta) \geq 6, 6$ and 9 for $E_6$, $E_7$ and $E_8$ respectively and so only those geometries with card$\Sigma = 2$ remain as possibilities. For card$\Sigma = 2$ the size of $\mathrm{ht}_\Sigma(\theta)$ is still $\geq 5$ except for the 2 cases (apart from the symmetry of the $E_6$ Dynkin diagram) listed below, all of which are rigid by direct calculation.

| Monge systems | $W^2_\Sigma$ | Weights of $\sigma_{ij}$ |
|---|---|---|
| $F_4\{\,\alpha_1,\ \alpha_2\,\}$ | $[\sigma_{12}, \sigma_{21}, \sigma_{23}, \sigma_{24}]$ | $[-1, 0, -3]$ |
| $E_6\{\,\alpha_5,\ \alpha_6\,\}$ | $[\sigma_{54}, \sigma_{56}, \sigma_{65}]$ | $[-1, 0, 0]$ |
| $E_7\{\,\alpha_6,\ \alpha_7\,\}$ | $[\sigma_{65}, \sigma_{67}, \sigma_{76}]$ | $[-1, 0, 0]$ |

□

We conclude this section with the description of $H^2(\mathfrak{g}_-, \mathfrak{g})_p$ with positive homogeneity weights as $\mathfrak{g}_0^{ss}$-representations. This gives a characterization of the curvature for the normal Cartan connection which will play an important role in our subsequent study of the Cartan equivalence problem for non-rigid parabolic geometries of Monge type. With this application in mind and in view of (1.3), we will only discuss the non-rigid parabolic geometries of Monge type in Theorem B with $H^1(\mathfrak{g}_-, \mathfrak{g})_p = 0$ for all $p \geq 0$. Therefore we will not discuss the cases **Ib**, **IIIb**, and **IIIa** with $\ell = 2$ in the following.

By Kostant's theorem, the irreducible components of $H^2(\mathfrak{g}_-, \mathfrak{g})_p$ are in one-to-one correspondence with $W^2_\Sigma$. The corresponding lowest weight vector is given by (3.7). We make the standard transformation from the lowest weight to the highest weight by the longest Weyl reflection.

Furthermore, if in (3.7) the $e_{-\sigma(\theta)} \in \mathfrak{p}$, then the corresponding cohomology class is called ***curvature*** and otherwise it is called ***torsion***. As mentioned in the introduction, all our second cohomology classes with positive homogeneities are *torsion* classes, except the case for the Hilbert–Cartan equation **Va**. We indicate this in our table by listing the homogeneity weight of $-\sigma(\theta)$, and it is strictly negative in all cases except one.

In the following table, the $\omega$ are the fundamental weights of $\mathfrak{g}_0^{ss}$, and the $V$ are the standard representations of $\mathfrak{g}_0^{ss}$ corresponding to its first fundamental weight $\omega_1$. The subscript tf stands for trace free, and $\underline{\otimes}$ means the Cartan component of the tensor product.

## 4. Standard differential systems for the non-rigid parabolic geometries of Monge type

In this section we use the standard matrix representations of the classical simple Lie algebras to explicitly calculate the structure equations for each negatively graded compo-

| | Non-rigid par. Monge | $\mathfrak{g}_0^{ss}$ | $W^2_\Sigma$ | Hom. wts | wts of $-\sigma(\theta)$ | Highest weights | Rep. spaces |
|---|---|---|---|---|---|---|---|
| **Ia** | $A_3\{\alpha_1,\alpha_2,\alpha_3\}$ | 0 | $\sigma_{12},\ \sigma_{32}$ | 1 | $-2$ | 0 | $\mathbb{R}$ |
| | | | $\sigma_{13}$ | 1 | $-1$ | | |
| | | | $\sigma_{21},\ \sigma_{23}$ | 2 | $-1$ | | |
| | $A_\ell\{\alpha_1,\alpha_2,\alpha_3\}$, $\ell \geq 4$ | $A_{\ell-3}$ | $\sigma_{12}$ | 1 | $-2$ | $\omega_1$ | $V$ |
| | | | $\sigma_{21}$ | 2 | $-1$ | | |
| **IIa** | $C_3\{\alpha_2,\alpha_3\}$ | $A_1$ | $\sigma_{21}$ | 1 | $-1$ | 0 | $\mathbb{R}$ |
| | | | $\sigma_{23}$ | 1 | $-3$ | $5\omega_1$ | $S^5(V)$ |
| | $C_\ell\{\alpha_{\ell-1},\alpha_\ell\}$ $\ell \geq 4$ | $A_{\ell-2}$ | $\sigma_{\ell-1,\ell}$ | 1 | $-3$ | $3\omega_1+2\omega_{\ell-2}$ | $S^3(V)\underline{\otimes}S^2(V^*)$ |
| **IIb** | $C_3\{\alpha_1,\alpha_2,\alpha_3\}$ | 0 | $\sigma_{21}$ | 2 | $-1$ | 0 | $\mathbb{R}$ |
| **IIIa** | $B_3\{\alpha_1,\alpha_2\}$ | $A_1$ | $\sigma_{12}$ | 2 | $-1$ | $4\omega_1$ | $S^4(V)$ |
| | | | $\sigma_{21}$ | 1 | $-2$ | $6\omega_1$ | $S^6(V)$ |
| | $B_\ell\{\alpha_1,\alpha_2\}$ $\ell \geq 4$ | $B_{\ell-2}$ | $\sigma_{12}$ | 2 | $-1$ | $2\omega_1$ | $S^2(V)_{\mathrm{tf}}$ |
| | | | $\sigma_{21}$ | 1 | $-2$ | $3\omega_1$ | $S^3(V)_{\mathrm{tf}}$ |
| **IIIc** | $B_3\{\alpha_2,\alpha_3\}$ | $A_1$ | $\sigma_{32}$ | 3 | $-1$ | $2\omega_1$ | $S^2(V)$ |
| **IIId** | $B_3\{\alpha_1,\alpha_2,\alpha_3\}$ | 0 | $\sigma_{32}$ | 2 | $-2$ | 0 | $\mathbb{R}$ |
| **IVa** | $D_4\{\alpha_1,\alpha_2\}$ | $A_1\oplus A_1$ | $\sigma_{12}$ | 2 | $-1$ | $[2\omega_1,2\omega_1]$ | $S^2(V_1)\otimes S^2(V_2)$ |
| | | | $\sigma_{21}$ | 1 | $-2$ | $[3\omega_1,3\omega_1]$ | $S^3(V_1)\otimes S^3(V_2)$ |
| | $D_\ell\{\alpha_1,\alpha_2\}$ $\ell \geq 5$ | $D_{\ell-2}$ | $\sigma_{12}$ | 2 | $-1$ | $2\omega_1$ | $S^2(V)_{\mathrm{tf}}$ |
| | | | $\sigma_{21}$ | 1 | $-2$ | $3\omega_1$ | $S^3(V)_{\mathrm{tf}}$ |
| **Va** | $G_2\{\alpha_1\}$ | $A_1$ | $\sigma_{12}$ | 4 | **0** | $4\omega_1$ | $S^4(V)$ |
| **Vb** | $G_2\{\alpha_1,\alpha_2\}$ | 0 | $\sigma_{12}$ | 4 | $-1$ | 0 | $\mathbb{R}$ |

nent $\mathfrak{g}_-$ of the non-rigid parabolic geometries of Monge type enumerated in Theorem B. We give the structure equations in terms of the dual 1-forms. In each case these structure equations are easily integrated to give the Maurer–Cartan forms on the nilpotent Lie group $N$ for the Lie algebra $\mathfrak{g}_-$ and the associated standard differential system is found.

**Ia. $A_\ell\{\,\alpha_1,\,\alpha_2,\,\alpha_3\,\}$, $\ell \geq 3$.** We use the standard matrix representation for the Lie algebra $A_\ell = \mathfrak{sl}(\ell+1)$. Then the Cartan subalgebra is defined by the trace-free diagonal matrices $H_i = E_{i,i} - E_{i+1,i+1}$, $1 \leq i \leq \ell$. Let $L_i$ be the linear function on the Cartan subalgebra taking the value of the $i$th entry. The simple roots are $\alpha_i = L_i - L_{i+1}$ for $1 \leq i \leq \ell$ and the positive roots are $\alpha_i + \cdots + \alpha_j$ for $1 \leq i \leq j \leq \ell$. Thus the positive roots of height 1 with respect to $\Sigma = \{\,\alpha_1,\,\alpha_2,\,\alpha_3\,\}$ are $\alpha_1$, $\alpha_2$ and $\alpha_3 + \cdots + \alpha_i$ for $3 \leq i \leq \ell$. The leader is $X = e_{-\alpha_2} = E_{3,2}$ and the remaining root vectors of height $-1$, which define a basis for the abelian subalgebra $\mathfrak{y}$ are $P_0 = e_{-\alpha_1} = E_{2,1}$ and $P_i = e_{-\alpha_3-\cdots-\alpha_{i+2}} = E_{i+3,3}$ for $1 \leq i \leq \ell - 2$. This somewhat obscure labeling of the basis vectors will be justified momentarily. It is easy to verify that the given matrices are indeed the required root vectors with respect to the above choice of Cartan subalgebra. These vectors define the weight $-1$ component $\mathfrak{g}_{-1}$ of the grading for $\mathfrak{sl}(\ell+1)$ defined by $\Sigma$. Since $[\mathfrak{g}_{-1}, \mathfrak{g}_{-i}] = \mathfrak{g}_{-i-1}$, we calculate the remaining vectors in $\mathfrak{g}_-$ to be

$$[\,P_0,\,X\,] = Y_0 = -E_{3,1}, \quad [\,P_i,\,X\,] = Y_i = E_{i+3,2},$$

$$[P_0, Y_i] = [P_i, Y_0] = Z_i = -E_{i+3,1}.$$

The grading of $\mathfrak{g}_-$ and full structure equations are therefore

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle P_0, P_1, P_2, \dots, P_{\ell-2}, X \rangle, \\ \mathfrak{g}_{-2} &= \langle Y_0, Y_1, \dots, Y_{\ell-2} \rangle, \\ \mathfrak{g}_{-3} &= \langle Z_1, \dots, Z_{\ell-2} \rangle, \end{aligned} \quad \text{and} \quad \begin{array}{c|cccccc} & P_0 & P_i & X & Y_0 & Y_i & Z_i \\ \hline P_0 & 0 & 0 & Y_0 & 0 & Z_i & 0 \\ P_i & & 0 & Y_i & Z_i & 0 & 0 \\ X & & & 0 & 0 & 0 & 0 \\ Y_0 & & & & 0 & 0 & 0 \\ Y_i & & & & & 0 & 0 \\ Z_i & & & & & & 0 \end{array}.$$

In terms of the dual basis $\{\theta^0_p, \theta^i_p, \theta_x, \theta^0_y, \theta^i_y, \theta^i_z\}$ for the Lie algebra these structure equations are

$$d\theta^0_p = 0, \quad d\theta^i_p = 0 \quad d\theta_x = 0,$$
$$d\theta^0_y = \theta_x \wedge \theta^0_p, \quad d\theta^i_y = \theta_x \wedge \theta^i_p, \quad d\theta^i_z = \theta^i_y \wedge \theta^0_p + \theta^0_y \wedge \theta^i_p.$$

These structure equations are easily integrated to obtain Maurer–Cartan forms on the nilpotent Lie group for $\mathfrak{g}_-$. The first 5 structure equations immediately give

$$\theta^0_p = dp^0, \quad \theta^i_p = dp^i, \quad \theta_x = dx, \quad \theta^0_y = dy^0 - p^0 dx, \quad \theta^i_y = dy^i - p^i dx,$$

so that the last structure equation becomes

$$d\,\theta^i_z = -dp^0 \wedge dy^i - dp^i \wedge dy^0 + (p^i dp^0 + p^0 dp^i) \wedge dx. \tag{4.1}$$

The first term can be written as either $d(-p^0 dy^i)$ or $d(y^i dp^0)$. Since our goal is to give the simplest possible form for span $\{\theta^0_y, \theta^i_y, \theta^i_z\}$, we chose to write the first term as $d(-p^0 dy^i)$ so that (4.1) integrates to

$$\theta^i_z = dz^i - p^0 dy^i - p^i dy^0 + p^0 p^i dx \equiv dz^i - p^0 p^i dx \mod \{\theta^0_y, \theta^i_y\}.$$

The standard Pfaffian system defined by the parabolic geometry $A_\ell\{\alpha_1, \alpha_2, \alpha_3\}$ is therefore

$$\begin{aligned} I_{A_\ell\{1,2,3\}} &= \operatorname{span}\{\theta^0_y, \theta^i_y, \theta^i_z\} \\ &= \operatorname{span}\{dy^0 - p^0 dx, dy^i - p^i dx, dz^i - p^0 p^i dx\}. \end{aligned}$$

This is the canonical Pfaffian system for the Monge equations (1.4). By Tanaka's theorem we are guaranteed that the symmetry algebra of the system is $\mathfrak{sl}(\ell+1)$.

**IIa. $C_\ell\{\alpha_{\ell-1}, \alpha_\ell\}$, $\ell \geq 3$.** The split real form for $C_\ell$ which we shall use is $\mathfrak{sp}(\ell, \mathbb{R}) = \{X \in \mathfrak{gl}(2\ell, \mathbb{R}) \mid X^t J + JX = 0\}$, where

$$J = \begin{bmatrix} 0 & K_\ell \\ -K_\ell & 0 \end{bmatrix} \quad \text{and} \quad K_\ell = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Each $X \in \mathfrak{sp}(\ell, \mathbb{R})$ may be written as $X = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix}$ where $A$, $B$, $C$ are $\ell \times \ell$ matrices, $A' = KA^tK$ and $B = B'$ and $C = C'$. The diagonal matrices $H_i = E_{i,i} - E_{2\ell+1-i,2\ell+1-i}$ define a Cartan subalgebra. The simple roots $\alpha_i = L_i - L_{i+1}$, $1 \leq i \leq \ell - 1$ and $\alpha_\ell = 2L_\ell$ and the positive roots are

$$\begin{cases} \alpha_i + \cdots + \alpha_{j-1} & \text{for } 1 \leq i < j \leq \ell, \quad \text{and} \\ (\alpha_i + \cdots + \alpha_{\ell-1}) + (\alpha_j + \cdots + \alpha_\ell) & \text{for } 1 \leq i \leq j \leq \ell. \end{cases} \tag{4.2}$$

(For the Lie algebras of type $B$, $C$ and $D$, we use the lists of positive roots from [16].) Therefore, for the choice of simple roots $\Sigma = \{\alpha_{\ell-1}, \alpha_\ell\}$, the roots of height 1 are $\alpha_\ell$ and $\alpha_i + \cdots + \alpha_{\ell-1}$, for $1 \leq i \leq \ell - 1$. The root $-\alpha_\ell$ is our leader with root vector $X = E_{\ell+1,\ell}$. A basis for the abelian subalgebra $\mathfrak{y}$, corresponding to the remaining roots of height $-1$ is given by $P_i = E_{\ell,i} - E_{2\ell+1-i,\ell+1}$. One easily checks that these matrices belong to $\mathfrak{sp}(\ell, \mathbb{R})$ and that they are indeed root vectors for the above choice of Cartan subalgebra. By direct calculation we then find that

$$[\,P_i,\, X\,] = Y_i = -E_{\ell+1,i} - E_{2\ell+1-i,\ell}, \quad \text{and}$$
$$[\,P_i,\, Y_i\,] = 2Z_{ii} = 2E_{2\ell+1-i,i} \quad \text{and } [\,P_i,\, Y_j\,] = Z_{ij} = E_{2\ell+1-i,j} + E_{2\ell+1-j,i}$$

Note that $Z_{ij} = Z_{ji}$. The grading and full structure equations for $\mathfrak{g}_-$ are therefore

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle\, P_1,\, P_2,\, \ldots\,, P_{\ell-1},\, Z\,\rangle, \\ \mathfrak{g}_{-2} &= \langle\, Y_1, \ldots, Y_{\ell-1}\,\rangle, \\ \mathfrak{g}_{-3} &= \langle\, Z_{11},\, Z_{12},\, \ldots\,, Z_{\ell\ell}\,\rangle, \end{aligned} \qquad \text{and} \qquad \begin{array}{c|cccc} & P_i & X & Y_i & Z_{ij} \\ \hline P_h & 0 & Y_h & \epsilon Z_{hi} & 0 \\ X & & 0 & 0 & 0 \\ Y_k & & & 0 & 0 \\ Z_{hk} & & & & 0 \end{array}$$

where $\epsilon = 2$ if $i = j$ and $\epsilon = 1$ otherwise. In terms of the dual basis $\{\,\theta_p^i,\, \theta_x,\, \theta_y^i,\, \theta_z^{ij}\}$ for $\mathfrak{g}_-$ these structure equations are

$$d\theta_p^i = 0 \quad d\theta_x = 0, \quad d\theta_y^i = \theta_x \wedge \theta_p^i,$$
$$d\theta_z^{ij} = \theta_y^i \wedge \theta_p^j + \theta_y^j \wedge \theta_p^i$$

These structure equations are easily integrated to give the following Maurer–Cartan forms

$$\theta_p^i = dp^i, \quad \theta_x = dx, \quad \theta_y^i = dy^i - p^i dx,$$
$$\theta_z^{ij} = dz^{ij} - p^i dy^j - p^j dy^i + p^i p^j\, dx.$$

The standard Pfaffian system defined by the parabolic geometry $C_\ell\{\,\alpha_{\ell-1},\,\alpha_\ell\,\}$ is therefore

$$I_{C_\ell\{\ell-1,\ell\}} = \operatorname{span}\{\,\theta^i_y,\,\theta^{ij}_z\,\} = \operatorname{span}\{\,dy^i - p^i dx,\; dz^{ij} - p^i p^j dx\,\}.$$

This is the canonical Pfaffian system for the Monge equations (1.5). By Tanaka's theorem we are guaranteed that the symmetry algebra of the system is $\mathfrak{sp}(\ell,\mathbb{R})$.

**IIIa. $B_\ell\{\alpha_1,\,\alpha_2\}$, $\ell \geq 3$.** The split real form for $B_\ell$ is $\mathfrak{so}(\ell+1,\ell)$ which we take to be the Lie algebra of $n\times n$ matrices, $n = 2\ell+1$, which are skew-symmetric with respect to the anti-diagonal matrix $K_n = [k_{ij}]$. The diagonal matrices $H_i = E_{i,i} - E_{n+1-i,n+1-i}$ define a Cartan subalgebra. The simple roots are $\alpha_i = L_i - L_{i+1}$, $1\leq i\leq \ell-1$ and $\alpha_\ell = L_\ell$ and the positive roots are

$$\begin{cases} \alpha_i + \cdots + \alpha_j & \text{for } 1\leq i\leq j\leq \ell, \quad \text{and} \\ (\alpha_i + \cdots + \alpha_\ell) + (\alpha_j + \cdots + \alpha_\ell) & \text{for } 1\leq i< j\leq \ell. \end{cases}$$

Therefore, for the choice of simple roots $\Sigma = \{\alpha_1,\,\alpha_2\}$, the roots of height 1 are

$$\begin{cases} \alpha_1 \\ \alpha_2 + \cdots + \alpha_j \quad \text{for } 2\leq j\leq \ell, \quad \text{and} \\ \alpha_2 + \cdots + \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_\ell \quad \text{for } 3\leq i\leq \ell. \end{cases}$$

The root $-\alpha_1$ is our leader with root vector $X = E_{2,1} - E_{n,n-1}$. A basis for the abelian subalgebra $\mathfrak{y}$, corresponding to the remaining roots of height $-1$ is given by $P_i = E_{i+2,2} - E_{n-1,n-i-1}$ for $1\leq i\leq n-4$. One easily checks that these matrices belong to $\mathfrak{so}(\ell+1,\ell)$ and that they are indeed root vectors for the above choice of Cartan subalgebra. By direct calculation we find that for $1\leq i\leq n-4$ and $1\leq j\leq n-4$

$$[\,P_i,\,X\,] = Y_i = E_{i+2,1} - E_{n,n-i-1}, \quad \text{and}$$
$$[\,P_i,\,Y_j\,] = \kappa_{ij} Z, \quad \text{where} \quad Z = E_{n,2} - E_{n-1,1} \text{ and } [\kappa_{ij}] = K_{n-4}.$$

The grading and full structure equations for $\mathfrak{g}_-$ are therefore

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle\, P_1,\,P_2,\,\ldots,\,P_{n-4},\,X\,\rangle, \\ \mathfrak{g}_{-2} &= \langle\, Y_1,\ldots,Y_{n-4}\rangle, \\ \mathfrak{g}_{-3} &= \langle\, Z\,\rangle, \end{aligned} \qquad \text{and} \qquad \begin{array}{c|cccc} & P_i & X & Y_i & Z \\ \hline P_h & 0 & Y_h & \kappa_{hi} Z & 0 \\ X & & 0 & 0 & 0 \\ Y_k & & & 0 & 0 \\ Z & & & & 0 \end{array}.$$

In terms of the dual basis $\{\,\theta^i_p,\,\theta_x,\,\theta^i_y,\,\theta_z\,\}$ for $\mathfrak{g}_-$ the structure equations are

$$d\theta^i_p = 0, \quad d\theta_x = 0, \quad d\theta^i_y = \theta_x\wedge\theta^i_p, \quad d\theta_z = \kappa_{ij}\theta^j_y\wedge\theta^i_p,$$

which are integrated to give the following Maurer–Cartan forms

$$\theta_p^i = dp^i, \quad \theta_x = dx, \quad \theta_y^i = dy^i - p^i dx,$$
$$\theta_z = dz - \kappa_{ij} p^i dy^j + \frac{1}{2}\kappa_{ij} p^i p^j \, dx.$$

The standard Pfaffian system defined by the parabolic geometry $B_\ell\{\,\alpha_1,\,\alpha_2\,\}$ is therefore

$$I_{B_\ell\{1,2\}} = \operatorname{span}\{\,\theta_y^i,\,\theta_z\,\} = \operatorname{span}\{\,dy^i - p^i dx,\, dz - \frac{1}{2}\kappa_{ij} p^i p^j \, dx\,\}. \tag{4.3}$$

This is the canonical Pfaffian system for the Monge equations (1.6). By Tanaka's theorem we are guaranteed that the symmetry algebra of the system is $\mathfrak{so}(\ell+1,\ell)$.

**IVa.** $\boldsymbol{D_\ell\{\alpha_1,\,\alpha_2\},\ \ell \geq 4.}$ In this case $n = 2\ell$ and the positive roots are

$$\begin{cases} \alpha_i + \cdots + \alpha_{j-1} & \text{for } 1 \leq i < j \leq \ell, \quad \text{and} \\ (\alpha_i + \cdots + \alpha_{\ell-2}) + (\alpha_j + \cdots + \alpha_\ell) & \text{for } 1 \leq i < j \leq \ell \end{cases} \tag{4.4}$$

but otherwise the formulas from **III** remain unchanged.

We now turn to the exceptional cases.

**Ib.** $\boldsymbol{A_\ell\{\alpha_1,\,\alpha_2\,\},\ \ell \geq 2.}$ We retain the notation used in **Ia**. In the present case the leader is $X = e_{-\alpha_1} = E_{2,1}$ and the matrices $P_i = e_{-\alpha_2-\cdots-\alpha_{i+2}} = E_{i+2,2}$, $1 \leq i \leq \ell - 1$ define a basis for $\mathfrak{y}$. The structure equations are $[\,P_i, X\,] = Y_i = E_{i+2,1}$ and the standard differential system is the contact system

$$I_{A_\ell\{\alpha_a,\alpha_2\}} = \{\,dy^1 - p^1 dx, dy^2 - p^2 dx, \ldots, dy^{\ell-1} - p^{\ell-1} dx\,\} \tag{4.5}$$

on the jet space $J^1(\mathbb{R}, \mathbb{R}^{\ell-1})$.

**IIb.** $\boldsymbol{C_3\{\alpha_1,\,\alpha_2,\alpha_3\,\}.}$ The roots of height 1 are $\Sigma = \{\alpha_1, \alpha_2, \alpha_3\}$. The root $-\alpha_2$ is our leader with root vector $X = E_{3,2} - E_{5,4}$. A basis for the abelian subalgebra $\mathfrak{y}$, corresponding to the roots $-\alpha_1$ and $-\alpha_3$, is $P_1 = E_{2,1} - E_{6,5}$ and $P_2 = E_{4,3}$ and we calculate

$$\begin{aligned} Y_1 &= [P_1, X] = E_{6,4} - E_{3,1}, & Y_2 &= [P_2, X] = E_{5,3} + E_{4,2}, \\ Z_1 &= [P_1, Y_2] = -E_{4,1} - E_{6,3}, & Z_2 &= [X, Y_2] = -2E_{5,2}, \\ Z_3 &= [X, Z_1] = E_{6,2} + E_{5,1}, & Z_4 &= [P_1, Z_3] = -2E_{6,1}. \end{aligned}$$

The grading and full structure equations for $\mathfrak{g}_-$ are therefore

$$
\begin{aligned}
\mathfrak{g}_{-1} &= \langle\, P_1,\, P_2,\, X \,\rangle, \\
\mathfrak{g}_{-2} &= \langle\, Y_1,\, Y_2 \,\rangle, \\
\mathfrak{g}_{-3} &= \langle\, Z_1,\, Z_2 \,\rangle, \\
\mathfrak{g}_{-4} &= \langle\, Z_3 \,\rangle, \\
\mathfrak{g}_{-5} &= \langle\, Z_4 \,\rangle
\end{aligned}
\qquad \text{and} \qquad
\begin{array}{c|ccccccccc}
 & P_1 & P_2 & X & Y_1 & Y_2 & Z_1 & Z_2 & Z_3 & Z_4 \\
\hline
P_1 & 0 & 0 & Y_1 & 0 & Z_1 & 0 & 2Z_3 & Z_4 & 0 \\
P_2 & & 0 & Y_2 & Z_1 & 0 & 0 & 0 & 0 & 0 \\
X & & & 0 & 0 & Z_2 & Z_3 & 0 & 0 & 0 \\
Y_1 & & & & 0 & Z_3 & Z_4 & 0 & 0 & 0 \\
Y_2 & & & & & 0 & 0 & 0 & 0 & 0 \\
Z_1 & & & & & & 0 & 0 & 0 & 0 \\
Z_2 & & & & & & & 0 & 0 & 0 \\
Z_3 & & & & & & & & 0 & 0 \\
Z_4 & & & & & & & & & 0
\end{array}.
$$

In terms of the dual basis $\{\theta_p^1,\, \theta_p^2,\, \theta_x,\, \theta_y^1,\, \theta_y^2,\, \theta_z^1,\, \theta_z^2,\, \theta_z^3,\, \theta_z^4\}$ for $\mathfrak{g}_-$ the structure equations are

$$
\begin{gathered}
d\theta_p^1 = 0, \quad d\theta_p^2 = 0, \quad d\theta_x = 0, \quad d\theta_y^1 = \theta_x \wedge \theta_p^1, \quad d\theta_y^2 = \theta_x \wedge \theta_p^2, \\
d\theta_z^1 = \theta_y^1 \wedge \theta_p^2 + \theta_y^2 \wedge \theta_p^1, \quad d\theta_z^2 = \theta_y^2 \wedge \theta_x, \\
d\theta_z^3 = -\theta_y^1 \wedge \theta_y^2 + \theta_z^1 \wedge \theta_x + 2\theta_z^2 \wedge \theta_p^1, \quad d\theta_z^4 = \theta_z^1 \wedge \theta_y^1 + \theta_z^3 \wedge \theta_p^1,
\end{gathered}
$$

which integrate to give

$$
\begin{gathered}
\theta_p^1 = dp^1, \quad \theta_p^2 = dp^2, \quad \theta_x = dx, \quad \theta_y^1 = dy^1 - p^1 dx, \quad \theta_y^2 = dy^2 - p^2 dx, \\
\theta_z^1 = dz^1 - p^2 dy^1 - p^1 dy^2 + p^1 p^2 dx, \quad \theta_z^2 = dz^2 - x dy^2, \\
\theta_z^3 = dz^3 - x dz^1 - 2p^1 dz^2 + (2xp^1 - y^1) dy^2 \\
\theta_z^4 = dz^4 + (xp^1 - y^1) dz^1 + (p^1)^2 dz^2 - p^1 dz^3 - p^1 (xp^1 - y^1) dy^2.
\end{gathered}
$$

The standard differential system for $C_3\{\alpha_1, \alpha_2, \alpha_3\}$ is therefore

$$
\begin{aligned}
I_{C_3\{1,2,3\}} &= \{\theta_y^1,\, \theta_y^2,\, \theta_z^1,\, \theta_z^2,\, \theta_z^3,\, \theta_z^4\} \\
&= \{dy^1 - p^1 dx, dy^2 - p^2 dx,\, dz^1 - p^1 p^2 dx,\, dz^2 - xp^2 dx, \\
&\qquad dz^3 - (y^1 p^2 + xp^1 p^2)\, dx,\, dz^4 - y^1 p^1 p^2 dx\}
\end{aligned}
$$

is the canonical differential system for the first order Monge system (1.8).

**IIIb. $\boldsymbol{B_2\{\,\alpha_2\,\}}$.** The roots of height 1 are $\alpha_2$ and $\alpha_1 + \alpha_2$, and the standard differential system is just the canonical differential system

$$
I_{B_2\{\,\alpha_2\,\}} = \{dy - p\, dx\} \tag{4.6}
$$

**IIIc. $\boldsymbol{B_3\{\,\alpha_2,\, \alpha_3\,\}}$.** The roots of height 1 are $\Sigma = \{\,\alpha_1 + \alpha_2,\, \alpha_2,\, \alpha_3\,\}$. The root $-\alpha_3$ is our leader with root vector $X = E_{4,3} - E_{5,4}$. A basis for the abelian subalgebra $\mathfrak{y}$, corresponding to the roots $-\alpha_2$ and $-\alpha_1 - \alpha_2$, is $Q_1 = E_{3,2} - E_{6,5}$ and $Q_2 = E_{3,1} - E_{7,5}$ and we calculate

$$\begin{aligned} P_1 &= [Q_1, X] &&= E_{6,4} - E_{4,2}, \quad & P_2 &= [Q_2, X] = E_{7,4} - E_{4,1}, \\ Y_1 &= [P_1, X] &&= E_{6,3} - E_{5,2}, \quad & Y_2 &= [P_2, X] = E_{7,3} - E_{5,1}, \\ Z &= [Q_1, Y_2] &&= E_{6,1} - E_{7,2}. \end{aligned}$$

The grading and full structure equations for $\mathfrak{g}_-$ are therefore

$$\begin{aligned} \mathfrak{g}_{-1} &= \langle\, Q_1,\, Q_2,\, X \,\rangle, \\ \mathfrak{g}_{-2} &= \langle\, P_1,\, P_2 \,\rangle, \\ \mathfrak{g}_{-3} &= \langle\, Y_1,\, Y_2 \,\rangle, \\ \mathfrak{g}_{-4} &= \langle\, Z \,\rangle, \end{aligned} \qquad \text{and} \qquad \begin{array}{c|cccccccc} & Q_1 & Q_2 & X & P_1 & P_2 & Y_1 & Y_2 & Z \\ \hline Q_1 & 0 & 0 & P_1 & 0 & 0 & 0 & Z & 0 \\ Q_2 & & 0 & P_2 & 0 & 0 & -Z & 0 & 0 \\ X & & 0 & 0 & -Y_1 & -Y_2 & 0 & 0 & 0 \\ P_1 & & & & 0 & -Z & 0 & 0 & 0 \\ P_2 & & & & & 0 & 0 & 0 & 0 \\ Y_1 & & & & & & 0 & 0 & 0 \\ Y_2 & & & & & & & 0 & 0 \\ Z & & & & & & & & 0 \end{array}.$$

In terms of the dual basis $\{\, \theta_q^1,\, \theta_q^2,\, \theta_x,\, \theta_p^1,\, \theta_p^2,\, \theta_y^1,\, \theta_y^2,\, \theta_z \}$ for $\mathfrak{g}_-$ the structure equations are

$$\begin{gathered} d\theta_q^1 = 0, \quad d\theta_q^2 = 0, \quad d\theta_x = 0, \quad d\theta_p^1 = \theta_x \wedge \theta_q^1, \quad d\theta_p^2 = \theta_x \wedge \theta_q^2, \\ d\theta_y^1 = \theta_x \wedge \theta_p^1, \quad d\theta_y^2 = \theta_x \wedge \theta_p^2, \quad d\theta_z = -\theta_y^1 \wedge \theta_q^2 + \theta_y^2 \wedge \theta_q^1 + \theta_p^1 \wedge \theta_p^2, \end{gathered}$$

and one finds that

$$\begin{gathered} \theta_q^1 = dq^1, \quad \theta_q^2 = dq^2 \quad \theta_x = dx, \quad \theta_p^1 = dp^1 - q^1 dx, \quad \theta_p^2 = dp^2 - q^2 dx, \\ \theta_y^1 = dy^1 - p^1 dx, \quad \theta_y^2 = dy^2 - p^2 dx, \\ \theta_z = dz - p^2 dp^1 + q^2 dy^1 - q^1 dy^2 + (p^2 q^1 - p^1 q^2) dx. \end{gathered}$$

The standard Pfaffian differential system for $B_2\{\alpha_2, \alpha_3\}$ is therefore

$$\begin{aligned} I_{B_2\{2,3\}} &= \{\, \theta_y^1,\, \theta_y^2,\, \theta_p^1,\, \theta_p^2,\, \theta_z \,\} \\ &= \{\, dy^1 - p^1 dx,\, dy^2 - p^2 dx,\, dp^1 - q^1 dx,\, dp^2 - q^2 dx,\, dz - p^2 q^1 dx \,\} \end{aligned}$$

which coincides with the differential system for the Monge equations (1.9). We remark that this Monge system may also be encoded on a 7-dimensional manifold without the coordinate $q^2$ by the Pfaffian system $\{\, \theta_y^1,\, \theta_y^2,\, \theta_p^1,\, \theta_z \,\}$ – however, the symmetry algebra of this latter Pfaffian system is only 16-dimensional.

**IIId.** $\boldsymbol{B_3, \Sigma = \{\alpha_1, \alpha_2, \alpha_3\}}$**.** The roots of height 1 are $\Sigma = \{\alpha_1, \alpha_2, \alpha_3\}$. The root $-\alpha_2$ is our leader with root vector $X = E_{3,2} - E_{6,5}$, a basis for the abelian subalgebra $\mathfrak{y}$, corresponding to the roots $-\alpha_1$ and $-\alpha_3$, is $P_1 = E_{2,1} - E_{7,6}$ and $P_2 = E_{4,3} - E_{5,4}$ and we calculate

$$
\begin{aligned}
Y_1 &= [P_1, X] = E_{7,5} - E_{3,1}, & Y_2 &= [P_2, X] = E_{4,2} - E_{6,4},\\
Z_1 &= [P_1, Y_2] = E_{7,4} - E_{4,1}, & Z_2 &= [P_2, Y_2] = E_{6,3} - E_{5,2},\\
Z_3 &= [P_1, Z_2] = E_{5,1} - E_{7,3}, & Z_4 &= [X, Z_3] = E_{7,2} - E_{6,1}.
\end{aligned}
$$

The grading and full structure equations for $\mathfrak{g}_-$ are therefore

$$
\begin{aligned}
\mathfrak{g}_{-1} &= \langle P_1,\ P_2,\ X \rangle,\\
\mathfrak{g}_{-2} &= \langle Y_1,\ Y_2 \rangle,\\
\mathfrak{g}_{-3} &= \langle Z_1,\ Z_2 \rangle,\\
\mathfrak{g}_{-4} &= \langle Z_3 \rangle,\\
\mathfrak{g}_{-5} &= \langle Z_4 \rangle
\end{aligned}
$$

and

| | $P_1$ | $P_2$ | $X$ | $Y_1$ | $Y_2$ | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ |
|---|---|---|---|---|---|---|---|---|---|
| $P_1$ | 0 | 0 | $Y_1$ | 0 | $Z_1$ | 0 | $Z_3$ | 0 | 0 |
| $P_2$ | | 0 | $Y_2$ | $Z_1$ | $Z_2$ | $Z_3$ | 0 | 0 | 0 |
| $X$ | | | 0 | 0 | 0 | 0 | 0 | $Z_4$ | 0 |
| $Y_1$ | | | | 0 | 0 | 0 | $-Z_4$ | 0 | 0 |
| $Y_2$ | | | | | 0 | $-Z_4$ | 0 | 0 | 0 |
| $Z_1$ | | | | | | 0 | 0 | 0 | 0 |
| $Z_2$ | | | | | | | 0 | 0 | 0 |
| $Z_3$ | | | | | | | | 0 | 0 |
| $Z_4$ | | | | | | | | | 0 |

In terms of the dual basis $\{\theta_p^1, \theta_p^2, \theta_x, \theta_y^1, \theta_y^2, \theta_z^1, \theta_z^2, \theta_z^3, \theta_z^4\}$ the structure equations for $\mathfrak{g}_-$ are

$$
\begin{gathered}
d\theta_p^1 = 0, \quad d\theta_p^2 = 0, \quad d\theta_x = 0, \quad d\theta_y^1 = \theta_x \wedge \theta_p^1, \quad d\theta_y^2 = \theta_x \wedge \theta_p^2,\\
d\theta_z^1 = \theta_y^2 \wedge \theta_p^1 + \theta_y^1 \wedge \theta_p^2, \quad d\theta_z^2 = \theta_y^2 \wedge \theta_p^2, \quad d\theta_z^3 = \theta_z^1 \wedge \theta_p^2 + \theta_z^2 \wedge \theta_p^1,\\
d\theta_z^4 = \theta_y^1 \wedge \theta_z^2 + \theta_y^2 \wedge \theta_z^1 + \theta_z^3 \wedge \theta_x.
\end{gathered}
$$

Integrating these equations, one finds that

$$
\begin{gathered}
\theta_p^1 = dp^1, \quad \theta_p^2 = dp^2 \quad \theta_x = dx, \quad \theta_y^1 = dy^1 - p^1 dx, \quad \theta_y^2 = dy^2 - p^2 dx,\\
\theta_z^1 = dz^1 - p^2 dy^1 - p^1 dy^2 + p^1 p^2 dx, \quad \theta_z^2 = dz^2 - p^2 dy^2 + \frac{1}{2}(p^2)^2 dx,\\
\theta_z^3 = dz^3 + \frac{1}{2}(p^2)^2 dy^1 + p^1 p^2 dy^2 - p^2 dz^1 - p^1 dz^2 - \frac{1}{2} p^1 (p^2)^2 dx,\\
\theta_z^4 = dz^4 + y^2 dz^1 + y^1 dz^2 - x\, dz^3.
\end{gathered}
$$

The standard Pfaffian differential system for the parabolic geometry $B_3\{\alpha_1, \alpha_2, \alpha_3\}$ is therefore

$$I_{B_3\{1,2,3\}} = \{\theta^1_y,\, \theta^2_y,\, \theta^1_z,\, \theta^2_z,\, \theta^3_z,\, \theta^4_z\,\}$$
$$= \{dy^1 - p^1dx,\, dy^2 - p^2dx, dz^1 - p^1p^2dx,\, dz^2 - \frac{1}{2}(p^2)^2dx,\, dz^3 - \frac{1}{2}p^1(p^2)^2dx,$$
$$dz^4 - \frac{1}{2}p^2(xp^1p^2 - y^1p^2 - 2y^2p^1)\, dx\}$$

which is the canonical Pfaffian system for the first order Monge equations (1.10). Given the visual asymmetry of these equations, it is a remarkable fact that the symmetry algebra is isomorphic to $\mathfrak{so}(4,3)$.

**Va. $G_2\{\alpha_1\}$.** Let $\{H_1, H_2\}$ be a Cartan subalgebra for $\mathfrak{g}_2$ and let $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ be bases for the root spaces for the negative roots $-\alpha_1$, $-\alpha_2$, $-\alpha_1 - \alpha_2$, $-2\alpha_1 - \alpha_2$, $-3\alpha_1 - \alpha_2$, $-3\alpha_1 - 2\alpha_2$. In terms of a Chevalley basis (see [8, p. 346]), the structure equations for $\mathfrak{g}_2$ are, in part,

| | $H_1$ | $H_2$ | $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
|---|---|---|---|---|---|---|---|---|
| $H_1$ | 0 | 0 | $-2Y_1$ | $3Y_2$ | $Y_3$ | $-Y_4$ | $-3Y_5$ | 0 |
| $H_2$ | | 0 | $Y_1$ | $-2Y_2$ | $-Y_3$ | 0 | $Y_5$ | $-Y_6$ |
| $Y_1$ | | | 0 | $-Y_3$ | $-2Y_4$ | $3Y_5$ | 0 | 0 |
| $Y_2$ | | | | 0 | 0 | 0 | $Y_6$ | 0 |
| $Y_3$ | | | | | 0 | $3Y_6$ | 0 | 0 |
| $Y_4$ | | | | | | 0 | 0 | 0 |
| $Y_5$ | | | | | | | 0 | 0 |
| $Y_6$ | | | | | | | | 0 |

For $\Sigma = \{\,\alpha_1\}$ the roots of height 1 are $\alpha_1$ and $\alpha_1 + \alpha_2$ and thus $\mathfrak{g}_-$ is spanned by the vectors

$$Q = Y_3,\ X = Y_1,\ P = [Q, X] = 2Y_4,\ Y = [P, X] = -6Y_5,\ Z = [Q, P] = 6Y_6.$$

The structure equations for the dual coframe $\{\,\theta^q, \theta^x, \theta^p, \theta^y, \theta^z\}$ are

$$d\theta^q = 0, \quad d\theta^x = 0, \quad d\theta^p = \theta^x \wedge \theta^q, \quad d\theta^y = \theta^x \wedge \theta^p, \quad d\theta^z = \theta^p \wedge \theta^q,$$

which are easily integrated to give

$$\theta^q = dq, \quad \theta^x = dx, \quad \theta^p = dp - q\, dx, \quad \theta^y = dy - p\, dx,$$
$$\theta^z = dz - q\, dp + \frac{1}{2}q^2\, dx.$$

The standard differential system for $\mathfrak{g}_2\{\alpha_1\}$ is therefore

$$I_{G_2\{1\}} = \text{span}\,\{\theta^y, \theta^p, \theta^z\} = \text{span}\ \{dy - p\, dx,\, dp - q\, dx, dz - \frac{1}{2}q^2\, dx\},$$

which is the canonical Pfaffian system for the Cartan–Hilbert equation (1.1).

**Vb. $G_2\{\alpha_1, \alpha_2\}$.** In this case the roots of height 1 are $\{\alpha_1, \alpha_2\}$ so that $\mathfrak{g}_-$ is the sum of all the negative root spaces. We set

$$R = Y_2, \quad X = Y_1, \quad Q = [R, X] = Y_3, \quad P = [Q, X] = 2Y_4,$$
$$Y = [P, X] = -6Y_5, \quad Z = [Y, R] = 6Y_6.$$

The structure equations for the dual coframe $\{\theta^r, \theta^x, \theta^q, \theta^p, \theta^y, \theta^z\}$ are

$$d\theta^r = 0, \quad d\theta^x = 0, \quad d\theta^q = \theta^x \wedge \theta^r, \quad d\theta^p = \theta^x \wedge \theta^q,$$
$$d\theta^y = \theta^x \wedge \theta^p, \quad d\theta^z = \theta^r \wedge \theta^y + \theta^p \wedge \theta^q$$

which are easily integrated to give

$$\theta^r = dr, \quad \theta^x = dx, \quad \theta^q = dq - r\,dx, \quad \theta^p = dp - q\,dx,$$
$$\theta^y = dy - p\,dx, \quad \theta^z = dz + r\,dy - q\,dp + (\frac{1}{2}q^2 - pr)\,dx.$$

The standard differential system for $\mathfrak{g}_2\{\alpha_1\}$ is therefore

$$I_{G_2\{1,2\}} = \text{span}\,\{\theta^y, \theta^p, \theta^q, \theta^z\} = \text{span}\ \{dy - p\,dx,\ dp - q\,dx, dq - r\,dx, dz - \frac{1}{2}q^2\,dx\},$$

which is the canonical Pfaffian system for the prolongation of the Pfaffian system for the Cartan–Hilbert equation (1.1) given in the previous case.

## 5. Infinitesimal symmetries for the standard models

In this section we give explicit formulas for the infinitesimal symmetries for the Monge equations in Theorem A. We find that these infinitesimal symmetries are all prolonged point symmetries and that coefficients of the vector fields for any symmetry are all quadratic functions of the variables $x$, $y^i$, $z^\alpha$.

The infinitesimal symmetries for any first order system of Monge equations

$$\dot{z}^\alpha = F^\alpha(x, y^i, \dot{y}^i, z^\alpha)$$

is, by definition, the Lie algebra of vector fields

$$X = A\frac{\partial}{\partial x} + B^i\frac{\partial}{\partial y^i} + C^\alpha\frac{\partial}{\partial z^\alpha} + D^i\frac{\partial}{\partial \dot{y}^i}, \tag{5.1}$$

where the coefficients $A$, $B^i$, $C^\alpha$, $D^i$ are functions of the variables $x$, $y^i$, $z^\alpha$, $\dot{y}^i$, which preserve the Pfaffian system

$$\mathcal{I} = \operatorname{span}\{\, \theta^i = dy^i - \dot{y}^i dx, \theta^\alpha = dz^\alpha - F^\alpha\, dx \,\}.$$

From the equation $\mathcal{L}_X \theta^i \equiv 0 \bmod \mathcal{I}$ one finds that the coefficients $A$ and $B^i$ are independent of the variables $\dot{y}^i$ and that

$$D^i = D_x B^i - \dot{y}^i D_x A, \quad \text{where} \quad D_x = \frac{\partial}{\partial x} + \dot{y}^i \frac{\partial}{\partial y^i} + F^\alpha \frac{\partial}{\partial z^\alpha}. \tag{5.2}$$

The equation $\mathcal{L}_X \theta^\alpha \equiv 0 \bmod \mathcal{I}$ then implies **[i]** that the coefficients $C^\alpha$ are also independent of the variables $\dot{y}^i$ so that $X$ is a prolonged point transformation, and **[ii]**

$$D_x C^\alpha - X(F^\alpha) - F^\alpha D_x(A) = 0. \tag{5.3}$$

To continue, we now take $F^\alpha = F^\alpha_{ij}\dot{y}^i\dot{y}^j$, where the coefficients $F^\alpha_{ij} = F^\alpha_{ji}$ are constant. Then, by Eq. (5.1), we find that (5.3) becomes

$$D_x C^\alpha - 2F^\alpha_{ij} D_x B^i \dot{y}^j + F^\alpha D_x(A) = 0.$$

This equation is a polynomial identity in the derivatives $\dot{y}^j$ of order 4. From the coefficients of $\dot{y}^i\dot{y}^j\dot{y}^h\dot{y}^k$ and 1 one finds that

$$\frac{\partial A}{\partial z^\alpha} = 0 \quad \text{and} \quad \frac{\partial C^\alpha}{\partial x} = 0. \tag{5.4}$$

The coefficients of $\dot{y}^i$, $\dot{y}^i\dot{y}^j$ and $\dot{y}^i\dot{y}^j\dot{y}^k$ give, respectively,

$$2F^\alpha_{\ell i}\frac{\partial B^\ell}{\partial x} = \frac{\partial C^\alpha}{\partial y^i}, \tag{5.5a}$$

$$F^\alpha_{\ell i}\frac{\partial B^\ell}{\partial y^j} + F^\alpha_{\ell j}\frac{\partial B^\ell}{\partial y^i} = F^\alpha_{ij}\frac{\partial A}{\partial x} + F^\beta_{ij}\frac{\partial C^\alpha}{\partial z^\beta}, \quad \text{and} \tag{5.5b}$$

$$2F^\alpha_{\ell(i}F^\beta_{jk)}\frac{\partial B^\ell}{\partial z^\beta} = F^\alpha_{(ij}\frac{\partial A}{\partial y^{k)}}. \tag{5.5c}$$

These are the determining equations for the symmetries of the Monge equations $\dot{z}^\alpha = F^\alpha_{ij}\dot{y}^i\dot{y}^j$.

The integrability conditions for (5.4) and (5.5) imply that all the coefficients $A, B^i$ and $C^\alpha$ are quadratic functions of the coordinates $\{x, y^i, z^\alpha\}$. Thus the determining equations reduce to purely algebraic equations. It is now a straightforward, albeit a slightly tedious, matter to explicitly construct a basis for all the solutions to the determining equations. The results of these calculations are summarized in the following table of symmetries for the Monge equations of type $A$, $BD$ and $C$.

| gr. | $\dot z^i = \dot y^0 \dot y^i$ | $\dot z = \frac{1}{2}\kappa_{ij}\dot y^i \dot y^j$ | $\dot z^{ij} = \dot y^i \dot y^j$ |
|---|---|---|---|
| $-1$ | $\partial_x,\ \partial_{y^0},\ \partial_{y^i},\ \partial_{z^i}$ | $\partial_x,\ \partial_{y^i},\ \partial_z$ | $\partial_x,\ \partial_{y^i},\ \partial_{z^{ij}}$ |
| 0 | $x\,\partial_x + \frac{1}{2}y^0\,\partial_{y^0} + \frac{1}{2}y^i\,\partial_{y^i}$ | $x\,\partial_x + \frac{1}{2}y^i\,\partial_{y^i}$ | $x\,\partial_x + \frac{1}{2}y^j\,\partial_{y^j}$ |
| | $y^0\,\partial_x + z^i\,\partial_{y^i}$ ,<br>$y^i\,\partial_x + z^i\,\partial_{y^0}$ | $y^i\,\partial_x + \frac{1}{2}z\kappa^{ij}\ \partial_{y^j}$ | $y^i\,\partial_x + \frac{1}{2}z^{ij}\,\partial_{y^j}$ |
| | $x\partial_{y^0} + y^i\partial_x$ ,<br>$x\partial_{y^i} + y^0\partial_{z^i}$ | $x\,\kappa^{ij}\partial_y^j + y^i\,\partial_z$ | $x\,\partial_{y^i} + 2y^j\,\partial_{ij}$<br>$\partial_{ij} = \frac{1+\delta_{ij}}{2}\partial_{z^{ij}}$ |
| | $y^0\partial_{y^0} + z^i\partial_{z^i}$ ,<br>$y^i\partial_{y^j} + z^i\partial_{z^j}$ | $y^i\,\partial_{y^i} + 2z\,\partial_z$ ,<br>$\kappa^{ik}b_{ij}y^j\,\partial_{y^k}$ ($b_{ij}$ skew) | $y^i\,\partial_{y^j} + 2z^{ik}\,\partial_{jk}$ |
| 1 | $x^2\partial_x + xy^0\partial_{y^0} + xy^i\partial_{y^i}$<br>$+ y^0y^i\partial_{z^i}$ | $x^2\partial_x + zy^i\partial_{y^i} + K\partial_z$<br>$K = \kappa_{ij}y^iy^j$ | $x^2\partial_x + xy^i\partial_{y^i} + y^iy^j\partial_{z^{ij}}$ |
| | $xy^0\partial_x + (y^0)^2\partial_{y^0} + y^0z^i\partial_{z^i}$ ,<br>$xy^i\partial_x + xz^i\partial_{y^0} + y^iy^j\partial_{y^j}$<br>$+ z^iy^j\partial_{z^j}$ | $xy^i\partial_x + (y^iy^j + (xz - K)\kappa^{ij})\partial_{y^j}$<br>$+ zy^i\partial_z$ | $xy^i\partial_x + y^jz^{ik}\partial_{jk}$<br>$+ \frac{1}{2}(xz^{ij} + y^iy^j)\partial_{y^i}$ |
| | $y^0y^i\partial_x + y^0z^i\partial_{y^0} + y^iz^j\partial_{y^j}$<br>$+ z^iz^k\partial_{z^k}$ | $K\partial_x + zy^i\partial_{y^i} + z^2\partial_z$ | $y^iy^j\partial_x + z^{ih}z^{jk}\partial_{hk}$<br>$+\frac{1}{2}(y^iz^{jk} + y^jz^{ik})\partial_{y^k}$ |

It is not difficult to see that *the above point symmetries for our Monge equations coincide with the infinitesimal generators for the $G$ action on $G/\tilde P$, where $\tilde P$ is the parabolic subgroup of $G$ defined by the $|1|$-gradings using the leader only.*


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