# Elliptic fibrations associated with the Einstein space–times

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Given a conformally nonflat Einstein space–time we define a fibration $\tilde{\mathscr{P}}$ over it. The fibers of this fibration are elliptic curves (two-dimensional tori) or their degenerate counterparts. Their topology depends on the algebraic type of the Weyl tensor of the Einstein metric. The fibration $\tilde{\mathscr{P}}$ is a double branched cover of the bundle $\mathscr{P}$ of null direction over the space–time and is equipped with six linearly independent one-forms which satisfy a certain relatively simple system of equations.  [S0022-2488(98)01210-9]

## I. DEFINITIONS

In Ref. 1 we defined a certain differential system $\mathscr{T}$ on an open set $U$ of $\mathbf{R}^6$. We showed that a pair $(U,\mathscr{T})$ naturally defines a four-dimensional conformally nonflat Lorentzian space–time $(\mathscr{M},g)$ which satisfies the Einstein equations $R_{ij}=\lambda g_{ij}$. In this paper we prove the converse statement. In particular, we give a construction which associates a certain six-dimensional elliptic fibration $\tilde{\mathscr{P}}$ with any conformally nonflat Lorentzian Einstein space–time. Moreover, we show how $\tilde{\mathscr{P}}$ may be equipped with a unique differential system which has all the properties of the system $\mathscr{T}$ on $U$.

We briefly recall the definitions of the geometrical objects we need in the following. Let $\mathscr{M}$ be a four-dimensional-oriented and time-oriented manifold equipped with a Lorentzian metric $g$ of signature $(+,+,+,-)$. It is convenient to introduce a null frame $(m,\bar{m},k,l)$ on $\mathscr{M}$ with a coframe $\theta^i=(\theta^1,\theta^2,\theta^3,\theta^4)=(M,\bar{M},K,L)$ so that

$$g=g_{ij}\theta^i\theta^j=M\bar{M}-KL. \tag{1}$$

[Such expressions as $\theta^i\theta^j$ mean the symmetrized tensor product, e.g., $\theta^i\theta^j=\frac{1}{2}(\theta^i\otimes\theta^j+\theta^j\otimes\theta^i)$. Also, we will denote by round (respectively, square) brackets the symmetrization (respectively, antisymmetrization) of indices, e.g., $a_{(ik)}=\frac{1}{2}(a_{ik}+a_{ki})$, $a_{[ik]}=\frac{1}{2}(a_{ik}-a_{ki})$, etc.]

The Lorentz group $\mathbf{L}$ consists of matrices $\lambda^i_{\ j}\in\mathbf{GL}(4,\mathbf{C})$ such that

$$g_{jl}=g_{ik}\lambda^i_{\ j}\lambda^k_{\ l},$$

$$\lambda^2_{\ 2}=\overline{\lambda^1_{\ 1}},\quad \lambda^2_{\ 1}=\overline{\lambda^1_{\ 2}},\quad \lambda^2_{\ 3}=\overline{\lambda^1_{\ 3}},\quad \lambda^2_{\ 4}=\overline{\lambda^1_{\ 4}}, \tag{2}$$

$$\lambda^3_{\ 2}=\overline{\lambda^3_{\ 1}},\quad \lambda^3_{\ 3}=\overline{\lambda^3_{\ 3}},\quad \lambda^3_{\ 4}=\overline{\lambda^3_{\ 4}},\quad \lambda^4_{\ 2}=\overline{\lambda^4_{\ 1}},\quad \lambda^4_{\ 3}=\overline{\lambda^4_{\ 3}},\quad \lambda^4_{\ 4}=\overline{\lambda^4_{\ 4}}.$$

We will denote the inverse of the Lorentz matrix $\lambda^i_{\ j}$ by $\tilde{\lambda}^i_{\ j}$.

The connected component of the identity element of $\mathbf{L}$ is the proper ortochroneous Lorentz group, which we denote by $\mathbf{L}_+^{\uparrow}$.

Given $g$ and $\theta^i$ the connection one-forms $\Gamma_{ij}=g_{ik}\Gamma^k_{\ j}$ are uniquely defined by

$$d\theta^i=-\Gamma^i_{\ j}\wedge\theta^j,\quad \Gamma_{ij}+\Gamma_{ji}=0. \tag{3}$$

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The connection coefficients $\Gamma_{ijk}$ are determined by $\Gamma_{ij}=\Gamma_{ijk}\theta^k$ (we lower and raise indices by means of the metric and its inverse). Using them we define the curvature two-forms $\mathscr{R}^k{}_i$, the Riemann tensor $R^i{}_{jkl}$, the Ricci tensor $R_{ij}$, and its scalar $R$ by

$$\mathscr{R}^k{}_i=\tfrac{1}{2}R^k{}_{imj}\theta^m\wedge\theta^j=d\Gamma^k{}_i+\Gamma^k{}_j\wedge\Gamma^j{}_i\,,\quad R_{ij}=R^k{}_{ikj}\,,\quad R=g^{ij}R_{ij}\,.$$

We also introduce the traceless Ricci tensor by

$$S_{ij}=R_{ij}-\tfrac{1}{4}g_{ij}R.$$

Note that the vanishing of $S_{ij}$ is equivalent to the Einstein equations $R_{ij}=\lambda g_{ij}$ for the metric $g$. We define the Weyl tensor $C^i{}_{jkl}$ by

$$C_{ijkl}=R_{ijkl}+\tfrac{1}{3}Rg_{i[k}g_{l]j}+R_{j[k}g_{l]i}+R_{i[l}g_{k]j}\,,$$

and its spinorial coefficients $\Psi_\mu$ by

$$\mathscr{R}_{23}=\Psi_4\bar{M}\wedge K+\Psi_3(L\wedge K-M\wedge\bar{M})+(\Psi_2+\tfrac{1}{12}R)L\wedge M$$
$$+\tfrac{1}{2}S_{33}M\wedge K+\tfrac{1}{2}S_{32}(L\wedge K+M\wedge\bar{M})+\tfrac{1}{2}S_{22}L\wedge\bar{M},$$

$$\mathscr{R}_{14}=(-\Psi_2-\tfrac{1}{12}R)\bar{M}\wedge K-\Psi_1(L\wedge K-M\wedge\bar{M})-\Psi_0L\wedge M$$
$$-\tfrac{1}{2}S_{11}M\wedge K-\tfrac{1}{2}S_{41}(L\wedge K+M\wedge\bar{M})-\tfrac{1}{2}S_{44}L\wedge\bar{M},$$

$$\tfrac{1}{2}(\mathscr{R}_{43}-\mathscr{R}_{12})=\Psi_3\bar{M}\wedge K+(\Psi_2-\tfrac{1}{24}R)(L\wedge K-M\wedge\bar{M})+\Psi_1L\wedge M$$
$$+\tfrac{1}{2}S_{31}M\wedge K+\tfrac{1}{4}(S_{12}+S_{34})(L\wedge K+M\wedge\bar{M})+\tfrac{1}{2}S_{42}L\wedge\bar{M}.$$

## II. THE BUNDLE OF NULL COFRAMES

Let $\mathscr{F}(\mathscr{M})$ denote the bundle of oriented and time oriented null coframes over $\mathscr{M}$. This means that $\mathscr{F}(\mathscr{M})$ is the set of all equally oriented null coframes $\theta^i$ at all points of $\mathscr{M}$. The mapping $\pi:\mathscr{F}(\mathscr{M})\rightarrow\mathscr{M}$, which maps a coframe $\theta^i$ at $x\in\mathscr{M}$ onto $x$, gives the canonical projection. A fiber $\pi^{-1}(x)$ in $\mathscr{F}(\mathscr{M})$ consists of all the null coframes at point $x$ which have the same orientation and time orientation. If $\theta^j$ is a null coframe at $x\in\mathscr{M}$, then any other equivalently oriented null coframe at $x$ is given by $\theta'^i=\lambda^i{}_j\theta^j$, where $\lambda^i{}_j$ is a certain element of $\mathbf{L}_+^\uparrow$. This defines an action of $\mathbf{L}_+^\uparrow$ on $\mathscr{F}(\mathscr{M})$. Thus, $\mathscr{F}(\mathscr{M})$ is a ten-dimensional principal fiber bundle with $\mathbf{L}_+^\uparrow$ as its structural group.

It is well known that the bundle $\mathscr{F}(\mathscr{M})$ is equipped with a natural four-covector-valued one-form $e^i$, $i=1,2,3,4$, the Cartan soldering form, which is defined as follows. Take any vector $v_c$ tangent to $\mathscr{F}(\mathscr{M})$ at a point $c$. Let $c$ be in the fiber $\pi^{-1}(x)$ over a point $x\in\mathscr{M}$. This means that $c$ may be identified with a certain null coframe $\theta^i_c$ at $x$.

Then, the formal definition of $e^i$ reads: $e^i(v_c)=\theta^i_c(\pi_*v_c)$. The first two components of $e^i$ are complex and mutually conjugated. The remaining two are real. Altogether, they constitute a system of four well-defined linearly independent one-forms on $\mathscr{F}(\mathscr{M})$. In the following we will denote them by

$$F=e^1=\overline{e^2},\quad T=\bar{T}=e^3,\quad \Lambda=\bar{\Lambda}=e^4. \tag{4}$$

A theorem which we present below is a null coframe reformulation of the Elie Cartan theorem on affine connections.

**Theorem 1:** *Let $e^i=(F,\bar{F},T,\Lambda)$ be the soldering form on $\mathscr{F}(\mathscr{M})$. Then*

*(i) the system of equations*

$$de^i+\omega^i{}_j\wedge e^j=0,\quad g_{ki}\omega^i{}_j+g_{ji}\omega^i{}_k=0 \tag{5}$$

*for a matrix of complex-valued one-forms $\omega^i{}_j$ $(i,j=1,2,3,4)$ on $\mathscr{F}(\mathscr{M})$ has a unique solution,* *(ii)* $\omega^i{}_j$ *uniquely defines six complex-valued one-forms* $(E,\bar{E},\Gamma,\bar{\Gamma},\Omega,\bar{\Omega})$ *by*

$$\omega^i{}_j=\begin{pmatrix}\bar{\Omega}-\Omega & 0 & -E & -\bar{\Gamma}\\ 0 & \Omega-\bar{\Omega} & -\bar{E} & -\Gamma\\ -\Gamma & -\bar{\Gamma} & \Omega+\bar{\Omega} & 0\\ -\bar{E} & -E & 0 & -\Omega-\bar{\Omega}\end{pmatrix}, \tag{6}$$

*(iii) the forms* $(F,\bar{F},T,\Lambda,E,\bar{E},\Gamma,\bar{\Gamma},\Omega,\bar{\Omega})$ *are linearly independent at each point of* $\mathcal{F}(\mathcal{M})$.

For completeness we sketch the proof.

First, we show that if there is a solution to (5) then it is unique. To do this we assume the existence of two solutions—$\omega^i{}_j$ and $\hat{\omega}^i{}_j$.

Subtracting $de^i+\hat{\omega}^i{}_j\wedge e^j=0$ from $de^i+\omega^i{}_j\wedge e^j=0$ we get

$$(\omega^i{}_j-\hat{\omega}^i{}_j)\wedge e^j=0. \tag{7}$$

Now, let $e^\mu$, $\mu=1$, 2, 3, 4, 5, 6, be a system of one-forms such that the ten one-forms $(e^i,e^\mu)$ constitute a basis of one-forms on $\mathcal{F}(\mathcal{M})$. Let $\omega^i{}_j=\omega^i{}_{jk}e^k+\omega^i{}_{j\mu}e^\mu$ and $\hat{\omega}^i{}_j=\hat{\omega}^i{}_{jk}e^k+\hat{\omega}^i{}_{j\mu}e^\mu$ be the corresponding decompositions of the solutions. Then (7) easily yields $\omega^i{}_{j\mu}=\hat{\omega}^i{}_{j\mu}$ and $\omega_{i[jk]}=\hat{\omega}_{i[jk]}$. The defining properties of the solutions give also $\omega_{(ij)k}=0=\hat{\omega}_{(ij)k}$. Now, due to the identity $A_{ijk}=A_{i[jk]}-A_{j[ik]}-A_{k[ij]}$, which is true for any $A_{ijk}$ such that $A_{(ij)k}=0$, we get $\omega_{ijk}=\hat{\omega}_{ijk}$. This shows that $\omega^i{}_j=\hat{\omega}^i{}_j$, hence the uniqueness.

We proceed to the construction of a solution.

Given a sufficiently small neighborhood $\mathcal{O}$ in $\mathcal{M}$ we identify $\pi^{-1}(\mathcal{O})$ with $\mathcal{O}\times\mathbf{L}_+^\uparrow$. Then, the soldering form may be written as

$$e^i=\lambda^i{}_j\theta^j. \tag{8}$$

Taking $de^i$ and using Eq. (3) one easily finds that

$$\omega^i{}_j=\lambda^i{}_k\Gamma^k{}_m\tilde{\lambda}^m_j-d\lambda^i{}_k\tilde{\lambda}^k{}_j \tag{9}$$

is a solution to (5). Given this solution one defines the forms ($E$, $\Gamma$, $\Omega$) by

$$E=-\omega^1{}_3,\quad \Gamma=-\omega^2{}_4,\quad \Omega=\tfrac{1}{2}(\omega^2{}_2+\omega^3{}_3). \tag{10}$$

This is in accordance with (6) due to $\omega_{(ij)}=0$ and the reality properties of $e^i$.

To prove the linear independence of the system $(F,\bar{F},T,\Lambda,E,\bar{E},\Gamma,\bar{\Gamma},\Omega,\bar{\Omega})$ it is enough to observe that the six one-forms $d\lambda^1{}_k\tilde{\lambda}^k{}_3$, $d\lambda^2{}_k\tilde{\lambda}^k{}_3$, $d\lambda^1{}_k\tilde{\lambda}^k{}_4$, $d\lambda^2{}_k\tilde{\lambda}^k{}_4$, $d\lambda^1{}_k\tilde{\lambda}^k{}_1$, and $d\lambda^3{}_k\tilde{\lambda}^k{}_3$ constitute a basis of right invariant forms on $\mathbf{L}_+^\uparrow$.

The theorem is proven.

## III. THE STRUCTURE EQUATIONS

Consider the ten well-defined forms $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ on $\mathcal{F}(\mathcal{M})$ given by (4) and (10). The differentials of the first four forms are given by (5) and (6). In the basis $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ they assume the form

$$dF=(\Omega-\bar{\Omega})\wedge F+E\wedge T+\bar{\Gamma}\wedge\Lambda, \tag{11}$$

$$dT=\Gamma\wedge F+\bar{\Gamma}\wedge\bar{F}-(\Omega+\bar{\Omega})\wedge T, \tag{12}$$

$$d\Lambda=\bar{E}\wedge F+E\wedge\bar{F}+(\Omega+\bar{\Omega})\wedge\Lambda. \tag{13}$$

The differentials of the other forms may be easily calculated using the local representation (9) and the well-known structure equations

$$d\omega^i{}_j=-\omega^i{}_k\wedge\omega^k{}_j+\tfrac{1}{2}R^k{}_{smn}\lambda^i{}_k\tilde{\lambda}^s{}_j\tilde{\lambda}^m_l\tilde{\lambda}^n{}_p e^l\wedge e^p. \tag{14}$$

These differentials are by far much more complicated than the differentials of $dF$, $dT$, and $d\Lambda$. In particular, the decompositions of $dF$, $dT$, and $d\Lambda$ onto the basis of two-forms associated with $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ have only constant coefficients. It turns out that in the differentials $dE$, $d\Omega$, and $d\Gamma$ coefficients which are functions appear. The zero sets of these functions have a well-defined geometrical meaning and define certain subsets of $\mathcal{F}(\mathcal{M})$. Now, the hope is that when we restrict ourselves to such subsets then the differentials of $E$, $\Omega$, and $\Gamma$ will have a much simpler form than their differentials on the whole $\mathcal{F}(\mathcal{M})$. Our aim now is to study this possibility.

## IV. EXPLICIT EXPRESSIONS FOR THE BASIC FORMS ON $\mathcal{F}(\mathcal{M})$

We concentrate on the analysis of $dE=-d\omega^1{}_3$.

Let us introduce the matrices $\lambda^i{}_j(w,z,y)$ and $\lambda^i{}_j(w',z',y')$ such that

$$\lambda^i{}_j(w,z,y)=\begin{pmatrix} |w|w^{-1}(1+\overline{yz}) & |w|w^{-1}\overline{yz} & |w|w^{-1}(1+\overline{yz})z & |w|w^{-1}\bar{y} \\ w|w|^{-1}y\bar{z} & w|w|^{-1}(1+yz) & w|w|^{-1}(1+yz)\bar{z} & w|w|^{-1}y \\ |w|(1+\overline{yz})y & |w|(1+yz)\bar{y} & |w||1+\overline{yz}|^2 & |w||y|^2 \\ \bar{z}|w|^{-1} & z|w|^{-1} & |z|^2|w|^{-1} & |w|^{-1} \end{pmatrix},$$

$$\lambda^i{}_j(w',z',y')=\begin{pmatrix} |w'|w'^{-1}\bar{y}'z' & |w'|w'^{-1}(1+\bar{y}'\bar{z}') & |w'|w'^{-1}\bar{y}' & |w'|w'^{-1}(1+\bar{y}'\bar{z}')z' \\ w'|w'|^{-1}(1+y'z') & w'|w'|^{-1}y'\bar{z}' & w'|w'|^{-1}y' & w'|w'|^{-1}(1+y'z')\bar{z}' \\ |w'|(1+y'z')\bar{y}' & |w'|(1+\bar{y}'\bar{z}') & |w'||y'|^2 & y'|w'||1+\bar{y}'\bar{z}'|^2 \\ z'|w'|^{-1} & \bar{z}'|w'|^{-1} & |w'|^{-1} & |z'|^2|w'|^{-1} \end{pmatrix}.$$

Then, it is well known that $\mathbf{L}_+^\uparrow$ can be represented by

$$\mathbf{L}_+^\uparrow=\mathcal{U}\cup\mathcal{U}', \tag{15}$$

where

$$\mathcal{U}=\{\lambda^i{}_j(w,z,y)\ \text{ such that }\ (w,z,y)\in\mathbf{C}^3, w\neq 0\}, \tag{16}$$

$$\mathcal{U}'=\{\lambda^i{}_j(w',z',y')\ \text{ such that }\ (w',z',y')\in\mathbf{C}^3, w'\neq 0\}. \tag{17}$$

On the intersection $\mathcal{U}\cap\mathcal{U}'$, the coordinates $(w,z,y)$ and $(w',z',y')$ shall be related by

$$w=-\frac{w'}{z'^2},\quad z=\frac{1}{z'},\quad y=-z'(1+y'z'), \tag{18}$$

$$w'=-\frac{w}{z^2},\quad z'=\frac{1}{z},\quad y'=-z(1+yz). \tag{19}$$

Thus, we can cover any $\pi^{-1}(\mathcal{O})\cong(\mathcal{O}\times\mathbf{L}_+^\uparrow)$ by the two charts $\mathcal{O}\times\mathcal{U}$ and $\mathcal{O}\times\mathcal{U}'$. Now, on $\mathcal{O}$ consider the coframe $\theta^i$ of (1). Inserting $\lambda^i{}_j=\lambda^i{}_j(w,z,y)$ or $\lambda^i{}_j(w',z',y')$ to the formulas (8), (9), (14) and using the definitions (1), (4), (10) we easily obtain the following two lemmas.

*Lemma 1: On $\mathcal{O}\times\mathcal{U}$ the forms $F$, $T$, $\Lambda$, $E$, $\Omega$ and $\Gamma$ read*

$$F=\frac{|w|}{w}\,[(1+\overline{yz})(M+zK)+\overline{yz}\bar{M}+\bar{y}L], \tag{20}$$

$$T=|w|[|1+\overline{yz}|^2K+y(1+\overline{yz})M+\bar{y}(1+yz)\bar{M}+y\bar{y}L], \tag{21}$$

$$\Lambda=\frac{1}{|w|}\,[L+z\bar{z}K+z\bar{M}+\bar{z}M], \tag{22}$$

$$E=\frac{1}{w}\,[dz+\Gamma_{32}+z(\Gamma_{21}+\Gamma_{43})+z^2\Gamma_{14}], \tag{23}$$

$$\Omega=-\frac{1}{2}\,\frac{dw}{w}-y\,dz-y\Gamma_{32}-\frac{1}{2}\,(1+2yz)(\Gamma_{21}+\Gamma_{43})-z(1+yz)\Gamma_{14}, \tag{24}$$

$$\Gamma=w[dy-y^2dz-y^2\Gamma_{32}-y(1+yz)(\Gamma_{43}-\Gamma_{12})-(1+yz)^2\Gamma_{14}]. \tag{25}$$

*Lemma 2: On* $\mathscr{O}\times\mathscr{U}'$ *the forms* $F$, $T$, $\Lambda$, $E$, $\Omega$ *and* $\Gamma$ *read*

$$F=\frac{|w'|}{w'}\,[(1+\bar{y'}\bar{z'})(\bar{M}+z'L)+\bar{y'}z'M+\bar{y'}K], \tag{26}$$

$$T=|w'|[|1+\bar{y'}\bar{z'}|^2L+y'(1+\bar{y'}\bar{z'})\bar{M}+\bar{y'}(1+y'z')M+y'\bar{y'}K], \tag{27}$$

$$\Lambda=\frac{1}{|w'|}\,[K+z'\bar{z'}L+z'M+\bar{z'}\bar{M}], \tag{28}$$

$$E=\frac{1}{w'}\,[dz'+\Gamma_{41}+z'(\Gamma_{12}+\Gamma_{34})+z'^2\Gamma_{23}], \tag{29}$$

$$\Omega=-\frac{1}{2}\,\frac{dw'}{w'}-y'dz'-y'\Gamma_{41}-\frac{1}{2}\,(1+2y'z')(\Gamma_{12}+\Gamma_{34})-z'(1+y'z')\Gamma_{23}, \tag{30}$$

$$\Gamma=w'[dy'-y'^2dz'-y'^2\Gamma_{41}-y'(1+y'z')(\Gamma_{34}-\Gamma_{21})-(1+y'z')^2\Gamma_{23}]. \tag{31}$$

Note that this Lemma follows from the previous one by applying transformations $M\leftrightarrow\bar{M}$, $K\leftrightarrow L$, $1\leftrightarrow 2$, and $3\leftrightarrow 4$, where the last two transformations refer to the tetrad indices.

Now one can easily find the differential of $dE$.

*Lemma 3: The decomposition of dE onto the basis* $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ *of one-forms on* $\pi^{-1}(\mathscr{O})$ *reads*

$$dE=2\Omega\wedge E+\phi T\wedge F+b(\Lambda\wedge T+F\wedge\bar{F})+\beta F\wedge\Lambda+\psi T\wedge\bar{F}+a(\Lambda\wedge T-F\wedge\bar{F})+\alpha\Lambda\wedge F,$$

*where* $\phi,b,\beta,\psi,a,\alpha$ *are well-defined functions on* $\pi^{-1}(\mathscr{O})$.

The functions $\phi,b,\beta,\psi,a,\alpha$ are given by

$$\phi=\frac{1}{|w|^2}\,\Phi,\quad b=\bar{w}(\tfrac{1}{2}\phi_{\bar{z}}+\bar{y}\phi),\quad \beta=\bar{w}^2(\tfrac{1}{2}\phi_{\bar{z}\bar{z}}+\bar{y}\phi_{\bar{z}}+\bar{y}^2\phi),$$

$$\psi=\frac{1}{w^2}\,\Psi,\quad a=w(\tfrac{1}{4}\psi_z+y\psi),\quad \alpha=-w^2(\tfrac{1}{12}\psi_{zz}+y^2\psi+\tfrac{1}{2}y\psi_z)-\tfrac{1}{12}R,$$

$$\Phi=\tfrac{1}{2}S_{33}-\bar{z}S_{23}-zS_{13}+z\bar{z}(S_{12}+S_{34})+\tfrac{1}{2}\bar{z}^2S_{22}+\tfrac{1}{2}z^2S_{11}-\bar{z}^2zS_{24}-z^2\bar{z}S_{14}+\tfrac{1}{2}z^2\bar{z}^2S_{44},$$

and

$$\Psi=\Psi_4-4\Psi_3z+6\Psi_2z^2-4\Psi_1z^3+\Psi_0z^4$$

on $\mathscr{O}\times\mathscr{U}$ and by

$$\phi=\frac{1}{|w'|^2}\,\Phi',\quad b=\overline{w'}(\tfrac{1}{2}\phi_{\overline{z'}}+\overline{y'}\,\phi),\quad \beta=\overline{w'}^2(\tfrac{1}{2}\phi_{\overline{z'}\,\overline{z'}}+\overline{y'}\,\phi_{\overline{z'}}+\overline{y'}^2\phi),$$

$$\psi=\frac{1}{w'^2}\,\Psi',\quad a=w'(\tfrac{1}{4}\psi_{z'}+y'\,\psi),\quad \alpha=-w'^2(\tfrac{1}{12}\psi_{z'z'}+y'^2\psi+\tfrac{1}{2}y'\,\psi_{z'})-\tfrac{1}{12}R,$$

$$\Phi'=\tfrac{1}{2}S_{44}-\overline{z'}S_{14}-z'S_{24}+z'\overline{z'}(S_{12}+S_{34})+\tfrac{1}{2}\overline{z'}^2S_{11}$$
$$+\tfrac{1}{2}z'^2S_{22}-\overline{z'}^2z'S_{13}-z'^2\overline{z'}S_{23}-\tfrac{1}{2}\overline{z'}^2\overline{z'}^2S_{33}$$

and

$$\Psi'=\Psi_0-4\Psi_1z'+6\Psi_2z'^2-4\Psi_3z'^3+\Psi_4z'^4$$

on $\mathscr{O}\times\mathscr{U}'$.

The following three cases are of particular interest.

(a) The metric $g$ of the four-manifold $\mathscr{M}$ satisfies the Einstein equations $R_{ij}=\lambda g_{ij}$ and is not conformally flat. This case is characterized by $\Phi\equiv0$ and $\Psi\not\equiv0$.
(b) The metric $g$ is conformally flat but not Einstein. This case corresponds to $\Psi\equiv0$, $\Phi\not\equiv0$.
(c) The metric $g$ is of constant curvature. This means that $\Psi\equiv\Phi\equiv0$.

In the first two cases there is a canonical choice of certain six-dimensional subsets in $\mathscr{F}(\mathscr{M})$. This is defined by the demand that on such sets certain components of $dE$ should identically vanish. This approach is impossible in case (c) since this implies an immediate reduction of $dE$ to the form

$$dE=2\Omega\wedge E+\tfrac{1}{12}R\Lambda\wedge F. \tag{32}$$

## V. DISTINGUISHED SUBSET OF $\mathscr{F}(\mathscr{M})$

From now on we consider case (a). This is the most interesting generic Einstein case.

Imposing the restrictions (a) on $dE$ we immediately see that

$$dE=2\Omega\wedge E+\psi T\wedge\bar{F}+a(\Lambda\wedge T-F\wedge\bar{F})+\alpha\Lambda\wedge F,$$

where $\psi,a,\alpha$ are the same as in the Lemma 1. Since we are in the not conformally flat case (a) we have $\psi\neq0$. This makes possible the restriction to such a set $\mathscr{W}\subset\pi^{-1}(\mathscr{O})$ in which $a$ identically vanish. Thus we consider

$$\mathscr{W}=\mathscr{W}_1\cup\mathscr{W}_2, \tag{33}$$

where

$$\mathscr{W}_1=\{(x;w,z,y)\in(\mathscr{O}\times\mathscr{U})\ \text{ such that }\ \psi y+\tfrac{1}{4}\psi_z=0\},$$

$$\mathscr{W}_2=\{(x;w',z',y')\in(\mathscr{O}\times\mathscr{U}')\ \text{ such that }\ \psi y'+\tfrac{1}{4}\psi_{z'}=0\},$$

or (what is the same due to the nonvanishing of $w$ and $w'$)

$$\mathscr{W}_1=\{(x;w,z,y)\in(\mathscr{O}\times\mathscr{U})\ \text{ such that }\ \Psi y+\tfrac{1}{4}\Psi_z=0\}, \tag{34}$$

$$\mathscr{W}_2=\{(x;w',z',y')\in(\mathscr{O}\times\mathscr{U}')\ \text{ such that }\ \Psi'y'+\tfrac{1}{4}\Psi'_{z'}=0\}. \tag{35}$$

On $\mathscr{W}$ we have

$$dE=2\Omega\wedge E+\psi T\wedge\bar{F}+\alpha\Lambda\wedge F.$$

One can still simplify this relation by restricting oneself to a subset $\tilde{\mathscr{P}}_0$ of $\mathscr{W}$ in which $\psi=-1$. Then, $\tilde{\mathscr{P}}_0$ is a subset of $\pi^{-1}(\mathscr{O})$, which is given by

$$\tilde{\mathscr{P}}_0=\tilde{\mathscr{P}}_{01}\cup\tilde{\mathscr{P}}_{02}, \tag{36}$$

where

$$\tilde{\mathscr{P}}_{01}=\{(x;w,z,y)\in(\mathscr{O}\times\mathscr{U})\ \text{ such that }\ \Psi y+\tfrac{1}{4}\Psi_z=0,\ \Psi+w^2=0\} \tag{37}$$

and

$$\tilde{\mathscr{P}}_{02}=\{(x;w',z',y')\in(\mathscr{O}\times\mathscr{U}')\ \text{ such that }\ \Psi' y'+\tfrac{1}{4}\Psi'_{z'}=0,\ \Psi'+w'^2=0\}. \tag{38}$$

It follows from the construction that on $\tilde{\mathscr{P}}_0$ we have

$$dE=2\Omega\wedge E+\bar{F}\wedge T+\alpha F\wedge\Lambda.$$

## VI. ELLIPTIC FIBRATION

We study the geometry and topology of the set $\tilde{\mathscr{P}}_0$.

The equations defining $\tilde{\mathscr{P}}_0$ may be written as

$$y=-\frac{1}{4}\frac{\Psi_z}{\Psi},\quad y'=-\frac{1}{4}\frac{\Psi'_{z'}}{\Psi'}, \tag{39}$$

$$\Psi+w^2=0,\quad \Psi'+w'^2=0. \tag{40}$$

We see that the first pair of equations uniquely subordinates $y$ to $z$ and $y'$ to $z'$. The second pair gives a relation between $w$ and $z$ and $w'$ and $z'$. Thus, locally among the parameters $(x;w,z,y)$ in $\mathscr{O}\times\mathscr{U}$ [respectively, $(x;w',z',y')$ in $\mathscr{O}\times\mathscr{U}'$], only $x$ and $z$ (respectively, $x$ and $z'$) are free. This shows that $\tilde{\mathscr{P}}_0$ is six dimensional. Moreover, $\tilde{\mathscr{P}}_0$ is fibered over $\mathscr{O}$ with two-dimensional fibers. These are locally parametrized by $z$ or $z'$. To discuss the topology of fibers one observes that the relation between $w$ and $z$ (respectively, $w'$ and $z'$) is purely polynomial. Over every point of $\mathscr{O}$ it has the form

$$w^2=-\Psi_4+4\Psi_3 z-6\Psi_2 z^2+4\Psi_1 z^3-\Psi_0 z^4 \tag{41}$$

or

$$w'^2=-\Psi_0+4\Psi_1 z'-6\Psi_2 z'^2+4\Psi_3 z'^3-\Psi_4 z'^4. \tag{42}$$

Assume for a moment (a) that $w=0$ and $w'=0$ are the allowed values of the parameters and (b) that equations

$$-\Psi_4+4\Psi_3 z-6\Psi_2 z^2+4\Psi_1 z^3-\Psi_0 z^4=0, \tag{43}$$

$$-\Psi_0+4\Psi_1 z'-6\Psi_2 z'^2+4\Psi_3 z'^3-\Psi_4 z'^4=0 \tag{44}$$

for $z$ and $z'$ have only distinct roots. Then, the relations (41)–(42), as being fourth order in the parameters $z$ and $z'$, describe a two-dimensional torus. (This is a well-known fact of classical algebraic geometry, see, e.g., Ref. 2. I am very grateful to Roger Penrose for clarifying this for me).[3]

Let us comment on (a) and (b).

(a) We know that $w$ and $w'$ cannot be zero by their definitions. So to have a torus fibration over the Einstein space–time we need to accept that $w$ and $w'$ may vanish. A price paid for this is that some of the forms $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ will be singular on the resulting fibration at these values of $w$ and $w'$. With these remarks, from now on, we accept that $w$ and $w'$ may vanish. This enables us to introduce the fibration $\tilde{\mathcal{P}}$ over $\mathcal{M}$ which, over the neighborhood $\mathcal{O}\subset\mathcal{M}$ is given by

$$\tilde{\mathcal{P}}=\tilde{\mathcal{P}}_1\cup\tilde{\mathcal{P}}_2, \tag{45}$$

where

$$\tilde{\mathcal{P}}_1=\{(x;w,z,y)\in(\mathcal{O}\times\mathbf{C}^3)\ \text{ such that }\ \Psi y+\tfrac{1}{4}\Psi_z=0,\ \Psi+w^2=0\}, \tag{46}$$

$$\tilde{\mathcal{P}}_2=\left\{(x;w',z',y')\in(\mathcal{O}\times\mathbf{C}^3)\ \text{ such that }\ \Psi'y'+\frac{1}{4}\Psi'_{z'}=0,\ \Psi'+w'^2=0\right\} \tag{47}$$

and the transition functions between $(w,z,y)$ and $(w',z',y')$ coordinates are given by (18).

(b) The discussion in (a) means that $\tilde{\mathcal{P}}$ is a torus fibration over $\mathcal{O}$ provided that Eqs. (43) and (44) have distinct roots. It is well known that the number of distinct roots in (43) and (44) is directly related to the algebraic (Cartan–Petrov–Penrose[4–6]) classification of space–times. Thus, if the space–time $\mathcal{M}$ is algebraically general in $\mathcal{O}$, then $\tilde{\mathcal{P}}$ is a torus fibration over $\mathcal{O}$. In the algebraically special cases the fibers of $\tilde{\mathcal{P}}$ are degenerate tori. These topologically are:

(II) a torus with one vanishing cycle in the Cartan–Petrov–Penrose type II,

(D) two spheres touching each other in two different points in the Cartan–Petrov–Penrose type D,

(III) a sphere with one singular point in the Cartan–Petrov–Penrose type III,

(N) two spheres touching each other in one point in the Cartan–Petrov–Penrose type N.

The pure situations considered so far may be a bit more complicated when the Cartan–Petrov–Penrose type of the Einstein space–time varies from point to point. Imagine, for example, that along a continuous path from $x$ to $x'$ in $\mathcal{O}$ the Cartan–Petrov–Penrose type of the Einstein space–time changes from I to II. Then the fiber of $\tilde{\mathcal{P}}$ over $x'$ is only a torus with one vanishing cycle although the fiber over the starting point $x$ was a torus. It is clear that more complicated situations may occur, and that the fibers of $\tilde{\mathcal{P}}$ over different points of $\mathcal{M}$ can have topologies II, D, III, and N. Fibrations of this type are widely used in algebraic geometry. They are called elliptic, since their fibers can be any kind (even degenerate) of an elliptic curve.

## VII. THE MAIN THEOREM

In Secs. IV–VI, for the clarity of presentation, we restricted ourselves to the neighborhood $\mathcal{O}$ of $\mathcal{M}$. We ended up with an elliptic fibration $\tilde{\mathcal{P}}$ over $\mathcal{O}$. This, however, can be easily prolonged to an elliptic fibration over the whole $\mathcal{M}$. To see this it is enough to observe that a fiber over any point in $\mathcal{O}$ is essentially defined by the Weyl tensor. Since the Weyl tensor is uniquely defined on the whole $\mathcal{M}$ then we can use equations like (41) to uniquely define the elliptic fibers over all $\mathcal{M}$.

Summing up the information from Secs. IV–VI we have the following theorem.

**Theorem 2:** *Given a four-dimensional conformally nonflat space–time $\mathcal{M}$ satisfying the Einstein equations $R_{ij}=\lambda g_{ij}$ one naturally defines a fibration $\Pi:\tilde{\mathcal{P}}\to\mathcal{M}$ with the following properties.*

*(1) A fiber $\Pi^{-1}(x)$ over a point $x\in\mathcal{M}$ is a (possibly degenerate) elliptic curve $\mathcal{C}$ given by*

$$\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2,$$

$$\mathcal{C}_1=\{(w,z)\in\mathbf{C}^2\ \text{ such that }\ w^2=-\Psi_4+4\Psi_3z-6\Psi_2z^2+4\Psi_1z^3-\Psi_0z^4\},$$

$$\mathcal{C}_2=\{(w',z')\in\mathbf{C}^2\ \text{ such that }\ w'^2=-\Psi_0+4\Psi_1z'-6\Psi_2z'^2+4\Psi_3z'^3-\Psi_4z'^4\},$$

*where the transition functions between $(w,z)$ and $(w',z')$ coordinates are given by*

$$w=-\frac{w'}{z'^2},\quad z=\frac{1}{z'}.$$

*(2) The degeneracy of a fiber depends on the algebraic type of the space–time metric and may change from point to point.*

*(3) There is a unique construction of a certain surface $\tilde{\mathscr{P}}_0$ of dimension six immersed in the bundle of null coframes $\mathscr{F}(\mathscr{M})$. $\tilde{\mathscr{P}}_0$ is fibered over $\mathscr{M}$ and $\tilde{\mathscr{P}}$ may be viewed as an extension of $\tilde{\mathscr{P}}_0$ achieved by adding to each fiber of $\tilde{\mathscr{P}}_0$ at most four points.*

*(4) There are ten one-forms $(F,\bar{F},T,\Lambda,E,\bar{E},\Omega,\bar{\Omega},\Gamma,\bar{\Gamma})$ on $\tilde{\mathscr{P}}$ with the following properties:*

- *(a) Forms $T,\Lambda$ are real, all the other are complex valued.*
- *(b) The forms are defined in two steps. First, by restricting the soldering form components $e^i$ and the Levi–Civita connection components $\omega^i_j$ from $\mathscr{F}(\mathscr{M})$ to $\tilde{\mathscr{P}}_0$ and second, by extending the restrictions to $\tilde{\mathscr{P}}$.*
- *(c) $(F,\bar{F},T,\Lambda,E,\bar{E})$ constitute the basis of one-forms on $\tilde{\mathscr{P}}$.*
- *(d) The forms satisfy the following equations on $\tilde{\mathscr{P}}$:*

$$
\begin{aligned}
dF&=(\Omega-\bar{\Omega})\wedge F+E\wedge T+\bar{\Gamma}\wedge\Lambda,\\
dT&=\Gamma\wedge F+\bar{\Gamma}\wedge\bar{F}-(\Omega+\bar{\Omega})\wedge T,\\
d\Lambda&=\bar{E}\wedge F+E\Lambda\bar{F}+(\Omega+\bar{\Omega})\wedge\Lambda,\\
dE&=2\Omega\wedge E+\bar{F}\wedge T+\alpha\Lambda\wedge F,
\end{aligned}
\tag{48}
$$

*with a certain function $\alpha$ on $\tilde{\mathscr{P}}$.*

The explicit formulas for the forms on $\tilde{\mathscr{P}}_1$ (respectively, on $\tilde{\mathscr{P}}_2$) may be obtained from the expressions of Lemma 1 (respectively, Lemma 2) by inserting the relations $y=-\Psi_z/(4\Psi)$ and $w^2+\Psi=0$ [respectively, $y'=-\Psi'_{z'}/(4\Psi')$ and $w'^2+\Psi'=0$].

## VIII. RELATION BETWEEN THE ELLIPTIC FIBRATION AND THE BUNDLE OF NULL DIRECTIONS

Finally we note that the bundle $\tilde{\mathscr{P}}$ constitutes a double branched cover of the Penrose bundle $\mathscr{P}$ of null directions over the space–time.
To see this consider the map $f$ given by

$$\tilde{\mathscr{P}}_1 \ni (x;w,z,y)\overset{f}{\rightarrow}(x,z)\in\mathscr{O}\times\mathbf{C},$$

$$\tilde{\mathscr{P}}_2 \ni (x;w',z',y')\overset{f}{\rightarrow}(x,z')\in\mathscr{O}\times\mathbf{C}.$$

Since on the intersection $\tilde{\mathscr{P}}_1\cap\tilde{\mathscr{P}}_2$ the coordinates $z$ and $z'$ are related by $z'=1/z$, then the two copies of $\mathbf{C}$, which appear in the above relations may be considered as two coordinate charts (say around the North and the South pole, respectively) on the two-dimensional sphere. This sphere is the sphere of null directions at a given point of $\mathscr{O}$ which can be seen as follows.

Consider directions of all the one-forms $L(z)=L+z\bar{z}K+z\bar{M}+\bar{z}M$ at $x\in\mathscr{O}$ for all the values of the complex parameter $z$. These directions are in one to one correspondence with null directions $k(z)=k+z\bar{z}l-zm-\bar{z}\bar{m}$ via $L(z)=-g(k(z))$. These null directions do not form a sphere yet, since the direction corresponding to the vector $l$ is missing. But the missing direction is included in the family of null directions corresponding to the directions of one-forms $K(z')=K+z'\bar{z'}L+z'\bar{M}+\bar{z'}M$ at $x$. Thus the space parametrized by $z$ and $z'$ subject to the relation $z'=1/z$ is in one to one correspondence with the sphere of null directions at $x$. This proves the assertion that $f$ is a cover of $\mathscr{P}$.

The map $f$ is a double cover since if $z$ (or $z'$) is not a root of (43) [respectively, (44)] then $f^{-1}(z)$ is a two-point set. In at most four cases when $z$ is a root $f^{-1}(z)$ is a one-point set. This proves that $f$ is a singular cover.

As we know the singular points are branch points, which proves the statement from the beginning of this section.

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