ON PENROSE'S 'BEFORE THE BIG BANG' IDEAS

C. DENSON HILL AND PAWEŁ NUROWSKI

Abstract. We point out that algebraically special Einstein fields with twisting rays exhibit the basic properties of conformal Universes considered recently by Roger Penrose.

MSC classification: 83C15, 83C20, 83C30, 83F05

According to Penrose [6], there might have been a *pre-big-bang era*, in which the Universe was equipped with a *conformal* Lorentzian structure, rather than with the *full* Lorentzian *metric* structure. This conformal structure was conformally flat, and pretty much the same as the conformal structure our Universe will have at the *very late stage* of its evolution.

Penrose argues that the observed *positive sign* of the cosmological constant, $\Lambda > 0$, forces our Universe to last forever. It will last long enough that all the massive particles will mannage to disintegrate, either by finding their antimatter counterparts with which to annihilate, or because of their finite half life. This will produce only *massless* particles, such as photons, which after all the matter has been disintegrated, will be the only content of the late Universe, except for massive *black holes*. These massive black holes will be the remnants of the galaxies, and perhaps, of very massive stars.

Since the Universe will last forever, and since it will be expanding, it will generally *cool down*, trying to reach the *absolute zero* temperature at its final state. Thus, there will be a time in its evolution such that the temperature of the Universe will be lower than the temperature of even the *most massive* black holes, which at this stage will hide the only *mass* of the Universe. This will create a thermodynamic instability forcing *all* the black holes to evaporate by radiating massless particles[1]. After these evaporations the dying Universe will be *totally* filled by *massless particles.*

Penrose argues that massless particles, whatever they are, have no way of defining clocks' ticks. This leads to the conclusion that the Universe at its late age, being filled with *only* massless observers, will ultimately lose the information about its conformal factor. Thus it will become similar to the Universe in the 'pre-big-bang' era that preceeded its creation. This late state of the Universe Penrose likes *to interprete* as the 'pre-big-bang' era of the *new* Universe. Let us adopt this point of view.

It is a well known fact that during the *big bang* (such as in the Friedmann-Lemaitre-Robertson-Walker models) the only metric *singularity* is in the *vanishing*

Date: October 23, 2007.
This research was supported by the KBN grant 1 P03B 07529.

[1]Penrose *assumes* that the black holes lose information about the properties of the masses hidden in them, and that in the process of evaporation only massless particles are being radiated.

of the conformal factor, leaving the conformally rescaled metric perfectly regular (actually *conformally flat*). Another well known fact is that the *field equations* for the *massless* particles are *conformally invariant*. This shows that the massless particles (the observers) of the dying conformal Universe, or as we interpret it now, of the 'pre-big-bang' era of the *new* conformal Universe, *will not feel* the big bang singularity. They will happily *pass through it* from the dying conformal Universe to another conformally flat manifold[2]. Penrose interprets this conformal object as an 'after big bang' conformal era of the new Universe. It will eventually acquire a *new conformal factor*, and possibly some distortion (meaning non-zero Weyl), promoting this conformal remnant of the old Universe to a *new* Lorentzian *metric* Universe.

Our *exact solutions of Einstein's equations*, discussed below, exhibit the main features of Penrose's 'before the big bang' argument. Although these solutions form a very *thin set* in all the possible Einstein metrics, and although they were obtained on *purely mathematical grounds* by the mere assumption that the corresponding spacetimes admit a *twisting congruence of null and shearfree geodesics*, it is a *remarkable coincidence* that the *pure mathematics* of Einstein's equations, imposed on such spacetimes, forces the solutions to fit to Penrose's ideas.

Let us discuss the mathematics of our solutions first. After we do it, we will indicate the parallels between our solutions and Penrose's Universes.

About twenty years ago we proved [2, 5] the following

Theorem 0.1. *Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian spacetime which satisfies the Einstein equations*

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + \Phi k_\mu k_\nu, \qquad \Lambda = \mathrm{const}, \tag{0.1}$$

with k_μ being a null vector tangent to a twisting *congruence of* null *and* shearfree geodesics*. Then its metric g* factorizes *as*

$$g = \Omega^{-2}\hat{g}, \qquad \Omega = \cos(\tfrac{r}{2}), \tag{0.2}$$

where $\hat{g}$ is periodic *in terms of the null cooordinate r along $k = \partial_r$.*

More explicitly, (see [2, 5] and, especially, [3] for details) we showed that if g satisfies (0.1), then $\mathcal{M}$ is a *circle bundle* $\mathbb{S}^1 \to \mathcal{M} \to M$ over a 3-dimensional strictly pseudoconvex CR manifold $(M, [(\lambda, \mu)])$, and that

$$\hat{g} = p^2[\mu\bar{\mu} + \lambda(\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda)], \tag{0.3}$$

with

$$W = ia\mathrm{e}^{-ir} + b, \quad H = \frac{n}{p^4}\mathrm{e}^{2ir} + \frac{\bar{n}}{p^4}\mathrm{e}^{-2ir} + q\mathrm{e}^{ir} + \bar{q}\mathrm{e}^{-ir} + h. \tag{0.4}$$

Here λ (real) and μ (complex) are 1-forms on $\mathcal{M}$ such that $k \lrcorner \lambda = k \lrcorner \mu = 0$, $k \lrcorner \mathrm{d}\lambda = k \lrcorner \mathrm{d}\mu = 0$, $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ and, as a result of the Einstein equations (0.1), the functions a, b, n, q (complex) and p, h (real) are *independent* of the null

[2]It should be noted that this passage is only possible *conformally*. In the *full* Lorentzian *metric* of the old Universe, the process of expansion will last *infinitely long*. But after conformal rescaling, this proces takes *finite time*, and enables us to speculate what will be after.

coordinate r: $a_r = b_r = n_r = q_r = p_r = h_r = 0$. A part of the Einstein equations in (0.1) can be explicitly integrated, obtaining:

$$\begin{aligned} a &= c + 2\partial \log p \\ b &= ic + 2i\partial \log p \\ q &= \frac{3n+\bar{n}}{p^4} + \frac{2}{3}\Lambda p^2 + \frac{2\partial p \bar{\partial} p - p(\partial\bar{\partial} p + \bar{\partial}\partial p)}{2p^2} - \frac{i}{2}\partial_0 \log p - \bar{\partial} c \\ h &= 3\frac{n+\bar{n}}{p^4} + 2\Lambda p^2 + \frac{2\partial p \bar{\partial} p - p(\partial\bar{\partial} p + \bar{\partial}\partial p)}{p^2} - \bar{\partial} c - \partial \bar{c}. \end{aligned} \tag{0.5}$$

Here the r-independent complex function c is defined via

$$\begin{aligned} \mathrm{d}\mu = 0, \qquad & \mathrm{d}\bar{\mu} = 0, \\ \mathrm{d}\lambda = \; & i\mu \wedge \bar{\mu} + (c\mu + \bar{c}\bar{\mu}) \wedge \lambda, \end{aligned} \tag{0.6}$$

and the operators $(\partial_0, \partial, \bar{\partial})$ are vector fields on M, which are algebraic dual to the coframe $(\lambda, \mu, \bar{\mu})$ on the CR manifold M. Note that the function c is *defined uniquely*, once a CR manifold $(M, [(\lambda, \mu)])$ has been chosen, and thus is considered as a *known* function in the process of solving (0.1).

The remaining Einstein equations in (0.1) reduce to a system of *two* PDEs on M, for the functions n and p, which are the only *unknowns*. These PDEs are:

$$\partial n + 3cn = 0, \tag{0.7}$$

$$[\; \partial\bar{\partial} + \bar{\partial}\partial + \bar{c}\partial + c\bar{\partial} + \tfrac{1}{2}c\bar{c} + \tfrac{3}{4}(\partial\bar{c} + \bar{\partial}c) \;]p = \frac{n+\bar{n}}{p^3} + \tfrac{2}{3}\Lambda p^3. \tag{0.8}$$

These are the only equations which need to be solved in order to make g satisfy (0.1). Once these equations are solved, the Einstein metric g has an energy momentum tensor describing the 'pure radiation' of a mixture of *massless* particles, moving with the speed of light along the null direction k, in a spacetime with cosmological constant Λ. The spacetime is algebraically special; the Weyl spin coefficients Ψ_0, Ψ_1, Ψ_2 being $\Psi_0 = \Psi_1 = 0$, $\Psi_2 = \frac{(1+e^{ir})^3}{2p^6}\, n$.

Note that at $r = \pm\pi$, where the *conformal factor* Ω for g becomes *zero*, the Weyl coefficient Ψ_2 *vanishes*,

$$\Psi_2(\pm\pi) = 0.$$

Although the formulae for Ψ_3 and Ψ_4 are quite complicated, they also share this property, i.e.

$$\Psi_3(\pm\pi) = \Psi_4(\pm\pi) = 0.$$

The above quoted result enables us to interpret the $r = \pm\pi$ hypersurfaces as the respective scris $\mathcal{I}^{\pm}$ of the spacetime $(\mathcal{M}, g)$. The Weyl tensor is conformally flat there: $\Psi_\mu(\pm\pi) = 0$ for all $\mu = 0, 1, 2, 3, 4$.

Since the conformally rescaled spacetime $(\mathcal{M}, \hat{g})$ is *periodic* and *regular* in r, it gives a full-Einstein-theory realization of Penrose's idea [6] that there was a 'before the big bang era' of the Universe.

In this context the following remarks are in order:

- To be in accordance with current observations, we concentrate on those solutions (0.2)-(0.3) of the Einstein equations (0.1) which have *positive* cosmological constant $\Lambda > 0$.

- It is known ([8], p. 353) that every solution of Einstein's equations (0.1) with $\Lambda > 0$ has *spacelike* scris $\mathcal{I}^\pm$. Thus, restricting to $\Lambda > 0$, we know that our scris $\mathcal{I}^\pm$ at $r = \pm\pi$ are *spacelike*.
- Moreover the scris are *conformally flat*, and therefore can be identified with the respective surfaces of the *big bang* ($r = -\pi$) and the surface of the *conformal infinity in the future* ($r = \pi$).
- The Universes corresponding to our solutions are either *empty* ($\Phi = 0$), or are filled with a dust of *massless* particles ($\Phi > 0$) moving with the speed of light along k.
- Since going from the 'big bang' to the 'infinity in the future' corresponds to a passage from $r = -\pi$ to $r = \pi$, and since the conformal metric $\hat{g}$ is *periodic* in r, we see that our *conformal* solutions are repetitive in the r variable.
- Thus our solutions give *conformal Universes which periodically reproduce themselves, and smoothly pass through the 'big bang' and the 'future infinity'.*

To be more explicit we discuss the following example[3] which is a solution to our equations (0.7)-(0.8).

We choose the CR manifold $(M, [(\lambda, \mu)])$ to be the Heisenberg group CR manifold. This may be represented by the 1-forms $\lambda = \mathrm{d}u + \frac{i}{2}(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z})$ and $\mu = \mathrm{d}z$. Here $(u, z, \bar{z})$ are the standard coordinates on the Heisenberg group (u is real, z is complex).

Obviously $\mathrm{d}\mu = 0$ and $\mathrm{d}\lambda = i\mu \wedge \bar{\mu}$, so that the function c in (0.6) is $c \equiv 0$. This immediately leads to a solution for (0.7)-(0.8). Indeed: take $n = const \in \mathbb{C}$ and $p = 1$, then equation (0.7) is automatically satisfied, and equation (0.8) gives $\Lambda = -\frac{3}{2}(n + \bar{n})$. Thus we take $n = -\frac{1}{3}\Lambda + im = const$, $\Lambda, m \in \mathbb{R}$. This leads to the *conformal metric*

$$\hat{g} = 2\cos^2(\tfrac{r}{2})g = 2\mathrm{d}z\mathrm{d}\bar{z} + 2\lambda\Big(\mathrm{d}r - 2\big(2m(1 + \cos r)\sin r + \frac{\Lambda}{3}(2\cos r + \cos 2r)\big)\lambda\Big),$$

with $\lambda = \mathrm{d}u + \frac{i}{2}(\bar{z}\mathrm{d}z - z\mathrm{d}\bar{z})$, and the *physical metric* g satisfying all the equations (0.1). Actually, the physical metric satisfies more: It is a solution to the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$, thus $\Phi = 0$. Its Weyl tensor has $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ everywhere, with the only nonvanishing Weyl coefficient $\Psi_2 = -\frac{1}{3}(1 + \mathrm{e}^{ir})^3(\Lambda - 3im)$. This means that the metric corresponding to this solution is of Petrov type D everywhere, except along the scris, $r = \pm\pi$, where it is conformally flat[4]. Restricting to $\Lambda > 0$ we get a 2-parameter family of solutions with spacelike scris, which has a periodic conformal metric $\hat{g}$. This, in addition to being periodic in r,

[3]Note that among our solutions there are many interesting, well known, solutions of Einstein's field equations. Actually our metrics include all *algebraically special* vacuums and the aligned pure radiation gravitational fields. Thus, our solutions include for example the celebrated *rotating black hole* solution of Kerr. This solution, however, is beyond the class of solutions relevant for Penrose's ideas since it has $\Lambda = 0$ (and $\Phi = 0$).

[4]This solution g is the classical Taub-NUT solution with cosmological constant[1, 9]. It includes the Taub-NUT ($\Lambda = 0$) solution [4] as a special case. When $m = 0$ the solution is a vacuum metric with a cosmological constant, which has two Robinson congruences [7] as two distinct principal null dierctions.

is regular everywhere on any hypersurface transversal to k.

As a more complicated example we take a CR structure parameterized by coordinates (x, y, u) and represented by forms

$$\lambda = -\frac{2}{f'(y)}\Big(\mathrm{d}u + f(y)\mathrm{d}x\Big), \qquad \mu = \mathrm{d}x + i\mathrm{d}y.$$

It has $c = \frac{if''(y)}{2f'(y)}$. Then assuming that $p = p(y)$ and $n = n(y)$, we immediately get the following solution for equation (0.7):

$$n = f'(y)^3 m, \qquad m = \mathrm{const} \in \mathbb{C}.$$

Having this, the only remaining Einstein equation to be solved is (0.8). It is equivalent to an ODE:

$$\tfrac{9}{4}pf'f''' + 3p'f'f'' - 3pf''^2 - 3p''f'^2 + 4\Lambda p^3 f'^2 + 6(m + \bar{m})f'^5 p^{-3} = 0,$$

for the functions $p = p(y)$ and $f = f(y)$. Since this is a *single* ODE for *two* real functions of one real variable y, one can use one of these functions to arrange the energy Φ of the corresponding pure radiation to be nonegative for positive Λ.

We believe that many more solutions with appealing physical properties may be found in our class, the main reason being that the class consists of *all* (known and unknown) algebraically special solutions with twisting rays.

We close the paper with the following mathematical comment.

It is interesting to give an interpretation to the only nontrivial Einstein equation (0.8). If one considers the metric $\tilde{g} = \sec^2(\frac{r}{2})\hat{g}$, with $\hat{g}$ as in (0.3), and functions W, H as in (0.4)-(0.5), then the equation (0.8) is the Yamabe equation (see e.g. [8], p. 332) saying that the rescaled metric $g = p^2\tilde{g}$ has constant Ricci scalar $R = 4\Lambda$.

References

[1] Carter B (1968) A new family of Einstein spaces *Phys. Lett.* **A 26**, 399-

[2] Lewandowski J, Nurowski P (1990) Algebraically special twisting gravitational fields and CR structures, *Class. Q. Grav.* **7** 309-328

[3] Hill C D, Lewandowski J, Nurowski P (2007) Einstein equations and the embedding of 3-dimensional CR manifolds, submited for publication, `arXiv:math.DG/0709.3660`

[4] Newman E T, Tamburino L A, Unti T (1963) Empty-space generalization of the Schwarzschild metric *Journ. Math. Phys.* **4**, 1467-

[5] Nurowski P (1993) "Einstein equations and Cauchy-Riemann geometry" PhD Thesis, International School for Advanced Studies, Trieste, also avaliable at `www.fuw.edu.pl/~nurowski/publikacje.html`

[6] Penrose R (2005) Before the Big Bang? A new perspective on the Weyl curvature hypothesis, http://www.newton.cam.ac.uk/webseminars/pg+ws/2005/gmr/gmrw04/1107/penrose/

[7] Penrose R (1987) On the origins of twistor theory, in *Gravitation and geometry*, Trautman A, Rindler W (eds), Bibliopolis, Napoli

[8] Penrose R, Rindler W (1986) *Spinors and spacetime* **vol. 2**, Cambridge University Press

[9] Plebański J F, Demiański M (1976) Rotating, charged and uniformly accelerated mass in general relativity, *Ann. Phys. USA* **98**, 98-

Department of Mathematics, Stony Brook University, Stony Brook, N.Y. 11794, USA
E-mail address: dhill@math.sunysb.edu

Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoza 69, 00-681 Warszawa, Poland
E-mail address: nurowski@fuw.edu.pl