Examples of explicit Fefferman-Graham metrics

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Plan

1. Ambient metrics and distributions
   - Fefferman-Graham construction
   - Conformal structures and Cartan's paper
2. The main theorem
   - An ansatz
   - The theorem
3. Examples of explicit ambient metrics
   - Solutions analytic in ρ
   - Nonanalytic in ρ solutions
   - Poincaré-Einstein picture

Plan

Conformal structure

A *conformal structure* $(M^n, [g])$ on an $n = n_+ + n_-$ dimensional manifold M^n is an equivalence $[g]$ class of (n_+, n_-)-signature metrics on M^n, such that two metrics g and $\hat{g}$ are in the same class $[g]$ if and only if there exists a function ϕ on M^n, such that

$$\hat{g} = \mathrm{e}^{2\phi} g.$$

Ambient metric

- Consider a conformal structure $(M^n, [g])$ as defined on the previous slide.
- An *ambient space* $\tilde{M}$ for $(M^n, [g])$ is locally a product

$$\tilde{M} =]0, +\infty[\times M^n \times]-\epsilon, \epsilon[, \qquad \epsilon > 0,$$

with respective coordinates (t, x^i, ρ), and the *ambient metric* $\tilde{g}$ for $(M^n, [g])$ is an $(n_+ + 1, n_- + 1)$-signature *Ricci flat* metric on $\tilde{M}$ given by:

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2 g(x^i, \rho)$$

such that

$$g(x^i, \rho)_{|\rho=0} = g(x^i),$$

for some metric $g = g(x^i)$ from the conformal structure $[g]$.

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Explicit ambient metrics?

Assuming that the metric $\tilde{g}$ admits a power series expansion with integer powers in ρ one can see that:

- If $[g]$ contains the *flat* metric g_0 than

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- If $[g]$ contains an *Einstein* metric g_0, $Ric(g_0) = \Lambda g_0$, then

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An old nontrivial example in five dimensions

Consider a function

$$f = f_0 + f_1 p + f_2 p^2 + f_3 p^3 + f_4 p^4 + f_5 p^5 + f_6 p^6, \qquad f_0, f_1, \ldots f_6 = \mathrm{const},$$

and a 5-manifold M^5 parametrized by (x, y, p, q, z), and equipped with a conformal structure $[g]$ represented by

$$g = 2\omega^1\omega^5 - 2\omega^2\omega^4 + (\omega^3)^2,$$

$$\omega^1 = \mathrm{d}y - p\mathrm{d}x, \quad \omega^2 = \mathrm{d}z - (q^2 + f + bz)\mathrm{d}x - \tfrac{\sqrt{2}}{2}q\omega^3, \quad \omega^3 = 2\sqrt{2}(\mathrm{d}p - q\mathrm{d}x), \quad \omega^4 = 3\mathrm{d}x,$$

$$\omega^5 = \tfrac{\sqrt{2}b}{2}\omega^3 - 6\mathrm{d}q + 3(2bq + f_p)\mathrm{d}x + \tfrac{1}{10}(9f_{pp} + 4b^2)\omega^1,$$

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with

$$A = \tfrac{27}{8}f_6\rho^2 - \tfrac{9}{5}(f_4 + 5pf_5 + 15p^2 f_6)\rho,$$

$$B = \tfrac{1}{16}(f_5 + 6pf_6)\rho^2 - \tfrac{3}{20}(f_3 + 4pf_4 + 10p^2 f_5 + 20p^3 f_6)\rho,$$

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$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\Big(g + A\cdot(\omega^1)^2 + 2B\cdot\omega^1\omega^4 + C\cdot(\omega^4)^2\Big),$$

with

$$A = \tfrac{27}{8} f_6 \rho^2 - \tfrac{9}{5}(f_4 + 5pf_5 + 15p^2 f_6)\rho,$$

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An old nontrivial example in five dimensions

Consider a function

$$f = f_0 + f_1 p + f_2 p^2 + f_3 p^3 + f_4 p^4 + f_5 p^5 + f_6 p^6, \qquad f_0, f_1, \dots f_6 = \mathrm{const},$$

and a 5-manifold M^5 parametrized by (x, y, p, q, z), and equipped with a conformal structure $[g]$ represented by

$$g = 2\omega^1\omega^5 - 2\omega^2\omega^4 + (\omega^3)^2,$$

$$\omega^1 = \mathrm{d}y - p\mathrm{d}x, \quad \omega^2 = \mathrm{d}z - (q^2 + f + bz)\mathrm{d}x - \tfrac{\sqrt{2}}{2}q\omega^3, \quad \omega^3 = 2\sqrt{2}(\mathrm{d}p - q\mathrm{d}x), \quad \omega^4 = 3\mathrm{d}x,$$

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Plan

Distributions associated with $z' = F(x, y, y', y'', z)$

Associated with a differential equation

$$z' = F(x, y, p, q, z),$$

where $p = y'$, $q = y''$, there is a 5-manifold M^5 parametrized by (x, y, p, q, z), and a distribution

$$\mathcal{D} = \mathrm{Span}\Big(\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z\Big),$$

whose differential invariants, when $F_{qq} \neq 0$, are in one-to-one correspondance with *conformal* invariants of a certain conformal class $[g_{\mathcal{D}}]$ of metrics of signature $(3, 2)$ on M^5.

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The conformal class for $F = q^2 + f(x,p) + bz$

If $F = q^2 + f(x,p) + bz$, where b is a real constant, the conformal class may be represented by a metric $g_{\mathcal{D}_f}$ in a relatively simple form:

$$g_{\mathcal{D}_f} = 8\Big(\mathrm{d}p - q\mathrm{d}x\Big)^2 - 6\Big(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\Big)\mathrm{d}x - \\ 2\Big(\mathrm{d}y - p\mathrm{d}x\Big)\Big(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\Big).$$

QUESTION: Can we find explicit formulae for Fefferman-Graham ambient metrics for the conformal class $(M^5, [g_{\mathcal{D}_f}])$?

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An ansatz

- **Observation:** The Schouten tensor for the class $[g_{\mathcal{D}_f}]$ has the form: $\mathsf{P} = \alpha \cdot (\omega^1)^2 + 2\beta \cdot \omega^1\omega^4 + \gamma \cdot (\omega^4)^2$, with $\omega^1 = \mathrm{d}y - p\mathrm{d}x$ and $\omega^4 = 3\mathrm{d}x$, and α, β, γ functions depending on f and its derivatives.
- Idea: Make an *ansatz* for the ambient metric $\tilde{g}_{\mathcal{D}_f}$ in which $g_{\mathcal{D}_f}(x^i, \rho)$ assumes a similar form.
- Explicitly, make the following ansatz for $\tilde{g}_{\mathcal{D}_f}$:

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Theorem (Ian Anderson + PN)

The metric $\tilde{g}_{\mathcal{D}_f}$, as above, is an ambient metric for the conformal class $(M^5, [g_{\mathcal{D}_f}])$, if and only if the unknown functions $A = A(x, p, \rho)$, $B = B(x, p, \rho)$ and $C = C(x, p, \rho)$, satisfy the initial conditions $A_{|\rho=0} \equiv 0$, $B_{|\rho=0} \equiv 0$, $C_{|\rho=0} \equiv 0$ and the following system of PDEs:

$$LA = \frac{9}{40} f_{pppp}$$

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with the *linear* operator L given by

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Power series expansion in ρ

One can solve the above equations, assuming power series expansion in ρ:

$$A=\sum_{k=1}^{\infty}a_k(x,p)\rho^k,\quad B=\sum_{k=1}^{\infty}b_k(x,p)\rho^k,\quad C=\sum_{k=1}^{\infty}c_k(x,p)\rho^k,$$

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$$A=\sum_{k=1}^{\infty}\frac{3}{5}\cdot\frac{(2k-1)(2k-3)}{2^{2k}(2k)!}\cdot\frac{\partial^{(2k+2)}f}{\partial p^{(2k+2)}}\cdot\rho^k,$$

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Solutions being ploynomials in ρ

An important feature of the analytic solutions is that their coefficients behave as:

$$a_k(x,p) \sim \frac{\partial^{(2k+2)} f}{\partial p^{(2k+2)}}, \quad b_k(x,p) \sim \frac{\partial^{(2k+1)} f}{\partial p^{(2k+1)}}, \quad c_k(x,p) \sim \frac{\partial^{(2k)} f}{\partial p^{(2k)}}.$$

Thus, if we want to have an example of an ambient metric that does not involve powers in ρ higher than k_0 we need to have $\frac{\partial^{(2k_0+2)} f}{\partial p^{(2k_0+2)}} \equiv 0$, i.e. the function $f = f(x,p)$ defining the distribution must be a polynomial of order no higher than $2k_0 + 1$. Because of $c_3(x,p) \equiv 0$, this statement can be improved, if we want to have ambient metrics truncated at order $k_0 = 2$. Here f must be a polynomial of order no higher than $2k_0 + 2 = 6$, which is the case of examples of Leistner and PN.

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Polynomial solutions have G_2 holonomy

It is a matter of checking that the so obtained analytic in ρ Fefferman-Graham metrics generically have full G_2 holonomy. As an example we give a formula for a Fefferman-Graham full G_2 holonomy metric that truncates at order 4 in ρ:

$$f = f_0 + f_1 p + f_2 p^2 + f_3 p^3 + f_4 p^4 + f_5 p^5 + f_6 p^6 + f_7 p^7 + f_8 p^8 + f_9 p^9,$$

$$\omega^1 = \mathrm{d}y - p\mathrm{d}x, \quad \omega^2 = \mathrm{d}z - (q^2 + f + bz)\mathrm{d}x - \tfrac{\sqrt{2}}{2} q\omega^3, \quad \omega^3 = 2\sqrt{2}(\mathrm{d}p - q\mathrm{d}x), \quad \omega^4 = 3\mathrm{d}x,$$

$$\omega^5 = \tfrac{\sqrt{2}b}{2}\omega^3 - 6\mathrm{d}q + 3(2bq + f_p)\mathrm{d}x + \tfrac{1}{10}(9f_{pp} + 4b^2)\omega^1,$$

$$\tilde{g}_{\mathcal{D}_f} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\Big(2\omega^1\omega^5 - 2\omega^2\omega^4 + (\omega^3)^2 + A\cdot(\omega^1)^2 + 2B\cdot\omega^1\omega^4 + C\cdot(\omega^4)^2\Big),$$

$$A = \tfrac{63}{8}(f_8 + 9pf_9)\rho^3 + \tfrac{27}{8}(f_6 + 7pf_7 + 28p^2 f_8 + 84p^3 f_9)\rho^2 - \tfrac{9}{5}(f_4 + 5pf_5 + 15p^2 f_6 + 35p^3 f_7 + 70p^4 f_8 + 126p^5 f_9)\rho,$$

$$B = -\tfrac{63}{256} f_9\rho^4 - \tfrac{7}{64}(f_7 + 8pf_8 + 36p^2 f_9)\rho^3 + \tfrac{1}{16}(f_5 + 6pf_6 + 21p^2 f_7 + 56p^3 f_8 + 126p^4 f_9)\rho^2 - \tfrac{3}{20}(f_3 + 4pf_4 + 10p^2 f_5 + 20p^3 f_6 + 35p^4 f_7 + 56p^5 f_8 + 84p^6 f_9)\rho,$$

$$C = \tfrac{7}{1152}(f_8 + 9pf_9)\rho^4 + \tfrac{1}{360}(f_4 + 5pf_5 + 15p^2 f_6 + 35p^3 f_7 + 70p^4 f_8 + 126p^5 f_9)\rho^2 + \tfrac{1}{45}(2b^2 - f_2 - 3pf_3 - 6p^2 f_4 - 10p^3 f_5 - 15p^4 f_6 - 21p^5 f_7 - 28p^6 f_8 - 36p^7 f_9)\rho.$$

Here all $f_0, f_1, \ldots f_9$ are arbitrary functions of the variable x.

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Polynomial solutions have G_2 holonomy

It is a matter of checking that the so obtained analytic in ρ Fefferman-Graham metrics generically have full G_2 holonomy. As an example we give a formula for a Fefferman-Graham full G_2 holonomy metric that truncates at order 4 in ρ:

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Plan

Indicial exponents

To find *all*, and in particular nonanalytic in ρ, solutions to the system

$$LA = \frac{9}{40} f_{pppp}, \qquad LB = -\frac{1}{36} A_p + \frac{3}{40} f_{ppp}$$
$$LC = -\frac{1}{18} B_p + \frac{1}{324} A + \frac{1}{30} f_{pp} - \frac{2}{15} b^2,$$

we first observe that the two independent solutions to $L(\rho^k) = 0$ are ρ^0 and $\rho^{5/2}$. Thus, the most general solution to the above system can be obtained by the following series:

$$A = \sum_{k=1}^{\infty} a_k(x,p)\rho^k + \rho^{5/2} \sum_{k=0}^{\infty} \alpha_k(x,p)\rho^k,$$

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Holonomy questions

- If $\alpha_0 \equiv \beta_0 \equiv \gamma_0 \equiv 0$ and for a randomly chosen f the holonomy of the corresponding ambient metric is equal to G_2 (Graham-Willse result).
- What about the holonomy of FG metrics corresponding to the solutions with nontrivial $\rho^{5/2+k}$ terms?
- Problems:
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Solutions with $f \equiv 0$, $b = 0$

In case of a flat distribution, i.e. when $f \equiv 0$ and $b = 0$ the solutions are:

$$A = \rho^{5/2} \sum_{k=0}^{\infty} 60 \cdot \frac{(k+2)(k+1)}{2^{2k}(2k+5)!} \cdot \frac{\partial^{2k}\alpha_0}{\partial p^{2k}} \cdot \rho^k,$$

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and, as in the general case, they depend on three arbitrary functions $\alpha_0, \beta_0, \gamma_0$ of variables x and p.

As an illustration we discuss holonomy properties of the corresponding ambient metrics on a very simple example, in which we have made a particular choice of these 3 functionsons.

We believe that the discussed behaviour is a typical one.

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Ambient metrics for the flat equation $z' = (y'')^2$

- Take

$$\alpha_0 = \beta(x) + p\alpha(x), \quad \beta_0 = f_0(x) + pf_1(x) + 252cp^2\alpha(x),$$
$$\gamma_0 = f_3(x) + pf_4(x) + \tfrac{1}{81}p^2(2268f_2(x) - \beta(x) + 18f_1(x)),$$

with c a real constant.

- This gives the following solution:

$$A = \Big(252p\alpha(x) + \beta(x)\Big)\rho^{5/2}$$
$$B = (9c-1)\alpha(x)\rho^{7/2} + \Big(f_0(x) + pf_1(x) + 252cp^2\alpha(x)\Big)\rho^{5/2}$$
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corresponding to the following Fefferman-Graham *family* of metrics for the *flat* conformal structure:

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- The family depends on *seven* arbitrary functions $\alpha = \alpha(x), \beta = \beta(x), f_0 = f_0(x), f_1 = f_1(x), \dots, f_4 = f_4(x)$ and a real constant c.
- $\tilde{g}$ is an ambient metric for the flat conformal structure represented by a flat metric

$$g = 8(\mathrm{d}p - q\mathrm{d}x)^2 - 6(\mathrm{d}z - 2q\mathrm{d}p + q^2\mathrm{d}x)\mathrm{d}x - 12(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}q.$$

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Holonomy of the ambient metrics from this family

Holonomy properties of this family are quite interesting:

- In general these metrics have *full* $SO(4,3)$ holonomy!!!!
- Even if we put: $\beta(x) \equiv f_0(x) \equiv f_1(x) \equiv \cdots \equiv f_4(x) \equiv 0$, and $\alpha(x) \equiv 1$, the holonomy algebra behaves as this:
  - the curvature defines 6 independent components of the holonomy algebra
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- Interestingly these metrics include, as special cases, metrics with full G_2 holonomy, which can not be extended to anything larger!!!
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Plan

Passing from ρ to r such that $\rho = r^2$

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- On doing this we first assume that $r > 0$, and bring the metric $\tilde{g}$ to the form

$$\tilde{g} = 2\mathrm{d}t\mathrm{d}(r^2 t) + t^2(8(\mathrm{d}p - q\mathrm{d}x)^2 - 6(\mathrm{d}z - 2q\mathrm{d}p + q^2\mathrm{d}x)\mathrm{d}x - 12(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}q +$$
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$$\tilde{g}_{\mathcal{D}_f} = 2\mathrm{d}t\mathrm{d}(\rho t) + t^2\Big(g_{\mathcal{D}_f} + A\cdot(\omega^1)^2 + 2B\cdot\omega^1\omega^4 + C\cdot(\omega^4)^2\Big),$$

$$Ric(\tilde{g}) = 0,$$

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- The metric g_{PE} is Einstein,

$$Ric(g_{PE}) = \frac{5}{2} g_{PE},$$

if and only if the functions A, B, C satisfy

$$LA = \frac{9}{40} f_{pppp}, \qquad LB = -\frac{1}{36} A_p + \frac{3}{40} f_{ppp}$$
$$LC = -\frac{1}{18} B_p + \frac{1}{324} A + \frac{1}{30} f_{pp} - \frac{2}{15} b^2,$$

i.e. iff they correspond to the Fefferman-Graham (Ricci flat) metric $\tilde{g}_{\mathcal{D}_f}$.

- Note that in coordinate r the linear operator is

$$L = \frac{1}{2}\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{8}\frac{\partial^2}{\partial p^2},$$

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Nontrivial Poincare-Einstein metrics associated with flat conformal structure

For example: a Poincaré-Einstein metric corresponding to the Fefferman-Graham metric associated with the flat conformal structure discussed few slides ago is then given by:

$$\tilde{g} = r^{-2}\Big(-2\mathrm{d}r^2 + 8(\mathrm{d}p - q\mathrm{d}x)^2 - 6(\mathrm{d}z - 2q\mathrm{d}p + q^2\mathrm{d}x)\mathrm{d}x - 12(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}q +$$
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Nonanalytic PE metrics and flat structure

The most general one associated with $f \equiv 0$ and $b = 0$ is:

$$\tilde{g} = r^{-2}\Big(-2\mathrm{d}r^2 + 8(\mathrm{d}p - q\mathrm{d}x)^2 - 6(\mathrm{d}z - 2q\mathrm{d}p + q^2\mathrm{d}x)\mathrm{d}x - 12(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}q +$$
$$A(\mathrm{d}y - p\mathrm{d}x)^2 + 6B(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}x + 9C\mathrm{d}x^2\Big)$$

$$A = \rho^{5/2}\sum_{k=0}^{\infty} 60 \cdot \frac{(k+2)(k+1)}{2^{2k}(2k+5)!} \cdot \frac{\partial^{2k}\alpha_0}{\partial p^{2k}} \cdot \rho^k,$$
$$B = \rho^{5/2}\sum_{k=0}^{\infty} \frac{20}{3} \cdot \frac{(k+1)(k+2)}{2^{2k}(2k+5)!} \cdot \Big(9\frac{\partial^{2k}\beta_0}{\partial p^{2k}} - 2k\frac{\partial^{(2k-1)}\alpha_0}{\partial p^{(2k-1)}}\Big) \cdot \rho^k,$$
$$C = \rho^{5/2}\sum_{k=0}^{\infty} \frac{20}{27} \cdot \frac{(k+1)(k+2)}{2^{2k}(2k+5)!} \cdot \Big(81\frac{\partial^{2k}\gamma_0}{\partial p^{2k}} - 36k\frac{\partial^{(2k-1)}\beta_0}{\partial p^{(2k-1)}} + 2k(2k-1)\frac{\partial^{(2k-2)}\alpha_0}{\partial p^{(2k-2)}}\Big) \cdot \rho^k.$$

This can be truncated at *any* half-integer order of ρ, starting at $\rho^{5/2}$ by choosing $\alpha_0, \beta_0, \gamma_0$ as polynomials in p of an apropriate order.

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TT tensors

The explicit solutions for the PE metrics can be used to calculate the trace-free, divergence-free tensors for each of the conformal structure $[g_{\mathcal{D}_f}]$:

$$TT = \alpha_0(x,p)(\mathrm{d}y - p\mathrm{d}x)^2 + 6\beta_0(x,p)(\mathrm{d}y - p\mathrm{d}x)\mathrm{d}x + 9\gamma_0(x,p)\mathrm{d}x^2.$$

For all choices of the free functions α_0, β_0 and γ_0 they are trace-free and divergence-free in the metric

$$\begin{aligned} g_{\mathcal{D}_f} =& 8\left(\mathrm{d}p - q\mathrm{d}x\right)^2 - 6\left(\mathrm{d}z - 2q\mathrm{d}p + (q^2 - f - bz)\mathrm{d}x\right)\mathrm{d}x - \\ & 2\left(\mathrm{d}y - p\mathrm{d}x\right)\left(6\mathrm{d}q - 2b\mathrm{d}p - (\tfrac{2}{5}b^2 + \tfrac{9}{10}f_{pp})(\mathrm{d}y - p\mathrm{d}x) - (4bq + 3f_p)\mathrm{d}x\right), \end{aligned}$$

associated with $z' = (y'')^2 + f(x,y') + bz$.

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THANK YOU!