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Non-vacuum twisting type-N metrics

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Abstract
We present a number of results for twisting type-N metrics.
(a) A maximally reduced system of equations corresponding to the twisting type-N Einstein metrics is given. When the cosmological constant $\lambda \to 0$ they reduce to the standard equations for the vacuum twisting type Ns.
(b) All the metrics which are conformally equivalent to the twisting type-N metrics and which admit three-dimensional conformal group of symmetries are presented.
(c) In the Feferman class of metrics an example is given of a twisting type-N metric which satisfies Bach's equations but is not Einstein.

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1. Introduction

The Weyl tensor of a Lorentzian conformally non-flat four-dimensional manifold distinguishes at each point at most four null directions. These directions, known already to Cartan [4], are nowadays called principal null directions (PNDs). In the case of vacuum spacetimes (with or without a cosmological constant) a PND is always geodesic and shear-free [8]. If the number of PNDs at each point of the spacetime is equal to one, then the Riemann tensor of such a spacetime is, by definition, of type N. This terminology also applies to the metric which, in such a case, is called type N.

It is a well known fact [24, 26] that the Riemann tensor of a radiating gravitational field far from the bounded sources is of type N. This fact raised the question of the existence of exact solutions to the Einstein field equations with metrics of type N.

The oldest vacuum solutions of this kind are pp-waves [3]. They are contained in a wider family of solutions called Kundt's class [12]. Kundt's family is characterized by the condition that the PND of the corresponding metrics has vanishing expansion and vanishing twist (in addition to its geodetic and shear-free property). Robinson and Trautman [23] found all type-N vacuums with an expanding but non-twisting PND. It is very difficult to find exact type-N vacuums solutions with twisting PNDs. The only explicit[3] one is due to Hauser [9].

[3] That is, expressible by transcendental functions.

In the case of vacuum Einstein equations with cosmological constant all type-N solutions with non-twisting PNDs are known explicitly [7]. As far as we know the twisting case has not yet been analysed.

Type Ns with an energy–momentum tensor of the perfect fluid do not obey the Goldberg–Sachs theorem. Their PND need not be either geodetic or shear-free. Type-N perfect fluid spacetimes with geodetic PND are only possible in the case of PNDs having shear [20].

In this paper we present a number of results on twisting type-N metrics. Section 2 gives a maximally reduced system of equations corresponding to type-N vacuums with a cosmological constant. The rest of the paper is devoted to the twisting type-N metrics which are not vacuum. Such metrics appear naturally in twistor theory [21, 25] and the theory of Cauchy–Riemann structures. They constitute the Feferman (non-vacuum) class of metrics [6] which we briefly describe in section 3. Section 4 presents twisting type-N metrics with a three-dimensional group of conformal symmetries; section 5 gives examples of twisting type-N metrics in the case when the group of conformal motions has dimension two. The metrics presented in sections 4 and 5 do not belong to the Feferman class, but are also non-vacuum. Section 6 gives an example of a twisting type-N metric which is non-vacuum but satisfies Bach's equations. It belongs to the Feferman class.

2. Vacuum type Ns with cosmological constant

A way of obtaining the maximally reduced system of equations corresponding to the metrics with the energy–momentum tensor of the form $T_{\mu\nu} = k_\mu k_\nu$, where k_μ is a quadruple principal null direction, was presented by one of us (JFP) in an unpublished paper [22]. The method[4] of [22] can also be applied to the vacuum type-N equations

$$R_{\mu\nu} = \lambda g_{\mu\nu} \tag{1}$$

with cosmological constant λ. It leads to the following result.

Theorem 1. *Any type-N metric satisfying the Einstein equations (1) is generated by a single complex function $L = L(u, z, \bar z)$ of variables u (real) and z (complex), subject to the following equations:*

$$\begin{aligned} &\bar{\mathcal{D}}^2\mathcal{D}L - \mathcal{D}^2\bar{\mathcal{D}}\bar L = \tfrac{1}{3}\lambda(\mathcal{D}\bar L - \bar{\mathcal{D}}L)^3 \\ &\bar{\mathcal{D}}\partial_u\mathcal{D}L = \tfrac{1}{2}\lambda(\mathcal{D}\bar L - \bar{\mathcal{D}}L)[\mathcal{D}^2\bar L - \bar{\mathcal{D}}\mathcal{D}L]. \end{aligned} \tag{2}$$

Here, $\mathcal{D} = \partial_z - L\partial_u$. In terms of L the metric reads

$$\begin{aligned} &g = 2(\theta^1\theta^2 - \theta^3\theta^4) \\ &\theta^3 = \overline{\theta^3} = \mathrm{d}u + L\,\mathrm{d}z + \bar L\,\mathrm{d}\bar z \\ &\theta^1 = \overline{\theta^2} = [r + \tfrac{1}{2}(\mathcal{D}\bar L - \bar{\mathcal{D}}L)]\,\mathrm{d}z + \bar L_u\theta^3 \\ &\theta^4 = \overline{\theta^4} = \mathrm{d}r + [L_u(\mathcal{D}\bar L - \bar{\mathcal{D}}L) - \tfrac{1}{2}\mathcal{D}(\mathcal{D}\bar L - \bar{\mathcal{D}}L)]\,\mathrm{d}z + [\bar L_u(\bar{\mathcal{D}}L - \mathcal{D}\bar L) \\ &\qquad -\tfrac{1}{2}\bar{\mathcal{D}}(\bar{\mathcal{D}}L - \mathcal{D}\bar L)]\,\mathrm{d}\bar z + \left[\tfrac{1}{6}\lambda(-r^2 + \tfrac{5}{4}(\mathcal{D}\bar L - \bar{\mathcal{D}}L)^2) - \tfrac{1}{2}\partial_u(\mathcal{D}\bar L + \bar{\mathcal{D}}L)\right]\theta^3, \end{aligned} \tag{3}$$

[4] The results of [22] are described in [11, pp 240–2].

and the four-dimensional spacetime is parametrized by $(r, u, z, \bar{z})$*, with* r *being a real coordinate. The only non-vanishing coefficient of the Weyl tensor is*

$$\Psi_4 = \frac{\mathcal{D}_u^2 \bar{\mathcal{D}} \bar{L} + \frac{1}{6}\lambda(3\bar{L}_u - \bar{\mathcal{D}})(\bar{\mathcal{D}}^2 L - \partial p \bar{\mathcal{D}} \bar{L})}{r - \frac{1}{2}(\mathcal{D}\bar{L} - \bar{\mathcal{D}} L)}. \tag{4}$$

The metric (3) has a quadruple principal null direction $k = \partial_r$*. It is twisting iff*

$$\mathcal{D}\bar{L} - \bar{\mathcal{D}} L \neq 0. \tag{5}$$

As far as we know the above system of maximally reduced equations for the type-N metrics with a cosmological constant has not been presented in the literature so far. It may lead to new solutions only in the case of twisting k (condition (5) satisfied). In the non-twisting case all the metrics corresponding to solutions of (2) are known [7]. We were unable to find any explicit solution to the system of theorem 1 in the case of condition (5) being satisfied. Even the Hauser explicit solution [9] is not easily expressible in terms of function L only.

3. The Feferman class

In this paper we relax the Einstein condition and search for twisting type-N metrics, which do not satisfy any additional curvature conditions. One class of such metrics is given by the Feferman conformal class [6], which in the context of GR was first studied by Sparling [25]. To describe the Feferman metrics one needs the notion of a Cauchy–Riemann structure.

Definition 1. *A Cauchy–Riemann (CR) structure* $(\mathcal{N}, [(\Omega, \Omega_1)])$ *is a three-dimensional manifold* $\mathcal{N}$ *equipped with a class of pairs of 1-forms* $[(\Omega, \Omega_1)]$ *such that* Ω *is real- and* Ω_1 *is complex-valued,* $\Omega \wedge \Omega_1 \wedge \bar{\Omega}_1 \neq 0$ *at each point of* $\mathcal{N}$*, two pairs* (Ω, Ω_1) *and* (Ω', Ω_1') *are equivalent iff there exists non-vanishing functions* f *(real) and* h *(complex) and a complex function* p *on* $\mathcal{N}$ *such that*

$$\Omega' = f\Omega \qquad \Omega_1' = h\Omega_1 + p\Omega. \tag{6}$$

A Cauchy–Riemann structure is non-degenerate iff $\mathrm{d}\Omega \wedge \Omega \neq 0$.

Given a non-degenerate CR structure $(\mathcal{N}, [(\Omega, \Omega_1)])$ one can always choose a representative (Ω, Ω_1) from the class $[(\Omega, \Omega_1)]$ such that [5, 18]

$$\mathrm{d}\Omega = \mathrm{i}\Omega_1 \wedge \bar{\Omega}_1. \tag{7}$$

Since $(\Omega, \Omega_1, \bar{\Omega}_1)$ constitutes a basis of 1-forms on $\mathcal{N}$ then the differential of Ω_1 uniquely defines functions α, θ (complex) and β (real) such that

$$\mathrm{d}\Omega_1 = \bar{\alpha}\Omega_1 \wedge \bar{\Omega}_1 + \mathrm{i}\beta\Omega \wedge \Omega_1 - \theta\Omega \wedge \bar{\Omega}_1. \tag{8}$$

Let $(\partial_0, \partial, \bar{\partial})$ be a basis of vector fields on $\mathcal{N}$ dual to $(\Omega, \Omega_1, \bar{\Omega}_1)$, respectively. Then the equation $\mathrm{d}^2\Omega_1 \equiv 0$ implies the following identity:

$$\partial_0\alpha - \mathrm{i}\partial\beta + \bar{\partial}\bar{\theta} + \mathrm{i}\beta\alpha - \bar{\alpha}\bar{\theta} = 0. \tag{9}$$

It is convenient to introduce the following operators:

$$\triangle = \partial - \alpha \qquad \delta = \partial_0 + \mathrm{i}\beta. \tag{10}$$

Condition (6) does not fix the forms (Ω, Ω_1) totally. They still can be transformed by means of the following transformations:

$$\Omega \to t\bar{t}\Omega \qquad \Omega_1 \to t[\Omega_1 + \mathrm{i}\bar{\partial}\log(t\bar{t})\Omega], \tag{11}$$

where t is a non-vanishing complex function on $\mathcal{N}$. The corresponding transformations of functions α, β, θ are

$$\begin{aligned} &\alpha \to \frac{1}{t}(\alpha - \partial\log(t\bar{t}^2)) \\ &\theta \to \frac{1}{\bar{t}^2}(\theta + \bar{\alpha}z + \bar{\partial}z + \mathrm{i}z^2) \\ &\beta \to \frac{1}{|t|^2}[\beta - \bar{\partial}\partial\log t - \partial\bar{\partial}\log\bar{t} - \alpha\bar{\partial}\log t - \bar{\alpha}\partial\log\bar{t}2(\bar{\partial}\log t\partial + \partial\log\bar{t}\bar{\partial})\log|t|], \end{aligned} \tag{12}$$

where $z = \mathrm{i}\bar{\partial}\log(t\bar{t})$.

Let $(\mathcal{N}, [(\Omega, \Omega_1)])$ be a CR-structure with (Ω, Ω_1) satisfying (7). One defines a manifold

$$\mathcal{M} = \mathbb{R} \times \mathcal{N} \tag{13}$$

with a canonical projection $\pi : \mathcal{M} \to \mathcal{N}$ and pull-backs the forms (Ω, Ω_1) to $\mathcal{M}$ by means of π. Then, using the same letters to denote the pull-backs, one equips $\mathcal{M}$ with a class of Lorentzian metrics of the form

$$g = \mathrm{e}^{2\phi}[\Omega_1\bar{\Omega}_1 - \Omega(\mathrm{d}r + W\Omega_1 + \bar{W}\bar{\Omega}_1 + H\Omega)]. \tag{14}$$

Here ϕ, H (real) and W (complex) are arbitrary functions on $\mathcal{N}$ and r is a real coordinate along the factor $\mathbb{R}$ in $\mathcal{M} = \mathbb{R} \times \mathcal{N}$.

It follows that the vector field $k = \partial_r$ is null geodesic and shear-free in any metric (14). It generates a congruence of shear-free and null geodesics in $\mathcal{M}$ which is always twisting due to condition (7). The converse is also true. Any spacetime admitting a twisting congruence of shear-free and null geodesics can be obtained in this way [27]. This, in particular, means that any such spacetime defines its corresponding CR-structure—the three-dimensional manifold of the lines of the congruence.

In the context of the present paper it is interesting to ask when the metrics (14) are of type N with k being a quadruple principal null direction. The answer is given by the following theorem [15, 17].

Theorem 2. *The metric (14) has $k = \partial_r$ as a quadruple principal null direction if and only if*

$$\begin{aligned} W &= 2a\mathrm{i}\mathrm{e}^{\mathrm{i}r} + b \\ H &= \mathrm{e}^{\mathrm{i}r}(\bar{\Delta} - \mathrm{i}\bar{b})a + \mathrm{e}^{-\mathrm{i}r}(\Delta + \mathrm{i}b)\bar{a} + h \\ h &= \mathrm{i}(\Delta\bar{b} - \bar{\Delta}b) - 6a\bar{a} - \tfrac{1}{2}(\Delta\bar{\alpha} + \bar{\Delta}\alpha + \beta) \end{aligned} \tag{15}$$

where the complex functions $a : \mathcal{N} \to \mathbb{C}$ and $b : \mathcal{N} \to \mathbb{C}$ satisfy

$$2\mathrm{i}ha - 2\delta a - \mathrm{i}\partial\bar{\Delta}a - \partial(\bar{b}a) - b\bar{\Delta}a + \mathrm{i}b\bar{b}a = 0 \tag{16}$$

$$4\bar{\alpha}\bar{\theta} - 2\bar{\partial}\bar{\theta} + 3\mathrm{i}(\delta b - \bar{\theta}\bar{b}) + \partial(\bar{\Delta}b - \Delta\bar{b} - 4\mathrm{i}h) + 8\mathrm{i}(a\Delta\bar{a} - \partial(a\bar{a}) + \mathrm{i}ba\bar{a}) = 0. \tag{17}$$

Take

$$a = 0, \qquad b = \tfrac{2}{3}\mathrm{i}\alpha. \tag{18}$$

Then equation (16) is automatically satisfied and equation (17) becomes the identity (9). Then, applying theorem 1 we have the following corollary.

Corollary 1. *Let* $(\mathcal{N}, [(\Omega, \Omega_1)])$ *be a CR-structure generated by forms* (Ω, Ω_1) *satisfying condition (7). Then the metric*

$$g = \mathrm{e}^{2\phi}\big\{\Omega_1\bar{\Omega}_1 - \Omega\big[\mathrm{d}r + \tfrac{2}{3}\mathrm{i}\alpha\Omega_1 - \tfrac{2}{3}\mathrm{i}\bar{\alpha}\bar{\Omega}_1 + \tfrac{1}{6}(\triangle\bar{\alpha} + \bar{\triangle}\alpha - 3\beta)\Omega\big]\big\} \tag{19}$$

on manifold $\mathcal{M} = \mathbb{R} \times \mathcal{N}$ *is of type N with twisting shear-free null geodesics generated by the quadruple principal null direction* $k = \partial_r$.

The metrics (19) are called the Feferman metrics [6]. Their main property is presented in the following theorem.

Theorem 3. *Let a pair* (Ω, Ω_1) *satisfying (7) undergo transformation (11). Then the metric* g *of (19) transforms according to*[5] $g \to t\bar{t}g$.

Thus, any non-degenerate CR-structure defines a conformal class of Feferman metrics (19). Each of the metrics in the class is of type N and its quadruple principal null direction defines a congruence of shear-free and null geodesics with twisting rays. We stress here that corollary 1 provides an effective method of evaluating Feferman twisting type-N metrics for each non-degenerate CR-structure. For example, if the CR-structure $\mathcal{N}$ is embeddable in $\mathbb{C}^2$ (cf [28, p 499], for definition of embeddability) it may be parametrized by coordinates $(u, z, \bar{z})$, u real, z complex, and generated by a free complex function $L = L(u, z, \bar{z})$ such that $\mathcal{D}\bar{L} - \bar{\mathcal{D}}L \neq 0$. The 1-forms (Ω, Ω_1) satisfying (7) may be chosen to be

$$\begin{aligned} \Omega &= \mathrm{i}\frac{\mathrm{d}u + L\,\mathrm{d}z + \bar{L}\,\mathrm{d}\bar{z}}{\mathcal{D}\bar{L} - \bar{\mathcal{D}}L} \\ \Omega_1 &= \mathrm{d}z - \mathrm{i}\bar{w}\Omega, \end{aligned} \tag{20}$$

where

$$w = L_u + \mathcal{D}\log(\mathcal{D}\bar{L} - \bar{\mathcal{D}}L). \tag{21}$$

Then, the Feferman metric (19) is

$$g = \mathrm{e}^{2\phi}\big\{\mathrm{d}z\,\mathrm{d}\bar{z} - \Omega\big[\mathrm{d}r - \tfrac{1}{3}\mathrm{i}w\,\mathrm{d}z + \tfrac{1}{3}\mathrm{i}\bar{w}\,\mathrm{d}\bar{z} - \tfrac{1}{6}\mathcal{D}\bar{w}\Omega\big]\big\}. \tag{22}$$

Inserting (20) and (21) into (22) gives an explicit form of the Feferman metric for each embeddable CR-structure.

Another characterization of the Feferman class of metrics is given by the following theorem [14, 25].

Theorem 4. *Feferman metrics* g *are the only metrics satisfying the following three conditions:*

(a) g *are of type N with a quadruple principal null direction generated by a vector field* k*;*
(b) k *is geodesic, shear-free and twisting;*
(c) k *is a conformal Killing vector field.*

A slightly disappointing property of the Feferman class is given below [13].

Theorem 5. *None of the Feferman metrics satisfies Einstein equations* $R_{\mu\nu} = \lambda g_{\mu\nu}$.

[5] To obtain this result one has to redefine the r coordinate according to $r \to r + \frac{1}{3}\mathrm{i}\log(t/\bar{t})$.

4. Twisting type-N metrics with a three-dimensional group of conformal symmetries

In this section we find a local form of all Lorentzian metrics g which satisfy the following assumptions:

(a) g are of type N with k being a quadruple principal null direction;
(b) k is geodesic, shear-free and twisting;
(c) g is not conformally equivalent to any of the Feferman metrics and is not conformally flat;
(d) g admit at least three conformal Killing vector fields.

The following theorem is implicit in sections 4–6 of [15].

Theorem 6. *All the metrics satisfying assumptions (a)–(d) can locally be represented by*

$$g = e^{2\phi}[\Omega_1\bar{\Omega}_1 - \Omega(\mathrm{d}r + W\Omega_1 + \bar{W}\bar{\Omega}_1 + H\Omega)], \tag{23}$$

where

$$W = 2a\mathrm{i}e^{\mathrm{i}r} + b \qquad H = -a[e^{\mathrm{i}r}(\bar{\alpha} + \mathrm{i}\bar{b}) + e^{-\mathrm{i}r}(\alpha - \mathrm{i}b)] + \mathrm{i}(\bar{\alpha}b - \alpha\bar{b}) - 6a^2 - \tfrac{1}{2}\beta + \alpha\bar{\alpha} \tag{24}$$

and the functions $a > 0, b, \alpha, \beta, \theta$ are constants[6]*, satisfying*

$$-12\mathrm{i}a^2 - \bar{\alpha}b + \mathrm{i}b\bar{b} + 2\alpha(\mathrm{i}\bar{\alpha} + \bar{b}) - 3\mathrm{i}\beta = 0 \tag{25}$$

$$-8\mathrm{i}a^2(\alpha - \mathrm{i}b) - 3b\beta + 4\bar{\alpha}\bar{\theta} - 3\mathrm{i}\bar{b}\bar{\theta} = 0. \tag{26}$$

Ω and Ω_1 are related to α, β, θ by (8) and satisfy (7). The spacetime is locally a Cartesian product $\mathcal{M} = \mathbb{R} \times \mathcal{N}$, with $(\mathcal{N}, (\Omega, \Omega_1))$ being a three-dimensional non-degenerate CR-structure. The coordinate r is chosen so that the orbits of the three conformal symmetries X_i, $i = 1, 2, 3$ are given by $r =$ constant and $X_i(r) = 0$. The three symmetries X_i are such that

$$\mathcal{L}_{X_i}\Omega = \mathcal{L}_{X_i}\Omega_1 = 0 \qquad \forall i = 1, 2, 3, \tag{27}$$

so that they also constitute three symmetries of the CR-structure $(\mathcal{N}, (\Omega, \Omega_1))$.

It follows from this theorem that all the metrics satisfying (a)–(d) can be obtained by inspecting the list [5, 15, 19] of all non-degenerate CR-structures admitting three symmetries. Such structures are classified according to the Bianchi type of the corresponding symmetries. For each Bianchi type the forms Ω, Ω_1 and the constants α, β, θ are presented in [15]. Using this list, one has to check whether a given Bianchi type represented by constants α, β, θ is admitted by the type-N equations (25) and (26). If it is, one finds the corresponding a and b.

It turns out that only CR-structures with symmetry groups of Bianchi types VI_h and *VIII* are admitted by equations (25) and (26). Below we describe the corresponding solutions.

4.1. Solutions for Bianchi type VIII

In this case one has a one-parameter family of non-equivalent CR-structures, parametrized by $k \geqslant 0$, $k \neq 1$. The manifold $\mathcal{N}$ of such CR-structures can be coordinatized by $(u, z, \bar{z})$,

[6] If $a \equiv 0$ then the corresponding metric is in the Feferman class.

(u-real, z-complex) and the forms Ω, Ω_1 and the constants α, β, θ of theorem 5 can be chosen so that

$$\begin{aligned}
&\Omega = \frac{2}{k^2-1}\lambda_0 \qquad \Omega_1 = \mu_0 - \frac{k}{k^2-1}\lambda_0 \\
&\mu_0 = \frac{2\mathrm{e}^{\mathrm{i}u}}{z\bar{z}-1}\,\mathrm{d}z \qquad \lambda_0 = \mathrm{d}u + \frac{k\mathrm{e}^{\mathrm{i}u}-\mathrm{i}\bar{z}}{z\bar{z}-1}\mathrm{d}z + \frac{k\mathrm{e}^{-\mathrm{i}u}+\mathrm{i}z}{z\bar{z}-1}\,\mathrm{d}\bar{z} \\
&\alpha = 0 \qquad \beta = \tfrac{1}{4}(k^2-2) \qquad \theta = -\tfrac{1}{4}\mathrm{i}k^2.
\end{aligned} \tag{28}$$

Inserting the above α, β, θ into the type-N equations (25) and (26) one finds two branches of solutions for a and b. The corresponding metrics are

$$g = \mathrm{e}^{2\phi}[\Omega_1\bar{\Omega}_1 - \Omega\nu] \tag{29}$$

where the forms Ω and Ω_1 are given by (28) and the real 1-form ν is given below for each branch (a) and (b) separately.

(a) If $0 \leqslant k \leqslant \sqrt{2}/2$ then

$$\begin{aligned}
\nu = \mathrm{d}r &+ \tfrac{1}{2}\mathrm{i}\sqrt{3(1-k^2)}\left(\mathrm{e}^{\mathrm{i}r} \pm \sqrt{\frac{1-2k^2}{1-k^2}}\right)\Omega_1 - \tfrac{1}{2}\mathrm{i}\sqrt{3(1-k^2)}\left(\mathrm{e}^{-\mathrm{i}r} \pm \sqrt{\frac{1-2k^2}{1-k^2}}\right)\bar{\Omega}_1 \\
&+\left(\mp\tfrac{3}{8}\sqrt{(1-2k^2)(1-k^2)}(\mathrm{e}^{\mathrm{i}r}+\mathrm{e}^{-\mathrm{i}r}) + k^2 - \tfrac{7}{8}\right)\Omega.
\end{aligned}$$

Solutions are not conformally flat iff $k \neq \frac{1}{2}\sqrt{2}$.

(b) If $k \geqslant 0, k \neq 1, k \neq \sqrt{3}$ then

$$\begin{aligned}
\nu = \mathrm{d}r &+ \frac{\sqrt{3}}{2}(\mathrm{i}\mathrm{e}^{\mathrm{i}r} \pm \sqrt{1+k^2})\Omega_1 + \frac{\sqrt{3}}{2}(-\mathrm{i}\mathrm{e}^{-\mathrm{i}r} \pm \sqrt{1+k^2})\bar{\Omega}_1 \\
&+\tfrac{1}{8}(\mp 3\mathrm{i}\sqrt{1+k^2}(\mathrm{e}^{\mathrm{i}r}-\mathrm{e}^{-\mathrm{i}r}) - k^2 - 7)\Omega.
\end{aligned}$$

Solutions are not conformally flat iff $k \neq 1, k \neq \sqrt{3}$.

4.2. *Solutions for Bianchi type VI$_h$*

In this case one has only one CR-structure for each value of the real parameter $h = -[(1-d)/(1+d)]^2$, $-1 < d \leqslant 1$. For each value of d the CR-manifold $\mathcal{N}$ can be coordinatized by real (u, x, y), and the forms Ω, Ω_1 and the constants α, β, θ of theorem 5 can be chosen so that

$$\begin{aligned}
&\Omega = -\frac{2}{d+1}\lambda_0 \qquad \Omega_1 = \mu_0 + \frac{d}{d+1}\lambda_0 \\
&\mu_0 = y^{-1}d(x+\mathrm{i}y) \qquad \lambda_0 = y^d\,\mathrm{d}u - y^{-1}\,\mathrm{d}x \\
&\alpha = \tfrac{1}{2}\mathrm{i}(d-1) \qquad \beta = -\tfrac{1}{4}d \qquad \theta = -\tfrac{1}{4}\mathrm{i}d.
\end{aligned} \tag{30}$$

For the above α, β, θ the type-N equations (25) and (26) imply that the constant b is real and satisfies

$$8b^3 + 8(d-1)b^2 - 2(1+4d+d^2)b - 2d^3 - 3d^2 + 3d + 2 = 0. \tag{31}$$

Note that equation (31) always admits at least one real solution. Once the real solution $b = b(d)$ of this equation is known one has to check, whether the quantity

$$A = \tfrac{1}{48}[4b^2 + 6(d-1)b + 2d^2 - d + 2]$$

is positive. If it is positive, the constant

$$a = \sqrt{A}. \tag{32}$$

Otherwise there is no solution to (25) and (26) corresponding to $b = b(d)$.

To describe an example of explicit solutions to equations (31)–(32) we choose

$$d = \tfrac{1}{2}. \tag{33}$$

Then, there are three different solutions for a and b:

$$b_1 = -\tfrac{1}{4}(\sqrt{6}+1) \qquad a_1 = \tfrac{1}{8}\sqrt{6+\tfrac{5}{3}\sqrt{6}}, \tag{34}$$

$$b_2 = 1 \qquad a_2 = \tfrac{1}{4}, \tag{35}$$

$$b_3 = \tfrac{1}{4}(\sqrt{6}-1) \qquad a_3 = \tfrac{1}{8}\sqrt{6-\tfrac{5}{3}\sqrt{6}}. \tag{36}$$

The corresponding metrics are given by

$$g = e^{2\phi}[\Omega_1\bar{\Omega}_1 - \Omega\nu] \tag{37}$$

where the forms Ω and Ω_1 are given by (30) with $d = \frac{1}{2}$, and the real 1-form ν is given below for each of the three above solutions by

$$\nu_1 = dr + \tfrac{1}{4}\big[i\sqrt{6+\tfrac{5}{6}\sqrt{6}}e^{ir} - \sqrt{6} - 1\big]\Omega_1 + \tfrac{1}{4}[-i\sqrt{6+\tfrac{5}{6}\sqrt{6}}e^{-ir} - \sqrt{6} - 1]\bar{\Omega}_1$$
$$+\tfrac{1}{32}\big[i\sqrt{36+10\sqrt{6}}(e^{ir} - e^{-ir}) - \sqrt{6} - 10\big]\Omega$$

$$\nu_2 = dr + \big[\tfrac{1}{2}ie^{ir} + 1\big]\Omega_1 + \big[-\tfrac{1}{2}ie^{-ir} + 1\big]\bar{\Omega}_1 + \big[\tfrac{5}{16}i(e^{-ir} - e^{ir}) - \tfrac{3}{4}\big]\Omega$$

$$\nu_3 = dr + \tfrac{1}{4}\big[i\sqrt{6-\tfrac{5}{6}\sqrt{6}}e^{ir} + \sqrt{6} - 1\big]\Omega_1 + \tfrac{1}{4}\big[-i\sqrt{6-\tfrac{5}{6}\sqrt{6}}e^{-ir} + \sqrt{6} - 1\big]\bar{\Omega}_1$$
$$+\tfrac{1}{32}\big[i\sqrt{36-10\sqrt{6}}(e^{-ir} - e^{ir}) + \sqrt{6} - 10\big]\Omega.$$

Each of these metrics is conformally non-flat.

Although equation (31) can be solved explicitly, the formula for b is not very useful in obtaining the explicit forms of the metrics. It is more convenient to solve equation (31) for particular values of d as we did above for $d = \frac{1}{2}$. Instead of giving further examples we present a qualitative description of the solutions, which was obtained by numerical analysis of (31) and (32). We have three possible branches of solutions, corresponding to three different roots $b_1(d)$, $b_2(d)$, $b_3(d)$ of (31). These branches are as follows.

- $b_1(d)$ The solutions corresponding to the first root $b_1(d)$ are only possible if $d \geqslant -0.511\,878$. It turns out then that for each value of $d \geqslant -0.511\,878$ there exists a type-N metric, which is not conformally flat iff $d \neq -\frac{1}{2}$ and $d \neq 0$.
- $b_2(d)$ The second root $b_2(d)$ admits solutions for each value of the parameter $d \geqslant 0$. For each such d there exists precisely one type-N metric, which is nonconformally flat iff $d \neq 0$.
- $b_3(d)$ The third root $b_3(d)$ admits solutions only for $-0.511\,878 \leqslant d \leqslant -0.220\,789$ or $d \geqslant 0$. For each such d there exists precisely one type-N metric which is conformally non-flat iff $d \neq -0.333\,347$, $d \neq -0.220\,789$, $d \neq 0$, $d \neq 1$.

We close this section with a note that the metrics satisfying assumptions (a)–(d) are not conformally equivalent to the Ricci flat metrics. This result follows directly from the analysis performed in [15].

5. Examples of twisting type-N metrics admitting two conformal symmetries

In this section we present examples of type-N metrics admitting two conformal symmetries. We additionally assume that the metrics are not conformally flat and that they do not belong to the Feferman class. The general solution for such a problem is rather hopeless to obtain but the following two examples can be given.

Case A. Consider a three-dimensional manifold parametrized by the real coordinates (u, x, y). The CR-structure $(\mathcal{N}, [(\Omega, \Omega_1)])$ is generated on $\mathcal{N}$ by the forms

$$\Omega = \mathrm{d}u - y\,\mathrm{d}x \qquad \Omega_1 = \frac{\sqrt{2}}{2}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y). \tag{38}$$

The forms (38) satisfy (7). On $\mathcal{M} = \mathbb{R} \times \mathcal{N}$ introduce a coordinate r along the $\mathbb{R}$ factor and consider the metric

$$g = \mathrm{e}^{2\phi}[\Omega_1\bar{\Omega}_1 - \Omega\nu] \tag{39}$$

with the 1-form ν defined by

$$\begin{aligned}\nu = \mathrm{d}r + y^{-1}\bigg[\bigg(\frac{\sqrt{2}}{2} - \sqrt{3} + \left(\tfrac{3}{2}\right)^{1/4}\mathrm{i}\mathrm{e}^{\mathrm{i}r}\bigg)\Omega_1 + \bigg(\frac{\sqrt{2}}{2} - \sqrt{3} - \left(\tfrac{3}{2}\right)^{1/4}\mathrm{i}\mathrm{e}^{-\mathrm{i}r}\bigg)\bar{\Omega}_1\bigg] \\ + y^{-2}\bigg[\frac{\mathrm{i}6^{1/4}}{4}(\sqrt{6} - 2)(\mathrm{e}^{\mathrm{i}r} - \mathrm{e}^{-\mathrm{i}r}) + \frac{\sqrt{6}}{4} - 1\bigg]\Omega.\end{aligned} \tag{40}$$

It is a matter of straightforward calculation to see that the so defined metric is of type N, admits a congruence of twisting shear-free and null geodesics aligned with the principal null direction and is never conformally flat. Moreover, it has only two conformal symmetries $X_1 = \partial_u$, $X_2 = \partial_x$.

Case B. Now the CR-structure $(\mathcal{N}, [(\Omega, \Omega_1)])$ is generated by the forms

$$\Omega = \mathrm{d}u - y^2\,\mathrm{d}x \qquad \Omega_1 = \sqrt{y}(\mathrm{d}x + \mathrm{i}\,\mathrm{d}y). \tag{41}$$

The forms (41) satisfy (7). The type-N metric is defined on $\mathcal{M} = \mathbb{R} \times \mathcal{N}$ by

$$g = \mathrm{e}^{2\phi}[\Omega_1\bar{\Omega}_1 - \Omega\nu] \tag{42}$$

with the 1-form ν defined by

$$\begin{aligned}\nu = \mathrm{d}r + \frac{1}{2y^{3/2}}\Big[\Big(\mathrm{i}\sqrt{5 + 2\sqrt{19}}\mathrm{e}^{\mathrm{i}r} - \sqrt{19}\Big)\Omega_1 + \Big(-\mathrm{i}\sqrt{5 + 2\sqrt{19}}\mathrm{e}^{-\mathrm{i}r} - \sqrt{19}\Big)\bar{\Omega}_1\Big] \\ + \frac{\mathrm{i}}{8y^3}\Big[\sqrt{5 + 2\sqrt{19}}\Big(\sqrt{19} - 1\Big)(\mathrm{e}^{\mathrm{i}r} - \mathrm{e}^{-\mathrm{i}r}) - 16 - 2\sqrt{19}\Big]\Omega.\end{aligned} \tag{43}$$

Here r is a coordinate r along the factor $\mathbb{R}$ in $\mathcal{M}$.

The above metric is of type N, admits a congruence of twisting shear-free and null geodesics aligned with the principal null direction and is never conformally flat. It has only two conformal symmetries $X_1 = \partial_u$, $X_2 = \partial_x$.

6. Example of a type-N metric with vanishing Bach tensor and not conformal to an Einstein metric

It is interesting to ask whether metrics (39), (40), (42), (43) are conformally equivalent to Einstein metrics. It is known that a necessary condition for a metric to be conformal to an Einstein metric is that its Bach tensor

$$B_{\mu\nu} = C_{\mu\rho\nu\sigma}^{;\sigma\rho} + \tfrac{1}{2} C_{\mu\rho\nu\sigma} R^{\rho\sigma} \tag{44}$$

vanishes identically [2, 10]. (We denoted the Weyl conformal curvature by $C_{\mu\rho\nu\sigma}$.)

With the help of an extremely powerful symbolic algebra package GRTensor [16] we calculated the Bach tensor for metrics (39), (40) and (42), (43). In both cases it is never vanishing, so we conclude that the metrics (39), (40) and (42), (43) are not conformally equivalent to any Einstein metric.

Surprisingly, within the Feferman class we found a metric which has vanishing Bach tensor. This metric reads as follows:

$$g = \mathrm{e}^{2\phi}\left[\mathrm{d}x^2 + \mathrm{d}y^2 - \tfrac{2}{3}(\mathrm{d}x + y^3\,\mathrm{d}u)(y\,\mathrm{d}r + \tfrac{11}{9}\,\mathrm{d}x - \tfrac{1}{9}y^3\,\mathrm{d}u)\right]. \tag{45}$$

This metric, like any other from the Feferman class, is of type N, admits a congruence of twisting shear-free and null geodesics aligned with the principal null direction and is not conformally flat. It is interesting that it satisfies the Bach equations and, being in the Feferman class, is not conformal to any Einstein metrics. To the best of our knowledge this is the only known example of a Lorentzian metric having this last property [1].

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