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New relations between G_2 geometries in dimensions 5 and 7

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There are two well-known parabolic split G_2 geometries in dimension 5, $(2,3,5)$ distributions and G_2 contact structures. Here we link these two geometries with yet another G_2 related contact structure, which lives on a 7-manifold. More precisely, we present a natural geometric construction that associates to a $(2,3,5)$ distribution a 7-dimensional bundle endowed with a canonical Lie contact structure. We further study the relation between the canonical normal Cartan connections associated with the two structures and we show that the Cartan holonomy of the induced Lie contact structure reduces to G_2. This motivates the study of the curved orbit decomposition associated with a G_2 reduced Lie contact structure on a 7-manifold. It is shown that, provided an additional curvature condition is satisfied, in a neighborhood of each point in the open curved orbit the structure descends to a $(2,3,5)$ distribution on a local leaf space. The closed orbit carries an induced G_2 contact structure.

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1. Introduction

A distribution $\mathcal{D}$ (locally) spanned by vector fields ξ_1 and ξ_2 on a smooth five manifold M is called a $(2,3,5)$ distribution, or generic, if the five vector fields

$$\xi_1, \quad \xi_2, \quad [\xi_1,\xi_2], \quad [\xi_1,[\xi_1,\xi_2]], \quad [\xi_2,[\xi_1,\xi_2]]$$

are linearly independent at each point. Two distributions $\mathcal{D}$ and $\mathcal{D}'$ are said to be equivalent if there exists a diffeomorphism $\phi : M \to M'$ such that $\phi_* \mathcal{D} = \mathcal{D}'$. It is a classical result [13] that every $(2,3,5)$ distribution is locally equivalent to the kernel $\mathcal{D}_F = \ker(\omega_1, \omega_2, \omega_3)$ of three 1-forms

$$\omega_1 = \mathrm{d}y - p\,\mathrm{d}x, \quad \omega_2 = \mathrm{d}p - q\,\mathrm{d}x, \quad \omega_3 = \mathrm{d}z - F\mathrm{d}x$$

for coordinates (x,y,p,q,z) on an open subset $U \subset \mathbb{R}^5$ around the origin and a smooth function $F = F(x,y,p,q,z)$ such that $F_{qq} \neq 0$. Infinitesimal symmetries of a distribution $\mathcal{D} \subset TM$ are vector fields $\eta \in \mathfrak{X}(M)$ that preserve the distribution, i.e. $\mathcal{L}_\eta \xi = [\eta, \xi] \in \Gamma(\mathcal{D})$ for all $\xi \in \Gamma(\mathcal{D})$.

Cartan and Engel found, independently but at the same time [10, 12], the first explicit realizations of the exceptional simple Lie algebra $\mathfrak{g}_2$. Each of them presented two different descriptions of $\mathfrak{g}_2$, and among them one that realized it as the Lie algebra of infinitesimal symmetries of a rank 2 distribution locally equivalent to the distribution $\mathcal{D}_{q^2}$ associated with the function $F = q^2$. Cartan's later fundamental work [11] shows that the Cartan–Engel distribution $\mathcal{D}_{q^2}$ can indeed be regarded as the *flat* and *maximally symmetric* model in the category of $(2,3,5)$ distributions. Flat, because he shows how to associate to any $(2,3,5)$ distribution a curvature tensor, called Cartan quartic $\mathcal{C} \in \Gamma(S^4\mathcal{D}^*)$, which vanishes if and only if the distribution is locally equivalent to $\mathcal{D}_{q^2}$. Maximally symmetric, because he proves that the symmetry algebra of a distribution with nonvanishing Cartan quartic has dimension smaller than $\dim(\mathfrak{g}_2) = 14$.

More recent work [23] associates to a $(2,3,5)$ distribution a canonical conformal structure of signature $(2,3)$, i.e. an equivalence class of pseudo-Riemannian metrics of signature $(2,3)$ where two metrics g and $\hat{g}$ are considered equivalent if one is a conformal rescaling of the other, meaning that $\hat{g} = e^{2f}\,g$. On the one hand, this allows to understand the geometry of $(2,3,5)$ distributions in terms of the more familiar conformal geometry. On the other hand, the construction provides an interesting class of conformal metrics given explicitly in terms of a single function F, see [24, 19, 14, 30]. From an algebraic point of view, the construction is based on the Lie algebra inclusion $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(4,3)$, see [15].

This paper is of a similar flavor. It links $(2,3,5)$ distributions with special types of contact geometries associated with the Lie algebras $\mathfrak{g}_2$ and $\mathfrak{so}(4,3)$. Associated with every simple Lie algebra, there is a *parabolic contact geometry*; it is given by a contact distribution (i.e. a corank one distribution that is locally given as the kernel of a 1-form θ such that $\theta \wedge (d\theta)^n \neq 0$) and additional geometric structure on the contact distribution. By Pfaff's theorem, all contact distributions are locally equivalent. However, by equipping the distribution with additional geometric structure, e.g. with a tensor field of some type or a decomposition of the contact distribution as a tensor product of auxiliary vector bundles, one again obtains an interesting geometry with nontrivial local invariants. Every parabolic contact geometry has a flat and maximally symmetric model and the infinitesimal symmetry algebra of the model realizes the simple Lie algebra in question. Parabolic contact geometries associated with special orthogonal Lie algebras $\mathfrak{so}(p+2, q+2)$ have been studied under the name Lie contact structures, [27, 28, 20, 21], by means of Tanaka theory [29].

It is now a natural question whether one can use the Lie algebra inclusion $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(4,3)$ to relate $(2,3,5)$ distributions to Lie contact geometry, as it is done in the construction of conformal structures from $(2,3,5)$ distributions. Inspecting the models of the two geometries shows that there is indeed such a natural geometric construction. More precisely, in Sec. 3 we show the following.

Theorem 1.1. *Let* $\mathcal{D} = \mathrm{span}(\xi_1, \xi_2)$ *be a* $(2,3,5)$ *distribution with derived rank* 3 *distribution* $[\mathcal{D},\mathcal{D}] = \mathrm{span}(\xi_1, \xi_2, [\xi_1, \xi_2])$ *and consider the* 7*-manifold* $\mathbb{T} = \mathbb{P}([\mathcal{D},\mathcal{D}]) \backslash \mathbb{P}(\mathcal{D})$ *of lines contained in the rank* 3 *distribution but not contained in the rank* 2 *distribution. Then* $\mathbb{T}$ *carries a naturally induced Lie contact structure. The induced Lie contact structure is flat if and only if the* $(2,3,5)$ *distribution is flat.*

The proof of the theorem is based on the equivalent descriptions of $(2,3,5)$ distributions and Lie contact structures, respectively, as particular types of Cartan geometries. It employs a functorial construction that assigns to the canonical Cartan geometry encoding a $(2,3,5)$ distribution a Cartan geometry encoding a Lie contact structure.

In Sec. 4, we use the structure equations for a class of $(2,3,5)$ distributions (for those that are encoded in terms of functions $F = h(q)$ of a single variable q) to construct the corresponding Lie contact structure explicitly in terms of a conformal symmetric rank 4 tensor $[\Upsilon]$ on the contact distribution. In particular, this enables us to find explicit generators for the symmetry algebras in the case that $F = \frac{1}{k(k-1)} q^k$ and thus examples of Lie contact structures with large symmetry algebras.

In Sec. 5, we analyze the relation between the canonical normal Cartan connections associated with the two structures. We show that the construction preserves normality, see Lemma 5.1, and as a consequence, we have the following.

Proposition 1.1. *The holonomy of the canonical normal Cartan connection associated with the induced Lie contact structure on* $\mathbb{T}$ *reduces to* G_2.

We then proceed to discuss, more generally, Lie contact structures in dimension 7 endowed with Cartan holonomy reductions to G_2. We show the following (see also Theorem 5.1).

Theorem 1.2. *Consider a Lie contact structure on a 7-manifold $\widetilde{M}$ with canonical normal $\mathfrak{so}(4,3)$-valued Cartan connection $\widetilde{\omega}$. A holonomy reduction of $\widetilde{\omega}$ to G_2 determines a distinguished rank 2 distribution $\mathcal{V}$ on an open dense subset $\widetilde{M}^o \subset \widetilde{M}$. If the curvature of the Cartan connection $\widetilde{\omega}$ annihilates the rank 2 distribution $\mathcal{V}$, then $\mathcal{V}$ is integrable and in a neighborhood of each point in $\widetilde{M}^o$ one can form a local 5-dimensional leaf space, which carries an induced $(2,3,5)$ distribution. Moreover, if $\widetilde{M}^o$ is a proper subset of the 7-manifold $\widetilde{M}$, then the complement carries an induced parabolic contact structure associated with the Lie algebra $\mathfrak{g}_2$.*

Our work combines a conceptual approach based on theory of parabolic geometries [4–6] with explicit calculations in terms of exterior differential systems (EDSs).

2. Algebraic and Geometric Background

A first step to understanding the construction from $(2,3,5)$ distributions to Lie contact structures is to understand the relationship between the homogeneous models of the two structures. In this section, we present the algebra behind the construction, and we further discuss the (curved) geometric structures we are interested in.

2.1. *Split octonions and* $\mathbf{G_2}$

The exceptional complex simple Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ has two real forms: the split real form and the compact real form. In this paper, we will be concerned with the split real form $\mathfrak{g}_2$ and the (connected) Lie group G_2 with Lie algebra $\mathfrak{g}_2$ that can be defined as the automorphism group of the split octonions $(\mathbb{O}', \cdot)$. For more algebraic background see e.g. [26, 1].

An algebra $(\mathcal{A}, \cdot)$ with unit 1 together with a non-degenerate quadratic form N that is multiplicative in the sense that

$$N(X \cdot Y) = N(X)N(Y)$$

is called a composition algebra. There are, up to isomorphism, precisely two 8-dimensional real composition algebras: the octonions $\mathbb{O}$ and the split octonions $\mathbb{O}'$. The two can be distinguished by the signature of their quadratic forms. The split octonions are the unique 8-dimensional real composition algebra with quadratic form $N : \mathbb{O}' \to \mathbb{R}$ of signature $(4,4)$.

Given a composition algebra, there is a notion of conjugation $\bar{X} = 2\langle X, 1\rangle 1 - X$, where $\langle , \rangle$ denotes the bilinear form determined by N via polarization. The space of imaginary split octonions is then defined as

$$\mathbb{V} = \mathrm{Im}\mathbb{O}' = \{X \in \mathbb{O}' : \bar{X} = -X\} = 1^{\perp}.$$

Since the unit 1 has norm one, $\langle , \rangle$ restricts to a bilinear form of signature $(3,4)$ on $\mathbb{V}$; we define H to be the *negative* of the restriction, which has thus signature $(4,3)$. One can further define a 3-form $\Phi \in \Lambda^3\mathbb{V}^*$ as

$$\Phi(X,Y,Z) := \langle X \cdot Y, Z\rangle = H(X \times Y, Z),$$

where

$$X \times Y = X \cdot Y + \langle X, Y\rangle\, 1$$

denotes the split octonionic cross product on $\mathbb{V}$. Since an algebra automorphism of a composition algebra preserves the corresponding bilinear form, G_2 preserves all these data. Indeed, it is known that G_2 is precisely the stabilizer of Φ in $\mathrm{GL}(\mathbb{V})$, and the representation on $\mathbb{V}$ defines an inclusion $\mathrm{G}_2 \hookrightarrow \mathrm{O}(H) = \mathrm{O}(4,3)$.

2.2. *Explicit matrix presentations of* $\mathfrak{g}_2$ *and* $\mathfrak{so}(4,3)$

Here we will present an explicit matrix realization of the inclusion

$$\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(4,3). \tag{1}$$

Let $e_1, \ldots, e_7$ be a basis for $\mathbb{V}$ with dual basis $e^1, \ldots, e^7$, i.e. $e^i(e_j) = \delta_{ij}$. Consider the bilinear form

$$H = 2e^1e^7 + 2e^2e^6 + 2e^3e^5 + e^4e^4, \tag{2}$$

defining

$$\mathfrak{so}(4,3) = \left\{ \begin{pmatrix} a^7 & -a^3 & -a^6 & a^{11} & -a^{16} & a^{19} & 0 \\ -a^{17} & a^{10} & a^9 & a^{15} & -a^{20} & 0 & -a^{19} \\ -a^{14} & a^8 & a^{13} & a^{18} & 0 & a^{20} & a^{16} \\ a^{12} & a^5 & a^2 & 0 & -a^{18} & -a^{15} & -a^{11} \\ -a^4 & -a^0 & 0 & -a^2 & -a^{13} & -a^9 & a^6 \\ a^1 & 0 & a^0 & -a^5 & -a^8 & -a^{10} & a^3 \\ 0 & -a^1 & a^4 & -a^{12} & a^{14} & a^{17} & -a^7 \end{pmatrix},\ a^0, \ldots, a^{20} \in \mathbb{R} \right\}. \tag{3}$$

Then the subalgebra of $\mathfrak{so}(4,3)$ preserving the 3-form

$$\Phi = 2e^1 \wedge e^4 \wedge e^7 + e^1 \wedge e^5 \wedge e^6 + 8e^2 \wedge e^3 \wedge e^7 - 2e^2 \wedge e^4 \wedge e^6 - 2e^3 \wedge e^4 \wedge e^5 \tag{4}$$

is the exceptional Lie algebra

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} -q^1 - q^4 & -2b^6 & -12b^5 & -2q^5 & q^6 & -6q^7 & 0 \\ -\frac{1}{2}b^3 & -q^4 & 6q^2 & -6b^5 & \frac{1}{2}q^5 & 0 & 6q^7 \\ -\frac{1}{12}b^4 & \frac{1}{3}q^3 & -q^1 & b^6 & 0 & -\frac{1}{2}q^5 & -q^6 \\ \frac{1}{3}b^0 & -\frac{1}{3}b^4 & 2b^3 & 0 & -b^6 & 6b^5 & 2q^5 \\ -b^1 & -\frac{2}{3}b^0 & 0 & -2b^3 & q^1 & -6q^2 & 12b^5 \\ \frac{1}{6}b^2 & 0 & \frac{2}{3}b^0 & \frac{1}{3}b^4 & -\frac{1}{3}q^3 & q^4 & 2b^6 \\ 0 & -\frac{1}{6}b^2 & b^1 & -\frac{1}{3}b^0 & \frac{1}{12}b^4 & \frac{1}{2}b^3 & q^1 + q^4 \end{pmatrix}, \ b^0, \dots, q^7 \in \mathbb{R} \right\}. \tag{5}$$

2.3. *Parabolic subalgebras of* $\mathfrak{g}_2$ *and* $\mathfrak{so}(4,3)$

A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ of a semisimple Lie algebra $\mathfrak{g}$ is a *parabolic subalgebra* if and only if its maximal nilpotent ideal $\mathfrak{p}_+$ coincides with the orthogonal complement $\mathfrak{p}^\perp$ of $\mathfrak{p}$ in $\mathfrak{g}$ with respect to the Killing form. In particular, this yields an identification $(\mathfrak{g}/\mathfrak{p})^* \cong \mathfrak{p}_+$. A parabolic subalgebra $\mathfrak{p}$ determines a filtration

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^0 \supset \cdots \supset \mathfrak{g}^k,$$

$[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, where $\mathfrak{g}^0 = \mathfrak{p}$, $\mathfrak{g}^1 = \mathfrak{p}^\perp$, $\mathfrak{g}^i = [\mathfrak{g}^1, \mathfrak{g}^{i-1}]$ and $\mathfrak{g}^{-i} = (\mathfrak{g}^i)^\perp$ for $i > 1$. For a choice of (reductive) subalgebra $\mathfrak{g}_0 \subset \mathfrak{p}$ isomorphic to $\mathfrak{p}/\mathfrak{p}_+$, called a Levi subalgebra, the filtration splits which determines a grading of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k,$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. Conversely, given such a grading,

$$\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

defines a parabolic subalgebra, and the filtration can be recovered from the grading as $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. We will now discuss the parabolic subalgebras of $\mathfrak{g} = \mathfrak{g}_2$ and $\tilde{\mathfrak{g}} = \mathfrak{so}(4,3)$ that are relevant for this paper.

Consider $\tilde{\mathfrak{g}} = \mathfrak{so}(4,3)$ in the matrix presentation (3). Let $\tilde{\mathfrak{p}} \subset \mathfrak{so}(4,3)$ be a parabolic subalgebra defined as the stabilizer of a totally null 2-plane $\mathbb{E}$ with respect to H as in (2). Let

$$\tilde{\mathfrak{g}}^{-2} \supset \tilde{\mathfrak{g}}^{-1} \supset \tilde{\mathfrak{g}}^0 \supset \tilde{\mathfrak{g}}^1 \supset \tilde{\mathfrak{g}}^2$$

be the filtration of $\tilde{\mathfrak{g}}$ determined by $\tilde{\mathfrak{p}}$, and let

$$\mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1, \tag{6}$$

be the $\tilde{\mathfrak{p}}$-invariant filtration of the standard representation, where $\mathbb{V}^1 = \mathbb{E}$, $\mathbb{V}^0 = \mathbb{E}^\perp$, $\mathbb{V}^{-1} = \mathbb{V}$.

Any two parabolic subalgebras of $\mathfrak{so}(4,3)$ defined as stabilizers of distinct totally null 2-planes are conjugated to each other by an inner automorphism of $\mathfrak{so}(4,3)$. Hence modulo conjugation, they define the same parabolic subalgebra and induced filtration. However, the parabolic subalgebras may be different concerning their position relative to the given subalgebra $\mathfrak{g}_2 \subset \mathfrak{so}(4,3)$. This observation will be relevant for the purpose of this paper.

Let us first choose the totally null plane $\mathbb{E}' = \mathrm{span}(e_1, e_2)$, where $e_1, \ldots, e_7$ denotes the basis of $\mathbb{V}$ as in Sec. 2.2. Consider the splitting $\mathbb{V}_1 \oplus \mathbb{V}_0 \oplus \mathbb{V}_{-1}$ of the filtration (6) of $\mathbb{V}$ given by $\mathbb{V}_1 = \mathbb{E}'$, $\mathbb{V}_0 = \mathrm{span}(e_3, e_4, e_5)$ and $\mathbb{V}_{-1} = \mathrm{span}(e_6, e_7)$, which corresponds to the grading

$$\left(\begin{array}{c|c|c} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 & \tilde{\mathfrak{g}}_2 \\ \hline \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 \\ \hline \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 \end{array}\right) \left(\begin{array}{c} \mathbb{V}_1 \\ \hline \mathbb{V}_0 \\ \hline \mathbb{V}_{-1} \end{array}\right), \tag{7}$$

of $\tilde{\mathfrak{g}} = \mathfrak{so}(4,3)$. Then the subalgebra $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{p}}$ of block diagonal matrices is a Levi subalgebra isomorphic to $\mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{so}(2,1)$ ($\widetilde{\mathbb{V}}_1$ is the defining representation for the $\mathfrak{gl}(2,\mathbb{R})$-summand and $\widetilde{\mathbb{V}}_0$ for the $\mathfrak{so}(2,1)$-summand of $\tilde{\mathfrak{g}}_0$). The parabolic subalgebra $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$ consists of upper block triangular matrices.

Next we choose a different totally null 2-plane, $\mathbb{E} = \mathrm{span}(e_2, e_3)$. The grading of $\tilde{\mathfrak{g}}$ corresponding to the splitting $\widetilde{\mathbb{V}}_1 = \mathrm{span}(e_2, e_3)$, $\widetilde{\mathbb{V}}_0 = \mathrm{span}(e_1, e_4, e_7)$, $\widetilde{\mathbb{V}}_{-1} = \mathrm{span}(e_5, e_6)$ looks as follows:

$$\left(\begin{array}{c|c|c|c|c} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 & 0 \\ \hline \tilde{\mathfrak{g}}_1 & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 & \tilde{\mathfrak{g}}_2 & \tilde{\mathfrak{g}}_1 \\ \hline \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 & \tilde{\mathfrak{g}}_0 \\ \hline \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_{-1} \\ \hline 0 & \tilde{\mathfrak{g}}_{-1} & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1 & \tilde{\mathfrak{g}}_0 \end{array}\right) \left(\begin{array}{c} \mathbb{V}_0 \\ \hline \mathbb{V}_1 \\ \hline \mathbb{V}_0 \\ \hline \mathbb{V}_{-1} \\ \hline \mathbb{V}_0 \end{array}\right). \tag{8}$$

Looking at the explicit form of the defining 3-form Φ for $\mathfrak{g}_2$ as in (4), we immediately notice that while $\mathbb{E}' \lrcorner\, \Phi = e_1 \lrcorner\, e_2 \lrcorner\, \Phi = 0$, this is not true for the 2-plane $\mathbb{E}$ as $e_2 \lrcorner\, e_3 \lrcorner\, \Phi = 8e^7 \neq 0$. Inserting the 2-plane $\mathbb{E}$ into Φ we obtain the line in $\mathbb{V}^*$ spanned by e^7, or using the isomorphism $\mathbb{V}^* \cong \mathbb{V}$ induced by the metric H, the line $\ell \subset \mathbb{V}$ spanned by e_1.

Definition 2.1. Let $\mathbb{V}$ be a 7-dimensional vector space with a bilinear form H of signature $(4,3)$, and let Φ be a defining 3-form for $\mathrm{G}_2 \subset \mathrm{O}(4,3)$. We call a 2-plane $\mathbb{E} = \mathrm{span}(V, W)$

- *special* if

$$\mathbb{E} \lrcorner\, \Phi = V \lrcorner\, W \lrcorner\, \Phi = 0,$$

- *generic* if

$$\mathbb{E} \lrcorner\, \Phi = V \lrcorner\, W \lrcorner\, \Phi \neq 0.$$

G_2 acts transitively on generic and special 2-planes, respectively. This follows, for instance, immediately from the fact that G_2 acts transitively on split octonionic null triples (see [1, Theorem 13 and Proposition 15]): these are ordered triples X, Y, Z of pairwise orthogonal null imaginary split octonions such that $\Phi(X, Y, Z) = \frac{1}{2}$.

Let us now discuss parabolic subalgebras of $\mathfrak{g}_2$; there are, up to conjugation by inner automorphisms of $\mathfrak{g}_2$, three of them. Consider $\mathfrak{g} = \mathfrak{g}_2$ in the matrix representation (5).

Let $\mathfrak{p} \subset \mathfrak{g}$ be the stabilizer of a null line ℓ; we take the line $\ell = \mathbb{R}e_1 \subset \mathbb{V}$ through the first basis vector e_1. (G_2 acts transitively on null lines, see e.g. [2], and thus different choices lead to conjugated subalgebras). Let

$$\mathfrak{g}^{-3} \supset \mathfrak{g}^{-2} \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^{0} \supset \mathfrak{g}^{1} \supset \mathfrak{g}^{2} \supset \mathfrak{g}^{3}$$

be the filtration determined by the parabolic $\mathfrak{p}$. Since $\mathfrak{p}$ preserves ℓ and Φ, it also preserves the filtration

$$\mathbb{V}^{-2} \supset \mathbb{V}^{-1} \supset \mathbb{V}^{0} \supset \mathbb{V}^{1} \supset \mathbb{V}^{2}, \tag{9}$$

where $\mathbb{V}^2 = \ell = \mathrm{span}(e_1)$, $\mathbb{V}^1 = \{Y \in \mathbb{V} : Y \lrcorner X \lrcorner \Phi = 0, \forall X \in \ell\} = \mathrm{span}(e_1, e_2, e_3)$, $\mathbb{V}^0 = \mathbb{V}^{1\perp} = \mathrm{span}(e_1, e_2, e_3, e_4)$ and $\mathbb{V}^1 = \mathbb{V}^{2\perp} = \mathrm{span}(e_1, e_2, e_3, e_4, e_5, e_6)$, $\mathbb{V}^{-2} = \mathbb{V}$. A choice of Levi subalgebra $\mathfrak{g}_0 \cong \mathfrak{p}/\mathfrak{p}^{\perp} \cong \mathfrak{gl}(2, \mathbb{R})$ is determined by the splitting $\mathbb{V}^2 = \mathrm{span}(e_1)$, $\mathbb{V}^1 = \mathrm{span}(e_2, e_3)$, $\mathbb{V}^0 = \mathrm{span}(e_4)$, $\mathbb{V}^{-1} = \mathrm{span}(e_5, e_6)$, $\mathbb{V}^{-2} = \mathrm{span}(e_7)$, and the corresponding grading of $\mathfrak{g}$ is depicted below:

$$\left(\begin{array}{c|c|c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 & 0 \\ \hline \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline 0 & \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{array}\right) \left(\begin{array}{c} \mathbb{V}_2 \\ \hline \mathbb{V}_1 \\ \hline \mathbb{V}_0 \\ \hline \mathbb{V}_{-1} \\ \hline \mathbb{V}_{-2} \end{array}\right). \tag{10}$$

The other maximal parabolic subalgebra $\bar{\mathfrak{p}}$ is the stabilizer in $\mathfrak{g}$ of a special totally null 2-plane $\mathbb{E}' \subset \mathbb{V}$, i.e. one such that $X \lrcorner Y \lrcorner \Phi = 0$ for all $X, Y \in \mathbb{E}'$ (see e.g. [18]). Let us take $\mathbb{E}' = \mathrm{span}(e_1, e_2)$. Then, as in the case of the special orthogonal algebra discussed earlier, the parabolic subalgebra preserves the filtration (6) of $\mathbb{V}$, and the induced filtration of $\mathfrak{g}$ is of the form

$$\bar{\mathfrak{g}}^{-2} \supset \bar{\mathfrak{g}}^{-1} \supset \bar{\mathfrak{g}}^{0} \supset \bar{\mathfrak{g}}^{1} \supset \bar{\mathfrak{g}}^{2}. \tag{11}$$

A splitting of the filtration of $\mathbb{V}$ determines a Levi subalgebra $\bar{\mathfrak{g}}_0 = \mathfrak{gl}(2, \mathbb{R})$ and corresponding grading

$$\left(\begin{array}{c|c|c} \bar{\mathfrak{g}}_0 & \bar{\mathfrak{g}}_1 & \bar{\mathfrak{g}}_2 \\ \hline \bar{\mathfrak{g}}_{-1} & \bar{\mathfrak{g}}_0 & \bar{\mathfrak{g}}_1 \\ \hline \bar{\mathfrak{g}}_{-2} & \bar{\mathfrak{g}}_{-1} & \bar{\mathfrak{g}}_0 \end{array}\right) \left(\begin{array}{c} \mathbb{V}_1 \\ \hline \mathbb{V}_0 \\ \hline \mathbb{V}_{-1} \end{array}\right). \tag{12}$$

Finally there is the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$; it is the stabilizer of a filtration $\ell \subset \mathbb{E}'$ consisting of a null line contained in a special totally null 2-plane $\mathbb{E}' \subset \mathbb{V}$.

We shall use analogous notation for the parabolic subgroups appearing in this paper:

- $\widetilde{P} \subset \mathrm{O}(4,3)$ denotes the stabilizer of a totally null 2-plane $\mathbb{E} \subset \mathbb{V}$,
- $P \subset \mathrm{G}_2$ denotes the stabilizer of a null line $\ell \subset \mathbb{V}$,
- $\bar{P} \subset \mathrm{G}_2$ denotes the stabilizer of a special totally null 2-plane $\mathbb{E}' \subset \mathbb{V}$,
- $B \subset \mathrm{G}_2$ denotes the stabilizer of a filtration $\ell \subset \mathbb{E}'$ consisting of a null line contained in a special totally null 2-plane.

For reasons that will become clear later, we call $\widetilde{P} \subset \mathrm{O}(4,3)$ the *Lie contact parabolic*, $P \subset \mathrm{G}_2$ the (2,3,5) *parabolic* and $\bar{P} \subset \mathrm{G}_2$ the G_2 *contact parabolic.*

2.4. *Parabolic geometries*

Here we provide a very brief summary of basic notions from parabolic geometry, mostly to set notation. For a comprehensive introduction to parabolic geometries see [8]. See also [29, 31, 22] for additional information.

A *Cartan geometry* of type (G,P) is given by

- a principal bundle $\mathcal{G} \to M$ with structure group P,
- a Cartan connection $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$, i.e. a P-equivariant Lie algebra valued 1-form such that $\omega(u)(\zeta_X) = X$ for all fundamental vector fields ζ_X and $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism.

The curvature of a Cartan connection ω is the 2-form in $\Omega^2(\mathcal{G},\mathfrak{g})$ defined as

$$K(\xi,\eta) = d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)],$$

for $\xi,\eta \in \mathfrak{X}(\mathcal{G})$. It is P-equivariant and horizontal, and thus equivalently encoded in the curvature function $K : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ given by

$$K(u)(X,Y) = d\omega\big(\omega^{-1}(u)(X),\omega^{-1}(u)(Y)\big) + [X,Y].$$

It is one of the basic results about Cartan connections that the curvature of a Cartan geometry vanishes, i.e. the geometry is flat if and only if it is locally equivalent to $G \to G/P$ equipped with the Maurer Cartan form ω_G. The latter geometry is referred to as the (homogeneous) model.

A Cartan geometry of type (G,P) is called a *parabolic geometry* if $\mathfrak{g}$ is a semisimple Lie algebra and $P \subset G$ a parabolic subgroup, i.e. a closed subgroup with Lie algebra a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Given a principal bundle $P \hookrightarrow \mathcal{G} \to M$ and Lie algebra $\mathfrak{g}$ there are *a priori* several choices of Cartan connections $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$. In pioneering work Tanaka established the following curvature conditions that pin down the Cartan connection uniquely: A parabolic geometry is called

- *regular* if the curvature K is of homogeneity ≥ 1, i.e. $K(u)(X,Y) \subset \mathfrak{g}^{i+j+1}$ for all $X \in \mathfrak{g}^i$, $Y \in \mathfrak{g}^j$ and $u \in \mathcal{G}$,
- *normal* if $\partial^* \circ K = 0$, where $\partial^* : \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is the (P-equivariant) Kostant codifferential. Identifying $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}_+$ via the Killing form, it is the boundary operator computing the Lie algebra homology $H_*(\mathfrak{p}_+, \mathfrak{g})$, given on a decomposable element as

$$\partial^*(Z_0 \wedge Z_1 \otimes A) = Z_0 \otimes [Z_1, A] - Z_1 \otimes [Z_0, A] - [Z_0, Z_1] \otimes A. \tag{13}$$

Projecting the curvature K of a regular, normal parabolic geometry to $\mathbb{H}^2 := \ker(\partial^*)/\mathrm{Im}(\partial^*)$ gives the *harmonic curvature* K_H, which is the fundamental curvature quantity of a regular, normal parabolic geometry.

2.5. $(2,3,5)$ *distributions*

A $(2,3,5)$ distribution $\mathcal{D} \subset TM$ is a rank 2 distribution on a 5-manifold that is bracket generating in a minimal number of steps, i.e. $[\mathcal{D},\mathcal{D}]$ is a subbundle of rank 3 and $[\mathcal{D},[\mathcal{D},\mathcal{D}]] = TM$. In other words, the (weak) derived flag $\mathcal{D} \subset [\mathcal{D},\mathcal{D}] \subset TM$ is a sequence of nested bundles of ranks 2, 3 and 5.

So, $(2,3,5)$ distributions are in a sense opposite to integrable distributions, and they are different in character. While integrable rank 2 distributions in dimension 5 are all locally equivalent, $(2,3,5)$ distributions have functional local invariants. A solution to the local equivalence problem was established in Cartan's influential 1910 paper [11]. His work also shows that the symmetry algebra of a $(2,3,5)$ distribution is finite-dimensional; for the most symmetric of these distributions it is the simple Lie algebra $\mathfrak{g}_2$.

Note that a relationship to $\mathfrak{g}_2$ can be seen immediately. At each point $x \in M$, the symbol algebra $\mathrm{gr}(T_xM)$ of a $(2,3,5)$ distribution, i.e. the associated graded $\mathcal{D}_x \oplus [\mathcal{D},\mathcal{D}]_x/\mathcal{D} \oplus T_xM/[\mathcal{D},\mathcal{D}]_x$ of the derived flag together with the bracket $\mathcal{L}_x$ induced by the Lie bracket of vector fields, is a nilpotent Lie algebra isomorphic to the negative part of the grading (10) of $\mathfrak{g}_2$. Indeed, we have the following (see e.g. [8]).

Theorem 2.1. *There is an equivalence of categories between* $(2,3,5)$ *distributions and parabolic geometries of type* (G_2, P), *where* $P \subset \mathrm{G}_2$ *is the parabolic subgroup defined as the stabilizer of a null line in the* 7*-dimensional irreducible representation of* G_2.

Based on the Cartan geometric interpretation of $(2,3,5)$ distributions, a relation to conformal geometry was observed in [23].

Theorem 2.2. *Every* $(2,3,5)$ *distribution* $\mathcal{D} \subset TM$ *determines a conformal class* $[g]_{\mathcal{D}}$ *of metrics of signature* $(2,3)$ *on* M. *The distribution* $\mathcal{D}$ *is totally null with respect to any metric from the conformal class* $[g]_{\mathcal{D}}$.

2.6. *Lie contact structures*

A contact distribution $\mathcal{H} \subset TM$ on a manifold of dimension $2n+1$ is a corank 1 subbundle such that the Levi-bracket

$$\mathcal{L} : \Lambda^2\mathcal{H} \to TM/\mathcal{H}, \quad \mathcal{L}(\xi_x \wedge \eta_x) = [\xi, \eta]_x + \mathcal{H}_x,$$

is non-degenerate at each point $x \in M$. In other words, locally, $\mathcal{H}$ is the kernel of a contact form θ. Contact distributions do not have local invariants; locally one may always find coordinates (t, q_i, p_j) such that $\theta = \mathrm{d}t - \sum_i p_i \mathrm{d}q_i$.

Lie contact structures have been introduced and studied by Sato and Yamaguchi [27, 28], and Miyaoka [20, 21]. Here we shall specialize to manifolds of dimension 7. Note that the symbol algebra $\mathrm{gr}(T_xM)$ of a contact distribution $\mathcal{H}$ on a 7-manifold is at each point $x \in M$ isomorphic to the negative part $\tilde{\mathfrak{g}}_- = \tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{g}}_{-2}$ of the grading (7) of $\tilde{\mathfrak{g}} = \mathfrak{so}(4,3)$ from Sec. 2.3 (i.e. to the 7-dimensional Heisenberg algebra). A contact distribution has a natural graded frame bundle $\mathcal{F}$ whose fiber $\mathcal{F}_x$ at a point $x \in M$ comprises all graded Lie algebra isomorphisms $\phi : \mathrm{gr}(T_xM) \to \tilde{\mathfrak{g}}_-$; its structure group is the group of grading preserving Lie algebra automorphisms $\mathrm{Aut}_{gr}(\tilde{\mathfrak{g}}_-)$, which is isomorphic to the conformal symplectic group $\mathrm{CSp}(3)$. Let $\widetilde{G}_0 \cong \mathrm{GL}(2) \times \mathrm{O}(2,1)$ be the Levi subgroup of $\widetilde{P}$ preserving the grading (7). A Lie contact structure can be defined as a contact distribution equipped with a reduction of structure group $\widetilde{\mathcal{G}}_0 \hookrightarrow \mathcal{F}$ of the graded frame bundle with respect to $\widetilde{G}_0 \to \mathrm{Aut}_{gr}(\tilde{\mathfrak{g}}_-)$.

Equivalently, see [8, 32]: A Lie contact structure of signature $(2,1)$ on a manifold M of dimension 7 is given by

- a contact distribution $\mathcal{H} \subset TM$,
- two auxiliary vector bundles, $E \to M$ of rank 2 and $F \to M$ of rank 3, and a bundle metric b of signature $(2,1)$ on F,
- an isomorphism $\mathcal{H} \cong E^* \otimes F$ such that the Levi bracket $\mathcal{L}$ is invariant under the induced action of the orthogonal group $O(b)$ on $\mathcal{H}$.

Theorem 2.3. *There is an equivalence of categories between Lie contact structures of signature* $(2,1)$ *and regular, normal parabolic geometries of type* $(\mathrm{O}(4,3), \widetilde{P})$, *where* $\widetilde{P} \subset \mathrm{O}(4,3)$ *is the stabilizer of a totally null* 2*-plane.*

Given a parabolic geometry $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ of type $(\mathrm{O}(4,3), \widetilde{P})$, vector bundles $E \to M$ and $F \to M$ as in the above description of Lie contact structures are obtained as associated bundles $E = \widetilde{\mathcal{G}} \times_{\widetilde{P}} \mathbb{E}$ and $F = \widetilde{\mathcal{G}} \times_{\widetilde{P}} (\mathbb{E}^\perp/\mathbb{E})$, where $\mathbb{E} \subset \mathbb{R}^{4,3}$ is the totally null 2-plane stabilized by the parabolic subgroup $\widetilde{P}$.

Remark 2.1. There are a number of (locally) equivalent ways to describe Lie contact structures in terms of geometric structures on M; for our purposes a description in terms of a (conformal) tensor field on the contact distribution will be most convenient. Since $\tilde{\mathfrak{g}}_0$ is a maximal subalgebra of $\mathfrak{csp}(\tilde{\mathfrak{g}}_{-1})$ (see e.g. [7, Proposition 4.2]) any tensor on $\tilde{\mathfrak{g}}_{-1}$ preserved up to scale by $\tilde{\mathfrak{g}}_0$ but not by all of $\mathfrak{csp}(\tilde{\mathfrak{g}}_{-1})$ can be used for a description.

In Sec. 4.2.2, we will encode our Lie contact structures in terms of a conformal symmetric rank 4 tensor defined on the contact distribution $\mathcal{H}$. In this particular case the semisimple part of $\tilde{\mathfrak{g}}_0$ is $\tilde{\mathfrak{g}}_0^{ss} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2,1)$, we can identify $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$, and as a $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$-representation $\tilde{\mathfrak{g}}_{-1} = \mathbb{E} \otimes S^2\dot{\mathbb{E}}$, where $\mathbb{E} = \mathbb{R}^2$ is the defining representation for one $\mathfrak{sl}(2,\mathbb{R})$ and $\dot{\mathbb{E}} = \mathbb{R}^2$ for the other $\mathfrak{sl}(2,\mathbb{R})$. Now one can verify that there is precisely one trivial summand in the decomposition of the $(\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}))$-representation $S^4(\mathbb{E} \otimes S^2\dot{\mathbb{E}})$ into irreducible components. To construct the invariant rank 4 tensor, write an element $\psi \in \mathbb{E} \otimes S^2\dot{\mathbb{E}}$ using index notation as $\psi^{A\dot{B}\dot{C}}$ and define a map

$$L(\psi) : \mathbb{R}^2 \to \mathbb{R}^2, \quad L(\psi)^{\dot{C}}{}_{\dot{H}} = \psi^{A\dot{B}\dot{C}}\psi^{D\dot{E}\dot{F}}\epsilon_{AD}\epsilon_{\dot{B}\dot{E}}\epsilon_{\dot{F}\dot{H}},$$

for volume forms $\epsilon_{AB} \in \Lambda^2\mathbb{E}^*$ and $\epsilon_{\dot{A}\dot{B}} \in \Lambda^2\dot{\mathbb{E}}^*$. It turns out that the trace of this map is zero, but the trace of its square is not, and the unique up to constants invariant symmetric rank 4 tensor is

$$\Upsilon(\psi) = \mathrm{Tr}(L(\psi) \circ L(\psi)).$$

Since Υ is invariant under $\tilde{\mathfrak{g}}_0^{ss}$ but rescales under the action of the center of $\tilde{\mathfrak{g}}_0$, it induces a conformal symmetric rank 4 tensor $[\Upsilon]$ on the contact distribution $\mathcal{H} = \widetilde{\mathcal{G}}_0 \times_{\widetilde{G}_0} \tilde{\mathfrak{g}}_{-1}$.

2.7. G_2 *contact structures*

A G_2 contact structure on a 5-manifold M is given by a contact distribution $\mathcal{H} \subset TM$ together with a reduction of structure group $\bar{\mathcal{G}}_0 \hookrightarrow \mathcal{F}$ of the graded frame bundle of $\mathcal{H}$ with respect to $\bar{G}_0 \to \mathrm{Aut}_{gr}(\bar{\mathfrak{g}}_-)$, where $\bar{G}_0 \cong \mathrm{GL}(2,\mathbb{R})$ is the Levi subgroup of $\bar{P}$ preserving the grading (12).

Equivalently, it is a contact distribution $\mathcal{H}$ together with an identification $\mathcal{H} \cong S^3E$, for some rank 2 bundle $E \to M$, such that the Levi bracket $\mathcal{L}$ is invariant under the induced action of $\mathrm{GL}(E)$, see [8]. Again, by the general theory, we have the following.

Theorem 2.4. *There is an equivalence of categories between* G_2 *contact structures and parabolic geometries of type* $(\mathrm{G}_2, \bar{P})$, *where* $\bar{P} \subset \mathrm{G}_2$ *is the parabolic subgroup defined as the stabilizer of a special totally null* 2*-plane.*

Remark 2.2. We can also encode a G_2 contact structure in terms of a conformal symmetric rank 4 tensor $[\Upsilon]$ defined on the contact distribution $\mathcal{H} \subset TM$. This can be seen completely analogously to the case of Lie contact structures explained in Remark 2.1.

2.8. *Relating the models*

The model for $(2,3,5)$ distributions is the homogeneous space G_2/P together with its canonical G_2-invariant distribution $\mathcal{D}$. Since G_2 acts transitively on the projective quadric $\mathbb{P}(\mathcal{C})$ of all null lines with respect to the invariant bilinear form H, and

P is the stabilizer of such a null line ℓ, we get an identification

$$\mathrm{G}_2/P \cong \mathbb{P}(\mathcal{C}).$$

The model for the Lie contact structures we are interested in is the homogeneous space $\mathrm{O}(4,3)/\widetilde{P}$ with its canonical left invariant Lie contact structure. Since $\widetilde{P}$ is the stabilizer of a totally null 2-plane and $\mathrm{O}(4,3)$ acts transitively on such 2-planes, this homogeneous space can be identified with the 7-dimensional orthogonal Grassmannian of totally null 2-planes,

$$\mathrm{O}(4,3)/\widetilde{P} \cong \mathrm{Gr}_0(2,\mathbb{R}^{4,3}).$$

Finally, the homogeneous model for G_2 contact structures is $\mathrm{G}_2/\bar{P}$, which is the 5-dimensional Grassmannian of special totally null 2-planes, with its canonical left-invariant G_2 contact structure.

The following proposition relates all of these models.

Proposition 2.1. *Let* $\mathbb{V}$ *be a* 7*-dimensional vector space with a bilinear form* H *of signature* $(4,3)$, *and consider the Grassmannian* $\mathrm{Gr}_0(2,\mathbb{R}^{4,3})$ *of totally null* 2*-planes in* $\mathbb{V}$. *Let* Φ *be a defining* 3*-form for* $\mathrm{G}_2 \subset \mathrm{O}(4,3)$. *Then* $\mathrm{Gr}_0(2,\mathbb{R}^{4,3})$ *decomposes into two* G_2*-orbits*:

- *a closed,* 5*-dimensional orbit of special* 2*-planes isomorphic to* $\mathrm{G}_2/\bar{P}$
- *an open orbit of generic* 2*-planes isomorphic to* G_2/Q, *where* Q *is the* 7*-dimensional stabilizer in* G_2 *of a generic totally null* 2*-plane.*

Insertion of a generic totally null 2*-plane* $\mathbb{E}$ *into the* 3*-form* Φ *defines a line* $\ell \subset \mathbb{V}$, *which is null. The stabilizer* Q *of* $\mathbb{E}$ *in* G_2 *is the subgroup* $Q = G_0 \ltimes \exp(\mathfrak{g}^2)$ *of the parabolic subgroup* $P = G_0 \ltimes \exp(\mathfrak{p}_+)$ *that stabilizes the null line* ℓ. *In particular, the open* G_2*-orbit fibers over* G_2/P:

$$\begin{tikzcd} P/Q \arrow[r, hook] & \mathrm{G}_2/Q \arrow[d] \\ & \mathrm{G}_2/P. \end{tikzcd}$$

The content of Proposition 2.1 is known, see e.g. [17, 25]. In Sec. 2.3, we have seen that G_2 acts transitively on generic and special 2-planes, respectively. We will outline the arguments for a proof of the remaining statements in Proposition 2.1.

First, observe that for any totally null 2-plane $\mathbb{E}$, $\mathbb{E} \lrcorner \Phi$ is either zero or defines a null line: Take $\mathbb{V} = \mathrm{Im}\mathbb{O}'$ and $\Phi(X,Y,Z) = H(X \times Y, Z)$. Consider a totally null 2-plane $\mathbb{E} = \mathrm{span}(W_1, W_2) \subset \mathbb{V}$, then

$$W_1 \cdot W_2 = W_1 \times W_2 - \langle W_1, W_2\rangle 1 = W_1 \times W_2,$$

since $\langle W_1, W_2\rangle = 0$. Hence $\mathbb{E}$ is special if and only if $W_1 \cdot W_2 = 0$ (i.e. it corresponds to a null subalgebra) and generic if and only if $W_1 \cdot W_2 \neq 0$. In the latter case

$$\ell = \mathrm{span}(W_1 \cdot W_2) \subset \mathbb{V}$$

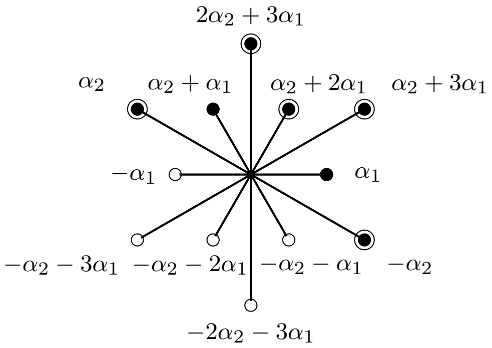
is a well-defined line determined by the plane $\mathbb{E}$, and it is null since $\langle W_1 \cdot W_2, W_1 \cdot W_2\rangle = \langle W_1, W_1\rangle\langle W_2, W_2\rangle$ and both W_1 and W_2 are null.

Next consider the stabilizer Q of a generic null 2-plane $\mathbb{E} = \mathrm{span}(W_1, W_2)$. First, it preserves the null line $\ell = \mathrm{span}(W_1 \cdot W_2)$ determined by $\mathbb{E}$. Hence, evidently, Q is contained in the parabolic subgroup P stabilizing ℓ. Next one can show that $\mathbb{E} \oplus \ell = \mathrm{span}(W_1, W_2, W_1 \cdot W_2)$ coincides with

$$\{Z \in \mathbb{V} : Z \cdot (W_1 \cdot W_2) = 0\} = \{Z : Z \lrcorner X \lrcorner \Phi = 0, \forall X \in \ell\} = \mathbb{V}^1,$$

the latter space being the 3-dimensional filtrand in the filtration (9) preserved by the parabolic P. So now we choose a subgroup $G_0 \subset Q$, $G_0 \cong P/\exp(\mathfrak{p}_+)$. Then $P = G_0 \ltimes \exp(\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3)$, where $\exp(\mathfrak{g}_1)$ acts by (nonzero) maps from $\mathbb{E}$ to ℓ, while $\exp(\mathfrak{g}_2 \oplus \mathfrak{g}_3)$ acts trivially on $\mathbb{E}$. Hence the subgroup Q, which preserves $\mathbb{E}$, is isomorphic to $G_0 \ltimes \exp(\mathfrak{g}_2 \oplus \mathfrak{g}_3)$.

Remark 2.3. In the following root diagram, all black dots correspond to root spaces contained in the standard parabolic $\mathfrak{p}$ and the ones with circles correspond to root spaces contained in the subalgebra $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \subset \mathfrak{p}$:



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3. From $(2, 3, 5)$ Distributions to Lie Contact Structures

In this section, we present a natural geometric construction of a 7-dimensional twistor bundle over a 5-manifold equipped with a $(2, 3, 5)$ distribution, and we investigate the induced geometric structure on the twistor bundle. In particular, we will prove Theorem 1.1.

3.1. *The* $(2, 3, 5)$ *twistor bundle*

Let $\mathcal{D}$ be a $(2, 3, 5)$ distribution on a 5-manifold M with derived flag $\mathcal{D} \subset [\mathcal{D}, \mathcal{D}] \subset TM$ and conformal class $[g]_{\mathcal{D}}$. Then we can form the bundle

$$\pi : \mathbb{P}([\mathcal{D}, \mathcal{D}]) = \bigcup_{x \in M} \{\ell_x \subset [\mathcal{D}, \mathcal{D}]_x\} \to M$$

of all lines contained in the rank 3 distribution. The 7-dimensional manifold $\mathbb{P}([\mathcal{D}, \mathcal{D}])$ decomposes as $\mathbb{P}([\mathcal{D}, \mathcal{D}]) = \mathbb{P}(\mathcal{D}) \cup \mathbb{T}$ into the 6-dimensional subset $\mathbb{P}(\mathcal{D})$

of all lines contained in $\mathcal{D}$, and the open subset $\mathbb{T}$ of all lines in $[\mathcal{D},\mathcal{D}]$ transversal to $\mathcal{D}$. The space $\mathbb{P}(\mathcal{D})$ has an interesting induced geometry, but here we are interested in the complement.

Definition 3.1. We call

$$\mathbb{T} = \mathbb{P}([\mathcal{D},\mathcal{D}])\backslash\mathbb{P}(\mathcal{D}) = \bigcup_{x\in M}\{\ell_x \subset [\mathcal{D},\mathcal{D}]_x : \ell_x \not\subset \mathcal{D}\}$$

the twistor bundle of the $(2,3,5)$ distribution $\mathcal{D}$.

Remark 3.1.

- Since $\mathcal{D}$ is totally null with respect to $[g]_{\mathcal{D}}$, we can equivalently describe $\mathbb{T}$ as the space of all non-null lines contained in $[\mathcal{D},\mathcal{D}]$.
- Via the conformal structure, we can identify $\mathcal{P}(TM)$ with $\mathcal{P}(T^*M)$. Under this identification, $\mathbb{T}$ corresponds to the space of lines in the cotangent space that annihilate $\mathcal{D}$ but do not annihilate $[\mathcal{D},\mathcal{D}]$:

$$\mathbb{T} = \mathbb{P}(\mathcal{D}^\perp)\backslash\mathbb{P}([\mathcal{D},\mathcal{D}]^\perp) = \bigcup_{x\in M}\{\ell_x \subset {\mathcal{D}_x}^\perp : \ell_x \not\subset {[\mathcal{D},\mathcal{D}]_x}^\perp\} \subset \mathbb{P}(T^*M).$$

Among the geometric structures that are naturally present on the twistor bundle $\mathbb{T}$ we are particularly interested in the rank 6 sub-bundle

$$\mathcal{H} = \bigcup_{\ell\in\mathbb{T}}\{\xi \in T_\ell\mathbb{T} : \pi_*(\xi) \in \ell^\perp\},$$

where the orthogonal complement $\ell^\perp$ is taken with respect to the conformal class $[g]_{\mathcal{D}}$ on M. Alternatively, if we realize $\mathbb{T}$ inside $\mathbb{P}(T^*M)$, then $\mathcal{H}$ is precisely the intersection of the canonical contact distribution on $\mathbb{P}(T^*M)$ with $T\mathbb{T}$. Now it is not difficult to see that $\mathcal{H} \subset T\mathbb{T}$ defines a contact structure on $\mathbb{T}$. In the following we will show more, we will prove that $\mathbb{T}$ has a naturally induced Lie contact structure of signature $(2,1)$.

3.2. *The induced Lie contact structure*

We shall prove Theorem 1.1 using the descriptions of $(2,3,5)$ distributions and Lie contact structures, respectively, in terms of Cartan geometries. There is a very general functorial construction that assigns to a Cartan geometry of some type (G,P) over a manifold M a Cartan geometry of a different type $(\widetilde{G},\widetilde{P})$ over a manifold $\widetilde{M}$. In the context of parabolic geometries these constructions are referred to as Fefferman-type constructions, see [5, 8]. We briefly recall the general principles.

Suppose we have an inclusion $i: G \hookrightarrow \widetilde{G}$ of Lie groups, and subgroups P and $\widetilde{P}$ such that the G-orbit of $o = e\widetilde{P} \in \widetilde{G}/\widetilde{P}$ is open and $Q := i^{-1}(\widetilde{P}) \subset G$ is contained in P. Then the construction proceeds in two steps. Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G,P). Now form the so-called *correspondence space*

$$\widetilde{M} = \mathcal{G}/Q, \tag{14}$$

and regard $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ as a Cartan connection on the Q-principal bundle $\mathcal{G} \to \widetilde{M}$. Then $(\mathcal{G} \to \widetilde{M}, \omega)$ is automatically a Cartan geometry of type (G, Q). In a second step, extend the structure group

$$\widetilde{\mathcal{G}} := \mathcal{G} \times_Q \widetilde{P},$$

such that $\widetilde{\mathcal{G}} \to \widetilde{M}$ is now a $\widetilde{P}$-principal bundle over $\widetilde{M}$. Let $j : \mathcal{G} \to \widetilde{\mathcal{G}}$ be corresponding bundle inclusion. Since the G-orbit of $e\widetilde{P}$ in $\widetilde{G}/\widetilde{P}$ is open, there is a unique extension of ω to a Cartan connection $\widetilde{\omega} \in \Omega^1(\widetilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $j^*\widetilde{\omega} = \omega$, see [8]. Thus, we obtain a Cartan geometry $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ of type $(\widetilde{G}, \widetilde{P})$.

The curvature functions $\widetilde{K} : \widetilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ and $K : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ of the respective Cartan geometries are related as

$$\widetilde{K} \circ j = (\Lambda^2 \varphi \otimes i') \circ K,$$

where $i' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is the derivative of the Lie group homomorphism i and $\varphi : (\mathfrak{g}/\mathfrak{p})^* \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$ is the dual map to the projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$.

Now we specialize to our groups. We take $G = \mathrm{G}_2$ and $\widetilde{G} = \mathrm{O}(4,3)$, so in particular we have an inclusion $i : G \hookrightarrow \widetilde{G}$. Then we take P to be the parabolic subgroup in G_2 that stabilizes a null line $\ell \subset \mathbb{R}^7$, and $\widetilde{P}$ to be the stabilizer in $\mathrm{O}(4,3)$ of a generic null 2-plane $\mathbb{E} \subset \mathbb{R}^7$ such that the null line determined by $\mathbb{E}$ is ℓ, i.e. $\iota_{\mathbb{E}}\Phi = \ell$. By Proposition 2.1 this means that the G-orbit of $o = e\widetilde{P} \in \widetilde{G}/\widetilde{P}$ is open and the subgroup $Q = i^{-1}(\widetilde{P})$ is contained in the parabolic P.

Given a $(2,3,5)$ distribution $\mathcal{D}$ with its canonical Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), it then follows immediately from the general considerations outlined above that there is a naturally associated Cartan geometry $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ of type $(\widetilde{G}, \widetilde{P})$. It remains to show that this Cartan geometry (which is of the right type) determines a Lie contact structure on $\widetilde{M}$. This is the case provided the curvature $\widetilde{K}$ is regular, i.e. $\widetilde{K}(u)(\tilde{\mathfrak{g}}^i, \tilde{\mathfrak{g}}^j) \subset \tilde{\mathfrak{g}}^{i+j+1}$ at any point $u \in \widetilde{\mathcal{G}}$.

Remark 3.2. To understand the geometric meaning of the regularity condition, note that the Cartan connection $\widetilde{\omega}$ determines an isomorphism

$$T\widetilde{M} \cong \widetilde{\mathcal{G}} \times_{\widetilde{P}} \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$$

and via this isomorphism the $\widetilde{P}$-invariant subspace $\tilde{\mathfrak{g}}^{-1}/\tilde{\mathfrak{p}} \subset \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ gives rise to a rank 6-subbundle

$$\mathcal{H} \cong \widetilde{\mathcal{G}} \times_{\widetilde{P}} \tilde{\mathfrak{g}}^{-1}/\tilde{\mathfrak{p}}.$$

Now the regularity condition ensures that the bundle map $\mathcal{L}$ on the graded bundle $\mathrm{gr}(T\widetilde{M}) = \mathcal{H} \oplus T\widetilde{M}/\mathcal{H}$ induced by the Lie bracket of vector fields coincides with the one induced by the algebraic Lie bracket on $\mathrm{gr}(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) = \tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1}$. Inspecting the Lie bracket on $\tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1}$ immediately shows that this implies that $\mathcal{L} : \Lambda^2\mathcal{H} \to T\widetilde{M}/\mathcal{H}$ is non-degenerate, i.e. $\mathcal{H}$ is a contact distribution. To see that one indeed gets an induced Lie contact structure, note that as a $\widetilde{P}$-representation $\tilde{\mathfrak{g}}^{-1}/\tilde{\mathfrak{p}} = \mathbb{E}^* \otimes \mathbb{E}^\perp/\mathbb{E}$. See also [8].

Proposition 3.1. *Suppose* $(\mathcal{G} \to M, \omega)$ *is a regular and normal parabolic geometry of type* (G, P), *then the induced parabolic geometry* $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ *of type* $(\widetilde{G}, \widetilde{P})$ *is regular. In particular, it determines a Lie contact structure on the manifold* $\widetilde{M} = \mathcal{G}/Q$.

Proof. It is known, see [23] or Theorem 4.1, that the regular, normal Cartan geometry $(\mathcal{G} \to M, \omega)$ associated with a $(2,3,5)$ distribution is torsion-free, i.e. the curvature function K takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$. Via the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$, the parabolic $\mathfrak{p}$ is contained in the $\widetilde{P}$-module $\tilde{\mathfrak{g}}^{-1}$, and so the curvature function $\widetilde{K}$ of the Cartan geometry $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ takes values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}^{-1}$. This implies that the curvature $\widetilde{K}$ is of homogeneity ≥ 1, i.e. the geometry is regular. □

Next we show that $\widetilde{M}$ is the twistor bundle $\mathbb{T}$ as introduced in Definition 3.1.

Proposition 3.2. *The manifold* $\widetilde{M} = \mathcal{G}/Q$ *can be naturally identified with the twistor bundle* $\mathbb{T} = \bigcup_{x \in M}\{\ell_x \in [\mathcal{D}, \mathcal{D}]_x : \ell_x \notin \mathcal{D}\}$ *of all lines in* $[\mathcal{D}, \mathcal{D}]$ *transversal to* $\mathcal{D}$.

Proof. By definition,

$$\widetilde{M} = \mathcal{G}/Q = \mathcal{G} \times_P P/Q.$$

Let $\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}^{-2}/\mathfrak{p} \subset \mathfrak{g}^{-3}/\mathfrak{p}$ be the P-invariant filtration on $\mathfrak{g}/\mathfrak{p}$. To prove the proposition it remains to identify the homogeneous space P/Q with the set of lines in $\mathfrak{g}^{-2}/\mathfrak{p}$ that are not contained in $\mathfrak{g}^{-1}/\mathfrak{p}$.

We have noticed in the proof of Proposition 2.1 that $Q = G_0 \ltimes \exp(\mathfrak{g}_2 \oplus \mathfrak{g}_3)$ for some subgroup $G_0 \cong P/P_+$ and corresponding G_0-invariant grading $\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. Now $\exp(\mathfrak{g}_2 \oplus \mathfrak{g}_3)$ acts trivially on $\mathfrak{g}^{-2}/\mathfrak{p}$ and G_0 preserves the line $\ell = (\mathfrak{g}_{-2} + \mathfrak{p})/\mathfrak{p}$ (and acts nontrivially on it). On the other hand, the action identifies $\exp(\mathfrak{g}_1)$ with the space of linear maps from ℓ to $\mathfrak{g}^{-1}/\mathfrak{p}$. It follows that the P-action is transitive on lines in $\mathfrak{g}^{-2}/\mathfrak{p}$ not contained in $\mathfrak{g}^{-1}/\mathfrak{p}$ and the stabilizer of ℓ as above is the subgroup Q. □

In particular, we have proven Theorem 1.1.

Remark 3.3. In [8], a construction from conformal structures to Lie contact structures is presented, which generalizes the work of Miyaoka, Sato and Yamaguchi [20, 21, 28]. Note that the Lie contact structure constructed here is different from the Lie contact structure associated with the conformal structure $[g_{\mathcal{D}}]$ following their construction. The latter one lives on a 9-dimensional manifold, ours on a 7-manifold.

3.3. *Additional structure on the twistor bundle*

One immediately observes that the Lie contact structures obtained from $(2,3,5)$ distributions are special. In particular, besides $\mathcal{H}$, there are several other naturally defined distributions on $\mathbb{T}$. First there is the vertical bundle

$$\mathcal{V} = \bigcup_{\ell \in \mathbb{T}} \{\xi \in T_\ell \mathbb{T} : \pi_*(\xi) = 0\}$$

for the projection $\pi : \mathbb{T} \to M$, which has rank 2. Then there are the lifts of $\mathcal{D}$ and $[\mathcal{D}, \mathcal{D}]$,

$$\widetilde{\mathcal{D}} = \bigcup_{\ell \in \mathbb{T}} \{\xi \in T_\ell \mathbb{T} : \pi_*(\xi) \in \mathcal{D}\},$$

and

$$\widetilde{[\mathcal{D}, \mathcal{D}]} = \bigcup_{\ell \in \mathbb{T}} \{\xi \in T_\ell \mathbb{T} : \pi_*(\xi) \in [\mathcal{D}, \mathcal{D}]\},$$

which are bundles of ranks 4 and 5, respectively. Finally, there is a rank 3 distribution

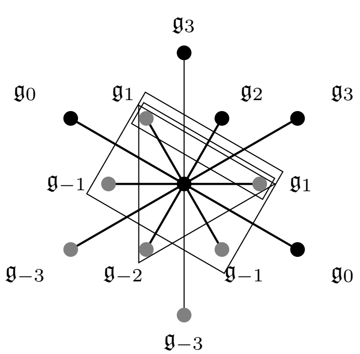
$$\mathcal{S} = \bigcup_{\ell \in \mathbb{T}} \{\xi \in T_\ell \mathbb{T} : \pi_*(\xi) \in \ell\},$$

called the prolongation of $\mathcal{D}$.

These distributions can be understood as follows: Since $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ arises as the extension of a Cartan geometry $(\mathcal{G} \to \widetilde{M}, \omega)$ of type (G, Q), we have an isomorphism

$$T\widetilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$$

via the Cartan connection ω. In particular, every Q-invariant subspace of $\mathfrak{g}/\mathfrak{q}$ corresponds to a natural subbundle of $T\widetilde{M}$. The vertical bundle $\mathcal{V}$ corresponds to $\mathfrak{p}/\mathfrak{q}$, the rank 3 bundle $\mathcal{S}$ corresponds to $(\mathfrak{g}_{-2} + \mathfrak{p})/\mathfrak{q}$, the contact subbundle $\mathcal{H}$ corresponds to $(\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{q}$, and $\widetilde{\mathcal{D}}$ and $\widetilde{[\mathcal{D}, \mathcal{D}]}$ correspond to $\mathfrak{g}^{-1}/\mathfrak{q}$ and $\mathfrak{g}^{-2}/\mathfrak{q}$, respectively. In the root diagram below the Q-submodules corresponding to the vertical bundle $\mathcal{V}$, the rank 3 bundle $\mathcal{S}$, and the rank 4 bundle $\widetilde{\mathcal{D}}$ are depicted.



4. The Exterior Differential System and Examples

Here we present a slightly different viewpoint on the construction of Lie contact structures from $(2, 3, 5)$ distributions, complementing the picture from the previous section. First we present the structure equations, or EDS, for $(2,3,5)$ distributions. Then we show how they can be applied to (locally) construct the induced Lie contact structures in terms of a conformal symmetric rank 4 tensor on the contact distribution $\mathcal{H}$. This viewpoint has the advantage that it leads to explicit formulae and enables us, for instance, to solve the symmetry equations for a given structure. This is carried out for a special class of distributions parametrized by functions $F(q) = \frac{q^k}{k(k-1)}$.

4.1. *The EDS for a* $(2,3,5)$ *distribution*

The EDS for a generic $(2,3,5)$ distribution was first introduced by Cartan in [11], and was then modified in [23] to get a form adapted to the corresponding (reduced to $\mathfrak{g}_2$) normal conformal Cartan connection. Here we have rewritten the system from [23] changing the notation to be more suitable to the contact structures we consider in this paper. The changes in notations with respect to [23] are as follows:

1-forms in [23]	the respective 1-forms in this paper
$\theta^1,\theta^2,\theta^3,\theta^4,\theta^5$	$\theta^1,\theta^2,\theta^0,\theta^3,\theta^4$
Ω_5,Ω_6	$3\theta^6, 3\theta^5$
$\Omega_7,\Omega_8,\Omega_9$	$\Omega_5,\Omega_6,\Omega_7$

Theorem 4.1. *A* $(2,3,5)$ *distribution* $\mathcal{D}$ *on a* 5*-manifold* M *uniquely defines a* 14*-dimensional bundle* $P \to \mathcal{G} \to M$ *together with a rigid coframe* $(\theta^0,\theta^1,\theta^2,\theta^3,\theta^4,\theta^5,\theta^6,\Omega_1,\Omega_2,\Omega_3,\Omega_4,\Omega_5,\Omega_6,\Omega_7)$ *on it satisfying the following EDS*:

$$\begin{aligned}
\mathrm{d}\theta^0 &= \theta^0\wedge(\Omega_1+\Omega_4) + 3\theta^1\wedge\theta^6 + 3\theta^2\wedge\theta^5 + \theta^3\wedge\theta^4,\\
\mathrm{d}\theta^1 &= \theta^0\wedge\theta^3 + \theta^1\wedge(2\Omega_1+\Omega_4) + \theta^2\wedge\Omega_2,\\
\mathrm{d}\theta^2 &= \theta^0\wedge\theta^4 + \theta^1\wedge\Omega_3 + \theta^2\wedge(\Omega_1+2\Omega_4),\\
\mathrm{d}\theta^3 &= 4\theta^0\wedge\theta^5 + \theta^1\wedge\Omega_5 + \theta^3\wedge\Omega_1 + \theta^4\wedge\Omega_2,\\
\mathrm{d}\theta^4 &= -4\theta^0\wedge\theta^6 + \theta^2\wedge\Omega_5 + \theta^3\wedge\Omega_3 + \theta^4\wedge\Omega_4,\\
\mathrm{d}\Omega_1 &= -\Omega_2\wedge\Omega_3 - \frac{1}{3}\Omega_5\wedge\theta^0 - \Omega_6\wedge\theta^1 - 2\theta^3\wedge\theta^6 + \theta^4\wedge\theta^5\\
&\quad - b_2\theta^0\wedge\theta^1 - b_3\theta^0\wedge\theta^2 + \frac{3}{8}c_2\theta^1\wedge\theta^2 + a_2\theta^1\wedge\theta^3\\
&\quad + a_3(\theta^1\wedge\theta^4+\theta^2\wedge\theta^3) + a_4\theta^2\wedge\theta^4,\\
\mathrm{d}\Omega_2 &= -\Omega_1\wedge\Omega_2 - \Omega_2\wedge\Omega_4 - \Omega_7\wedge\theta^1 - 3\theta^3\wedge\theta^5\\
&\quad - b_3\theta^0\wedge\theta^1 - b_4\theta^0\wedge\theta^2 + \frac{3}{8}c_3\theta^1\wedge\theta^2 + a_3\theta^1\wedge\theta^3\\
&\quad + a_4(\theta^1\wedge\theta^4+\theta^2\wedge\theta^3) + a_5\theta^2\wedge\theta^4,
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\Omega_3 &= \Omega_1 \wedge \Omega_3 + \Omega_3 \wedge \Omega_4 - \Omega_6 \wedge \theta^2 - 3\theta^4 \wedge \theta^6 \\
&\quad + b_1\theta^0 \wedge \theta^1 + b_2\theta^0 \wedge \theta^2 - \frac{3}{8}c_1\theta^1 \wedge \theta^2 - a_1\theta^1 \wedge \theta^3 \\
&\quad - a_2(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) - a_3\theta^2 \wedge \theta^4, \\
\mathrm{d}\Omega_4 &= \Omega_2 \wedge \Omega_3 - \frac{1}{3}\Omega_5 \wedge \theta^0 - \Omega_7 \wedge \theta^2 + \theta^3 \wedge \theta^6 - 2\theta^4 \wedge \theta^5 \\
&\quad + b_2\theta^0 \wedge \theta^1 + b_3\theta^0 \wedge \theta^2 - \frac{3}{8}c_2\theta^1 \wedge \theta^2 - a_2\theta^1 \wedge \theta^3 \\
&\quad - a_3(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) - a_4\theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^5 &= \Omega_2 \wedge \theta^6 + \Omega_4 \wedge \theta^5 - \frac{1}{3}\Omega_5 \wedge \theta^3 - \frac{1}{3}\Omega_7 \wedge \theta^0 \\
&\quad - \frac{1}{4}c_2\theta^0 \wedge \theta^1 - \frac{1}{4}c_3\theta^0 \wedge \theta^2 + e_1\theta^1 \wedge \theta^2 \\
&\quad + \frac{1}{4}b_2\theta^1 \wedge \theta^3 + \frac{1}{4}b_3(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) + \frac{1}{4}b_4\theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^6 &= \Omega_1 \wedge \theta^6 + \Omega_3 \wedge \theta^5 + \frac{1}{3}\Omega_5 \wedge \theta^4 - \frac{1}{3}\Omega_6 \wedge \theta^0 \\
&\quad - \frac{1}{4}c_1\theta^0 \wedge \theta^1 - \frac{1}{4}c_2\theta^0 \wedge \theta^2 + e_2\theta^1 \wedge \theta^2 \\
&\quad + \frac{1}{4}b_1\theta^1 \wedge \theta^3 + \frac{1}{4}b_2(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) + \frac{1}{4}b_3\theta^2 \wedge \theta^4, \\
\mathrm{d}\Omega_5 &= \Omega_1 \wedge \Omega_5 + \Omega_4 \wedge \Omega_5 - \Omega_6 \wedge \theta^3 - \Omega_7 \wedge \theta^4 - 12\theta^5 \wedge \theta^6 \\
&\quad + 4e_2\theta^0 \wedge \theta^1 + 4e_1\theta^0 \wedge \theta^2 + f\theta^1 \wedge \theta^2 - \frac{3}{8}c_1\theta^1 \wedge \theta^3 \\
&\quad - \frac{3}{8}c_2(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) - \frac{3}{8}c_3\theta^2 \wedge \theta^4, \\
\mathrm{d}\Omega_6 &= 2\Omega_1 \wedge \Omega_6 + \Omega_3 \wedge \Omega_7 + \Omega_4 \wedge \Omega_6 - 3\Omega_5 \wedge \theta^6 \\
&\quad - p_1\theta^0 \wedge \theta^1 - p_2\theta^0 \wedge \theta^2 + q_1\theta^1 \wedge \theta^2 + h_1\theta^1 \wedge \theta^3 \\
&\quad + h_2(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) + h_3\theta^2 \wedge \theta^4, \\
\mathrm{d}\Omega_7 &= \Omega_1 \wedge \Omega_7 + \Omega_2 \wedge \Omega_6 + 2\Omega_4 \wedge \Omega_7 - 3\Omega_5 \wedge \theta^5 \\
&\quad - \frac{1}{3}(2f + 3p_2)\theta^0 \wedge \theta^1 - p_3\theta^0 \wedge \theta^2 + q_2\theta^1 \wedge \theta^2 + (h_2 - e_2)\theta^1 \wedge \theta^3 \\
&\quad + (h_3 - e_1)(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) + h_4\theta^2 \wedge \theta^4.
\end{aligned}$$

The functions a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, c_1, c_2, c_3, e_1, e_2, f, q_1, q_2, p_1, p_2, p_3, h_1, h_2, h_3, h_4 *appearing in the EDS can be understood as the curvature coefficients of the normal Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g}_2)$ *associated with the distribution* $\mathcal{D}$.

In terms of the rigid coframe the Cartan normal connection ω *reads*

$$\omega = \begin{pmatrix} -\Omega_1-\Omega_4 & -2\theta^6 & -12\theta^5 & -2\Omega_5 & \Omega_6 & -6\Omega_7 & 0 \\ -\frac{1}{2}\theta^3 & -\Omega_4 & 6\Omega_2 & -6\theta^5 & \frac{1}{2}\Omega_5 & 0 & 6\Omega_7 \\ -\frac{1}{12}\theta^4 & \frac{1}{3}\Omega^3 & -\Omega^1 & \theta^6 & 0 & -\frac{1}{2}\Omega_5 & -\Omega_6 \\ \frac{1}{3}\theta^0 & -\frac{1}{3}\theta^4 & 2\theta^3 & 0 & -\theta^6 & 6\theta^5 & 2\Omega_5 \\ -\theta^1 & -\frac{2}{3}\theta^0 & 0 & -2\theta^3 & \Omega_1 & -6\Omega_2 & 12\theta^5 \\ \frac{1}{6}\theta^2 & 0 & \frac{2}{3}\theta^0 & \frac{1}{3}\theta^4 & -\frac{1}{3}\Omega_3 & \Omega^4 & 2\theta^6 \\ 0 & -\frac{1}{6}\theta^2 & \theta^1 & -\frac{1}{3}\theta^0 & \frac{1}{12}\theta^4 & \frac{1}{2}\theta^3 & \Omega_1+\Omega_4 \end{pmatrix}. \tag{15}$$

The curvature K *of the connection* ω *is of the form*

$$K = \frac{1}{2}K_{ij}\theta^i \wedge \theta^j, \quad \textit{where } i,j = 0,1,2,3,4,$$

and the above EDS is the same as

$$d\omega = -\omega \wedge \omega + \frac{1}{2}K_{ij}\theta^i \wedge \theta^j.$$

4.2. *From the EDS to underlying structures*

Suppose that the fourteen 1-forms $(\theta^0, \dots, \theta^6, \Omega_1, \dots, \Omega_7)$ on $\mathcal{G}$ are linearly independent at each point, $\theta^0 \wedge \cdots \theta^6 \wedge \Omega_1 \wedge \cdots \wedge \Omega_7 \neq 0$, and satisfy the EDS as in Theorem 4.1.

4.2.1. *The underlying* $(2,3,5)$ *distribution and conformal metric*

On the one hand, we easily conclude the following:

- $\mathcal{G}$ is locally foliated by 9-dimensional submanifolds tangent to the distribution $\mathcal{P}$ defined as the annihilator of the basis 1-forms $(\theta^0, \theta^1, \theta^2, \theta^3, \theta^4)$. That $\mathcal{P}$ is integrable follows immediately from the EDS, since it guarantees that

$$\mathrm{d}\theta^k \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0, \quad \forall\, k = 0,1,2,3,4.$$

- The rank 2 distribution $\bar{\mathcal{D}}$ on $\mathcal{G}$ annihilated by the forms $(\theta^0, \theta^1, \theta^2, \theta^5, \theta^6, \Omega_1, \dots, \Omega_7)$,

$$\bar{\mathcal{D}} = \ker(\theta^0, \theta^1, \theta^2, \theta^5, \theta^6, \Omega_1, \dots, \Omega_7),$$

  descends to a well-defined rank 2 distribution $\mathcal{D} = \pi_* \bar{\mathcal{D}}$ on the space $M = \mathcal{G}/P$ of leaves of the distribution $\mathcal{P}$. To see that this is the case, consider the frame $(X_0, \dots, X_6, Y_1, \dots, Y_7)$ dual to the coframe forms on $\mathcal{G}$. Then $\bar{\mathcal{D}}$ is spanned by X_3 and X_4,

$$\bar{\mathcal{D}} = \mathrm{Span}(X_3, X_4).$$

  To show that $\bar{\mathcal{D}}$ projects to a well-defined rank 2 distribution M it is enough to show that, at each point of $\mathcal{G}$, the Lie derivatives of X_3 and X_4 with respect to the fiber directions $X_5, X_6, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$ are spanned by no other vectors than the distribution vectors X_3, X_4, and the vertical vectors $X_5, X_6, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$. Dually, this precisely means that in the considered EDS the terms $\theta^3 \wedge \theta^5$, $\theta^3 \wedge \theta^6$, $\Omega_i \wedge \theta^3$, $\theta^4 \wedge \theta^5$, $\theta^4 \wedge \theta^6$, $\Omega_i \wedge \theta^4$, $i = 1, 2, \ldots, 7$, cannot appear in the exterior derivatives of the forms θ^0, θ^1 and θ^2. This is the case for the EDS from Theorem 4.1.

The distribution $\mathcal{D} = \pi_* \bar{\mathcal{D}}$ on M is $(2, 3, 5)$, since the EDS from Theorem 4.1 guarantees the following expressions for the commutators $[X_3, X_4] = -X_0$, $[X_3, X_0] = X_1$ and $[X_4, X_0] = X_2$, where equality is considered *modulo terms vertical with respect to* π.

- The conformal class of $(3, 2)$ signature metrics $[g_{\mathcal{D}}]$ is represented by the bilinear form
$$g_{\mathcal{D}} = \frac{4}{3}(\theta^0)^2 + 2\theta^1\theta^4 - 2\theta^2\theta^3.$$
The EDS from Theorem 4.1 guarantees that the Lie derivatives of $g_{\mathcal{D}}$ with respect to its degenerate directions spanned by $X_5, X_6, Y_1, \ldots, Y_7$ are always multiples of $g_{\mathcal{D}}$. Thus $g_{\mathcal{D}}$ descends to a well-defined conformal class $[g_{\mathcal{D}}]$ of $(3, 2)$ signature metrics on $M = \mathcal{G}/\mathcal{P}$.

4.2.2. *The corresponding Lie contact structure and* $(3, 5, 7)$ *distribution*

On the other hand, the EDS from Theorem 4.1 can be viewed quite differently:

- Consider the rank 7 distribution $\mathcal{Q}$ on $\mathcal{G}$ defined as the annihilator of the seven linearly independent 1-forms θ^A, $A = 0, 1, 2, 3, 4, 5, 6$. This distribution is integrable due to
$$\mathrm{d}\theta^A \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 \wedge \theta^6 = 0, \quad \forall\, A = 0, 1, 2, 3, 4, 5, 6.$$
As such, it defines a foliation of $\mathcal{G}$ by 7-dimensional leaves, and a fibration
$$Q \to \mathcal{G} \xrightarrow{\sigma} \widetilde{M} = \mathcal{G}/Q,$$
over the 7-dimensional leaf space $\widetilde{M} = \mathcal{G}/Q$.
- The rank 6 distribution $\bar{\mathcal{H}}$ on $\mathcal{G}$ annihilated by the forms $(\theta^0, \Omega_1, \ldots, \Omega_7)$,
$$\bar{\mathcal{H}} = \ker(\theta^0, \Omega_1, \ldots, \Omega_7),$$
descends to a well-defined rank 6 distribution $\mathcal{H} = \sigma_* \bar{\mathcal{H}}$ on the leaf space $\widetilde{M}$.

Moreover, using the EDS from Theorem 4.1 and a similar reasoning as before show that the rank 6 distribution $\mathcal{H} = \sigma_* \bar{\mathcal{H}}$ is indeed a *contact* distribution on $\widetilde{M}$. The one-form θ^0 descends from $\mathcal{G}$ to an equivalence class $[\lambda]$ of contact forms on $\widetilde{M}$, where two contact forms are in the same class if one is a functional multiple of the other; they span a well-defined line subbundle in $T^*\widetilde{M}$.
- Again using the EDS from Theorem 4.1, we show that the contact distribution $\mathcal{H}$ on $\widetilde{M}$ is equipped with additional structure. Consider the 2-form
$$\rho = 3\theta^1 \wedge \theta^6 + 3\theta^2 \wedge \theta^5 + \theta^3 \wedge \theta^4, \tag{16}$$

and the symmetric rank 4 tensor

$$\begin{aligned}\Upsilon = {} & 2\theta^2(\theta^3)^2\theta^6 - 3(\theta^1)^2(\theta^6)^2 - 2\theta^1\theta^3\theta^4\theta^6 - 6\theta^1\theta^2\theta^5\theta^6 \\ & + 2\theta^2\theta^3\theta^4\theta^5 - 2\theta^1(\theta^4)^2\theta^5 - 3(\theta^2)^2(\theta^5)^2.\end{aligned} \tag{17}$$

Then the Lie derivatives of ρ and Υ with respect to the fiber directions Y_A are

$$\mathcal{L}_{Y_A}\rho = u_A\rho + \theta^0 \wedge \alpha_A \quad \text{and} \quad \mathcal{L}_{Y_A}\Upsilon = v_A\Upsilon + \theta^0 \odot \gamma_A,$$

where u_A, v_A are functions, α_A are 1-forms, and γ_A are symmetric rank 3 tensors. Since θ^0 annihilates the distribution $\mathcal{H}$, ρ and Υ descend to the respective objects $[\rho]$ and $[\Upsilon]$ on the distribution $\mathcal{H}$, where they are defined up to scale, because some of the u_A, v_A are nonzero. (In fact, the class of ρ on $\mathcal{H}$ can be represented by $\mathrm{d}\theta^0|_{\mathcal{H}}$, so this defines a line subbundle of $\Lambda^2\mathcal{H}^*$ spanned by symplectic forms on $\mathcal{H}$.)

- The rank 3 distribution $\bar{\mathcal{S}}$ on the Cartan bundle $\mathcal{G}$ defined as

$$\bar{\mathcal{S}} = \ker(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \ldots, \Omega_7) = \operatorname{Span}(X_0, X_5, X_6),$$

descends to a well-defined rank 3 distribution $\mathcal{S} = \sigma_*\bar{\mathcal{S}}$ on $\widetilde{M}$. This can be seen from the fact that in the EDS from Theorem 4.1 the exterior derivatives of the forms θ^1, θ^2, θ^3 and θ^4 do not contain terms of the form $\theta^0 \wedge \Omega_i$, $\theta^5 \wedge \Omega_i$ and $\theta^6 \wedge \Omega_i$.

One easily checks using the system that $[X_5, X_6] = 0$, $[X_0, X_5] = -4X_3$, $[X_0, X_6] = 4X_4$, $[X_0, X_3] = -X_1$, $[X_0, X_4] = -X_2$ *modulo vertical terms*. This shows that the first commutator $[\mathcal{S}, \mathcal{S}]$ has rank 5 (and is equal to the lift $\widetilde{[\mathcal{D}, \mathcal{D}]}$ of $[\mathcal{D}, \mathcal{D}]$), and $[\mathcal{S}, [\mathcal{S}, \mathcal{S}]] = T\widetilde{M}$. In particular, the distribution $\mathcal{S}$ has growth vector $(3, 5, 7)$.

Remark 4.1. Locally, the structure on $\widetilde{M}$ described above in terms of the contact distribution $\mathcal{H}$ equipped with the equivalence class of symmetric rank 4 tensors $[\Upsilon]$ is equivalent to a Lie contact structure as introduced in Sec. 2.6. To see this, one shows that $[\Upsilon]$ reduces the structure group of the natural frame bundle $\mathcal{F}$ of the contact distribution to the correct group $\widetilde{G}_0 \subset \mathrm{CSp}(3)$, see also Remark 2.1.

4.3. *A class of examples*

Next we construct the 1-forms $(\theta^0, \ldots, \theta^6)$ explicitly with respect to a section for a special class of distributions. In particular, this yields an explicit local description of the induced Lie contact structure.

4.3.1. *A particular solution to the EDS in dimension* 7

Recall that we can specify a $(2, 3, 5)$ distribution $\mathcal{D}_F$ defined in a neighborhood $\mathcal{U}^5$ around the origin of $\mathbb{R}^5$ with local coordinates (x, y, p, q, z) by specifying a single function of five variables $F = F(x, y, p, q, z)$ such that $F_{qq} \neq 0$. Let us consider a differentiable function $F = h(q)$ of one variable q only. We assume that $h'' \neq 0$. Then the distribution $\mathcal{D}_h$ is given as the kernel of the three 1-forms

$$\omega^0 = \mathrm{d}p - q\mathrm{d}x, \quad \omega^1 = \mathrm{d}y - p\mathrm{d}x, \quad \omega^2 = \mathrm{d}z - h\mathrm{d}x.$$

The 1-forms $(\omega^0, \omega^1, \omega^2)$ can be supplemented to a coframe (ω^i), $i = 0, 1, 2, 3, 4$, on $\mathcal{U}^5$ given by

$$\begin{aligned}
\omega^0 &= \mathrm{d}p - q\mathrm{d}x,\\
\omega^1 &= \mathrm{d}y - p\mathrm{d}x,\\
\omega^2 &= \mathrm{d}z - h\mathrm{d}x,\\
\omega^3 &= \mathrm{d}q,\\
\omega^4 &= \mathrm{d}x.
\end{aligned} \tag{18}$$

Now one introduces forms

$$\begin{pmatrix} \theta^0 \\ \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 & 0 \\ u_7 & u_8 & u_9 & 0 & 0 \\ u_{10} & u_{11} & u_{12} & u_{13} & u_{14} \\ u_{15} & u_{16} & u_{17} & u_{18} & u_{19} \end{pmatrix} \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{pmatrix}, \tag{19}$$

with the 19 free parameters $(u_1, u_2, \ldots, u_{19})$. It follows that there exists a choice of these parameters, in which the forms $(\theta^0, \theta^1, \ldots, \theta^4)$ satisfy the EDS as in Theorem 4.1, with corresponding functions $(a_1, a_2, \ldots, h_3, h_4)$ and 1-forms $(\theta^5, \theta^6, \Omega_1, \Omega_2, \ldots, \Omega_7)$, such that

$$\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 \wedge \theta^6 \neq 0,$$

and

$$\Omega_i \wedge \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5 \wedge \theta^6 \equiv 0, \quad \forall\, i = 1, 2, \ldots, 7.$$

This means that there is an effective algorithm of solving the EDS of Theorem 4.1 for forms $(\theta^0, \theta^1, \ldots, \theta^6, \Omega_1, \Omega_2, \ldots, \Omega_7)$ and the coefficients $(a_1, a_2, \ldots, h_3, h_4)$ on a certain 7-dimensional manifold, which we below parametrized by (x, y, p, q, z, v, w). Explicitly, the forms corresponding to this choice are given below (we use the notation $h^{(n)}$ for higher derivatives of the function $h = h(q)$):

$$\begin{aligned}
\theta^0 &= \frac{vh''^{4/3}}{9w^4}\mathrm{d}y + \frac{h''^{4/3}}{9w^4}\mathrm{d}z - \frac{(w + h')h''^{4/3}}{9w^4}\mathrm{d}p - \frac{(vp - wq + h - qh')h''^{4/3}}{9w^4}\mathrm{d}x,\\
\theta^1 &= -\frac{ph''^{4/3}}{27w^4}\mathrm{d}x + \frac{h''^{4/3}}{27w^4}\mathrm{d}y,\\
\theta^2 &= \frac{vh''^{5/3}}{27w^5}\mathrm{d}y + \frac{h''^{5/3}}{27w^5}\mathrm{d}z - \frac{h'h''^{5/3}}{27w^5}\mathrm{d}p - \frac{(vp + h - qh')h''^{5/3}}{27w^5}\mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
\theta^3 = {} & \frac{vh''}{3w^3}\mathrm{d}y - \frac{(-20h''^4 - 4w^2{h^{(3)}}^2 + 3w^2h''h^{(4)})}{90w^3h''^3}\mathrm{d}z \\
& - \frac{(40wh''^4 + 20h'h''^4 + 10w^2h''^2h^{(3)} + 4w^2h'{h^{(3)}}^2 - 3w^2h'h''h^{(4)})}{90w^3h''^3}\mathrm{d}p \\
& + \frac{1}{90w^3h''^3}(30w^2h''^3 - 30vph''^4 + 40wqh''^4 - 20hh''^4 + 20qh'h''^4 \\
& + 10w^2qh''^2h^{(3)} - 4w^2h{h^{(3)}}^2 + 4w^2qh'{h^{(3)}}^2 + 3w^2hh''h^{(4)} \\
& - 3w^2qh'h''h^{(4)})\mathrm{d}x, \\
\theta^4 = {} & -\frac{h''^{4/3}}{3w^2}\mathrm{d}q + \frac{v^2h''^{4/3}}{9w^4}\mathrm{d}y - \frac{v(-10h''^4 - 4w^2{h^{(3)}}^2 + 3w^2h''h^{(4)})}{90w^4h''^{8/3}}\mathrm{d}z \\
& + \frac{v(-10h'h''^4 - 10w^2h''^2h^{(3)} - 4w^2h'{h^{(3)}}^2 + 3w^2h'h''h^{(4)})}{90w^4h''^{8/3}}\mathrm{d}p \\
& - \frac{v}{90w^4h''^{8/3}}(-30w^2h''^3 + 10vph''^4 + 10hh''^4 - 10qh'h''^4 - 10w^2qh''^2h^{(3)} \\
& + 4w^2h{h^{(3)}}^2 - 4w^2qh'{h^{(3)}}^2 - 3w^2hh''h^{(4)} + 3w^2qh'h''h^{(4)})\mathrm{d}x, \\
\theta^5 = {} & \frac{\mathrm{d}w}{h''^{1/3}} + \frac{(10h''^4 - 10wh''^2h^{(3)} + 4w^2{h^{(3)}}^2 - 3w^2h''h^{(4)})}{30h''^{10/3}}\mathrm{d}q \\
& + \frac{v(-5h''^6 + 40w^3{h^{(3)}}^3 - 45w^3h''h^{(3)}h^{(4)} + 9w^3h''^2h^{(5)})}{90w^2h''^{16/3}}\mathrm{d}z \\
& + \frac{v}{90w^2h''^{16/3}}(-15wh''^6 + 5h'h''^6 - 12w^3h''^2{h^{(3)}}^2 - 40w^3h'{h^{(3)}}^3 \\
& + 9w^3h''^3h^{(4)} + 45w^3h'h''h^{(3)}h^{(4)} - 9w^3h'h''^2h^{(5)})\mathrm{d}p \\
& - \frac{v}{90w^2h''^{16/3}}(-15wqh''^6 - 5hh''^6 + 5qh'h''^6 - 12w^3qh''^2{h^{(3)}}^2 \\
& + 40w^3h{h^{(3)}}^3 - 40w^3qh'{h^{(3)}}^3 + 9w^3qh''^3h^{(4)} \\
& - 45w^3hh''h^{(3)}h^{(4)} + 45w^3qh'h''h^{(3)}h^{(4)} + 9w^3hh''^2h^{(5)} \\
& - 9w^3qh'h''^2h^{(5)})\mathrm{d}x, \\
\theta^6 = {} & -\mathrm{d}v + \frac{v^2(-4{h^{(3)}}^2 + 3h''h^{(4)})}{90wh''^3}\mathrm{d}z + \frac{v(10h''^4 - 4w^2{h^{(3)}}^2 + 3w^2h''h^{(4)})}{30wh''^3}\mathrm{d}q \\
& + \frac{v^3(40{h^{(3)}}^3 - 45h''h^{(3)}h^{(4)} + 9h''^2h^{(5)})}{90h''^5}\mathrm{d}y - \frac{v^2}{90w^2h''^5}(5h''^6 - 10wh''^4h^{(3)} \\
& - 12w^2h''^2{h^{(3)}}^2 - 4wh'h''^2{h^{(3)}}^2 + 40w^3{h^{(3)}}^3 + 9w^2h''^3h^{(4)}
\end{aligned}
$$

$$+ 3wh'h''^3h^{(4)} - 45w^3h''h^{(3)}h^{(4)} + 9w^3h''^2h^{(5)})\mathrm{d}p - \frac{v^2}{90w^2h''^5}(30wh''^5$$
$$- 5qh''^6 + 10wqh''^4h^{(3)} + 12w^2qh''^2{h^{(3)}}^2 - 4whh''^2{h^{(3)}}^2 + 4wqh'h''^2{h^{(3)}}^2$$
$$+ 40vw^2p{h^{(3)}}^3 - 40w^3q{h^{(3)}}^3 - 9w^2qh''^3h^{(4)} + 3whh''^3h^{(4)} - 3wqh'h''^3h^{(4)}$$
$$- 45vw^2ph''h^{(3)}h^{(4)} + 45w^3qh''h^{(3)}h^{(4)} + 9vw^2ph''^2h^{(5)} - 9w^3qh''^2h^{(5)})\mathrm{d}x. \tag{20}$$

We could also write down the remaining forms $(\Omega_1, \Omega_2, \ldots, \Omega_7)$ that together with the above $(\theta^0, \theta^1, \ldots, \theta^6)$ satisfy the EDS from Theorem 4.1, but since they are not interesting for the rest of our paper we will skip them.

The particular solution $(\theta^0, \theta^1, \ldots, \theta^6)$ constructed above enables us to write down the structural tensors associated with the $(2, 3, 5)$ distribution

$$\mathcal{D}_h = \mathrm{Span}(\partial_x + p\partial_y + q\partial_p + h(q)\partial_z,\ \partial_q), \tag{21}$$

explicitly in the coordinates $(x, y, p, q, z; v, w)$.

It should be clear that the coordinates (x, y, p, q, z) parametrize the 5-manifold M on which the distribution $\mathcal{D}_h$ resides, and that (v, w) are the fiber coordinates of the bundle $\widetilde{M} \to M$. In particular, (v, w) locally parameterize directions $\ell(v, w) = \mathrm{dir}(\xi(v, w))$ in the 3-distribution $[\mathcal{D}_h, \mathcal{D}_h]$ as follows:

$$\xi(v, w) = \partial_x + p\partial_y + q\partial_p + h\partial_z + \frac{v}{h''}\partial_q + \frac{w}{h''}(\partial_p + h'\partial_z).$$

Note that in this parametrization the directions *transverse* to the 2-distribution $\mathcal{D}_h$ have $w \neq 0$, and that $w \equiv 0$ corresponds to the directions in the 2-distribution $\mathcal{D}_h$. Thus, when the coordinate $w \to 0$ we approach points (x, y, p, q, z, v) of the 6-dimensional boundary $\mathbb{P}(\mathcal{D}_h)$ of $\widetilde{M} \cong \mathbb{P}([\mathcal{D}_h, \mathcal{D}_h]) \backslash \mathbb{P}(\mathcal{D}_h)$.

In the remainder of Sec. 4.3 we will restrict our examples to the distributions $\mathcal{D}_h$ with

$$h(q) = \frac{1}{k(k-1)}q^k, \quad \text{where } k \in \mathbb{R},\ \ k \neq 0, 1. \tag{22}$$

Since in such case $\mathcal{D}_h$ is totally determined by a real number k, we will denote these distributions by $\mathcal{D}_k$. We have excluded the cases $k = 0, 1$ because they do not correspond to $(2, 3, 5)$ distributions.

4.3.2. *Conformal metric on M*

For the class of examples given by (22) the conformal class of metrics $[g_{\mathcal{D}_k}]$ may be represented by

$$g_{\mathcal{D}_k} = (k-1)^2(9k^2 - 9k + 2)q^2\mathrm{d}x^2 - 2k(k-1)(9k^2 - 9k - 8)q\mathrm{d}x\mathrm{d}p$$
$$+ 30k^2(k-1)^2p\mathrm{d}x\mathrm{d}q - 4k(k-1)^2(3k^2 + 2k - 1)q^{2-k}\mathrm{d}x\mathrm{d}z$$

$$
\begin{aligned}
&- 30k^2(k-1)^2\mathrm{d}y\mathrm{d}q + k^2(9k^2-9k+2)\mathrm{d}p^2 \\
&+ 4k^2(k-1)(3k^2-8k+4)q^{1-k}\mathrm{d}p\mathrm{d}z - k^2(k-1)^2(k^2-k-2)q^{2-2k}\mathrm{d}z^2.
\end{aligned} \tag{23}
$$

It is well known [23] that this metric is conformally flat if and only if the corresponding distribution $\mathcal{D}_k$ is flat, and this happens [11] precisely in the four cases when $k \in \{2, \frac{2}{3}, \frac{1}{3}, -1\}$. For example for $k = 2$ we get the conformally flat metric

$$
g_{\mathcal{D}_2} = 4(30\mathrm{d}x\mathrm{d}z - 5q^2\mathrm{d}x^2 - 20\mathrm{d}p^2 + 10q\mathrm{d}p\mathrm{d}x - 30p\mathrm{d}q\mathrm{d}x + 30\mathrm{d}q\mathrm{d}y). \tag{24}
$$

Now if $k \notin \{2, \frac{2}{3}, \frac{1}{3}, -1\}$ the distribution $\mathcal{D}_k$ has 7-dimensional symmetry algebra (the submaximal dimension) spanned by

$$
\begin{aligned}
&X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z, \quad X_4 = \partial_p + x\partial_y, \\
&X_5 = x\partial_x - p\partial_p - 2q\partial_q + (1-2k)z\partial_z, \quad X_6 = y\partial_y + p\partial_p + q\partial_q + kz\partial_z, \\
&X_7 = q^{k-1}\partial_x + (pq^{k-1} + (1-k)z)\partial_y + \frac{k-1}{k}q^k\partial_p + \frac{q^{2k-1}}{k(2k-1)}\partial_z.
\end{aligned}
$$

The conformal class represented by (23) has 9-dimensional symmetry algebra, spanned by $X_1, \ldots, X_7$ and the two *additional* generators

$$
\begin{aligned}
X_8 = q^{-\frac{1}{2}+\frac{\sqrt{10k^2-10k+5}}{10}} \Bigg(&\partial_x + p\partial_y + \frac{3k^2 - 2\sqrt{10k^2-10k+5} - 3k + 4}{(3k-2)(k-2)}q\partial_p \\
&+ 2\frac{4k^2-4k+2-k\sqrt{10k^2-10k+5}}{(3k-2)(k-2)k(k-1)}q^{-k}\partial_z\Bigg), \\
X_9 = q^{-\frac{1}{2}+\frac{\sqrt{10k^2-10k+5}}{10}} \Bigg(&\partial_x + p\partial_y + \frac{3k^2 + 2\sqrt{10k^2-10k+5} - 3k + 4}{(3k-2)(k-2)}q\partial_p \\
&+ 2\frac{4k^2-4k+2+k\sqrt{10k^2-10k+5}}{(3k-2)(k-2)k(k-1)}q^{k}\partial_z\Bigg).
\end{aligned}
$$

It is instructive to look at the symmetries in one of the flat cases, say $k = 2$. One sees that in this case X_8 and X_9 are singular, but the rescaling by a factor $(k-2)$ regularizes them at $k = 2$. These however, in the limit $k \to 2$, lead to one symmetry only, namely to $Z_1 = \lim_{k\to 2} X_9 = \partial_p + q\partial_z$, since the limit of the regularized X_8 is zero. In this case the eight conformal symmetries $(X_1, X_2, \ldots, X_7, X_9{}')$ are of course extendible to the full 21-dimensional algebra of symmetries $\mathfrak{so}(4,3)$.

We close this section providing the full algebra of symmetries of the distribution $\mathcal{D}_k$ with $k = 2$ and the full algebra of conformal symmetries of $[g_{\mathcal{D}_k}]$ in such case. In addition to the 7 symmetries $(X_1, X_2, \ldots, X_7)$ with $k = 2$ this distribution has additional seven symmetries, so that its full algebra of symmetries has dimension 14.

The remaining seven symmetries are

$$
\begin{aligned}
Y_1 &= \frac{1}{2}x^2\partial_y + x\partial_p + \partial_q + p\partial_z,\\
Y_2 &= \frac{1}{6}x^3\partial_y + \frac{1}{2}x^2\partial_p + x\partial_q + (xp - y)\partial_z,\\
Y_3 &= x^2\partial_x + 3xy\partial_y + (3y + xp)\partial_p + (4p - qx)\partial_q + 2p^2\partial_z,\\
Y_4 &= (8p - 6qx)\partial_x + (4p^2 + 6xz - 6pqx)\partial_y\\
&\quad + (6z - 3q^2x)\partial_p - 2q^2\partial_q - q^3x\partial_z,\\
Y_5 &= (16xp - 12y - 6qx^2)\partial_x + (6x^2z + 8p^2x - 6pqx^2)\partial_y\\
&\quad + (12xz + 4p^2 - 3q^2x^2)\partial_p + (12z + 4pq - 4q^2x)\partial_q\\
&\quad + (12pz - q^3x^2)\partial_z,\\
Y_6 &= (24px^2 - 6qx^3 - 36xy)\partial_x + (12p^2x^2 + 6x^3z - 36y^2 - 6pqx^3)\partial_y\\
&\quad + (12p^2x + 18x^2z - 3q^2x^3 - 36py)\partial_p\\
&\quad + (12pqx - 6q^2x^2 - 24p^2 + 36xz)\partial_q\\
&\quad + (36pxz - 8p^3 - q^3x^3 - 36yz)\partial_z,\\
Y_7 &= (12p^2 - 18qy)\partial_x + (8p^3 - 18pqy + 18yz)\partial_y + (18pz - 9q^2y)\partial_p\\
&\quad + (18qz - 6pq^2)\partial_q + (18z^2 - 3q^3y)\partial_z.
\end{aligned}
\tag{25}
$$

The 14-dimensional Lie algebra spanned by $(X_1, X_2, \ldots, X_7, Y_1, Y_2, \ldots, Y_7)$ is isomorphic to the split real form of the exceptional simple Lie algebra $\mathfrak{g}_2$. As for the conformal symmetries of $[g_{\mathcal{D}_2}]$: we have the 14 conformal symmetries of the distribution, $(X_1, X_2, \ldots, X_7, Y_1, Y_2, \ldots, Y_7)$, forming the Lie algebra of $\mathfrak{g}_2$, but also seven additional conformal symmetries given by

$$
\begin{aligned}
Z_1 &= \partial_p + q\partial_z,\\
Z_2 &= \partial_x + p\partial_y + \frac{3}{4}q\partial_p + \frac{1}{4}q^2\partial_z,\\
Z_3 &= \partial_q + \frac{1}{4}x\partial_p + \frac{1}{4}qx\partial_z,\\
Z_4 &= 4px\partial_y + (3qx + 6p)\partial_p + 12q\partial_q + (q^2x + 2qp + 12z)\partial_z,\\
Z_5 &= 4x^2\partial_x + 4px^2\partial_y + (3qx^2 + 4px - 6y)\partial_p + 8(qx - p)\partial_q\\
&\quad + (q^2x^2 + 4xqp - 6qy)\partial_z,\\
Z_6 &= 12qx\partial_x + (12xqp + 8p^2 - 12xz)\partial_y + (6q^2x + 12qp)\partial_p + 12q^2\partial_q\\
&\quad + (2q^3x + 4q^2p + 12qz)\partial_z,\\
Z_7 &= 4(px - 3y)\partial_x + 4p(px - 3y)\partial_y + (3xqp - 2p^2 - 9qy + 3xz)\partial_p\\
&\quad + (2q^2x - 8qp + 12z)\partial_q + (xpq^2 - 2p^2q - 3yq^2 + 3zqx)\partial_z.
\end{aligned}
$$

The 21-dimensional algebra generated by $X_1, \dots, X_7$, $Y_1, \dots, Y_7$, $Z_1, \dots, Z_7$ is isomorphic to the Lie algebra $\mathfrak{so}(4,3)$.

4.3.3. *The Lie contact structure*

The Lie contact structure $([\lambda], [\Upsilon])$ on $\widetilde{M}$ associated with the distribution $\mathcal{D}_h$ is totally expressible in terms of the forms $(\theta^0, \theta^1, \dots, \theta^6)$ as written in Sec. 4.3.1, formulas (16), (17). For $h(q) = \frac{q^k}{k(k-1)}$, the line of contact forms can be represented by

$$\lambda = \mathrm{d}z - \left(w + \frac{q^{k-1}}{k-1}\right)\mathrm{d}p + v\mathrm{d}y + \left(wq - vp + \frac{q^k}{k}\right)\mathrm{d}x. \tag{26}$$

To get this, we took θ^0 from (20), calculated it for $h = \frac{q^k}{k(k-1)}$ and rescaled, so that the term at $\mathrm{d}z$ is equal to one. One easily checks that

$$\mathrm{d}\lambda \wedge \mathrm{d}\lambda \wedge \mathrm{d}\lambda \wedge \lambda = -6w\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}p \wedge \mathrm{d}q \wedge \mathrm{d}z \wedge \mathrm{d}v \wedge \mathrm{d}w,$$

so λ is a contact form everywhere on $\widetilde{M}$ except the boundary $w = 0$. Even in the simple case that we are considering here, we found that the structural tensor Υ on $\mathcal{D}_k$, when written via the formula (17) in coordinates (x, y, p, q, z, v, w), is very ugly. For this reason we will not write it here. Instead we determine the symmetries of the Lie contact structure $([\lambda], [\Upsilon])$ on $\widetilde{M} = \mathbb{P}([\mathcal{D}_k, \mathcal{D}_k]) \backslash \mathbb{P}(\mathcal{D}_k)$ with this ugly Υ.

In general, an infinitesimal symmetry of a Lie contact structure $([\lambda], [\Upsilon])$ on $\widetilde{M}$ is a vector field X on $\widetilde{M}$ such that

$$(\mathcal{L}_X \lambda) \wedge \lambda = 0, \quad \text{and} \quad \mathcal{L}_X \Upsilon = f\Upsilon + \lambda \odot \tau, \tag{27}$$

where τ is a rank 3 tensor and f is a function on $\widetilde{M}$. We calculated the infinitesimal symmetries of the Lie contact structure $([\lambda], [\Upsilon])$ with λ as in (26) and Υ determined by (17), (20) with $h = \frac{q^k}{k(k-1)}$, obtaining the following proposition.

Proposition 4.1. *If* $k \notin \{2, \frac{2}{3}, \frac{1}{3}, 0, 1, -1\}$ *the algebra of infinitesimal symmetries of the Lie contact structure* $([\lambda], [\Upsilon])$ *on* $\widetilde{M} = \mathbb{P}([\mathcal{D}_k, \mathcal{D}_k]) \backslash \mathbb{P}(\mathcal{D}_k)$ *is* 7*-dimensional and is spanned by the infinitesimal symmetries*:

$$\tilde{X}_1 = \partial_x, \quad \tilde{X}_2 = \partial_y, \quad \tilde{X}_3 = \partial_z, \quad \tilde{X}_4 = \partial_p + x\partial_y,$$

$$\tilde{X}_5 = x\partial_x - p\partial_p - 2q\partial_q + (1-2k)z\partial_z + (1-2k)v\partial_v + 2(1-k)w\partial_w,$$

$$\tilde{X}_6 = y\partial_y + p\partial_p + q\partial_q + kz\partial_z + (k-1)v\partial_v + (k-1)w\partial_w,$$

$$\tilde{X}_7 = q^{k-1}\partial_x + (pq^{k-1} + (1-k)z)\partial_y + \frac{k-1}{k}q^k\partial_p + \frac{q^{2k-1}}{k(2k-1)}\partial_z + (1-k)v^2\partial_v + (1-k)vw\partial_w.$$

Remark 4.2. Note that the seven symmetries $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_7)$ above correspond to the seven symmetries $(X_1, X_2, \dots, X_7)$ of the distribution $\mathcal{D}_k$ defining the Lie contact structure $([\lambda], [\Upsilon])$. Explicitly note that we have: $\tilde{X}_i = X_i + a_i\partial_v + b_i\partial_w$,

$i = 1, 2, \ldots, 7$, with specific functional coefficients a_i and b_i. We remark that we obtained $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_7)$ by directly solving the symmetry equations (27), and *not* by assuming that the symmetry $\tilde{X}_i$ have the form $\tilde{X}_i = X_i + a_i\partial_v + b_i\partial_w$.

There is, however, more direct way of getting these seven symmetries. This is related to the general fact that every symmetry of a $(2, 3, 5)$ distribution $\mathcal{D}$ induces a symmetry of its twistor Lie contact structure, since the construction is natural. The simplest way of seeing this is via the *prolongation lift* (or simply prolongation) $\tilde{X}$ of an infinitesimal symmetry X of $\mathcal{D}$. This is done point by point as follows: Suppose that we want to lift X_p, i.e. the vector defined by an infinitesimal symmetry X at $p \in M$, from point p to a point (p, ℓ) in the fiber in $\widetilde{M}$ over p. At p the point (p, ℓ) defines a direction ℓ in the 3-distribution $[\mathcal{D}, \mathcal{D}]$. We transport this direction by a flow $\phi(t)$ of X along its integral curve $p(t)$ passing through p, $p(0) = p$. This defines a direction $\ell(t) = \phi_*(t)\ell$ at every point of the curve $p(t)$. Thus starting with $\ell(0) = \ell$ at $p(0) = p$, we have a direction $\ell(t)$ at $p(t)$ for every t. Since X is a symmetry of a $(2, 3, 5)$ distribution, its flow preserves the 3-distribution, so for any value of t the direction $\ell(t)$ sits in the 3-distribution. Thus, choosing a point ℓ at a fiber of p, at each point $p(t)$ of an integral curve of a symmetry vector field X we have a direction $\ell(t)$ in the 3-distribution. We thus have a curve $(p(t), \ell(t))$ in the bundle $\widetilde{M}$, which starts at (p, ℓ) and which projects to $p(t)$. The tangent vector $\tilde{X}_{(p,\ell)}$ to this curve at $t = 0$ is the *prolongation lift* of the symmetry vector X_p from $p \in M$ to $(p, \ell) \in \widetilde{M}$. By repeating this procedure for all pairs $(p, \ell) \in \widetilde{M}$ we define a vector field $\tilde{X}$ on $\widetilde{M}$ consisting of vectors $\tilde{X}_{(p,\ell)}$. We call $\tilde{X}$ the prolongation of X. It follows from the construction that the prolongation $\tilde{X}$ of an infinitesimal symmetry X of a $(2, 3, 5)$ distribution $\mathcal{D}$ is an infinitesimal symmetry of the corresponding Lie contact structure $([\lambda], [\Upsilon])$ on $\widetilde{M}$.

Finishing the remark we stress that *all* infinitesimal symmetries of the Lie contact structure $([\lambda], [\Upsilon])$ on $\widetilde{M} = \mathbb{P}([\mathcal{D}_k, \mathcal{D}_k])\backslash\mathbb{P}(\mathcal{D}_k)$ with all $k \notin \{2, \frac{2}{3}, \frac{1}{3}, 0, 1, -1\}$ are just prolongations of infinitesimal symmetries of the distribution $\mathcal{D}_k$. We have proven this by explicitly solving the symmetry equations and finding all their solutions.

4.3.4. *The* $(3, 5, 7)$ *distribution*

It is also interesting to look at the infinitesimal symmetries of the prolongation

$$\mathcal{S} = \mathrm{Span}\left(\partial_x + p\,\partial_y + q\,\partial_p + h\,\partial_z + \frac{v}{h''}\partial_q + \frac{w}{h''}(\partial_p + q\,\partial_z), \partial_v, \partial_w\right)$$

of $\mathcal{D}_h$. For (22) and $k \notin \{2, \frac{2}{3}, \frac{1}{3}, -1\}$, the seven lifts of infinitesimal symmetries of the distribution $\mathcal{D}_k$ from Proposition 4.1 clearly preserve the $(3, 5, 7)$ distribution $\mathcal{S}$. We calculated that all infinitesimal symmetries of $\mathcal{S} = \mathcal{S}_k$ are contained in the span of these seven symmetries. We further calculated that in the flat case $k = 2$, the symmetry algebra of the distribution $\mathcal{S}_2$ is precisely $\mathfrak{g}_2$ (the symmetry algebra of the Lie contact structure is of course $\mathfrak{so}(4, 3)$ in this case; see the end of this section for details). It turns out that for any $(2,3,5)$ distribution, the prolongation $\mathcal{S}$ has the same symmetry algebra as the underlying $(2,3,5)$ distribution $\mathcal{D}$.

Proposition 4.2. *For any* $(2,3,5)$ *distribution* $\mathcal{D} \subset TM$*, the infinitesimal symmetries of the prolongation* $\mathcal{S} \subset T\widetilde{M}$ *are precisely the lifts of the infinitesimal symmetries of* $\mathcal{D}$*.*

Proof. Since the construction is natural, every infinitesimal symmetry $X \in \mathfrak{X}(M)$ of the $(2,3,5)$ distribution $\mathcal{D}$ lifts to a vector field $\widetilde{X} \in \mathfrak{X}(\widetilde{M})$ that preserves the induced geometric structure on the twistor bundle. In particular, it defines a symmetry of the $(3,5,7)$ distribution $\mathcal{S}$. It remains to show that every infinitesimal symmetry of $\mathcal{S}$ projects to an infinitesimal symmetry of $\mathcal{D}$. Consider the tensorial map $\Lambda^2\mathcal{S} \to [\mathcal{S},\mathcal{S}]/\mathcal{S}$ induced by the Lie bracket. At every point this is a surjective map from a 3-dimensional to a 2-dimensional space and thus it has a 1-dimensional kernel spanned by a decomposable element. So this defines a rank 2 distribution on $\widetilde{M}$. Since the vertical distribution $\mathcal{V}$ for $\widetilde{M} \to M$ is evidently contained in this rank 2 distribution and of the same dimension, the two coincide. Note that this means that the vertical bundle is characterized as the unique rank 2 subbundle in $\mathcal{S}$ such that Lie brackets of its sections are again contained in $\mathcal{S}$. This in particular implies that any infinitesimal symmetry $\widetilde{X} \in \mathfrak{X}(\widetilde{M})$ of $\mathcal{S}$ also preserves the vertical bundle $\mathcal{V}$ and thus it is projectable to a vector field $X \in \mathfrak{X}(M)$. Moreover, $\widetilde{\xi}$ also preserves $[\mathcal{S},\mathcal{S}]$, and since $[\mathcal{S},\mathcal{S}] = \widetilde{[\mathcal{D},\mathcal{D}]}$, then naturality of the Lie bracket implies that X preserves $[\mathcal{D},\mathcal{D}]$. By the same line of argument as above, $\mathcal{D}$ can be characterized as the unique rank 2 subbundle in $[\mathcal{D},\mathcal{D}]$ such that Lie brackets of its sections are again contained in $[\mathcal{D},\mathcal{D}]$, and this implies that ξ is an infinitesimal symmetry for the $(2,3,5)$ distribution $\mathcal{D}$. □

4.3.5. *Flat Lie contact structure*

We conclude this section with a discussion of the flat case, corresponding to the $(2,3,5)$ distribution with $h(q) = \frac{1}{2}q^2$, i.e. $k = 2$. In this case we have

$$\lambda = \mathrm{d}z - (w+q)\mathrm{d}p + v\mathrm{d}y + \left(wq - vp + \frac{q^2}{2}\right)\mathrm{d}x$$

and the conformal tensor Υ on $\ker(\lambda)$ can be represented by

$$\begin{aligned}\Upsilon = &-3v^2(2vp - 2wq - q^2)\mathrm{d}x^4 + 6v^3\mathrm{d}x^3\mathrm{d}y - 6v^2(w+q)\mathrm{d}p\mathrm{d}x^3\\ &+ 6v(3vp - 2wq - q^2)\mathrm{d}q\mathrm{d}x^3 - 2(-9w^2q - 12wq^2 + 9vpw + 9pvq - 4q^3)\mathrm{d}v\mathrm{d}x^3\\ &+ 6v(-q^2 + 3vp - 3wq)\mathrm{d}w\mathrm{d}x^3 + 3v^2\mathrm{d}p^2\mathrm{d}x^2 - 3(-2wq - q^2 + 6vp)\mathrm{d}q^2\mathrm{d}x^2\\ &- 9p^2\mathrm{d}v^2\mathrm{d}x^2 - 324q^2\mathrm{d}w^2\mathrm{d}x^2 + 9v(w+q)\mathrm{d}p\mathrm{d}q\mathrm{d}x^2\\ &+ 6(-3w^2 - 8wq - 4q^2 + 3vp)\mathrm{d}p\mathrm{d}v\mathrm{d}x^2 + 6v(3w + 2q)\mathrm{d}p\mathrm{d}w\mathrm{d}x^2\end{aligned}$$

$$
\begin{aligned}
&+18p(w+q)\mathrm{d}q\mathrm{d}v\mathrm{d}x^2-6(-3wq+6vp-q^2)\mathrm{d}q\mathrm{d}w\mathrm{d}x^2-18v^2\mathrm{d}q\mathrm{d}y\mathrm{d}x^2\\
&+18pq\mathrm{d}v\mathrm{d}w\mathrm{d}x^2+18v(w+q)\mathrm{d}v\mathrm{d}y\mathrm{d}x^2-18v^2\mathrm{d}w\mathrm{d}y\mathrm{d}x^2-6v\mathrm{d}p^2\mathrm{d}q\mathrm{d}x\\
&+24(w+q)\mathrm{d}p^2\mathrm{d}v\mathrm{d}x-6v\mathrm{d}p^2\mathrm{d}w\mathrm{d}x-6(w+q)\mathrm{d}p\mathrm{d}q^2\mathrm{d}x\\
&-18p\mathrm{d}p\mathrm{d}q\mathrm{d}v\mathrm{d}x-6(3w+2q)\mathrm{d}p\mathrm{d}q\mathrm{d}w\mathrm{d}x-18p\mathrm{d}p\mathrm{d}v\mathrm{d}w\mathrm{d}x\\
&-18v\mathrm{d}p\mathrm{d}v\mathrm{d}y\mathrm{d}x+18q\mathrm{d}p\mathrm{d}w^2\mathrm{d}x+6p\mathrm{d}q^3\mathrm{d}x+18p\mathrm{d}q^2\mathrm{d}w\mathrm{d}x\\
&+18v\mathrm{d}q^2\mathrm{d}x\mathrm{d}y-18(w+q)\mathrm{d}q\mathrm{d}v\mathrm{d}y\mathrm{d}x+36v\mathrm{d}q\mathrm{d}w\mathrm{d}x\mathrm{d}y+18p\mathrm{d}v^2\mathrm{d}x\mathrm{d}y\\
&-18q\mathrm{d}v\mathrm{d}w\mathrm{d}x\mathrm{d}y-8\mathrm{d}p^3\mathrm{d}v+3\mathrm{d}p^2\mathrm{d}q^2+6\mathrm{d}p^2\mathrm{d}q\mathrm{d}w-9\mathrm{d}p^2\mathrm{d}w^2\\
&+18\mathrm{d}p\mathrm{d}q\mathrm{d}v\mathrm{d}y+18\mathrm{d}p\mathrm{d}v\mathrm{d}w\mathrm{d}y-6\mathrm{d}q^3\mathrm{d}y-18\mathrm{d}q^2\mathrm{d}w\mathrm{d}y-9\mathrm{d}v^2\mathrm{d}y^2.
\end{aligned}
$$

The infinitesimal symmetries of the Lie contact structure $([\lambda],[\Upsilon])$ form a Lie algebra $\mathfrak{so}(3,4)$ and are naturally grouped as $(\tilde{X}_1,\dots,\tilde{X}_7)$, $(\tilde{Y}_1,\dots,\tilde{Y}_7)$ and $(\hat{Z}_1,\dots,\hat{Z}_7)$, where we have.

The first seven symmetries are just prolongations $(\tilde{X}_1,\dots,\tilde{X}_7)$ of the seven symmetries $(X_1,\dots,X_7)$ of the distribution $\mathcal{D}_k$, as given in Proposition 4.1, and restricted to the case $k=2$:

$$
\begin{aligned}
&\tilde{X}_1=\partial_x,\quad \tilde{X}_2=\partial_y,\quad \tilde{X}_3=\partial_z,\quad \tilde{X}_4=\partial_p+x\partial_y,\\
&\tilde{X}_5=x\partial_x-p\partial_p-2q\partial_q-3z\partial_z-3v\partial_v-2w\partial_w,\\
&\tilde{X}_6=y\partial_y+p\partial_p+q\partial_q+2z\partial_z+v\partial_v+w\partial_w,\\
&\tilde{X}_7=q\partial_x+(pq-z)\partial_y+\frac{1}{2}q^2\partial_p+\frac{1}{6}q^3\partial_z-v^2\partial_v-vw\partial_w.
\end{aligned}
$$

The second group of symmetries are the lifts of the seven symmetries $(Y_1,\dots,Y_7)$ of the flat distribution $\mathcal{D}_2$ given in (25).

$$
\begin{aligned}
\tilde{Y}_1&=\frac{1}{2}x^2\partial_y+x\partial_p+\partial_q+p\partial_z,\\
\tilde{Y}_2&=\frac{1}{6}x^3\partial_y+\frac{1}{2}x^2\partial_p+x\partial_q+(xp-y)\partial_z+\partial_v,\\
\tilde{Y}_3&=x^2\partial_x+3xy\partial_y+(3y+xp)\partial_p+(4p-qx)\partial_q+2p^2\partial_z\\
&\quad-(3vx-3w-3q)\partial_v-wx\partial_w,\\
\tilde{Y}_4&=(8p-6qx)\partial_x+(4p^2+6xz-6pqx)\partial_y+(6z-3q^2x)\partial_p-2q^2\partial_q-q^3x\partial_z\\
&\quad+(6v^2x-6vw-6vq)\partial_v+(6vwx-6w^2-4wq)\partial_w,\\
\tilde{Y}_5&=(16xp-12y-6qx^2)\partial_x+(6x^2z+8p^2x-6pqx^2)\partial_y+(12xz+4p^2-3q^2x^2)\partial_p\\
&\quad+(12z+4pq-4q^2x)\partial_q+(12pz-q^3x^2)\partial_z+(6v^2x^2-12vwx-12vqx\\
&\quad+12wq+6q^2)\partial_v+(6vwx^2-12w^2x-8wqx+4wp)\partial_w,
\end{aligned}
$$

$$\begin{aligned}\tilde{Y}_6 = {} & (24px^2 - 6qx^3 - 36xy)\partial_x + (12p^2x^2 + 6x^3z - 36y^2 - 6pqx^3)\partial_y \\ & + (12p^2x + 18x^2z - 3q^2x^3 - 36py)\partial_p + (12pqx - 6q^2x^2 - 24p^2 + 36xz)\partial_q \\ & + (36pxz - 8p^3 - q^3x^3 - 36yz)\partial_z + (6v^2x^3 - 18vwx^2 - 18vqx^2 + 36wqx \\ & + 18q^2x + 36vy - 36wp - 36pq + 36z)\partial_v \\ & + (6vwx^3 - 18w^2x^2 - 12wqx^2 + 12wpx)\partial_w, \\ \tilde{Y}_7 = {} & (12p^2 - 18qy)\partial_x + (8p^3 - 18pqy + 18yz)\partial_y + (18pz - 9q^2y)\partial_p \\ & + (18qz - 6pq^2)\partial_q + (18z^2 - 3q^3y)\partial_z + (18v^2y - 18vwp - 18vpq \\ & + 9wq^2 + 3q^3 + 18vz)\partial_w + (18vwy - 18w^2p - 12wpq + 18wz)\partial_w.\end{aligned}$$

The 14 symmetries $(\tilde{X}_1, \dots, \tilde{X}_7, \tilde{Y}_1, \dots, \tilde{Y}_7)$ form a Lie algebra isomorphic to the split real form of the exceptional Lie algebra $\mathfrak{g}_2$.

The third group of seven symmetries is given by

$$\begin{aligned}\hat{Z}_1 = {} & \frac{1}{w}\left(\partial_x + (p - wx)\partial_y + q\partial_p + v\partial_q + \left(qw + \frac{1}{2}q^2\right)\partial_z\right), \\ \hat{Z}_2 = {} & \frac{1}{w}\left(x\partial_x + \left(px - \frac{1}{2}wx^2\right)\partial_y + qx\partial_p + vx\partial_q + \left(wqx - wp + \frac{1}{2}q^2x\right)\partial_z\right) - \partial_w, \\ \hat{Z}_3 = {} & \frac{1}{w}\left(q\partial_x + (pq - pw)\partial_y + q^2\partial_p + vq\partial_q + \frac{1}{2}(q^2w + q^3)\partial_z\right) - v\partial_w, \\ \hat{Z}_4 = {} & \frac{1}{w}((2p - qx)\partial_x + (2p^2 - pqx - 3wy + wpx)\partial_y \\ & + q(2p - qx)\partial_p + v(2p - qx)\partial_q \\ & + (2pq^2 - q^3x - 6zw + 4pqw - q^2xw)\partial_z) + (vx - 3w - q)\partial_w, \\ \hat{Z}_5 = {} & \frac{1}{w}\Big((4px - qx^2 - 6y)\partial_x + (wpx^2 - pqx^2 - 3wxy + 4p^2x - 6py)\partial_y \\ & + (3wpx - q^2x^2 + 4pqx - 9wy - 6qy)\partial_p + (4vpx - vqx^2 + 3wqx - 6vy \\ & - 6wp)\partial_q + \frac{1}{2}(8wpqx - q^3x^2 - wq^2x^2 + 4pq^2x - 4wp^2 - 12wqy - 6q^2y)\partial_z\Big) \\ & + (3vx - 9w - 3q)\partial_v + (vx^2 - 3wx - 2qx + 2p)\partial_w, \\ \hat{Z}_6 = {} & \frac{1}{w}\Big((4pq - 3wqx - 12z)\partial_x + (3wxz - 3wpqx - 2wp^2 + 4p^2q - 12pz)\partial_y \\ & + \frac{1}{2}(8pq^2 - 3wq^2x - 18wz - 24qz)\partial_p + (4vpq - 3wq^2 - 12vz)\partial_q \\ & + \frac{1}{2}q(4wpq - wq^2x + 4pq^2 - 24wz - 12qz)\partial_z\Big) + (3v^2x + 9vw - 3vq)\partial_v\end{aligned}$$

$$+ (3vwx - 4vp + 9w^2 + 6wq + 2q^2)\partial_w,$$

$$\hat{Z}_7 = \frac{1}{w}\Big((3wqx^2 - 8pqx - 18wy - 16p^2 + 24qy + 24xz)\partial_x$$

$$+ (3wpqx^2 + 4wp^2x - 3wx^2z - 8p^2qx - 24wpy - 16p^3 + 24pqy + 24pxz)\partial_y$$

$$+ \frac{1}{2}(3wq^2x^2 - 16pq^2x - 36wp^2 + 36wxz - 32p^2q + 48q^2y + 48qxz)\partial_p$$

$$+ 2(3wq^2x - 4vpqx - 8vp^2 + 12vqy + 12vxz - 9wpq + 9wz)\partial_q$$

$$+ \frac{1}{2}(wq^3x^2 - 8wpq^2x - 8pq^3x - 32wp^2q + 24wq^2y + 48wqxz - 16p^2q^2$$

$$+ 24q^3y + 24q^2xz - 12wpz)\partial_z\Big) + (6vqx - 3v^2x^2 - 18vwx + 18wq - 3q^2)\partial_v$$

$$+ (8vpx - 3vwx^2 - 18w^2x - 12wqx - 4q^2x - 24vy + 30wp + 16pq - 24z)\partial_w.$$

These symmetries are *not* lifts of vector fields from M. In particular, they are not lifts of conformal symmetries of the conformal class $[g_{\mathcal{D}_2}]$ of the distribution.

4.3.6. *Geometry on the boundary* $\mathbb{P}(\mathcal{D}_2)$ *of* $\mathbb{P}([\mathcal{D}_2, \mathcal{D}_2])$

Next we observe what happens if we pass to the 6-dimensional boundary $\mathbb{P}(\mathcal{D}_2)$, which in our parametrization is given by $w = 0$. This is done by considering an inclusion

$$\iota : \mathbb{P}(\mathcal{D}_2) \hookrightarrow \mathbb{P}([\mathcal{D}_2, \mathcal{D}_2]), \quad \iota(x, y, p, q, z, v) = (x, y, p, q, z, v, 0),$$

of the boundary $\mathbb{P}(\mathcal{D}_2)$ into $\mathbb{P}([\mathcal{D}_2, \mathcal{D}_2])$ and by pulling the structural objects λ and Υ back to the boundary. Taking λ as in (26) with $k = 2$ gives

$$\lambda_0 = \iota^*\lambda = \mathrm{d}z - q\mathrm{d}p + v\mathrm{d}y + \left(\frac{1}{2}q^2 - vp\right)\mathrm{d}x.$$

This defines a 5-distribution $\mathcal{H}_0$ on $\mathbb{P}(\mathcal{D}_2)$ via $\mathcal{H}_0 = \ker(\lambda_0)$.

Let us recall the following definition: Given a contact distribution $\mathcal{D} = \ker(\lambda)$ defined in terms of a 1-form λ on a manifold M, a nonzero vector field X on M is called its *Cauchy characteristic* if $X \lrcorner \lambda = 0$ and $X \lrcorner \mathrm{d}\lambda = 0 \bmod \lambda$. A Cauchy characteristic is a particular infinitesimal symmetry of $\mathcal{D}$, since the definition implies $\mathcal{L}_X\lambda \wedge \lambda = 0$. It follows that, in general, distributions have no Cauchy characteristics. However, it turns out that the distribution $\mathcal{H}_0$ on $\mathbb{P}(\mathcal{D}_2)$ has a Cauchy characteristic

$$X = \partial_x + p\,\partial_y + q\,\partial_p + \frac{q^2}{2}\partial_z + v\partial_q.$$

This characteristic preserves Υ_0 also, we have $\mathcal{L}_{fX}\Upsilon_0 = 0$. To explicitly see this we adapt coordinates in such a way that five of them are invariant with respect to X and the sixth one is chosen so that it ramifies X. Explicitly we pass from coordinates (x, y, p, q, z, v) to coordinates $(x_0, x_1, x_2, x_3, x_4, x_5)$, where

$$x = x_5, \quad y = \frac{1}{6}x_1{x_5}^3 + \frac{6^{1/3}}{2}x_2{x_5}^2 + \frac{6^{2/3}}{2}x_3x_5 + x_4,$$
$$p = \frac{1}{2}x_1{x_5}^2 + 6^{1/3}x_2x_5 + \frac{6^{2/3}}{2}x_3, \quad q = x_1x_5 + 6^{1/3}x_2,$$
$$z = \frac{6^{2/3}}{2}{x_2}^2x_5 + \frac{6^{1/3}}{2}x_1x_2{x_5}^2 + \frac{1}{6}{x_1}^2{x_5}^3 + x_0, \quad v = x_1.$$

In these new coordinates

$$X = \partial_{x_5}, \quad \lambda_0 = \mathrm{d}x_0 - 3x_2\mathrm{d}x_3 + x_1\mathrm{d}x_4,$$

and the pullback of the conformal symmetric rank 4 tensor is represented by

$$\Upsilon_0 = -3\mathrm{d}x_2^2\mathrm{d}x_3^2 + 4\mathrm{d}x_1\mathrm{d}x_3^3 + 4\mathrm{d}x_2^3\mathrm{d}x_4 - 6\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 + \mathrm{d}x_1^2\mathrm{d}x_4^2.$$

This suggests to consider the 5-dimensional quotient $N = \mathbb{P}(\mathcal{D}_2)/X$ of $\mathbb{P}(\mathcal{D}_2)$ by the foliation given by X.

4.3.7. *Associated flat contact* G_2 *geometry in dimension* 5

The above formulae show that λ_0 and Υ_0 descend to N. Moreover, we have

$$\mathrm{d}\lambda_0 \wedge \mathrm{d}\lambda_0 \wedge \lambda_0 = 2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}x_0,$$

so λ_0 defines a contact distribution $\mathcal{H}_0 = \ker\lambda_0$ on N. We equip this contact distribution with the line $[\Upsilon_0]$ of symmetric rank 4 tensors on $\mathcal{H}_0$ spanned by Υ_0. Then one finds that the pointwise common stabilizer of $[\Upsilon_0]$ and $[(\mathrm{d}\lambda_0)_{|\mathcal{H}_0}]$ is isomorphic to $\mathrm{GL}(2,\mathbb{R})$ in the irreducible 4-dimensional representation. That means that $([\lambda_0], [\Upsilon_0])$ describes a G_2 contact structure on N as introduced in Sec. 2.7.

The algebra of infinitesimal symmetries of the structure $([\lambda_0], [\Upsilon_0])$ is then defined as the set of vector fields $X \in \mathfrak{X}(N)$ such that

$$(\mathcal{L}_X\lambda_0) \wedge \lambda_0 = 0, \quad \text{and} \quad \mathcal{L}_X\Upsilon_0 = f\Upsilon_0 + \lambda_0 \odot \tau,$$

where τ is a rank 3 tensor and f is a function on M. The algebra of infinitesimal symmetries of $([\lambda_0], [\Upsilon_0])$ is the exceptional Lie algebra $\mathfrak{g}_2$, as described in the following proposition.

Proposition 4.3. *All symmetries* X *of the structure* $([\lambda_0], [\Upsilon_0])$ *defined by the representatives*:

$$\lambda_0 = \mathrm{d}x_0 - 3x_2\mathrm{d}x_3 + x_1\mathrm{d}x_4 \tag{28}$$

and

$$\Upsilon_0 = -3\mathrm{d}x_2^2\mathrm{d}x_3^2 + 4\mathrm{d}x_1\mathrm{d}x_3^3 + 4\mathrm{d}x_2^3\mathrm{d}x_4 - 6\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 + \mathrm{d}x_1^2\mathrm{d}x_4^2 \tag{29}$$

are $\mathbb{R}$*-linear combinations of the following* 14 *vector fields*:

$$\begin{aligned}
X_1 &= (x_0^2 + 3x_3^3 x_1 - 3x_3 x_1 x_4 x_2 - x_4 x_2^3 - 3x_3^2 x_2^2)\partial_0 + (x_1^2 x_4 + x_0 x_1 + x_2^3)\partial_1 \\
&\quad + (2x_3 x_2^2 + x_2 x_1 x_4 + x_0 x_2 - x_1 x_3^2)\partial_2 + (x_0 x_3 + x_2 x_3^2 + x_4 x_2^2)\partial_3 \\
&\quad + (x_0 x_4 + 3x_3 x_2 x_4 - x_3^3)\partial_4, \\
X_2 &= -(x_0 x_4 - 2x_3^3)\partial_0 + (x_1 x_4 + x_0)\partial_1 - x_3^2\partial_2 - x_3 x_4 \partial_3 - x_4^2 \partial_4, \\
X_3 &= -\left(\frac{1}{2}x_3 x_1 x_4 + \frac{1}{2}x_4 x_2^2 + x_2 x_3^2\right)\partial_0 + \frac{1}{2}x_2^2\partial_1 + \left(\frac{2}{3}x_3 x_2 + \frac{1}{6}x_1 x_4 + \frac{1}{6}x_0\right)\partial_2 \\
&\quad + \left(\frac{1}{6}x_3^2 + \frac{1}{3}x_2 x_4\right)\partial_3 + \frac{1}{2}x_3 x_4 \partial_4, \\
X_4 &= -(x_2 x_4 + x_3^2)\partial_0 + x_2\partial_1 + \frac{2}{3}x_3\partial_2 + \frac{1}{3}x_4\partial_3, \\
X_5 &= -x_4\partial_0 + \partial_1, \\
X_6 &= (x_0 x_2 - 2x_1 x_3^2)\partial_0 + x_1 x_2 \partial_1 + \left(\frac{1}{3}x_2^2 + \frac{2}{3}x_3 x_1\right)\partial_2 \\
&\quad + \left(\frac{1}{3}x_3 x_2 - \frac{1}{3}x_0\right)\partial_3 + x_3^2\partial_4, \\
X_7 &= x_0\partial_0 + x_1\partial_1 + \frac{2}{3}x_2\partial_2 + \frac{1}{3}x_3\partial_3, \\
X_8 &= -\left(\frac{3}{2}x_3 x_1 x_2 + \frac{1}{2}x_2^3\right)\partial_0 + \frac{1}{2}x_1^2\partial_1 + \frac{1}{2}x_1 x_2 \partial_2 \\
&\quad + \frac{1}{2}x_2^2\partial_3 + \left(\frac{3}{2}x_3 x_2 + \left(\frac{1}{2}x_0\right)\right)\partial_4, \\
X_9 &= -(x_3 x_1 + x_2^2)\partial_0 + \frac{1}{3}x_1\partial_2 + \frac{2}{3}x_2\partial_3 + x_3\partial_4, \\
X_{10} &= \partial_4, \\
X_{11} &= x_0\partial_0 + \frac{1}{3}x_2\partial_2 + \frac{2}{3}x_3\partial_3 + x_4\partial_4, \\
X_{12} &= -3x_2\partial_0 + \partial_3, \\
X_{13} &= \partial_2, \\
X_{14} &= \partial_0.
\end{aligned}$$

Here the symbols $\partial_\mu = \frac{\partial}{\partial x_\mu}$ *denote the partial derivatives with respect to the variables* x_μ, $\mu = 0, 1, 2, 3, 4$. *The Lie algebra generated by the* 14 *vector fields* X_A, $A = 1, 2, \ldots, 14$, *is isomorphic to the split real form of the exceptional simple Lie algebra* $\mathfrak{g}_2$, *and thus the* G_2 *contact structure* $([\lambda_0], [\Upsilon_0])$ *is flat.*

Remark 4.3. We remark that the previous discussion describes in local coordinates the well-known double fibration for G_2, see e.g. [3],

$$\begin{tikzcd} & \mathrm{G}_2/B \arrow[dl, "\pi"'] \arrow[dr, "\bar{\pi}"] & \\ \mathrm{G}_2/P & & \mathrm{G}_2/\bar{P}, \end{tikzcd} \tag{30}$$

where $B \subset \mathrm{G}_2$ is the Borel subgroup introduced in Sec. 2.3. The boundary $\mathbb{P}(\mathcal{D}_2)$ can be identified with G_2/B, the Cauchy characteristic X spans the vertical bundle for the projection $\mathrm{G}_2/B \to \mathrm{G}_2/\bar{P}$. What is new here is that the model geometry of type G_2/B is viewed as a natural compactification of the special twistorial Lie contact geometry on $\mathbb{P}([\mathcal{D}_2, \mathcal{D}_2])\backslash\mathbb{P}(\mathcal{D}_2)$ over the flat $(2,3,5)$-geometry.

5. $\mathbf{G_2}$-Reduced Lie Contact Structures

Here we show that the Lie contact structures on $\mathbb{P}([\mathcal{D}, \mathcal{D}])\backslash\mathbb{P}(\mathcal{D})$ associated with $(2,3,5)$ distributions $\mathcal{D}$ have holonomy reduced to $\mathrm{G}_2 \subset \mathrm{O}(4,3)$. We further study, more generally, Lie contact structures in dimension 7 whose holonomy is reduced to G_2. In particular, we prove Proposition 1.1 and Theorem 1.2.

5.1. *Normality of the induced Cartan connection*

We start this section with a technical result: we will prove that the curvature $\widetilde{K}$ of the induced Lie contact Cartan connection $\widetilde{\omega}$ satisfies the normality condition $\widetilde{\partial}^*\widetilde{K} = 0$. Note that we did not need this information to show that the twistor bundle of a $(2,3,5)$ distribution carries an induced Lie contact structure. However, the fact that $\widetilde{\omega}$ is the canonical normal Cartan connection will be of importance for further applications, in particular Proposition 1.1.

Given Theorem 4.1, proving normality of $\widetilde{\omega}$ is a straightforward task, although computationally involved. The following alternative proof uses methods from parabolic geometry, in particular Kostant's theorem [16] and [4, Corollary 3.2], which we will use to derive information about the full curvature of regular, normal parabolic geometries associated with $(2,3,5)$ distributions from information about their harmonic curvature space.

The Kostant codifferential can be written in terms of bases as follows: Let $X_1, \ldots, X_n \in \mathfrak{g}$ project to a basis for $\mathfrak{g}/\mathfrak{p}$ and let $Z_1, \ldots, Z_n \in \mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ the dual basis, then for any $\phi \in \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ and $X \in \mathfrak{g}$,

$$\partial^*\phi(X + \mathfrak{p}) = 2\sum_i [\phi(X_i + \mathfrak{p}, X + \mathfrak{p}), Z_i] + \sum_i \phi(X_i + \mathfrak{p}, [Z_i, X] + \mathfrak{p}),$$

see [8, Lemma 3.1.11].

Lemma 5.1. *Suppose* $(\mathcal{G} \to M, \omega)$ *is a regular and normal parabolic geometry of type* (G_2, P), *then the induced parabolic geometry* $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ *of type* $(\mathrm{O}(4,3), \widetilde{P})$ *is normal.*

Proof. Let $K : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ be the curvature function of ω and let $\widetilde{K} : \widetilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ be the curvature function of $\widetilde{\omega}$. By $\widetilde{P}$-equivariancy of $\widetilde{\partial}^* \widetilde{K}$ it suffices to prove that $\widetilde{\partial}^* \widetilde{K}(u) = 0$ for any $u \in \mathcal{G}$ (rather than $u \in \widetilde{\mathcal{G}}$) in order to show that the induced geometry is normal. Recall that, for $u \in \mathcal{G}$,

$$\widetilde{K}(u) = (\Lambda^2\varphi \otimes i')(K(u)),$$

where $i' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is the Lie algebra inclusion and $\varphi : (\mathfrak{g}/\mathfrak{p})^* \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$ is the dual map to the projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$.

Next let us recall some facts from the general theory of parabolic geometries, see [16, 4] for details. One can, as a G_0-representation, identify the harmonic curvature space $\ker(\partial^*)/\mathrm{im}(\partial^*)$ with the kernel of the so-called Kostant Laplacian $\ker(\Box) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. A lowest weight vector of $\ker(\Box)$ can be algorithmically determined using Kostant's theorem. Consider the grading (10) of $\mathfrak{g}$, then in our case the lowest weight vector is an element of the form

$$\phi_1 = Z_1 \wedge Z_4 \otimes A \in \mathfrak{g}_1 \wedge \mathfrak{g}_3 \otimes \mathfrak{g}_0.$$

Now, since regular, normal parabolic geometries of type (G_2, P) are torsion-free, [4, Corollary 3.2] implies that the curvature function K takes values in the P-module generated by successively raising this lowest weight vector via the action of $\mathfrak{p}_+$. Note that this implies, for instance, that $K(u)(X + \mathfrak{p}, Y + \mathfrak{p}) = 0$ whenever both X and Y are contained in $\mathfrak{g}^{-2}$.

Now to prove the lemma, pick an arbitrary map $\phi \in \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ contained in the P-module generated by raising the lowest weight vector in $\ker(\Box)$; in particular $\partial^*\phi = 0$. Let

$$\widetilde{\phi} = (\Lambda^2\varphi \otimes i')(\phi)$$

be the corresponding element in $\Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$. Choose elements $X_1, X_2 \in \mathfrak{g}_{-1}$, $X_3 \in \mathfrak{g}_{-2}$, $X_4, X_5 \in \mathfrak{g}_{-3}$ defining a basis for $\mathfrak{g}/\mathfrak{p}$, supplement them by $X_6, X_7 \in \mathfrak{g}_1$ to obtain a basis for $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$. Use the Killing form on $\tilde{\mathfrak{g}}$, which restricts to a multiple of the Killing form on $\mathfrak{g}$, to identify $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ and $\tilde{\mathfrak{p}}_+ \cong (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \cong (\mathfrak{g}/\mathfrak{q})^*$, and let $Z_1, \ldots, Z_5 \in \mathfrak{p}_+$ and $\widetilde{Z}_1, \ldots, \widetilde{Z}_7$ be the respective dual bases. By construction $\widetilde{\phi}$ vanishes upon insertion of elements of $\mathfrak{p}$, hence $\widetilde{\phi}(\cdot, X_i) = 0$ for $i = 6, 7$. Thus,

$$\widetilde{\partial}^*\widetilde{\phi}(X + \tilde{\mathfrak{p}}) = 2 \sum_{i=1,\ldots,5} [\widetilde{Z}_i, \widetilde{\phi}(X + \tilde{\mathfrak{p}}, X_i + \tilde{\mathfrak{p}})] - \sum_{i=1,\ldots,5} \widetilde{\phi}([\widetilde{Z}_i, X] + \tilde{\mathfrak{p}}, X_i + \tilde{\mathfrak{p}})$$

for any $X \in \mathfrak{g}$. Using that $\partial^*\phi = 0$ this can also be written as

$$\begin{aligned}\widetilde{\partial}^*\widetilde{\phi}(X + \tilde{\mathfrak{p}}) = 2 &\sum_{i=1,\ldots,5} [\widetilde{Z}_i - Z_i, \widetilde{\phi}(X + \tilde{\mathfrak{p}}, X_i + \tilde{\mathfrak{p}})] \\ &- \sum_{i=1,\ldots,5} \widetilde{\phi}([\widetilde{Z}_i - Z_i, X] + \tilde{\mathfrak{p}}, X_i + \tilde{\mathfrak{p}}). \qquad (31)\end{aligned}$$

Let us first show that the second term in the above expression vanishes. Note that $\tilde{\mathfrak{g}}$ splits into the direct sum of $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ and its orthogonal complement $\mathfrak{g}^\perp \subset \tilde{\mathfrak{g}}$ with respect to the Killing form, which can be identified as a $\mathfrak{g}$-representation with the 7-dimensional fundamental representation $\mathbb{V}$ of $\mathfrak{g}$. By construction, the differences $\widetilde{Z}_i - Z_i$ are contained in the orthogonal complement to $\mathfrak{g}$, i.e. in $\mathbb{V} = \mathfrak{g}^\perp$. Now $\mathbb{V}$ is $\mathfrak{g}$-invariant, hence $[\widetilde{Z}_i - Z_i, X] \subset \mathbb{V}$ for any $X \in \mathfrak{g}$. More precisely, $\tilde{Z}_i - Z_i \in \mathbb{V}_1$ for $i = 1, 2$, $\tilde{Z}_3 - Z_3 \in \mathbb{V}_2$, and $\tilde{Z}_i - Z_i = 0$ for $i = 4, 5$, where we use the grading from (10). Moreover, $\mathbb{V} = \bigoplus_{i=-2,\ldots,2} \mathbb{V}_i \subset \mathfrak{g}^{-2} + \tilde{\mathfrak{p}}$. Since $\phi(X + \mathfrak{p}, Y + \mathfrak{p}) = 0$ and hence $\widetilde{\phi}(X + \tilde{\mathfrak{p}}, Y + \tilde{\mathfrak{p}}) = 0$ for any $X, Y \in \mathfrak{g}^{-2}$, this implies that

$$\sum_{i=1,\ldots,5} \widetilde{\phi}([\widetilde{Z}_i - Z_i, X] + \tilde{\mathfrak{p}}, X_i + \tilde{\mathfrak{p}}) = 0.$$

Now for the first term in (31), consider the $\mathfrak{g}_0$-invariant decomposition of $\Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ according to homogeneity with respect to the grading (10) on $\mathfrak{g}$ (in the sense that an element $\phi \in \mathfrak{g}_i \wedge \mathfrak{g}_j \otimes \mathfrak{g}_k$ has homogeneity $i + j + k$). Since $[\mathfrak{g}_i, \mathbb{V}_j] \subset \mathbb{V}_{i+j}$ and $\mathbb{V}^3 = \bigoplus_{i \geq 3} \mathbb{V}_i = \{0\}$, one sees that

$$\sum_{i=1,\ldots,5} [\widetilde{Z}_i - Z_i, V] = 0 \quad \text{and} \quad \sum_{i=3,\ldots,5} [\widetilde{Z}_i - Z_i, W] = 0,$$

for any $V \in \mathfrak{g}^2 = \mathfrak{g}_2 \oplus \mathfrak{g}_3$ and $W \in \mathfrak{g}^1 = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. Keeping in mind also that $\widetilde{Z}_4 - Z_4 = \widetilde{Z}_5 - Z_5 = 0$ and that $\phi(X + \mathfrak{p}, Y + \mathfrak{p}) = 0$ if both X and Y are contained in $\mathfrak{g}^{-2}$, one concludes that it remains to inspect ϕ's contained in the P-module generated by the lowest weight vector ϕ_1 intersected with

$$(\mathfrak{g}_1 \wedge \mathfrak{g}_3 \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_2 \wedge \mathfrak{g}_3 \otimes \mathfrak{g}_0) \oplus (\mathfrak{g}_1 \wedge \mathfrak{g}_3 \otimes \mathfrak{g}_1)$$

(i.e. of homogeneity 4 or 5). Indeed, by Schur's Lemma and since G_0 includes into $\tilde{P}$, it suffices to compute $\widetilde{\partial}^* \widetilde{\phi}$ for one representative ϕ in each irreducible G_0-submodule of that space. One easily sees that there are only two such G_0-submodules: The lowest weight vector

$$\phi_1 = Z_1 \wedge Z_4 \otimes A$$

generates the first one, and raising it we obtain a generator of the second one of the form

$$\phi_2 = Z_3 \wedge Z_4 \otimes A + Z_1 \wedge Z_4 \otimes Z_1;$$

here $Z_1 \in \mathfrak{g}_1$, $Z_3 \in \mathfrak{g}_2$, $Z_4 \in \mathfrak{g}_3$, $A \in \mathfrak{g}_0$ and, since $\partial^* \phi_1 = \partial^* \phi_2 = 0$, $[Z_1, A] = [Z_3, A] = [Z_4, A] = 0$. Using that this implies that $[\tilde{Z}_1, A] = [\tilde{Z}_3, A] = [\tilde{Z}_4, A] = 0$ and the facts $\tilde{Z}_4 - Z_4 = 0$ and $[Z_1, \tilde{Z}_1] = 0$, which can be verified directly, we immediately conclude that the corresponding elements $\widetilde{\phi}_1 = \tilde{Z}_1 \wedge \tilde{Z}_4 \otimes A$ and $\widetilde{\phi}_2 = \tilde{Z}_3 \wedge \tilde{Z}_4 \otimes A + \tilde{Z}_1 \wedge \tilde{Z}_4 \otimes Z_1$ are contained in the kernel of $\widetilde{\partial}^*$. This completes the proof. □

5.2. *Holonomy in* $\mathbf{G_2}$ *and a parallel tractor* 3*-form*

Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) and let $\hat{\omega}$ be the canonical extension of ω to a principal connection on the extended G-principal bundle $\hat{\mathcal{G}} := \mathcal{G} \times_P G$. Assume that M is connected. The holonomy group of the Cartan geometry at a point $u \in \hat{\mathcal{G}}$ is then defined to be the *holonomy group*

$$\mathrm{Hol}_u(\omega) := \mathrm{Hol}_u(\hat{\omega}) \subset G$$

of the principal connection $\hat{\omega}$ at that point. Since different choices of base points u lead to conjugate subgroups within G, we will disregard the base point and speak of the holonomy $\mathrm{Hol}(\omega)$ of the Cartan connection ω (keeping in mind that it is well-defined only up to conjugacy in G). If $(\mathcal{G} \to M, \omega)$ is a normal, regular parabolic geometry encoding an underlying structure (e.g. a $(2, 3, 5)$ distribution or a Lie contact structure) then the holonomy of the underlying structure is defined to be the holonomy of the associated normal Cartan connection.

Holonomy reductions of Cartan connections are related to parallel sections of so-called tractor bundles. Given a G-representation $\mathbb{W}$, the principal connection $\hat{\omega} \in \Omega^1(\hat{\mathcal{G}}, \mathfrak{g})$ induces a linear connection ∇ on the associated bundle

$$\mathcal{W} := \mathcal{G} \times_P \mathbb{W} = \hat{\mathcal{G}} \times_G \mathbb{W}.$$

Vector bundles arising that way are called *tractor bundles* and the induced linear connections are called *tractor connections*. If the Cartan connection ω is normal, the induced tractor connection is said to be normal. By definition of $\mathcal{W}$ as an associated bundle, sections $s \in \Gamma(\mathcal{W})$ correspond to smooth equivariant maps $f_s : \hat{\mathcal{G}} \to \mathbb{W}$. A section s is parallel for the tractor connection if and only if the corresponding function is constant along all horizontal curves $c : I \to \hat{\mathcal{G}},\ \hat{\omega}(c'(t)) = 0$. The holonomy group $\mathrm{Hol}(\omega)$ is then contained in the pointwise stabilizer of the parallel section s.

Now consider a Lie contact structure of signature $(2, 1)$ on a manifold $\widetilde{M}$ with associated regular, normal parabolic geometry of type $(\mathrm{O}(4, 3), \widetilde{P})$. Let $\mathbb{V}$ be the standard representation for $\mathrm{O}(4, 3)$ and $\mathcal{T}$ the associated tractor bundle with its normal tractor connection. The constant map f_H from the Cartan bundle onto the (unique up to constants) $\mathrm{O}(4, 3)$-invariant bilinear form defines a parallel section $\mathbf{H} \in \Gamma(S^2\mathcal{T}^*)$ called the *tractor metric*.

Next recall the following (well-known) characterization of the Lie group G_2. Consider a 7-dimensional vector space $\mathbb{V}$ with bilinear form H of signature $(4, 3)$. Let $\Phi \in \Lambda^3\mathbb{V}^*$ be a 3-form, then $(X, Y) \mapsto (X \lrcorner \Phi) \wedge (Y \lrcorner \Phi) \wedge \Phi$ defines a symmetric $\Lambda^7\mathbb{V}^*$-valued bilinear form on $\mathbb{V}$. If this bilinear form is non-degenerate, then it determines a volume form vol_Φ and thus a $\mathbb{R}$-valued symmetric bilinear form H_Φ. Now suppose that H_Φ is a multiple of H, i.e.

$$H_\Phi(X, Y)\mathrm{vol}_\Phi := (X \lrcorner \Phi) \wedge (Y \lrcorner \Phi) \wedge \Phi = \lambda H(X, Y)\mathrm{vol}_\Phi, \tag{32}$$

for a constant λ. Then the stabilizer of Φ is a copy of $\mathrm{G}_2 \subset \mathrm{SO}(4, 3) = \mathrm{SO}(H)$. We will call a 3-form satisfying the above condition *compatible*, and we will use the same terminology on the level of tractors.

As an immediate consequence of the construction and Lemma 5.1, we obtain the following.

Corollary 5.1. *The Lie contact structure on* $\widetilde{M}$ *induced by a* $(2,3,5)$ *distribution* $\mathcal{D}$ *admits a compatible tractor* 3*-form* $\Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ *which is parallel for the normal tractor connection, and the holonomy of the Lie contact structure reduces to* G_2*.*

Proof. Let $(\mathcal{G} \to M, \omega)$ be the regular, normal parabolic geometry of type (G_2, P) associated with the $(2,3,5)$ distribution $\mathcal{D}$. Let $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ be the induced parabolic geometry of type $(\mathrm{O}(4,3), \widetilde{P})$ on the twistor bundle. Then, by construction, the principal connection $\hat{\widetilde{\omega}}$ on the extended bundle $\widetilde{\mathcal{G}} \times_{\tilde{P}} \mathrm{O}(4,3)$ reduces to the G_2-principal bundle connection $\hat{\omega}$ on $\mathcal{G} \times_P \mathrm{G}_2$.

Now let $\Phi \in \Lambda^3\mathbb{V}^*$ be a defining 3-form for $\mathrm{G}_2 \subset \mathrm{O}(4,3)$. Then the constant G_2-equivariant map $f_\Phi : \hat{\mathcal{G}} \to \Lambda^3\mathbb{V}^*$ onto Φ defines a section $\Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ of the Lie contact tractor bundle, which is compatible with $\mathbf{H}$. Since $\hat{\widetilde{\omega}}$ is the extension of the G_2-principal connection $\hat{\omega}$, Φ is parallel for the tractor connection induced by $\hat{\widetilde{\omega}}$, and by Lemma 5.1 this is the normal tractor connection on $\Lambda^3\mathcal{T}^*$. Moreover,

$$\mathrm{Hol}(\widetilde{\omega}) = \mathrm{Hol}(\hat{\widetilde{\omega}}) \subset \mathrm{G}_2 \subset \mathrm{O}(4,3)$$

and, again by normality of $\widetilde{\omega}$, this is the holonomy of the underlying Lie contact structure. $\square$

In particular, we have proven Proposition 1.1.

5.3. *A curved orbit decomposition*

Next we consider the more general situation of a Lie contact structure of signature $(2,1)$ together with a tractor 3-form $\Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ that is compatible in the sense of (32) and parallel for the normal tractor connection. Then the pointwise stabilizer of Φ is G_2 and the holonomy of the Lie contact structure is reduced, $\mathrm{Hol}(\widetilde{\omega}) \subset \mathrm{G}_2$. In order to formulate the geometric implications of this set-up, we will apply the curved orbit decomposition theorem discussed below.

Let $(\mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) and let $s \in \Gamma(\mathcal{W})$ be a parallel section of some tractor bundle $\mathcal{W}$ with corresponding G-equivariant function $f_s : \hat{\mathcal{G}} \to \mathbb{W}$. Assuming that M is connected, the image $f_s(\hat{\mathcal{G}})$ is a G-orbit $\mathcal{O} \subset \mathbb{W}$. In [6], the following pointwise invariant of s is introduced: the image $f_s(\mathcal{G}_x) \subset \mathcal{O}$ of a fiber is a P-orbit called the P-type of x with respect to s. The manifold M then decomposes according to the P-type of points into a disjoint union of *curved orbits* M_i,

$$M = \bigsqcup_{i \in P \backslash \mathcal{O}} M_i,$$

where $P\backslash\mathcal{O}$ denotes the set of P-orbits of the G-orbit $\mathcal{O}$. Fix an element in $\mathcal{O}$ and let $H \subset G$ be its stabilizer. Then the set of P-orbits of $\mathcal{O} \cong G/H$ is in bijective correspondence with the set of H-orbits of G/P via $PgH \mapsto Hg^{-1}P$. In particular, the set of curved orbits can be parametrized by H-orbits of G/P. Now suppose that M_i is a nonempty curved orbit and let α_i be a representative of the corresponding H-orbit $H \cdot \alpha_i \subset G/P$. Then it is shown in [6] that:

- for any $x \in M_i$ there are neighborhoods $U \subset M$ of x and $V \subset G/P$ of α_i and a diffeomorphism $\psi : U \to V$ such that $\psi(U \cap M_i) = V \cap (H \cdot \alpha_i)$.
- M_i carries an induced Cartan geometry $(\mathcal{G}_i \to M_i, \omega_i)$ of the same type as the corresponding H-orbit in G/P. The Cartan bundle can be realized as a subbundle $\mathcal{G}_i \subset \mathcal{G}|_{M_i}$ and the Cartan connection ω_i is the pullback of ω with respect to the corresponding inclusion.

In the following we apply this result in the case of interest for us, i.e. when the Cartan geometry is of type $(\mathrm{O}(4,3), \widetilde{P})$, the section $s = \Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ is a parallel compatible tractor 3-form and the stabilizer $H = \mathrm{G}_2$. As before, $\widetilde{P} \subset \mathrm{O}(4,3)$ denotes the Lie contact parabolic, $P \subset \mathrm{G}_2$ the $(2,3,5)$ parabolic and $\bar{P} \subset \mathrm{G}_2$ the G_2 contact parabolic as introduced in Sec. 2.3.

Theorem 5.1. *Suppose $\widetilde{M}$ is a 7-manifold endowed with a Lie contact structure of signature $(2,1)$ and let $(\widetilde{\mathcal{G}} \to \widetilde{M}, \widetilde{\omega})$ be the corresponding regular, normal parabolic geometry. Let $\Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ be a parallel compatible tractor 3-form that defines a holonomy reduction to G_2.*

Then the corresponding curved orbit decomposition is of the form

$$\widetilde{M} = \widetilde{M}^o \cup \widetilde{M}',$$

where $\widetilde{M}^o$ is open and $\widetilde{M}'$ (if nonempty) is a 5-dimensional submanifold of $\widetilde{M}$.

(1) *If $\widetilde{M}'$ is nonempty, then it carries an induced G_2 contact structure.*
(2) *$\widetilde{M}^o$ carries an induced Cartan geometry $(\mathcal{G} \to \widetilde{M}^o, \omega^o)$ of type (G, Q). Suppose further that the curvature of this Cartan geometry satisfies $K^o(u)(X, Y) = 0$ for all $X \in \mathfrak{p}$ and $Y \in \mathfrak{g}$. Then the rank 2 bundle $\mathcal{V} \subset T\widetilde{M}^o$ corresponding to $\mathfrak{p}/\mathfrak{q}$ is integrable and around each point $x \in \widetilde{M}^o$ we can form a local 5-dimensional leaf space which inherits a $(2,3,5)$ distribution.*

Proof. The first statement is an immediate consequence of Proposition 2.1, which describes the G_2-orbit decomposition of $\mathrm{O}(4,3)/\widetilde{P}$, and the curved orbit decomposition theorem. Combining these results shows that the manifold $\widetilde{M}$ decomposes into an open submanifold $\widetilde{M}^o$ and a complement $\widetilde{M}'$, which is either empty or a 5-dimensional submanifold. $\widetilde{M}'$ carries an induced Cartan geometry $(\mathcal{G}' \to \widetilde{M}', \omega')$ of type $(G, \bar{P})$ and $\widetilde{M}^o$ carries an induced Cartan geometry $(\mathcal{G}^o \to \widetilde{M}^o, \omega^o)$ of type (G, Q). These can be realized as subbundles in $\widetilde{\mathcal{G}}|_{\widetilde{M}'}$ and $\widetilde{\mathcal{G}}|_{\widetilde{M}^o}$, respectively, and the Cartan connections, ω' and ω^o, and their curvatures, K' and K^o, are the pullbacks of the Cartan connection $\widetilde{\omega}$ and curvature $\widetilde{K}$ with respect to the inclusions.

Using this, we next show that the induced Cartan connection on $\widetilde{M}'$ is regular. First, the curvature $\widetilde{K}$ of the regular, normal Lie contact Cartan connection takes values in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}^{-1}$, which follows from the structure of the harmonic curvature $\widetilde{K}_H$ and an application of the Bianchi identity. This implies that the curvature K' of the reduced connection takes values in $\Lambda^2(\mathfrak{g}/\bar{\mathfrak{p}})^* \otimes (\tilde{\mathfrak{g}}^{-1} \cap \mathfrak{g})$. Now $\tilde{\mathfrak{g}}^{-1} \cap \mathfrak{g}$ coincides with the filtration component $\bar{\mathfrak{g}}^{-1}$ for the G_2' contact grading (11), and this implies that K' is of homogeneity ≥ 1, i.e. the Cartan connection ω' is regular. In particular, $\widetilde{M}'$ carries an induced G_2' contact structure.

Next we investigate the Cartan geometry of type (G, Q) on $\widetilde{M}^o$. Via the Cartan connection, the Q-submodule $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ determines a distinguished rank 2 subbundle $\mathcal{V}$ in $T\widetilde{M}^o$. Now suppose that the curvature function satisfies $K^o(u)(X, \cdot) = 0$ for all $X \in \mathfrak{p}$. It is proven in [4], see also [8, Theorem 1.5.14], that this implies that the subbundle $\mathcal{V}$ is integrable, and locally around each point one can form a corresponding leaf space M, which carries an induced Cartan geometry of type (G_2, P).

To see that the Cartan geometry of type (G_2, P) determines a $(2,3,5)$ distribution on the leaf space M, it remains to see that the Cartan connection is regular. Arguing as before shows that the Q-equivariant curvature function K^o takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{q})^* \otimes (\tilde{\mathfrak{g}}^{-1} \cap \mathfrak{g})$. Looking at the gradings (8) and (10) shows that $\tilde{\mathfrak{g}}^{-1} \cap \mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{p}$. Note that this space is a Q-module, but not a P-module. The condition $K^o(u)(X, \cdot) = 0$ for all $X \in \mathfrak{p}/\mathfrak{q}$ in particular implies that, locally, $\mathcal{G} \to M$ is a P-principal bundle and the curvature function K^o is P-equivariant. Now suppose that for some $u \in \mathcal{G}$ and $X, Y \in \mathfrak{g}$, $K^o(u)(X, Y)$ has a nontrivial component in $\mathfrak{g}_{-3}$. Then we can find some $g \in \exp(\mathfrak{g}_1) \subset P$ such that $K^o(u \cdot g^{-1})(\mathrm{Ad}(g) \cdot X, \mathrm{Ad}(g) \cdot Y) = \mathrm{Ad}(g) \cdot K^o(u)(X, Y)$ has a nontrivial component in $\mathfrak{g}_{-2}$. But this is a contradiction to the assumptions on the values of K^o. Hence under the additional curvature condition, the curvature function takes indeed values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}^{-1}$, which implies that the (G_2, P) geometry on M is regular. □

We conclude with a number of remarks and open questions.

Remark 5.1. One can show that the resulting Cartan connection $\omega \in \Omega^1(\mathcal{G} \to M, \mathfrak{g}_2)$ is indeed the normal Cartan connection associated with the induced distribution on the local leaf space. However, this requires more information on the curvature of the regular, normal Lie contact Cartan connection, and will be discussed elsewhere.

Remark 5.2. The decomposition into curved orbits can also be described using the so-called normal BGG solution determined by the parallel tractor 3-form Φ.

Recall (see Sec. 2.3) that the parabolic subgroup $\widetilde{P}$ preserves a filtration $\widetilde{\mathbb{V}}^{-1} \supset \widetilde{\mathbb{V}}^0 \supset \widetilde{\mathbb{V}}^1$ of the standard representation, where $\widetilde{\mathbb{V}}^1 = \mathbb{E}$, $\widetilde{\mathbb{V}}^0 = \mathbb{E}^\perp$ and $\widetilde{\mathbb{V}}^{-1} = \mathbb{V}$. Correspondingly, the standard tractor bundle is filtered

$$\mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1,$$

where $\mathcal{T}^1 = E$, $\mathcal{T}^0/\mathcal{T}^1 \cong F$ and $\mathcal{T}/\mathcal{T}^1 \cong E^*$. There is an induced filtration of $\Lambda^3\mathcal{T}^*$, and a natural projection onto the quotient by the largest proper subbundle in this filtration,

$$\Pi : \Lambda^3\mathcal{T}^* \to \Lambda^3\mathcal{T}^*/(\Lambda^3\mathcal{T}^*)^0 \cong \Lambda^2 E^* \otimes F^* \cong \Lambda^2 E^* \otimes F.$$

The image of a tractor $\Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ under this projection defines an element

$$\phi \in \Gamma(\Lambda^2 E^* \otimes F),$$

i.e. a weighted section of F. By the general theory of parabolic geometries, if Φ is a *parallel* tractor 3-form, then the underlying section $\phi \in \Gamma(\Lambda^2 E^* \otimes F)$ is contained in the kernel of a first-order linear differential operator, called *first BGG operator for* $\Lambda^3\mathcal{T}^*$. Solutions of the corresponding overdetermined system of PDEs that are obtained in that way are called normal *BGG* solutions. See [9, 6] for more details.

Now suppose that $\Phi \in \Gamma(\Lambda^3\mathcal{T}^*)$ is a parallel compatible tractor 3-form. Recall (see Sec. 2.3 and Proposition 2.1) that inserting a totally null 2-plane $\mathbb{E}$ into a defining 3-form for G_2 gives either zero or a null line $\ell \in \mathbb{E}^\perp$ transversal to $\mathbb{E}$. Hence, for a parallel compatible tractor 3-form Φ, at any point $x \in M$ either $\phi_x = 0$ or ϕ_x defines a null line in F with respect to the bundle metric b. The decomposition of $\widetilde{M}$ into $\widetilde{P}$-types of Φ corresponds to the decomposition into the zero locus M' of ϕ and the open subset M^o where ϕ is nonvanishing. On M^o, via the isomorphism $\mathcal{H} = E^* \otimes F$, the filtration $\phi \subset \phi^\perp \subset F$ determines a distinguished filtration of a rank 2 subbundle contained in a rank 4 subbundle contained in the contact subbundle

$$\mathcal{V}_\phi \subset \mathcal{D}_\phi \subset \mathcal{H}.$$

Looking at the explicit matrices (8) and (10), it can be seen that via the isomorphisms $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{g}}^{-1}/\tilde{\mathfrak{p}} \cong \mathbb{E}^* \otimes \mathbb{E}^\perp/\mathbb{E}$, the subspace $\mathfrak{p}/\mathfrak{q}$ corresponds to $\mathbb{E}^* \otimes \ell$, where ℓ now denotes the projection of $\mathbb{E} \lrcorner\, \Phi$ to $\mathbb{E}^\perp/\mathbb{E}$, and $\mathfrak{g}^{-1}/\mathfrak{q}$ corresponds to $\mathbb{E}^* \otimes \ell^\perp$. Hence, for Lie contact structures coming from $(2, 3, 5)$ distributions via the twistor construction, $\mathcal{V}_\phi$ is the vertical bundle for the projection $\widetilde{M} \to M$, and $\mathcal{D}_\phi = \widetilde{\mathcal{D}}$ projects to the downstairs $(2, 3, 5)$ distribution.

Remark 5.3. The twistorial construction of Lie contact structures from $(2, 3, 5)$ distributions provides many non-flat examples of holonomy reductions to G_2. However, by construction, in these cases the corresponding parallel tractor 3-form Φ has only one $\widetilde{P}$-type and the underlying *BGG* solution ϕ is nowhere vanishing. It would be interesting to find examples of non-flat Lie contact structures admitting ϕ's as above with nonempty zero sets that carry induced G_2 contact structures and thus to provide non-flat examples of curved orbit decompositions as in Theorem 5.1 for which the 5-dimensional orbit is nonempty.

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