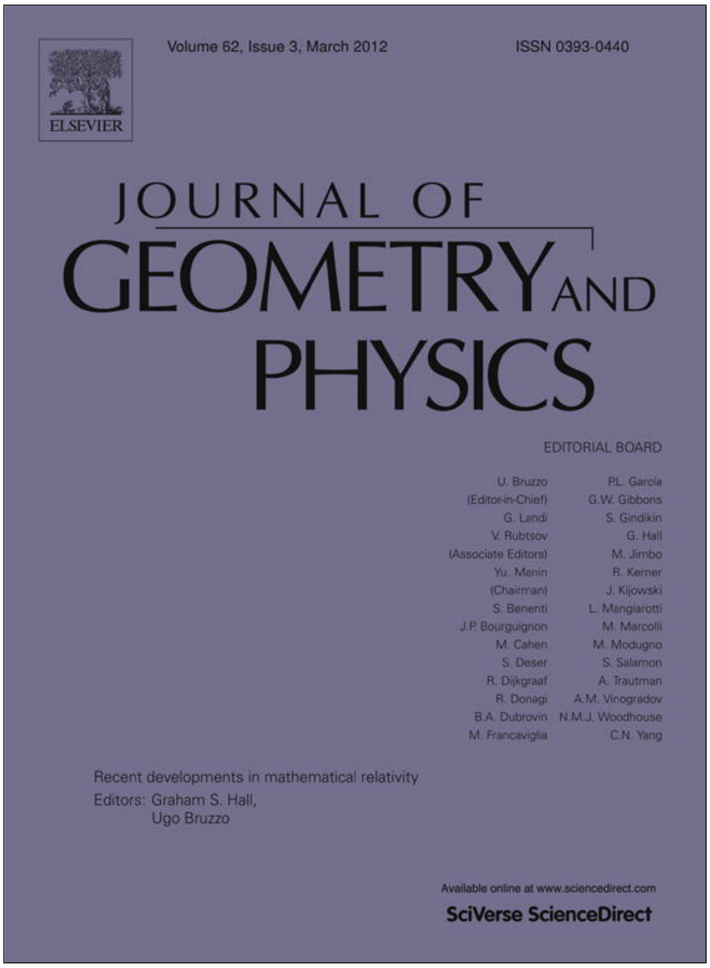

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Projective versus metric structures

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ABSTRACT

We present a number of conditions which are necessary for an n-dimensional projective structure $(M,[\nabla])$ to include the Levi-Civita connection ∇ of some metric on M. We provide an algorithm, which effectively checks whether a Levi-Civita connection is in the projective class and, which finds this connection and the metric, when it is possible. The article also provides basic information on invariants of projective structures, including the treatment via the Cartan normal projective connection. In particular we show that there are a number of Fefferman-like conformal structures, defined on a subbundle of the Cartan bundle of the projective structure, which encode the projectively invariant information about $(M,[\nabla])$.

1. Projective structures and their invariants

1.1. Definition of a projective structure

A *projective structure* on an n-dimensional manifold M is an equivalence class of *torsionless* connections $[\nabla]$ with an equivalence relation identifying every two connections $\hat{\nabla}$ and ∇ for which

$$\hat{\nabla}_X Y = \nabla_X Y + A(X)Y + A(Y)X, \quad \forall X, Y \in \mathrm{T}M, \tag{1}$$

with some 1-form A on M.

Two connections from a projective class have *the same unparameterised geodesics* in M, and the converse is also true: two torsionless connections have the same unparameterised geodesics in M if they belong to the same projective class.

The main purpose of this article is to answer the following question:

'When does a given projective class of connections $[\nabla]$ on M include a Levi-Civita connection of some metric g on M?'

This problem has a long history; see e.g. [1–3]. It was recently solved in $\dim M = 2$ in a beautiful paper [4], which also, in its last section, indicates how to treat the problem in $\dim M \geq 3$. In the present paper we follow [4] and treat the problem in full generality[2] in $\dim M \geq 3$. For doing this we need the *invariants* of projective structures.

The system of local invariants for projective structures was constructed by Cartan [6] (see also [7]). We briefly present it here for completeness (see e.g. [8–10] for more details).

For our purposes it is convenient to describe a connection ∇ in terms of the connection coefficients $\Gamma^a_{\ bc}$ associated with any frame (X_a) on M. This is possible via the formula:

$$\nabla_a X_b = \Gamma^c_{\ ba} X_c, \qquad \nabla_a := \nabla_{X_a}.$$

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2 I have been recently informed by Dunajski that the problem is also being considered by him and Casey [5].

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Given a frame (X_a) these relations provide a one-to-one correspondence between connections ∇ and the connection coefficients $\Gamma^a_{\ bc}$. In particular, a connection is torsionless iff

$$\Gamma^c_{\ ab} - \Gamma^c_{\ ba} = -\theta^c([X_a, X_b]),$$

where (θ^a) is a coframe dual to (X_a),

$$\theta^b(X_a) = \delta^b_{\ a}.$$

Moreover, two connections $\hat{\nabla}$ and ∇ are in the same projective class iff there exists a coframe in which

$$\hat{\Gamma}^c_{\ ab} = \Gamma^c_{\ ab} + \delta^c_{\ a}A_b + \delta^c_{\ b}A_a,$$

for some 1-form $A = A_a\theta^a$.

In the following, rather than using the connection coefficients, we will use a collective object

$$\Gamma^a_{\ b} = \Gamma^a_{\ bc}\theta^c,$$

which we call connection 1-forms. In terms of them the projective equivalence reads:

$$\hat{\Gamma}^a_{\ b} = \Gamma^a_{\ b} + \delta^a_{\ b}A + A_b\theta^a. \qquad (2)$$

1.2. Projective Weyl, Schouten and Cotton tensors

Now, given a projective structure $[\nabla]$ on M, we take connection 1-forms $(\Gamma^i_{\ j})$ of a particular representative ∇. Because of no torsion we have:

$$\mathrm{d}\theta^a + \Gamma^a_{\ b} \wedge \theta^b = 0. \qquad (3)$$

The curvature of this connection

$$\Omega^a_{\ b} = \mathrm{d}\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b}, \qquad (4)$$

which defines the curvature tensor $R^a_{\ bcd}$ via:

$$\Omega^a_{\ b} = \frac{1}{2}R^a_{\ bcd}\theta^c \wedge \theta^d,$$

is now decomposed onto the irreducible components with respect to the action of $\mathbf{GL}(n, \mathbb{R})$ group:

$$\Omega^a_{\ b} = W^a_{\ b} + \theta^a \wedge \omega_b + \delta^a_{\ b}\theta^c \wedge \omega_c. \qquad (5)$$

Here $W^a_{\ b}$ is endomorphism-valued 2-form:

$$W^a_{\ b} = \frac{1}{2}W^a_{\ bcd}\theta^c \wedge \theta^d,$$

which is totally traceless:

$$W^a_{\ a} = 0, \qquad W^a_{\ bac} = 0,$$

and has all the symmetries of $R^a_{\ bcd}$. Quantity ω_a is a covector-valued 1-form. It defines a tensor P_{ab} via

$$\omega_b = \theta^a\mathsf{P}_{ab}. \qquad (6)$$

The tensors $W^a_{\ bcd}$ and P_{ab} are called the (projective) Weyl tensor, and the (projective) Schouten tensor, respectively. They are related to the curvature tensor $R^a_{\ bcd}$ via:

$$R^a_{\ bcd} = W^a_{\ bcd} + \delta^a_{\ c}\mathsf{P}_{db} - \delta^a_{\ d}\mathsf{P}_{cb} - 2\delta^a_{\ b}\mathsf{P}_{[cd]}.$$

In particular, we have also the relation between the Schouten tensor P_{ab} and the Ricci tensor

$$R_{ab} = R^c_{\ acb},$$

which reads:

$$\mathsf{P}_{ab} = \frac{1}{n-1}R_{(ab)} - \frac{1}{n+1}R_{[ab]}. \qquad (7)$$

One also introduces the Cotton tensor Y_{bca}, which is defined via the covector-valued 2-form

$$Y_a = \frac{1}{2}Y_{bca}\theta^b \wedge \theta^c, \qquad (8)$$

by

$$Y_a = \mathrm{d}\omega_a + \omega_b \wedge \Gamma^b_{\ a}. \qquad (9)$$

Note that Y_{bca} is antisymmetric in $\{bc\}$.

Now, combining Eqs. (3)–(5), (8) and (9), we get the *Cartan structure equations*:

$$\begin{aligned} &\mathrm{d}\theta^a + \Gamma^a_{\ b} \wedge \theta^b = 0 \\ &\mathrm{d}\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b} = W^a_{\ b} + \theta^a \wedge \omega_b + \delta^a_{\ b}\theta^c \wedge \omega_c \\ &\mathrm{d}\omega_a + \omega_b \wedge \Gamma^b_{\ a} = Y_a. \end{aligned} \tag{10}$$

It is convenient to introduce the *covariant exterior differential* D, which on tensor-valued k-forms acts as:

$$\mathrm{D}K^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \mathrm{d}K^{a_1\ldots a_r}{}_{b_1\ldots b_s} + \sum_i \Gamma^{a_i}_{\ a} \wedge K^{a_1\ldots a\ldots a_r}{}_{b_1\ldots b_s} - \sum_i \Gamma^b_{\ b_i} \wedge K^{a_1\ldots a_r}{}_{b_1\ldots b\ldots b_s}.$$

This, in particular satisfies the Ricci identity:

$$\mathrm{D}^2K^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \sum_i \Omega^{a_i}_{\ a} \wedge K^{a_1\ldots a\ldots a_r}{}_{b_1\ldots b_s} - \sum_i \Omega^b_{\ b_i} \wedge K^{a_1\ldots a_r}{}_{b_1\ldots b\ldots b_s}. \tag{11}$$

This identity will be crucial in the rest of the paper.

Using D we can write the first and the third Cartan structure equation in respective compact forms:

$$\begin{aligned} &\mathrm{D}\theta^a = 0, \\ &\mathrm{D}\omega_a = Y_a. \end{aligned} \tag{12}$$

Noting that on tensor-valued 0-forms we have:

$$\mathrm{D}K^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \theta^c \nabla_c K^{a_1\ldots a_r}{}_{b_1\ldots b_s},$$

and comparing with definition (6) one sees that the second Eq. (12) is equivalent to:

$$Y_{bca} = 2\nabla_{[b}\mathsf{P}_{c]a}. \tag{13}$$

1.3. Bianchi identities

We now apply D on both sides of the Cartan structure Eqs. (10) and use the Ricci formula (11) to obtain the Bianchi identities.

Applying D on the first of (10) we get

$$0 = \mathrm{D}^2\theta^a = \Omega^a_{\ b} \wedge \theta^b,$$

i.e., tensorially:

$$R^a_{\ [bcd]} = 0.$$

This, because the Weyl tensor has the same symmetries as $R^a_{\ bcd}$, also means that

$$W^a_{\ [bcd]} = 0. \tag{14}$$

Next, applying D on the second of (10) we get:

$$\mathrm{D}W^a_{\ b} = \theta^a \wedge Y_b + \delta^a_{\ b}\theta^c \wedge Y_c.$$

This, when written in terms of the tensors $W^a_{\ bcd}$ and Y_{abc}, reads:

$$\nabla_a W^d_{\ ebc} + \nabla_c W^d_{\ eab} + \nabla_b W^d_{\ eca} = \delta^d_{\ a}Y_{bce} + \delta^d_{\ c}Y_{abe} + \delta^d_{\ b}Y_{cae} + \delta^d_{\ e}(Y_{abc} + Y_{cab} + Y_{bca}). \tag{15}$$

This, when contracted in $\{ad\}$, and compared with (14), implies in particular that:

$$\nabla_d W^d_{\ abc} = (n-2)Y_{bca} \tag{16}$$

and

$$Y_{[abc]} = 0. \tag{17}$$

Thus when $n > 2$ the Cotton tensor is determined by the divergence of the Weyl tensor.

It is also worthwhile to note that because of (17) the identity (15) simplifies to:

$$\nabla_a W^d_{\ ebc} + \nabla_c W^d_{\ eab} + \nabla_b W^d_{\ eca} = \delta^d_{\ a}Y_{bce} + \delta^d_{\ c}Y_{abe} + \delta^d_{\ b}Y_{cae}. \tag{18}$$

Another immediate but useful consequence of the identity (17) is

$$\nabla_{[a}\mathsf{P}_{bc]} = 0. \tag{19}$$

This fact suggests an introduction of a 2-form

$$\beta = \frac{1}{2}\mathsf{P}_{[ab]}\theta^a \wedge \theta^b.$$

Since β is a *scalar* 2-form we have:

$$\begin{aligned}
{\rm d}\beta &= {\rm D}\beta = {\rm D}\left(\frac{1}{2}{\sf P}_{[ab]}\theta^a\wedge\theta^b\right)\\
&= \frac{1}{2}({\rm D}{\sf P}_{[ab]})\theta^a\wedge\theta^b\\
&= \frac{1}{2}(\nabla_c{\sf P}_{[ab]})\theta^c\wedge\theta^a\wedge\theta^b\\
&= \frac{1}{2}(\nabla_{[c}{\sf P}_{ab]})\theta^c\wedge\theta^a\wedge\theta^b = 0.
\end{aligned}$$

Thus, due to the Bianchi identity (19) and the first structure Eq. (12), the 2-form β is *closed*.

Finally, applying D on the last Cartan equation (10) we get

$${\rm D}Y_a + \omega_b\wedge W^b{}_a = 0.$$

This relates the first derivatives of the Cotton tensor to a bilinear combination of the Weyl and the Schouten tensors:

$$\nabla_a Y_{bcd} + \nabla_c Y_{abd} + \nabla_b Y_{cad} = {\sf P}_{ae}W^e{}_{dcb} + {\sf P}_{be}W^e{}_{dac} + {\sf P}_{ce}W^e{}_{dba}. \tag{20}$$

1.4. Gauge transformations

It is a matter of checking that if we take another connection $\hat\nabla$ from the projective class $[\nabla]$, i.e., if we start with connection 1-forms $\hat\Gamma^i_j$ related to Γ^i_j via

$$\hat\Gamma^a_b = \Gamma^a_b + \delta^a_b A + A_b\theta^a,$$

then the basic objects ω_a, W^a_b and Y_a transform as:

$$\begin{aligned}
\hat\omega_a &= \omega_a - {\rm D}A_a + AA_a\\
\hat\beta &= \beta - {\rm d}A\\
\hat W^a_b &= W^a_b\\
\hat Y_a &= Y_a + A_b W^b{}_a.
\end{aligned}\tag{21}$$

Equivalently:

$$\begin{aligned}
\hat\Gamma^a_{bc} &= \Gamma^a_{bc} + \delta^a_c A_b + \delta^a_b A_c\\
\hat P_{ab} &= P_{ab} - \nabla_a A_b + A_a A_b\\
\hat P_{[ab]} &= P_{[ab]} - \nabla_{[a}A_{b]}\\
\hat W^a{}_{bcd} &= W^a{}_{bcd}\\
\hat Y_{abc} &= Y_{abc} + A_d W^d{}_{cab}.
\end{aligned}\tag{22}$$

This in particular means that *the Weyl tensor is a projectively invariant object*. We also note that the 2-form β *transforms modulo addition of a total differential.*

Corollary 1.1. *Locally in every projective class* $[\nabla]$ *there exists a torsionless connection* ∇^0 *for which the Schouten tensor is symmetric,* ${\sf P}_{ab} = {\sf P}_{(ab)}$.

Proof. We know that due to the Bianchi identities (19) the 2-form β encoding the antisymmetric part of ${\sf P}_{ab}$ is closed, ${\rm d}\beta = 0$. Thus, using the Poincaré lemma, we know that there exists a 1-form Υ such that locally $\beta = {\rm d}\Upsilon$. It is therefore sufficient to take $A = \Upsilon$ and $\hat\Gamma^a_b = \Gamma^a_b + \delta^a_b\Upsilon + \theta^a\Upsilon_b$, to get $\hat\beta = 0$, by the second relation in (21). This proves that in the connection $\hat\Gamma^a_b$ projectively equivalent to Γ^a_b, we have $\hat{\sf P}_{[ab]} = 0$. □

Remark 1.2. Note that if Γ^a_b is a connection for which ${\sf P}_{ab}$ is symmetric then it is also symmetric in any projectively equivalent connection for which $A = {\rm d}\phi$, where ϕ is a function.

Definition 1.3. A subclass of projectively equivalent connections for which the Schouten tensor is *symmetric* is called a *special* projective class.

Mutatis mutandis we have:

Corollary 1.4. *Locally every projective class contains a special projective subclass. This subclass is given modulo transformations* (2) *with A being a gradient,* $A = {\rm d}\phi$.

Corollary 1.5. *The curvature $\Omega^a{}_b$ of any connection from a special projective subclass of projective connections $[\nabla]$ is traceless, $\Omega^a{}_a = 0$.*

Proof. For the connections from a special projective subclass we have $\mathsf{P}_{ab} = \mathsf{P}_{ba}$. Hence $\theta^a \wedge \omega_a = \theta^a \wedge \mathsf{P}_{ba}\theta^b \wedge = 0$, and $\Omega^a{}_b = W^a{}_b + \theta^a \wedge \omega_b$. Thus

$$\Omega^a{}_a = W^a{}_a + \theta^a \wedge \omega_a = 0,$$

because the Weyl form $W^a{}_b$ is traceless. □

Remark 1.6. We also remark that *in dimension $n = 2$ the Weyl tensor of a projective structure is identically zero*. In this dimension the Cotton tensor provides the lowest order projective invariant (see the last equation in (22)). In dimension $n = 3$ the Weyl tensor is generically nonzero, and may have as much as *fifteen* independent components. It is also generically nonzero in dimensions higher than three.

Given an open set $\mathcal{U}$ with coordinates (x^a) surely the simplest projective structure $[\nabla]$ is the one represented by the connection $\nabla_a = \frac{\partial}{\partial x^a}$. This is called the *flat* projective structure on $\mathcal{U}$. The following theorem is well known [6,7]:

Theorem 1.7. *In dimension $n \geq 3$ a projective structure $[\nabla]$ is locally projectively equivalent to the flat projective structure if and only if its projective Weyl tensor vanishes identically, $W^a{}_{bcd} \equiv 0$. In dimension $n = 2$, a projective structure $[\nabla]$ is locally projectively equivalent to the flat projective structure if and only if its projective Schouten tensor vanishes identically, $Y_{abc} \equiv 0$.*

1.5. Cartan connection

Objects $(\theta^a, \Gamma^b{}_c, \omega_d)$ can be collected to the *Cartan connection* on an H principal fiber bundle $H \to P \to M$ over $(M, [\nabla])$. Here H is a subgroup of the $\mathbf{SL}(n+1,\mathbb{R})$ group defined by:

$$H = \left\{ b \in \mathbf{SL}(n+1,\mathbb{R}) \mid b = \begin{pmatrix} A^a{}_b & 0 \\ A_b & a^{-1} \end{pmatrix},\ A^a{}_b \in \mathbf{GL}(n,\mathbb{R}),\ A_a \in (\mathbb{R}^n)^*,\ a = \det(A^a{}_b) \right\}.$$

Using $(\theta^a, \Gamma^b{}_c, \omega_d)$ we define an $\mathfrak{sl}(n+1,\mathbb{R})$-valued 1-form

$$\mathcal{A} = b^{-1} \begin{pmatrix} \Gamma^a{}_b - \frac{1}{n+1}\Gamma^c{}_c\delta^a{}_b & \theta^a \\ \omega_b & -\frac{1}{n+1}\Gamma^c{}_c \end{pmatrix} b + b^{-1}\mathrm{d}b.$$

This can be also written as

$$\mathcal{A} = \begin{pmatrix} \hat\Gamma^a{}_b - \frac{1}{n+1}\hat\Gamma^c{}_c\delta^a{}_b & \hat\theta^a \\ \hat\omega_b & -\frac{1}{n+1}\hat\Gamma^c{}_c \end{pmatrix},$$

from which, knowing b, one can deduce the transformation rules

$$(\theta^a, \Gamma^b{}_c, \omega_d) \to (\hat\theta^a, \hat\Gamma^b{}_c, \hat\omega_d);$$

see e.g. [10]. Note that when the coframe θ^a is fixed, i.e., when $A^a{}_b = \delta^a{}_b$, these transformations coincide with (2) and (21); the above setup extends these transformations to the situation when we allow the frame to change under the action of the $\mathbf{GL}(n,\mathbb{R})$ group.

The form $\mathcal{A}$ defines an $\mathfrak{sl}(n+1,\mathbb{R})$ Cartan connection on $H \to P \to M$. Its curvature

$$\mathcal{R} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A},$$

satisfies

$$\mathcal{R} = b^{-1}\begin{pmatrix} W^a{}_b & 0 \\ Y_b & 0 \end{pmatrix} b = \begin{pmatrix} \hat W^a{}_b & 0 \\ \hat Y_b & 0 \end{pmatrix},$$

and consists of the 2-forms $W^a{}_b$, Y_b as defined in (10). In particular we have $\hat W^a{}_b = \frac12 \hat W^a{}_{bcd}\hat\theta^c \wedge \hat\theta^d$, and $\hat Y_a = \frac12 \hat Y_{abc}\hat\theta^b \wedge \hat\theta^c$, where $\hat W^a{}_{bcd}$ and $\hat Y_{abc}$ are the transformed Weyl and Cotton tensors.

Note that the $(n+n^2+n)$ 1-forms $(\hat\theta^a, \hat\Gamma^b{}_c, \hat\omega_d)$ constitute a *coframe* on the (n^2+2n)-dimensional bundle $H \to P \to M$; in particular these forms *are linearly independent* at each point of P. They satisfy the transformed Cartan structure equations

$$\begin{aligned} &\mathrm{d}\hat\theta^a + \hat\Gamma^a{}_b \wedge \hat\theta^b = 0 \\ &\mathrm{d}\hat\Gamma^a{}_b + \hat\Gamma^a{}_c \wedge \hat\Gamma^c{}_b = \hat W^a{}_b + \hat\theta^a \wedge \hat\omega_b + \delta^a{}_b\hat\theta^c \wedge \hat\omega_c \\ &\mathrm{d}\hat\omega_a + \hat\omega_b \wedge \hat\Gamma^b{}_a = \hat Y_a. \end{aligned} \tag{23}$$

1.6. Fefferman metrics

In Ref. [11], with *any* point equivalence class of second order ODEs $y''=Q(x,y,y')$, we associated a certain 4-dimensional manifold $P/\sim$ equipped with a *conformal* class of metrics of *split* signature $[g_F]$, whose *conformal invariants* encoded all the *point invariants* of the ODEs from the point equivalent class. By analogy with the theory of 3-dimensional CR structures we called the class $[g_F]$ the *Fefferman class*. The manifold P from $P/\sim$ was a principal fiber bundle $\mathcal{H}\to P\to N$ over a 3-dimensional manifold N, which was identified with the first jet space $\mathcal{J}^1$ of an ODE from the equivalence class. The bundle P was eight dimensional, and $\mathcal{H}$ was a 5-dimensional parabolic subgroup of $\mathbf{SL}(3,\mathbb{R})$. For each point equivalence class of ODEs $y''=Q(x,y,y')$, the *Cartan normal conformal connection* of the corresponding Fefferman metrics $[g_F]$, was reduced to a certain $\mathfrak{sl}(3,\mathbb{R})$ Cartan connection $\mathcal{A}$ on P. The *two main components of the curvature* of this connection were the *two classical basic point invariants* of the class $y''=Q(x,y,y')$, namely:

$$w_1=D^2Q_{y'y'}-4DQ_{yy'}-DQ_{y'y'}Q_{y'}+4Q_{y'}Q_{yy'}-3Q_{y'y'}Q_y+6Q_{yy},$$

and

$$w_2=Q_{y'y'y'y'}.$$

If *both of these invariants were nonvanishing* the Cartan bundle that encoded the structure of a point equivalence class of ODEs $y''=Q(x,y,y')$ was just $\mathcal{H}\to\mathcal{P}\to N$ with the Cartan connection $\mathcal{A}$. The nonvanishing of w_1w_2, was reflected in the fact that the corresponding Fefferman metrics were always of the Petrov type $N\times N'$, and never selfdual nor antiselfdual.

In the case of $w_1w_2\equiv 0$, the situation was more special [10]: the Cartan bundle $\mathcal{H}\to P\to N$ was also defining a Cartan bundle $H\to P\to M$, over a 2-dimensional manifold M, with the 6-dimensional parabolic subgroup H of $\mathbf{SL}(3,\mathbb{R})$ as the structure group. The manifold M was identified with the solution space of an ODE representing the point equivalent class. Furthermore the space M was naturally equipped with a *projective structure* $[\nabla]$, whose invariants were in one-to-one correspondence with the point invariants of the ODE. This one-to-one correspondence was realised in terms of the $\mathfrak{sl}(3,\mathbb{R})$ connection $\mathcal{A}$. This, although initially defined as a canonical $\mathfrak{sl}(3,\mathbb{R})$ connection on $\mathcal{H}\to P\to N$, in the special case of $w_1w_2\equiv 0$ became the $\mathfrak{sl}(3,\mathbb{R})$-valued *Cartan normal projective connection* of the structure $(M,[\nabla])$ on the Cartan bundle $H\to P\to M$. In such a case the corresponding Fefferman class $[g_F]$ on $P/\sim$ became *selfdual* or *antiselfdual* depending on which of the invariants w_1 or w_2 vanished.

What we have overlooked in the discussions in [10,11] was that in the case of $w_2\equiv 0$, $w_1\neq 0$ we could have defined *two, a priori different* Fefferman classes $[g_F]$ and $[g_F']$. As we see below the construction of these classes totally relies on the fact that we had a canonical projective structure $[\nabla]$ on M. Actually we have the following theorem.

Theorem 1.8. *Every n-dimensional manifold M with a projective structure* $[\nabla]$ *uniquely defines* a number *n of conformal metrics* $[g^a]$, *each of* split *signature* (n,n), *and each defined on its own natural* $2n$*-dimensional subbundle* $P_a=P/(\sim_a)$ *of the Cartan projective bundle* $H\to P\to M$.

Proof. Given $(M,[\nabla])$ we will construct the pair $(P_a,[g^a])$ for each $a=1,\dots,n$. We use the notation of Section 1.5.

Let $(X_a,X^b{}_c,X^d)$ be a frame of vector fields on P dual to the coframe $(\hat\theta^a,\hat\Gamma^b{}_c,\hat\omega_d)$. This means that

$$X_a\lrcorner\hat\theta^b=\delta^b{}_a,\qquad X^a{}_b\lrcorner\hat\Gamma^c{}_d=\delta^a{}_d\delta^c{}_b,\qquad X^a\lrcorner\hat\omega_b=\delta^a{}_b,\tag{24}$$

at each point, with *all* other contractions being zero.

We now define a number of n bilinear forms $\hat g^a$ on P defined by

$$\hat g^a=\left(\hat\Gamma^a{}_b-\frac{2}{n+1}\hat\Gamma^c{}_c\delta^a{}_b\right)\otimes\hat\theta^b+\hat\theta^b\otimes\left(\hat\Gamma^a{}_b-\frac{2}{n+1}\hat\Gamma^c{}_c\delta^a{}_b\right),$$

or

$$\hat g^a=2\left(\hat\Gamma^a{}_b-\frac{2}{n+1}\hat\Gamma^c{}_c\delta^a{}_b\right)\hat\theta^b,$$

for short. In this second formula we have used the classical notation, such as for example in $g=g_{ab}\theta^a\theta^b$, which abbreviates the symmetrised tensor product of two 1-forms λ and μ on P to $\lambda\otimes\mu+\mu\otimes\lambda=2\lambda\mu$.

We note that the formula for $\hat g^a$, when written in terms of the Cartan connection $\mathcal{A}$, reads[3]:

$$\hat g^a=2\mathcal{A}^a{}_\mu\mathcal{A}^\mu{}_{n+1},$$

where the index μ is summed over $\mu=1,\dots,n,n+1$. Indeed:

$$2\mathcal{A}^a{}_\mu\mathcal{A}^\mu{}_{n+1}=2\left(\hat\Gamma^a{}_b-\frac{1}{n+1}\hat\Gamma^c{}_c\delta^a{}_b\right)\hat\theta^b+2\hat\theta^a\left(-\frac{1}{n+1}\hat\Gamma^c{}_c\right)=2\left(\hat\Gamma^a{}_b-\frac{2}{n+1}\hat\Gamma^c{}_c\delta^a{}_b\right)\hat\theta^b=\hat g^a.$$

[3] Compare with the defining formula for G in [11].

The bilinear forms $\hat{g}^a$ are *degenerate* on P. For each *fixed* value of the index a, $a = 1, \dots, n$, they have n^2 degenerate directions spanned by $(X^b, Z^c_{\ D})$, where $b, c = 1, \dots, n$ and $D = 1, \dots, n$ *without* $D = a$. The $n(n-1)$ vector fields $Z^c_{\ D}$ are defined to be

$$Z^c_{\ D} = X^c_{\ D} - \frac{2}{n-1} X^d_{\ d} \delta^c_{\ D}.$$

Obviously $(X^b, Z^c_{\ D})$ annihilate all θ^bs. Also obviously all X^bs annihilate all $\left(\hat{\Gamma}^a_{\ b} - \frac{1}{n+1}\hat{\Gamma}^c_{\ c}\delta^a_{\ b}\right)$s. To see that all $Z^c_{\ D}$s annihilate all $\left(\hat{\Gamma}^a_{\ b} - \frac{1}{n+1}\hat{\Gamma}^c_{\ c}\delta^a_{\ b}\right)$s we extend the definition of $Z^c_{\ D}$s to

$$Z^c_{\ f} = X^c_{\ f} - \frac{2}{n-1} X^d_{\ d} \delta^c_{\ f},$$

where now $f = 1, \dots, n$. For these we get:

$$Z^c_{\ d} \lrcorner \left(\hat{\Gamma}^a_{\ b} - \frac{2}{n+1}\hat{\Gamma}^h_{\ h}\delta^a_{\ b}\right) = \delta^c_{\ b}\delta^a_{\ d}.$$

Thus, if $d \neq a$ we see that each $Z^c_{\ d}$ annihilates $\hat{\Gamma}^a_{\ b} - \frac{2}{n+1}\hat{\Gamma}^h_{\ h}\delta^a_{\ b}$. Hence $n(n-1)$ directions $Z^a_{\ D}$ are degenerate directions for $\hat{g}^a$.

Another observation is that the n^2 degenerate directions $(X^b, Z^c_{\ D})$ form an *integrable* distribution. This is simplest to see by considering their annihilator. At each point this is spanned by the $2n$ 1-forms $\left(\hat{\theta}^b, \hat{\Gamma}^{(a)}_{\ b} - \frac{2}{n+1}\hat{\Gamma}^h_{\ h}\delta^{(a)}_{\ b}\right)$, where the index (a) in brackets says that it is the fixed a which is not present in the range of indices D. Now using (23) it is straightforward to see that the forms $(\hat{\theta}^b, \hat{\tau}^{(a)}_{\ b}) = \left(\hat{\theta}^b, \hat{\Gamma}^{(a)}_{\ b} - \frac{2}{n+1}\hat{\Gamma}^h_{\ h}\delta^{(a)}_{\ b}\right)$ satisfy the Frobenius condition

$$\mathrm{d}\hat{\theta}^a \wedge \hat{\theta}^1 \wedge \dots \wedge \hat{\theta}^n = 0,$$
$$\mathrm{d}\hat{\tau}^{(a)}_{\ b} \wedge \hat{\tau}^{(a)}_{\ 1} \wedge \dots \wedge \hat{\tau}^{(a)}_{\ n} \wedge \hat{\theta}^1 \wedge \dots \wedge \hat{\theta}^n = 0.$$

Thus the n^2-dimensional distribution spanned by $(X^b, Z^c_{\ D})$ is integrable.

Now, using (23) we calculate the Lie derivatives of $\hat{g}^a$ with respect to the directions $(X^b, Z^c_{\ D})$. It is easy to see that:

$$\mathcal{L}_{X^b}\hat{g}^a = 0$$

and

$$\mathcal{L}_{Z^c_{\ d}}\hat{g}^a = -\delta^a_{\ d}\hat{g}^c + \frac{2}{n-1}\delta^c_{\ d}\hat{g}^a.$$

The last equation means also that

$$\mathcal{L}_{Z^c_{\ D}}\hat{g}^a = \frac{2}{n-1}\delta^c_{\ D}\hat{g}^a.$$

Thus, the bilinear form $\hat{g}^a$ transforms *conformally* when Lie transported along the integrable distribution spanned by $(X^b, Z^c_{\ D})$.

Now, for each fixed $a = 1, \dots, n$, we introduce an equivalence relation $\sim_a$ on P, which identifies points on the same integral leaf of $\mathrm{Span}(X^b, Z^c_{\ D})$. On the $2n$-dimensional leaf space $P_a = P/(\sim_a)$ the n^2 degenerate directions for $\hat{g}^a$ are squeezed to points. Since the remainder of $\hat{g}^a$ is given up to a conformal rescalling on each leaf, the bilinear form $\hat{g}^a$ *descends* to a *unique conformal class* $[g^a]$ of metrics, which on P_a have *split signature* (n, n). Thus, for each $a = 1, \dots, n$ we have constructed the $2n$-dimensional split signature conformal structure $(P_a, [g^a])$. It follows from the construction that P_a may be identified with *any* $2n$-dimensional submanifold $\tilde{P}_a$ of P, which is *transversal to the leaves of* $\mathrm{Span}(X^b, Z^c_{\ D})$. The conformal class $[g^a]$ is represented on each $\tilde{P}^a$ by the *restriction* $g^a = \hat{g}^a_{\ |\tilde{P}_a}$. This completes the proof of the theorem. □

One can calculate the Cartan normal conformal connection for the conformal structures (P_a, g^a). This is a lengthy, but straightforward calculation. The result is given in the following theorem.

Theorem 1.9. *In the null frame $(\hat{\tau}^{(a)}_{\ b}, \hat{\theta}^c)$ the Cartan normal conformal connection for the metric $\hat{g}^a$ is given by:*

$$G = \begin{pmatrix} -\frac{1}{n+1}\hat{\Gamma}^d_{\ d} & 0 & -\hat{\omega}_c & 0 \\ \hat{\tau}^{(a)}_{\ b} & -\hat{\Gamma}^e_{\ b} + \frac{1}{n+1}\hat{\Gamma}^d_{\ d}\delta^e_{\ b} & \hat{\theta}^d \hat{R}^{(a)}_{\ dcb} & -\hat{\omega}_b \\ \hat{\theta}^f & 0 & \hat{\Gamma}^f_{\ c} - \frac{1}{n+1}\hat{\Gamma}^d_{\ d}\delta^f_{\ c} & 0 \\ 0 & \hat{\theta}^e & \hat{\tau}^{(a)}_{\ c} & \frac{1}{n+1}\hat{\Gamma}^d_{\ d} \end{pmatrix}.$$

Its curvature $R = \mathrm{d}G + G \wedge G$ *is given by:*

$$R = \begin{pmatrix} 0 & 0 & -\hat{Y}_c & 0 \\ 0 & -\hat{W}^e{}_b & \hat{S}_{cb} & -\hat{Y}_b \\ 0 & 0 & \hat{W}^f{}_c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} \hat{S}_{cb} &= -\hat{\theta}^d(\hat{D}\hat{R}^{(a)}_{dcb} - \hat{\tau}^{(a)}_s \hat{W}^s{}_{dcb}) \\ &= -\hat{\theta}^d(\hat{D}\hat{W}^{(a)}_{d\,cb} - \hat{\tau}^{(a)}_s \hat{W}^s{}_{dcb}) + \delta^{(a)}_b \hat{Y}_c - \delta^{(a)}_c \hat{Y}_b. \end{aligned}$$

2. When a projective class includes a Levi-Civita connection?

2.1. Projective structures of the Levi-Civita connection

Let us now assume that an n-dimensional manifold M is equipped with a (pseudo-)Riemannian metric $\hat{g}$. We denote its *Levi-Civita connection* by $\hat{\nabla}$. The Levi-Civita connection $\hat{\nabla}$ defines its *projective class* $[\nabla]$ with connections ∇ such that (1) holds. Now, with the Levi-Civita representative $\hat{\nabla}$ of $[\nabla]$ we can define its curvature $\hat{\Omega}^a{}_b$, as in (4), and decompose it into the *projective* Weyl and Schouten tensors $\hat{W}^a{}_{bcd}$, $\hat{\mathsf{P}}_{ab}$, as in (5):

$$\hat{\Omega}^a{}_b = \hat{W}^a{}_b + \theta^a \wedge \hat{\omega}_b + \delta^a{}_b \theta^c \wedge \hat{\omega}_c. \tag{25}$$

However, since now M has an additional metric structure $\hat{g} = \hat{g}_{ab}\theta^a\theta^b$, with the inverse $\hat{g}^{ab}$ such that $\hat{g}_{ab}\hat{g}^{bc} = \delta^c_a$, another decomposition of the curvature is possible. This is the decomposition onto the *metric* Weyl and Schouten tensors $\overset{LC}{W}{}^a{}_{bcd}$, $\overset{LC}{\mathsf{P}}_{ab}$, given by:

$$\hat{\Omega}^a{}_b = \overset{LC}{W}{}^a{}_b + \hat{g}^{ac}\hat{g}_{bd}\overset{LC}{\omega}_c \wedge \theta^d + \theta^a \wedge \overset{LC}{\omega}_b. \tag{26}$$

The tensor counterparts of formulae (25)–(26) are respectively:

$$\begin{aligned} \hat{R}^a{}_{bcd} &= \hat{W}^a{}_{bcd} + \delta^a{}_c \hat{\mathsf{P}}_{db} - \delta^a{}_d \hat{\mathsf{P}}_{cb} - 2\delta^a{}_b \hat{\mathsf{P}}_{[cd]} \\ \hat{R}^a{}_{bcd} &= \overset{LC}{W}{}^a{}_{bcd} + \delta^a{}_c \overset{LC}{\mathsf{P}}_{db} - \delta^a{}_d \overset{LC}{\mathsf{P}}_{cb} + \hat{g}_{bd}\hat{g}^{ae}\overset{LC}{\mathsf{P}}_{ec} - \hat{g}_{bc}\hat{g}^{ae}\overset{LC}{\mathsf{P}}_{ed}. \end{aligned} \tag{27}$$

To find relations between the projective and the metric Weyl and Schouten tensors one compares the right-hand sides of (27). For example, because of the equality on the left-hand sides of (27), the projective and the Levi-Civita–Ricci tensors are equal:

$$\hat{R}_{bd} = \hat{R}^a_{bad} = \overset{LC}{R}_{bd}.$$

Thus, via (7), we get

$$\hat{\mathsf{P}}_{ab} = \frac{1}{n-1}\overset{LC}{R}_{ab}. \tag{28}$$

Further relations between the projective and Levi-Civita objects can be obtained by recalling that:

$$\overset{LC}{R}_{ab} = (n-2)\overset{LC}{\mathsf{P}}_{ab} + \hat{g}_{ab}\overset{LC}{\mathsf{P}},$$

where

$$\overset{LC}{\mathsf{P}} = \hat{g}^{ab}\overset{LC}{\mathsf{P}}_{ab},$$

and that the Levi-Civita–Ricci scalar is given by:

$$\overset{LC}{R} = \hat{g}^{ab}\overset{LC}{R}_{ab}.$$

After some algebra we get the following proposition.

Proposition 2.1. *The projective Schouten tensor* $\hat{\mathsf{P}}_{ab}$ *for the Levi-Civita connection* $\hat{\nabla}$ is related*to the metric Schouten tensor* $\overset{LC}{\mathsf{P}}_{ab}$ *via:*

$$\hat{\mathsf{P}}_{ab} = \overset{LC}{\mathsf{P}}_{ab} - \frac{1}{(n-1)(n-2)}G_{ab},$$

where G_{ab} *is the Einstein tensor for the Levi-Civita connection:*

$$G_{ab} = \overset{LC}{R}_{ab} - \frac{1}{2}\hat{g}_{ab}\overset{LC}{R}.$$

The projective Weyl tensor $\hat{W}^a{}_{bcd}$ *for the Levi-Civita connection* $\hat{\nabla}$ *is related to the metric Weyl tensor* $\overset{LC}{W}{}^a{}_{bcd}$ *via:*

$$\hat{W}^a{}_{bcd} = \overset{LC}{W}{}^a{}_{bcd} + \frac{1}{n-2}(\hat{g}_{bd}\hat{g}^{ae}\overset{LC}{R}_{ec} - \hat{g}_{bc}\hat{g}^{ae}\overset{LC}{R}_{ed}) + \frac{1}{(n-1)(n-2)}(\delta^a{}_c\overset{LC}{R}_{db} - \delta^a{}_d\overset{LC}{R}_{cb}) + \frac{\overset{LC}{R}}{(n-1)(n-2)}(\delta^a{}_d\hat{g}_{bc} - \delta^a{}_c\hat{g}_{bd}). \quad (29)$$

In particular we have the following corollary:

Corollary 2.2. *The projective Schouten tensor* $\hat{\mathsf{P}}_{ab}$ *of the Levi-Civita connection* $\hat{\nabla}$ *is symmetric*

$$\hat{\mathsf{P}}_{ab} = \hat{\mathsf{P}}_{ba}.$$

Moreover, the projective Weyl tensor $W^a{}_{bcd}$ *of any connection* ∇ *from the projective class* $[\nabla]$ *of a Levi-Civita connection satisfies*

$$\hat{g}_{ae}\hat{g}^{bc}W^e{}_{bcd} = \hat{g}_{de}\hat{g}^{bc}W^e{}_{bca}. \quad (30)$$

Proof. The first part of the corollary is an immediate consequence of the fact that the metric Schouten tensor of the Levi-Civita connection as well as the Einstein tensor are symmetric. The second part follows from the relation (29), which yields:

$$(n-1)\hat{g}_{ae}\hat{g}^{bc}\hat{W}^e{}_{bcd} = -n\overset{LC}{R}_{ad} + \overset{LC}{R}\,\hat{g}_{ad}.$$

Since $\overset{LC}{R}_{ab}$ is symmetric we get $\hat{g}_{ae}\hat{g}^{bc}\hat{W}^e{}_{bcd} = \hat{g}_{de}\hat{g}^{bc}\hat{W}^e{}_{bca}$. But according to the fourth transformation law in (22) the Weyl tensor is invariant under the projective transformations, $\hat{W}^a{}_{bcd} = W^a{}_{bcd}$. Thus (30) holds, for all connections ∇ from the projective class of $\hat{\nabla}$. This ends the proof. □

The above corollary is obviously related to the question in the title of this Section. It gives the first, very simple, obstruction for a projective structure $[\nabla]$ to include a Levi-Civita connection of some metric. We reformulate it to the following theorem.

Theorem 2.3. *A necessary condition for a projective structure* $(M, [\nabla])$ *to include a connection* $\hat{\nabla}$*, which is the Levi-Civita connection of some metric* $\hat{g}_{ab}$*, is an existence of a symmetric nondegenerate bilinear form* g^{ab} *on* M*, such that the Weyl tensor* $W^a{}_{bcd}$ *of the projective structure satisfies*

$$g_{ae}g^{bc}W^e{}_{bcd} = g_{de}g^{bc}W^e{}_{bca}, \quad (31)$$

with g_{ab} *being the inverse of* g^{ab}, $g_{ac}g^{cb} = \delta^b{}_a$*. If the Levi-Civita connection* $\hat{\nabla}$ *from the projective class* $[\nabla]$ *exists, then its corresponding metric* $\hat{g}_{ab}$ *must be conformal to the inverse* g_{ab} *of some solution* g^{ab} *of Eq.* (31)*, i.e.,* $\hat{g}_{ab} = \mathrm{e}^{2\phi}g_{ab}$*, for a solution* g^{ab} *of* (31) *and some function* ϕ *on* M*.*

As an example we consider a projective structure $[\nabla]$ on a 3-dimensional manifold M parameterised by three real coordinates (x, y, z). We choose a holonomic coframe $(\theta^1, \theta^2, \theta^3) = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, and generate a projective structure from the connection 1-forms

$$\Gamma^a{}_b = \begin{pmatrix} 0 & a\mathrm{d}z & a\mathrm{d}y \\ b\mathrm{d}z & 0 & b\mathrm{d}x \\ c\mathrm{d}y & c\mathrm{d}x & 0 \end{pmatrix}, \quad \text{with } a = a(z),\ b = b(z),\ c = c(z), \quad (32)$$

via (2).

It is easy to calculate the projective Weyl forms $W^a{}_b$, and the projective Schouten forms ω_b, for this connection. They read:

$$W^a{}_b = \begin{pmatrix} -\frac{1}{2}c'\mathrm{d}x\wedge\mathrm{d}y & 0 & -a'\mathrm{d}y\wedge\mathrm{d}z \\ 0 & \frac{1}{2}c'\mathrm{d}x\wedge\mathrm{d}y & -b'\mathrm{d}x\wedge\mathrm{d}z \\ -\frac{1}{2}c'\mathrm{d}y\wedge\mathrm{d}z & -\frac{1}{2}c'\mathrm{d}x\wedge\mathrm{d}z & 0 \end{pmatrix},$$

and

$$\omega_a = \left(-bc\mathrm{d}x + \frac{1}{2}c'\mathrm{d}y,\quad -ac\mathrm{d}y + \frac{1}{2}c'\mathrm{d}x,\quad -ab\mathrm{d}z\right).$$

With this information in mind it is easy to check that

$$g^{ab}=\begin{pmatrix}-fa' & g^{12} & 0\\ g^{12} & -fb' & 0\\ 0 & 0 & g^{33}\end{pmatrix},\tag{33}$$

with some undetermined functions $f=f(x,y,z)$, $g^{12}=g^{12}(x,y,z)$, $g^{33}=g^{33}(x,y,z)$, satisfies (31). Thus the connection Γ^a_b may, in principle, be the Levi-Civita connection of some metric $\hat{g}_{ab}$. According to Theorem 2.3 we may expect that the inverse of this g^{ab} is proportional to $\hat{g}_{ab}$.

2.2. Comparing natural projective and (pseudo-)Riemannian tensors

Proposition 2.1 in an obvious way implies the following corollary:

Corollary 2.4. *The Levi-Civita connection $\hat{\nabla}$ of a metric $\hat{g}_{ab}$ has its projective Schouten tensor equal to the Levi-Civita one, $\hat{\mathsf{P}}_{ab}=\overset{LC}{\mathsf{P}}_{ab}$, if and only if its Einstein (hence the Ricci) tensor vanishes. If this happens $\hat{\mathsf{P}}_{ab}\equiv 0$, and both the projective and the Levi-Civita–Weyl tensors are equal, $\hat{W}^a{}_{bcd}=\overset{LC}{W}{}^a{}_{bcd}$.*

Now we answer the question whether there are Ricci non-flat metrics having equal projective and Levi-Civita–Weyl tensors. We use (29). The requirement that $\hat{W}^a{}_{bcd}=\overset{LC}{W}{}^a{}_{bcd}$ yields the following proposition.

Proposition 2.5. *The Levi-Civita connection $\hat{\nabla}$ of a metric $\hat{g}_{ab}$ has its projective Weyl tensor equal to the Levi-Civita one, $\hat{W}^a{}_{bcd}=\overset{LC}{W}{}^a{}_{bcd}$, if and only if its Levi-Civita–Ricci tensor satisfies*

$$M_{abcd}{}^{ef}\,\overset{LC}{R}_{ef}=0,\tag{34}$$

where

$$M_{abcd}{}^{ef}=\hat{g}_{ac}\delta^e_d\delta^f{}_b-\hat{g}_{ad}\delta^e{}_c\delta^f{}_b+\hat{g}_{ad}\hat{g}_{cb}\hat{g}^{ef}-\hat{g}_{ac}\hat{g}_{db}\hat{g}^{ef}+(n-1)(\hat{g}_{bd}\delta^e{}_a\delta^f{}_c-\hat{g}_{bc}\delta^e{}_a\delta^f{}_d).$$

One easily checks that the *Einstein metrics*, i.e., the metrics for which

$$\overset{LC}{R}_{ab}=\Lambda\hat{g}_{ab},$$

satisfy (34). Therefore we have the following corollary:

Corollary 2.6. *The projective and the Levi-Civita–Weyl tensors of Einstein metrics are equal. In particular, all conformally flat Einstein metrics (metrics of constant curvature) are projectively equivalent.*

It is interesting to know if there are non-Einstein metrics satisfying condition (34).

2.3. Formulation a'la Roger Liouville

In this subsection we shall link our work with the approach of [1].

If ∇ is in the projective class of the Levi-Civita connection $\hat{\nabla}$ of a metric $\hat{g}$ we have:

$$0=\hat{\mathrm{D}}\hat{g}_{ab}=\mathrm{D}\hat{g}_{ab}-2A\hat{g}_{ab}-A_a\theta^c\hat{g}_{cb}-A_b\theta^c\hat{g}_{ac},$$

for some 1-form $A=A_a\theta^a$. Thus the condition that a torsionless connection ∇ is projectively equivalent to the Levi-Civita connection of some metric, is equivalent to the existence of a pair $(\hat{g}_{ab},A_a)$ such that

$$\mathrm{D}\hat{g}_{ab}=2A\hat{g}_{ab}+\theta^c(A_a\hat{g}_{cb}+A_b\hat{g}_{ac}),$$

with an *invertible* symmetric tensor $\hat{g}_{ab}$. Dually this last means that a torsionless connection ∇ is projectively equivalent to a Levi-Civita connection of some metric, iff there exists a pair $(\hat{g}^{ab},A_a)$ such that

$$\mathrm{D}\hat{g}^{ab}=-2A\hat{g}^{ab}-A_c(\theta^b\hat{g}^{ca}+\theta^a\hat{g}^{cb}),\tag{35}$$

with an invertible $\hat{g}^{ab}$.

The unknown A can be easily eliminated from these equations by contracting with the inverse $\hat{g}_{ab}$:

$$A=-\frac{\hat{g}_{ab}\mathrm{D}\hat{g}^{ab}}{2(n+1)},$$

so that the 'if an only if' condition for ∇ to be in a projective class of a Levi-Civita connection $\hat{\nabla}$ is the existence of $\hat{g}^{ab}$ such that

$$2(n+1)\mathrm{D}\hat{g}^{ab} = 2(\hat{g}_{cd}\mathrm{D}\hat{g}^{cd})\hat{g}^{ab} + (\hat{g}_{ef}\nabla_c\hat{g}^{ef})(\theta^b\hat{g}^{ca} + \theta^a\hat{g}^{cb}), \qquad \hat{g}_{ac}\hat{g}^{cb} = \delta^b{}_c.$$

This is an unpleasant-to-analyse, *nonlinear* system of PDEs, for the unknown $\hat{g}^{ab}$. It follows that it is more convenient to discuss the equivalent system (35) for the unknowns $(\hat{g}^{ab}, A_a)$, which we will do in the following.

The aim of this subsection is to prove the following theorem:

Theorem 2.7. *A torsionless connection $\tilde{\nabla}$ on an n-dimensional manifold M is projectively equivalent to a Levi-Civita connection $\hat{\nabla}$ of a metric $\hat{g}_{ab}$ if and only if its projective class $[\tilde{\nabla}]$ contains a special projective subclass $[\nabla]$ whose connections ∇ satisfy the following: for every $\nabla \in [\nabla]$ there exists a nondegenerate symmetric tensor g^{ab} and a vector field μ^a on M such that*

$$\nabla_c g^{ab} = \mu^a\delta^b{}_c + \mu^b\delta^a{}_c,$$

or, which is the same, there exists a nondegenerate g^{ab} and μ^a such that:

$$\mathrm{D}g^{ab} = \mu^a\theta^b + \mu^b\theta^a. \tag{36}$$

Proof. If $\hat{\nabla}$ is the Levi-Civita connection of a metric $\hat{g} = \hat{g}_{ab}\theta^a\theta^b$, we consider connections ∇ associated with $\hat{\nabla}$ via (1), in which $A = \mathrm{d}\phi$, with arbitrary functions (potentials) on M. This is a *special* class of connections, since the projective Schouten tensor $\hat{\mathsf{P}}_{ab}$ for $\hat{\nabla}$ is *symmetric* (see Corollary 2.2), and the transformation (22) with gradient As, preserves the symmetry of the projective Schouten tensor (see Remark 1.2).

Any connection ∇ from this special class satisfies (35) with $A = \mathrm{d}\phi$, and therefore is characterised by the potential ϕ, $\nabla = \nabla(\phi)$.

We now take the inverse $\hat{g}^{ab}$ of the metric $\hat{g}^{ab}$, $\hat{g}_{ac}\hat{g}^{cb} = \delta_a{}^b$, and *rescale* it to

$$g^{ab} = \mathrm{e}^{2f}\hat{g}^{ab},$$

where f is a function on M. Using (35) with $A = \mathrm{d}\phi$, after a short algebra, we get:

$$\mathrm{D}g^{ab} = -2(\mathrm{d}\phi - \mathrm{d}f)g^{ab} - (\nabla_c\phi)(\theta^b g^{ca} + \theta^a g^{cb}).$$

Thus taking

$$f = \phi + \text{const},$$

for each $\nabla = \nabla(\phi)$ from the special class $[\nabla]$, we associate $g^{ab} = \mathrm{e}^{2f}\hat{g}^{ab}$ satisfying

$$\mathrm{D}g^{ab} = -(\nabla_c\phi)(\theta^b g^{ca} + \theta^a g^{cb}).$$

Defining $\mu^a = -A_c g^{ca} = -\mathrm{e}^{2f}(\nabla_c\phi)\hat{g}^{ca}$ we get (36). Obviously g^{ab} is symmetric and nondegenerate since $\hat{g}^{ab}$ was.

The proof in the opposite direction is as follows:

We start with (∇, g^{ab}, μ^a) satisfying (36). In particular, connection ∇ is *special*, i.e., it has symmetric projective Schouten tensor and, by Corollary 1.5, its curvature satisfies

$$\Omega^a{}_a = 0.$$

Since g^{ab} is invertible, we have a symmetric g_{ab} such that $g_{ac}g^{cb} = \delta^b{}_a$. We define

$$A = -g_{ab}\mu^b\theta^a. \tag{37}$$

Contracting with (36) we get:

$$g_{ab}\mathrm{D}g^{ab} = -2A, \quad \text{or} \quad A = -\frac{1}{2}g_{ab}\mathrm{D}g^{ab}.$$

Now this last equation implies that:

$$\mathrm{d}A = -\frac{1}{2}\mathrm{D}g_{ab}\wedge\mathrm{D}g^{ab} - \frac{1}{2}g_{ab}\mathrm{D}^2g^{ab}.$$

This compared with the Ricci identity $\mathrm{D}^2g^{ab} = \Omega^a{}_c g^{cb} + \Omega^b{}_c g^{ac}$, the defining Eq. (35), and its dual

$$\mathrm{D}g_{ab} = -g_{ac}g_{bd}(\mu^c\theta^d + \mu^d\theta^c),$$

yields

$$\mathrm{d}A = -\Omega^a{}_a = 0.$$

Thus the 1-form A defined by (37) is *locally a gradient* of a function ϕ_0 on M, $A = \mathrm{d}\phi_0$. The potential ϕ_0 is defined by (∇, g^{ab}, μ^a) up to $\phi_0 \to \phi = \phi_0 + \text{const}$,

$$A = \mathrm{d}\phi.$$

We use it to rescale the inverse g_{ab} of g^{ab}. We define

$$\hat{g}_{ab} = e^{2\phi} g_{ab}.$$

This is a nondegenerate symmetric tensor on M.

Using our definitions we finally get

$$\begin{aligned} D\hat{g}_{ab} &= 2d\phi\hat{g}_{ab} - e^{2\phi} g_{ac} g_{bd}(\mu^c\theta^d + \mu^d\theta^c) \\ &= 2A\hat{g}_{ab} + A_a\hat{g}_{bc}\theta^c + A_b\hat{g}_{ac}\theta^c. \end{aligned}$$

This means that the new torsionless connection $\hat{\nabla}$ defined by (1), with A as above, satisfies

$$\hat{D}\hat{g}_{ab} = D\hat{g}_{ab} - 2A\hat{g}_{ab} - A_a\hat{g}_{bc}\theta^c - A_b\hat{g}_{ac}\theta^c = 0,$$

and thus is the Levi-Civita connection for a metric $\hat{g} = \hat{g}_{ab}\theta^a\theta^b$. Since $A = d\phi$ this shows that in the special projective class defined by ∇ there is a Levi-Civita connection $\hat{\nabla}$. This completes the proof. □

We also have the following corollary, which can be traced back to Roger Liouville [1], (see also [4,12,2,3]):

Corollary 2.8. *A projective structure $[\hat{\nabla}]$ on an n-dimensional manifold M contains a Levi-Civita connection of some metric if and only if at least one special connection ∇ in $[\hat{\nabla}]$ admits a solution to the equation*

$$\nabla_c g^{ab} - \frac{1}{n+1}\delta^a{}_c\nabla_d g^{bd} - \frac{1}{n+1}\delta^b{}_c\nabla_d g^{ad} = 0. \tag{38}$$

with a symmetric and nondegenerate tensor g^{ab}.

Proof. We use Theorem 2.7.

If (∇, g^{ab}, μ^a) satisfies (36) it is a simple calculation to show that (38) holds.

The other way around: if (38) holds for a special connection ∇ and an invertible g^{ab}, then defining μ^a by $\mu^a = \frac{1}{n+1}\nabla_d g^{ad}$ we get $\nabla_c g^{ab} = \mu^a\delta^b{}_c + \mu^b\delta^a{}_c$, i.e., Eq. (36), after contracting with θ^c. Now, if we take any other special connection $\hat{\nabla}$, then it is related to ∇ via $\hat{\nabla}_X(Y) = \nabla_X(Y) + X(\phi)Y + Y(\phi)X$. Rescalling g^{ab} to $\hat{g}^{ab} = e^{-2\phi}g^{ab}$ one checks that $\hat{\nabla}_c\hat{g}^{ab} - \frac{1}{n+1}\delta^a{}_c\hat{\nabla}_d\hat{g}^{bd} - \frac{1}{n+1}\delta^b{}_c\hat{\nabla}_d g^{ad} = 0$. Thus in any special connection $\hat{\nabla}$ we find an invertible $\hat{g}^{ab} = e^{-2\phi}g^{ab}$ with $\hat{\mu}^a = \frac{1}{n+1}\hat{\nabla}_d\hat{g}^{ad}$ satisfying $\hat{\nabla}_c\hat{g}^{ab} = \hat{\mu}^a\delta^b{}_c + \hat{\mu}^b\delta^a{}_c$. □

Remark 2.9. It is worthwhile to note that $\hat{\mu}^a$ and μ^b as in the above proof are related by

$$\hat{\mu}^a = e^{-2\phi}(\mu^a + g^{da}\nabla_d\phi).$$

2.4. Prolongation and obstructions

In this section, given a projective structure $[\nabla]$, we restrict it to a corresponding *special* projective subclass. All the calculations below are performed assuming that ∇_a is in this special projective subclass.

We will find consequences of the necessary and sufficient conditions (36) for this special class to include a Levi-Civita connection.

Applying D on both sides of (36), and using the Ricci identity (11) we get as a consequence:

$$\Omega^b{}_a g^{ac} + \Omega^c{}_a g^{ba} = D\mu^c \wedge \theta^b + D\mu^b \wedge \theta^c. \tag{39}$$

This expands to the following tensorial equation:

$$\delta^b{}_d\nabla_a\mu^c - \delta^b{}_a\nabla_d\mu^c + \delta^c{}_d\nabla_a\mu^b - \delta^c{}_a\nabla_d\mu^b = R^b{}_{ead}g^{ec} + R^c{}_{ead}g^{be}. \tag{40}$$

Now contracting this equation in $\{ac\}$ we get:

$$\nabla_a\mu^b = \delta^b{}_a\rho - \mathsf{P}_{ac}g^{bc} - \frac{1}{n}W^b{}_{cda}g^{cd} \tag{41}$$

with some function ρ on M. This is the prolonged equation (36). It can be also written as:

$$D\mu^b = \rho\theta^b - \omega_c g^{bc} - \frac{1}{n}W^b{}_{cda}g^{cd}\theta^a. \tag{42}$$

Applying D on both sides of this equation, after some manipulations, one gets the equation for the function ρ:

$$\nabla_a\rho = -2\mathsf{P}_{ab}\mu^b + \frac{2}{n}Y_{abc}g^{bc}. \tag{43}$$

This is the last prolonged equation implied by (36). It can be also written as:

$$D\rho = -2\omega_b\mu^b + \frac{2}{n}Y_{abc}g^{bc}\theta^a. \tag{44}$$

Thus we have the following theorem [12]:

Theorem 2.10. *Eq. (38) admits a solution for g^{ab} if and only if the following system*

$$\begin{aligned} \mathrm{D}g^{bc} &= \mu^c\theta^b + \mu^b\theta^c \\ \mathrm{D}\mu^b &= \rho\theta^b - \omega_c g^{bc} - \frac{1}{n}W^b{}_{cda}g^{cd}\theta^a \\ \mathrm{D}\rho &= -2\omega_b\mu^b + \frac{2}{n}Y_{abc}g^{bc}\theta^a, \end{aligned} \tag{45}$$

has a solution for (g^{ab}, μ^c, ρ).

Simple obstructions for having solutions to (45) are obtained by inserting $\mathrm{D}\mu^b$ from (42) into the integrability conditions (39), or what is the same, into (40). This insertion, after some algebra, yields the following proposition.

Proposition 2.11. *Eq. (42) is compatible with the integrability conditions (39)–(40) only if g^{ab} satisfies the following algebraic equation:*

$$T_{[ed]}{}^{cb}{}_{af}g^{af} = 0, \tag{46}$$

where

$$T_{[ed]}{}^{cb}{}_{af} = \frac{1}{2}\delta^c{}_{(a}W^b_{f)ed} + \frac{1}{2}\delta^b{}_{(a}W^c_{f)ed} + \frac{1}{n}W^c{}_{(af)[e}\delta^b_{d]} + \frac{1}{n}W^b{}_{(af)[e}\delta^c_{d]}. \tag{47}$$

Remark 2.12. Note that although the integrability condition (46) was derived in the special gauge when the connection ∇ was special, it is gauge independent. This is because the condition involves the *projectively invariant* Weyl tensor, and because it is homogeneous in g^{ab}.

For each pair of distinct indices $[ed]$ the tensor $T_{[ed]}{}^{cb}{}_{af}$ provides a map

$$S^2M \ni \kappa^{ab} \xrightarrow{\mathcal{T}_{[ed]}} \kappa'^{ab} = T_{[ed]}{}^{ab}{}_{cd}\kappa^{cd} \in S^2M, \tag{48}$$

which is an endomorphism $\mathcal{T}_{[ed]}$ of the space S^2M of symmetric 2-tensors on M. It is therefore clear that Eq. (46) has a *nonzero* solution for g^{ab} only if *each* of these endomorphisms *is singular*. Therefore we have the following theorem (see also the last section in [4]):

Theorem 2.13. *A necessary condition for a projective structure $[\nabla]$ to include a Levi-Civita connection of some metric g is that all the endomorphisms $\mathcal{T}_{[ed]} : S^2M \to S^2M$, built from its Weyl tensor, as in (47), have nonvanishing determinants. In dimension $n \geq 3$ this gives in general $\frac{n(n-1)}{2}$ obstructions to metrisability.*

Remark 2.14 (*Puzzle*)**.** Note that here we have $I = \frac{n(n-1)}{2}$ obstructions, whereas the naive count, as adapted from [4], yields $I' = \frac{1}{4}(n^4 - 7n^2 - 6n + 4)$. For $n = 3$, we see that we constructed $I = 3$ invariants, wheres I' says that there is only one. Why?

Remark 2.15. Note that Remark 2.12 enabled us to use *any* connection from the projective class, not only the special ones, in this theorem.

Further integrability conditions for (36) may be obtained by applying D on both sides of (42) and (44). Applying it on (42), using again the Ricci identity (11), after some algebra, we get the following proposition.

Proposition 2.16. *The integrability condition $\mathrm{D}^2\mu^b = \Omega^b{}_a\mu^a$, for (g^{ab}, μ^c, ρ) satisfying (45), is equivalent to:*

$$S_{[ae]}{}^b{}_{cd}g^{cd} = \left(\frac{n+4}{2}W^b{}_{cae} + W^b{}_{[ae]c}\right)\mu^c, \tag{49}$$

where the tensor $S_{[ae]}{}^b{}_{cd}$ is given by:

$$S_{[ae]}{}^b{}_{cd} = \frac{n-2}{2}Y_{ea(c}\delta^b_{d)} + \nabla_{(c}W^b_{d)ea} + W^b_{(cd)[e;a]}.$$

Here, in the last term, for simplicity of the notation, we have used the semicolon to denote the covariant derivative, $\nabla_e f = f_{;e}$.

Remark 2.17. Note that in dimension $n = 2$, where $W^a_{bcd} \equiv 0$, the integrability conditions (46) and (49) are automatically satisfied.

The last integrability condition $\mathrm{D}^2\rho = 0$ yields:

Proposition 2.18. *The integrability condition* $\mathrm{D}^2\rho = 0$*, for* (g^{ab}, μ^c, ρ)*, satisfying* (45) *is equivalent to:*

$$U_{[ab]cd}g^{cd} = -\frac{n+3}{2}Y_{bac}\mu^c, \tag{50}$$

where the tensor $U_{[ab](cd)}$ *reads:*

$$U_{[ab]cd} = \nabla_{[a}Y_{b](cd)} + W^e_{(cd)[a}\mathsf{P}_{b]e}.$$

Remark 2.19. For the sufficiency of conditions (46), (49) and (50) see Remark 4.1.

3. Metrisability of a projective structure check list

Here, based on Theorems 2.3, 2.7, 2.10 and 2.13 and Propositions 2.11, 2.16 and 2.18, we outline a procedure how to check if a given projective structure contains a Levi-Civita connection of some metric. The procedure is valid for the dimension $n \geq 3$.

Given a projective structure $(M, [\nabla])$ on an n-dimensional manifold M:

(1) calculate its Weyl tensor $W^a_{\ bcd}$ and the corresponding operators $\mathcal{T}_{[ed]}$ as in (48). If at least one of the determinants $\tau_{ed} = \det(\mathcal{T}_{[ed]})$, $e < d = 1, 2, \ldots, n$, is not zero the projective structure $(M, [\nabla])$ does *not* include any Levi-Civita connection.
(2) If all the determinants τ_{ed} vanish, find a special connection ∇^0 in $[\nabla]$, and restrict to a special projective subclass $[\nabla^0] \subset [\nabla]$.
(3) Now taking any connection ∇ from $[\nabla^0]$ calculate the Weyl, (symmetric) Schouten, and Cotton tensors, and the tensors $T_{[ed]}{}^{cb}{}_{af}$, $S_{[ae]}{}^{b}{}_{cd}$, $U_{[ab]cd}$ of Propositions 2.11, 2.16 and 2.18.
(4) Solve the *linear algebraic* equations (46), (49) and (50) for the unknown symmetric tensor g^{ab} and vector field μ^a.
(5) If these equations *have no solutions*, or the $n \times n$ symmetric matrix g^{ab} *has vanishing determinant*, then $(M, [\nabla])$ does *not* include any Levi-Civita connection.
(6) If Eqs. (46), (49) and (50) admit solutions with nondegenerate g^{ab}, find the inverse g_{ab} of the general solution for g^{ab}, and check whether Eq. (30) is satisfied. If this equation cannot be satisfied by restricting the free functions in the general solution g^{ab} of Eqs. (46), (49) and (50), then $(M, [\nabla])$ does *not* include any Levi-Civita connection.
(7) If (30) may be satisfied, restrict the general solution g^{ab} of (46), (49) and (50) to only g^{ab}s satisfying (30), and insert (g^{ab}, μ^a), with such g^{ab}s and the most general μ^a solving (46), (49) and (50), in Eqs. (45).
(8) Find the general solution to Eqs. (45) for (g^{ab}, μ^a, ρ), with (g^{ab}, μ^a) from the ansatz described in point (7).
(9) If the solution for such (g^{ab}, μ^a, ρ) does not exist, or the symmetric tensor g^{ab} is degenerate, then $(M, [\nabla])$ does *not* include any Levi-Civita connection.
(10) Otherwise find the inverse g_{ab} of g^{ab} from the solution (g^{ab}, μ^a, ρ), and solve for a function ϕ on M such that $\mathrm{d}\phi = -g_{ab}\mu^a\theta^b$.
(11) The metric $\hat{g} = \mathrm{e}^{2\phi}g_{ab}\theta^a\theta^b$ has the Levi-Civita connection $\hat{\nabla}$ which is in the special projective class $[\nabla^0] \subset [\nabla]$.

4. 3-dimensional examples

Example 1. Here, as the first example, we consider a 3-dimensional projective structure $(M, [\nabla])$ with the projective class represented by the connection 1-forms:

$$\Gamma^a_{\ b} = \begin{pmatrix} \frac{1}{2}a\mathrm{d}x - \frac{1}{4}b\mathrm{d}y & -\frac{1}{4}b\mathrm{d}x & 0 \\ -\frac{1}{4}a\mathrm{d}y & -\frac{1}{4}a\mathrm{d}x + \frac{1}{2}b\mathrm{d}y & 0 \\ c\mathrm{d}y - \frac{1}{4}a\mathrm{d}z & c\mathrm{d}x - \frac{1}{4}b\mathrm{d}z & -\frac{1}{4}a\mathrm{d}x - \frac{1}{4}b\mathrm{d}y \end{pmatrix}. \tag{51}$$

The 3-manifold M is parameterised by (x, y, z), and $a = a(z)$, $b = b(z)$, $c = c(z)$ are sufficiently smooth real functions of z. In addition we assume that

$$a \neq 0, \qquad b \neq 0, \qquad c \neq \text{const}.$$

It can be checked that this connection is special. More specifically we have:

$$W^a_{\ b}=\begin{pmatrix}-\frac12 c'\mathrm{d}xy-\frac38 a'\mathrm{d}xz+\frac14 b'\mathrm{d}yz & \frac38 b'\mathrm{d}xz & \frac18 b'\mathrm{d}xy\\ \frac38 a'\mathrm{d}yz & \frac12 c'\mathrm{d}xy+\frac14 a'\mathrm{d}xz-\frac38 b'\mathrm{d}yz & -\frac18 a'\mathrm{d}xy\\ -ac\mathrm{d}xy-\frac12 c'\mathrm{d}yz & bc\mathrm{d}xy-\frac12 c'\mathrm{d}xz & \frac18 a'\mathrm{d}xz+\frac18 b'\mathrm{d}yz\end{pmatrix},$$

where $(\mathrm{d}xy, \mathrm{d}xz, \mathrm{d}yz)$ is an abbreviation for $(\mathrm{d}x\wedge\mathrm{d}y, \mathrm{d}x\wedge\mathrm{d}z, \mathrm{d}y\wedge\mathrm{d}z)$, and

$$\omega_a=\Big(-\frac{3}{16}a^2\mathrm{d}x+\frac{1}{16}(8c'+ab)\mathrm{d}y-\frac18 a'\mathrm{d}z,\quad \frac{1}{16}(8c'+ab)\mathrm{d}x-\frac{3}{16}b^2\mathrm{d}y-\frac18 b'\mathrm{d}z,\quad -\frac18 a'\mathrm{d}x-\frac18 b'\mathrm{d}y\Big).$$

Having these relations we easily calculate the obstructions $\tau_{[ed]}$. These are:

$$\tau_{13}=-\frac{9}{8192}(a')^6,\qquad \tau_{23}=-\frac{9}{8192}(b')^6,$$

and

$$\tau_{12}=-\frac{3}{128}c^2(c')^2(ba'-ab')^2.$$

This shows that $(M,[\nabla])$ may be metrisable only if

$$a=\mathrm{const},\qquad b=\mathrm{const}.$$

For such a and b all the obstructions $\tau_{[ed]}$ vanish. Assuming this we pass to point (4) of our procedure from Section 3.

It follows that with our assumptions, the general solution of Eq. (46) is:

$$g^{11}=g^{22}=0,\qquad g^{13}=\frac{bc}{c'}g^{12},\qquad g^{23}=\frac{ac}{c'}g^{12}. \tag{52}$$

Inserting this in (49) shows that its general solution is given by the above relations for g^{ab} and

$$\mu^1=\frac{1}{12}\Big(1-\frac{4cc''}{(c')^2}\Big)bg^{12},\qquad \mu^2=\frac{1}{12}\Big(1-\frac{4cc''}{(c')^2}\Big)ag^{12}. \tag{53}$$

The general solution (52) and (53) of (46) and (49) is compatible with the last integrability condition (50) if and only if the function $c=c(z)$ defining our projective structure $(M,[\nabla])$ satisfies a third order ODE:

$$c^{(3)}c'c+\Big((c')^2-2cc''\Big)c''=0. \tag{54}$$

If this condition for $c=c(z)$ is satisfied then (52) and (53) is the general solution of (46), (49) and (50). Moreover, it follows that the solution (52) and (53) also satisfies (30), and the tensor g^{ab} is nondegenerate for this solution provided that $g^{12}\neq 0$.

This means that (i) the projective structure $(M,[\nabla])$ with $a\neq 0$, $b\neq 0$, $c\neq\mathrm{const}$ may include a Levi-Civita connection only if (54) holds, and (ii) if it holds, the integrability conditions (46), (49) and (50) are all satisfied with the general solution (52) and (53), with $g^{12}\neq 0$.

We now pass to point (8) of the procedure from Section 3: assuming that (54) holds, we want to solve (45) for (g^{ab},μ^a) satisfying (52) and (53).

It follows that the $\{11\}$ component of the first of Eq. (45) gives a further restriction on the function c. Namely, if (g^{ab},μ^a) are as in (52) and (53), then $\mathrm{D}g^{11}=2\mu^1\theta^1$ iff $c''c-(c')^2=0$, i.e., iff

$$c=c_1\mathrm{e}^{c_2z},\quad \text{where } c_1,\ c_2 \text{ are constants s.t. } c_1c_2\neq 0.$$

Luckily this c satisfies (54). Looking at the next component, $\{12\}$, of the first Eq. (45), we additionally get $\mathrm{d}g^{12}=-\frac12(a\mathrm{d}x+b\mathrm{d}y)g^{12}$. And now, this is compatible with the $\{13\}$ component of the first Eq. (45), if and only if $b=0$ or $g^{12}=0$. We have to exclude $g^{12}=0$, since in such a case g^{ab} is degenerate. On the other hand $b=0$ contradicts our assumptions about the function b. Thus, according to the procedure from Section 3, we conclude that $(M,[\nabla])$ with the connection represented by (51) with $ab\neq 0$ and $c\neq\mathrm{const}$ *never* includes a Levi-Civita connection.

Remark 4.1. Note that this example shows that even if all the integrability conditions (30), (46), (49) and (50) are satisfied Eqs. (45) may have no solutions with nondegenerate g^{ab}. Thus *conditions* (46), (49) *and* (50) *and* (30) *are not sufficient* for the existence of a Levi-Civita connection in the projective class.

Example 2. As a next example we consider the same 3-dimensional manifold M as above, and equip it with a projective structure $[\nabla]$ corresponding to $\Gamma^a_{\ b}$ as in (51), but now assuming that the functions $a = a(z)$ and $b = b(z)$ satisfy

$$a \equiv 0 \quad \text{and} \quad b \equiv 0.$$

For further convenience we change the variable $c = c(z)$ to the new function $h = h(z) \neq 0$ such that $c(z) = h'(z)$.

When running through the procedure of Section 3, which enables us to say if such a structure includes a Levi-Civita connection, everything goes in the same way as in the previous example, up to Eqs. (53). Thus applying our procedure of Section 3 we get that the general solution to (46) and (49) is given by

$$g^{11} = g^{22} = g^{13} = g^{23} = \mu^1 = \mu^2 = 0.$$

It follows that this general solution to (46) and (49), automatically satisfies (50) and (30).

Now, with $g^{11} = g^{22} = g^{13} = g^{23} = \mu^1 = \mu^2 = 0$, the first of Eq. (45) gives:

$$g^{12} = \text{const}, \qquad \mathrm{d}g^{33} = 2h'g^{12}\mathrm{d}z, \qquad \mu^3 = h'g^{12},$$

and the second, in addition, gives:

$$\rho = \frac{2}{3}h''g^{12}.$$

This makes the last of Eqs. (45) automatically satisfied.

The only differential equation to be solved is $\mathrm{d}g^{33} = 2h'g^{12}\mathrm{d}z$, which after a simple integration yields:

$$g^{33} = 2g^{12}h.$$

Thus we have

$$g^{ab} = g^{12}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2h \end{pmatrix},$$

with the inverse

$$g_{ab} = \frac{1}{g^{12}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2h} \end{pmatrix}, \quad g^{12} = \text{const} \neq 0,\ h = h(z) \neq 0.$$

Now, realising point (10) of the procedure of Section 3, we define

$$A = -g_{ab}\mu^a\theta^b = -\frac{h'}{2h}\mathrm{d}z = -\frac{1}{2}\mathrm{d}\log(h). \tag{55}$$

This means that the potential $\phi = -\frac{1}{2}\log(h)$, and that the metric $\hat{g}_{ab}$ whose Levi-Civita connection is in the projective class of

$$\Gamma^a_{\ b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ h'\mathrm{d}y & h'\mathrm{d}x & 0 \end{pmatrix}, \tag{56}$$

is given by

$$\hat{g}_{ab} = -\frac{1}{g^{12}}\begin{pmatrix} 0 & \frac{1}{h} & 0 \\ \frac{1}{h} & 0 & 0 \\ 0 & 0 & \frac{1}{2h^2} \end{pmatrix}, \quad g^{12} = \text{const} \neq 0,\ h = h(z) \neq 0,$$

or what is the same by:

$$\hat{g} = -\frac{1}{g^{12}h^2}\left(2h\mathrm{d}x\mathrm{d}y + \mathrm{d}z^2\right), \quad g^{12} = \text{const} \neq 0,\ h = h(z) \neq 0.$$

It is easy to check that in the coframe $(\theta^1, \theta^2, \theta^3) = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, the Levi-Civita connection 1-forms for the metric $\hat{g}$ as above is given by

$$\hat{\Gamma}^a_{\ b} = \begin{pmatrix} -\frac{h'}{2h}\mathrm{d}z & 0 & -\frac{h'}{2h}\mathrm{d}x \\ 0 & -\frac{h'}{2h}\mathrm{d}z & -\frac{h'}{2h}\mathrm{d}y \\ h'\mathrm{d}y & h'\mathrm{d}x & -\frac{h'}{2h}\mathrm{d}z \end{pmatrix},$$

which satisfies (2) with $\Gamma^a_{\ b}$ given by (56) and A given by (55).

Remark 4.2. Thus we have shown that the projective structure $[\nabla]$ generated by the connection 1-forms (56) is metrisable, and that modulo rescalling, $\hat{g} \to \text{const}\,\hat{g}$, there is a *unique* metric, whose Levi-Civita connection is in the projective structure $[\nabla]$. Note that the metric $\hat{g}$ has *Lorentzian* signature.

Example 3. Now we continue with the example of a projective structure defined in Section 2.1 by formula (32). Calculating the projective Cotton tensor for this structure we find that it is *projectively flat* if and only if

$$c'' = 0 \quad \text{and} \quad 2cb' + 3bc' = 0 \quad \text{and} \quad 2ca' + 3ac' = 0.$$

This happens when $a' = b' = c' = 0$, but also e.g. when $c = z$, $b = s_1 z^{-\frac{3}{2}}$ and $a = s_2 z^{-\frac{3}{2}}$, with s_1, s_2 being constants. If the structure is *not* projectively flat the most general nondegenerate solution to Eq. (46) is

$$g^{ab} = \begin{pmatrix} -\frac{g^{33}}{c'}a' & g^{12} & 0 \\ g^{12} & -\frac{g^{33}}{c'}b' & 0 \\ 0 & 0 & g^{33} \end{pmatrix}. \tag{57}$$

It follows that if $c' = 0$, projectively non-flat structures which are metrisable do not exist. In formula (57) we recognise (33) with $f = \frac{g^{33}}{c'}$. Looking for projectively non-flat structures, we now pass to Eq. (49). With g^{ab} as in (57) this, in particular, yields

$$\mu^1 = \mu^2 = 0 \quad \text{and} \quad ba' - ab' = 0.$$

Thus only the structures satisfying this last equation can be metrisable. In the following we assume that both a and b are *not* constant. Then

$$b = s_1 a,$$

with s_a a constant. This solution satisfies all the other Eqs. (49) if and only if

$$\mu^3 = \frac{2g^{12}(2cc'a' + ac'^2) + g^{33}(a'c'' - c'a'')}{6a'c'}.$$

Now, with all these choices Eqs. (50) are also satisfied. Thus we may pass to the differential equation (36) for the remaining undetermined g^{ab}. It follows that these equations can be satisfied if and only if

$$c = s_2 a$$

with $s_2 = \text{const}$. Now, the remaining Eqs. (36) are satisfied provided that the unknown functions g^{12} and g^{33} satisfy:

$$g^{12}_z = 2\,\frac{s_1}{s_2}\,a\,g^{33} \quad \text{and} \quad g^{33}_z = 2s_2\,a\,g^{12} \tag{58}$$

and are independent of the variables x and y. If g^{12} and g^{33} solve (58) then all the other Eqs. (45) are satisfied if and only if

$$\rho = s_1\,a^2\,g^{33} + \frac{2}{3}\,s_2\,a'\,g^{12}.$$

System (58) can be solved explicitly (the solution is not particularly interesting), showing that in this case also our procedure defined in Section 3 leads effectively to the solution of metrisability problem.

Example 4. Our last example goes beyond three dimensions. It deals with the so-called (anti-)de Sitter spaces.

Let X^a be a *constant* vector, and η_{ab} be a nondegenerate symmetric $n \times n$ *constant* matrix. We focus on an example when

$$\eta_{ab} = \text{diag}(1, \dots, 1, -1, \dots, -1),$$

with p '+1's, and q '−1's.

In

$$\mathcal{U} = \{(x^a) \in \mathbb{R}^n \mid \eta_{cd}X^c x^d \neq 0\}$$

we consider metrics $\hat{g}$ of the form

$$\hat{g} = \frac{\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b}{(\eta_{cd}X^c x^d)^2}. \tag{59}$$

We analyse these metrics in an orthonormal coframe

$$\theta^a = \frac{\mathrm{d}x^a}{\eta_{bc}X^b x^c}, \tag{60}$$

in which

$$\hat{g} = \eta_{ab}\theta^a\theta^b.$$

In the following we will use a convenient notation such that:

$$\eta_{fg}X^fX^g = \eta(X,X).$$

We call the vector X *timelike* iff $\eta(X,X) > 0$, *spacelike* iff $\eta(X,X) < 0$, and *null* iff $\eta(X,X) = 0$.

It is an easy exercise to find that in the coframe (60) the Levi-Civita connection 1-forms $\hat{\Gamma}^a_{\ b}$ associated with metrics (59) are:

$$\hat{\Gamma}^a_{\ b} = \eta_{bd}\,(X^a\theta^d - X^d\theta^a).$$

Thus the Levi-Civita connection curvature, $\hat{\Omega}^a_{\ b} = \mathrm{d}\hat{\Gamma}^a_{\ b} + \hat{\Gamma}^a_{\ c}\wedge\hat{\Gamma}^c_{\ b}$, is given by

$$\hat{\Omega}^a_{\ b} = -\eta(X,X)\theta^a\wedge\theta^d\eta_{bd}.$$

This, in particular, means that the Levi-Civita curvature tensor, $\hat{R}^a_{\ bcd}$, the Levi-Civita–Weyl tensor, $\overset{LC}{W}{}^a_{\ bcd}$, and the Ricci tensor $\overset{LC}{R}_{ab}$, look, respectively, as:

$$\hat{R}^a_{\ bcd} = \eta(X,X)\,(\eta_{bc}\delta^a_{\ d} - \eta_{bd}\delta^a_{\ c}),$$
$$\overset{LC}{W}{}^a_{\ bcd} = 0,$$

and

$$\overset{LC}{R}_{bd} = (1-n)\eta(X,X)\eta_{bd}.$$

This proves the following proposition:

Proposition 4.3. *The metrics*

$$\hat{g} = \frac{\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b}{(\eta_{cd}X^cx^d)^2}$$

are the metrics of constant curvature. Their curvature is totally determined by their constant Ricci scalar $\overset{LC}{R} = n(1-n)\eta(X,X)$. *It is positive, vanishing or negative depending on the causal properties of the vector* X. *Hence if* X *is spacelike* $(\mathcal{U},\hat{g})$ *is locally the de Sitter space, if* X *is timelike* $(\mathcal{U},\hat{g})$ *is locally the anti-de Sitter space, and if* X *is null* $(\mathcal{U},\hat{g})$ *is flat.*

Using this proposition and Corollary 2.6 we see that metrics (59) are all projectively equivalent. This fact may have some relevance in cosmology, as discussed e.g. in [13–16]. We discuss this point in more detail in a separate paper [17].

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