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LETTER TO THE EDITOR

Generalized exterior forms, geometry and space-time

P Nurowski[1] **and D C Robinson**[2]

[1] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul. Hoza 69, Warszawa, Poland
[2] Mathematics Department, King's College London, Strand, London WC2R 2LS

E-mail: nurowski@fuw.edu.pl and david.c.robinson@kcl.ac.uk

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Abstract
The properties of generalized p-forms, first introduced by Sparling, are discussed and developed. Generalized Cartan structure equations for generalized affine connections are introduced. A new representation of Einstein's equations, using generalized forms, is given.

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In this letter a development of a *generalized* exterior algebra and calculus of p-forms will be presented. This type of extension of the ordinary calculus and algebra of differential forms was first introduced by Sparling in order to associate an abstract twistor structure with any real analytic Einstein space-time [1–4]. However it is clear that it is a tool which can be employed in more general physical and geometrical contexts. Here the aim is to show how such a formalism can be further developed by constructing generalized affine connections, and by providing a simple formulation of Einstein's vacuum equations.

A generalized p-form, $\overset{p}{\mathbf{a}}$, is defined to be an ordered pair of ordinary p- and $p+1$-forms, that is

$$\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}) \in \Lambda^p \times \Lambda^{p+1}, \tag{1}$$

where Λ^p denotes the module of p-forms on a differentiable manifold M of dimension n. By defining a minus one-form to be an ordered pair

$$\overset{-1}{\mathbf{a}} = (0, \overset{0}{\alpha}), \tag{2}$$

where $\overset{0}{\alpha}$ is a function on M, the range of p can be taken to be $-1 \leqslant p \leqslant n$. The manifold and forms may be real or complex but here n is taken to be the real dimension of M. The module of generalized p-forms will be denoted by Λ^p_G and the formal sum $\sum^p_{-1\leqslant p\leqslant n} \Lambda^p_G$ will be denoted by Λ_G. The letters over the forms indicate the degrees of the forms. Whenever these degrees are obvious they will be omitted. In the following, bold Latin letters will be used for generalized forms and normal Greek letters for ordinary forms. A generalized form given by a pair $(\overset{p}{\alpha}, 0)$ will be identified with the ordinary p-form $\overset{p}{\alpha}$. Hence, for example, a function on M will be identified with the generalized 0-form $(\overset{0}{\alpha}, 0)$ while the pair $(0, \overset{0}{\alpha})$ defines a generalized minus one-form.

If $\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ and $\overset{q}{\mathbf{b}} \equiv (\overset{q}{\beta}, \overset{q+1}{\beta})$, then the (*left*) generalized exterior product, $\wedge : \Lambda_G^p \times \Lambda_G^q \to \Lambda_G^{p+q}$, and the (*left*) generalized exterior derivative, d: $\Lambda_G^p \to \Lambda_G^{p+1}$, are defined to be:

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} \equiv (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}) \tag{3}$$

and

$$\mathrm{d}\, \overset{p}{\mathbf{a}} \equiv (\mathrm{d}\, \overset{p}{\alpha} + (-1)^{p+1} k \overset{p+1}{\alpha}, \mathrm{d}\, \overset{p+1}{\alpha}), \tag{4}$$

where k is a constant which, in the following, is assumed to be non-zero. It should be noted that it follows that

$$\overset{-1}{\mathbf{a}} \wedge \overset{-1}{\mathbf{b}} = (0, 0). \tag{5}$$

These exterior products and derivatives of generalized forms can easily be shown to satisfy the standard rules of exterior algebra and calculus. This exterior product is associative and distributive. If $\mathbf{a}$ and $\mathbf{b}$ are generalized p-forms and q-forms respectively, then $\mathbf{a} \wedge \mathbf{b} = (-1)^{pq} \mathbf{b} \wedge \mathbf{a}$. The exterior derivative is an anti-derivation from p-forms to $(p+1)$-forms, that is $\mathrm{d}(\mathbf{a} \wedge \mathbf{b}) = \mathrm{d}\mathbf{a} \wedge \mathbf{b} + (-1)^p \mathbf{a} \wedge \mathrm{d}\mathbf{b}$. Furthermore, $\mathrm{d}^2 = 0$.

In Sparling's original use of minus one-forms his approach was to add to ordinary forms a form of degree minus one, ς say, which satisfied all the basic standard rules of exterior algebra and calculus, together with the condition that $\mathrm{d}\,\varsigma$ was constant. The rules for exterior multiplication and exterior differentiation presented above, follow when the generalized p-form $\overset{p}{\mathbf{a}} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$, is identified with $\overset{p}{\alpha} + \overset{p+1}{\alpha} \wedge \varsigma$, with $\mathrm{d}\varsigma = k$, and then such expressions are added, multiplied and differentiated by using the ordinary rules of exterior algebra and calculus[3].

The following generalized Poincaré lemma holds. Let $\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ be a closed generalized p-form, so that $\mathrm{d}\, \overset{p}{\mathbf{a}} = 0$. Then,

(a) $\mathrm{d}\, \overset{-1}{\mathbf{a}} = 0$ if and only if $\overset{-1}{\mathbf{a}} = (0, 0)$,

(b) in any simply connected neighbourhood of any point of M,

1. $\mathrm{d}\, \overset{0}{\mathbf{a}} = 0$ if and only if there exist ordinary 0-forms $\overset{0}{\beta}$ such that $\overset{0}{\mathbf{a}} = (\overset{0}{\beta}, k^{-1} \mathrm{d}\, \overset{0}{\beta})$ or $\overset{0}{\mathbf{a}} = \mathrm{d}\, \overset{-1}{\mathbf{b}}$, where $\overset{-1}{\mathbf{b}} = (0, k^{-1} \overset{0}{\beta})$.
2. $\mathrm{d}\, \overset{\mathrm{p}}{\mathbf{a}} = 0$, $1 \leqslant \mathrm{p} \leqslant \mathrm{n}$, if and only if there exist ordinary $(p-1)$- and p-forms $\overset{p-1}{\beta}$ and $\overset{p}{\beta}$ such that $\overset{p}{\mathbf{a}} = (\mathrm{d}\, \overset{p-1}{\beta} + (-1)^p k \overset{p}{\beta}, \mathrm{d}\, \overset{p}{\beta})$. Hence $\overset{p}{\mathbf{a}} = \mathrm{d}\, \overset{p-1}{\mathbf{b}}$, where $\overset{p-1}{\mathbf{b}} = (\overset{p-1}{\beta}, \overset{p}{\beta})$.

Next consider Lie groups and Lie algebras and let $G = Gl(n)$ or one of its sub-groups. In the present context it is natural to associate with G the semi-direct product of G and the Lie algebra of G (viewed as an additive abelian group). Define the (associated) Lie group $\mathbf{G}$ by

$$\mathbf{G} = \{\mathbf{a} \mid \mathbf{a} = \alpha(1, A)\}, \tag{6}$$
$$\alpha(1, A) \equiv (\alpha, 0) \wedge (1, A) = (\alpha, \alpha A),$$

[3] Similarly rules for *right* exterior multiplication and *right* exterior derivatives can be obtained by identifying a generalized p-form $\overset{p}{\mathbf{a}} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ with $\overset{p}{\alpha} + \varsigma \wedge \overset{p+1}{\alpha}$. The resulting *right* algebra is isomorphic to the *left* exterior algebra and the isomorphism can be used to identify the calculi. It is the *left* exterior algebra and the associated exterior calculus which is always used in this article.

where **a** is a generalized 0-form, α belongs to the Lie group G (GL(n) or a subgroup), with identity 1, and A is an ordinary 1-form with values in the Lie algebra of G. The product of two elements of **G**, $\mathbf{a} = \alpha(1, A)$ and $\mathbf{b} = \beta(1, B)$ is given by the above rules for left exterior multiplication, and is $\mathbf{ab} = \alpha\beta(1, B + \beta^{-1}A\beta)$. The inverse of **a** is $\mathbf{a}^{-1} = \alpha^{-1}(1, -\alpha A\alpha^{-1})$ and the identity is $(1, 0)$, where 0 is the zero 1-form. Right fundamental 1-forms **r** are formally defined to be forms of the type

$$\mathrm{d}\mathbf{a} \wedge \mathbf{a}^{-1} = (\mathrm{d}\alpha \wedge \alpha^{-1} - k\alpha A\alpha^{-1}, \alpha[\mathrm{d}A + kA \wedge A]\alpha^{-1}), \tag{7}$$

and satisfy the Maurer–Cartan equation

$$\mathrm{d}\mathbf{r} - \mathbf{r} \wedge \mathbf{r} = 0. \tag{8}$$

Similarly left fundamental 1-forms **l** are formally defined by

$$\mathbf{a}^{-1} \wedge \mathrm{d}\mathbf{a} = (\alpha^{-1}\mathrm{d}\alpha - kA, \mathrm{d}A - kA \wedge A + \alpha^{-1}\mathrm{d}\alpha \wedge A + A \wedge \alpha^{-1}\mathrm{d}\alpha). \tag{9}$$

When k is non-zero **l** can be neatly written in the form

$$\begin{aligned} \mathbf{l} &= (\lambda, -k^{-1}[\mathrm{d}\lambda + \lambda \wedge \lambda]), \\ \lambda &= \alpha^{-1}\mathrm{d}\alpha - kA, \end{aligned} \tag{10}$$

and **l** satisfies the Maurer–Cartan equation

$$\mathrm{d}\mathbf{l} + \mathbf{l} \wedge \mathbf{l} = 0. \tag{11}$$

Next, in order to construct generalized Cartan structure equations for generalized connection and curvature forms on M, a generalized moving co-frame of 1-forms, $\mathbf{e}^a = (\theta^a, -\Theta^a)$, and a generalized Lie algebra valued 1-form $\Gamma^a_b = (\omega^a_b, -\Omega^a_b)$ are introduced. When the aim is to identify the latter as a generalized affine connection the Lie algebra corresponding to the generalized structure group, **G,** is gl(n) or a sub-algebra. The lower case Latin indices range and sum over 1 to n. It will be convenient to use covariant exterior derivatives and the generalized covariant exterior derivative is denoted by **D**. The first generalized Cartan structure equation is given by

$$\mathbf{T}^a = \mathbf{D}\mathbf{e}^a \equiv \mathrm{d}\mathbf{e}^a - \mathbf{e}^b \wedge \Gamma^a_b. \tag{12}$$

The generalized torsion, the 2-form $\mathbf{T}^a$, is in fact given by

$$\mathbf{T}^a = (\mathrm{d}\theta^a - \theta^b \wedge \omega^a_b - k\Theta^a, -\mathrm{D}\Theta^a + \theta^b \wedge \Omega^a_b),$$

where

$$\mathrm{D}\Theta^a = \mathrm{d}\Theta^a + \Theta^b \wedge \omega^a_b. \tag{13}$$

Here D denotes an ordinary covariant exterior derivative. The second generalized Cartan structure equation is given by

$$\mathbf{F}^a_b = \mathrm{d}\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b, \tag{14}$$

where $\mathbf{F}^a_b$ is the generalized curvature of Γ^a_b. A short computation shows that

$$\mathbf{F}^a_b = (\mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b - k\Omega^a_b, -\mathrm{D}\Omega^a_b),$$

where

$$\mathrm{D}\Omega^a_b = \mathrm{d}\Omega^a_b + \Omega^c_b \wedge \omega^a_c - \Omega^a_c \wedge \omega^c_b. \tag{15}$$

When θ^a is a co-frame on M, and ω^a_b are connection 1-forms with torsion $k\Theta^a$ and curvature $k\Omega^a_b$, the ordinary Cartan structure equations

$$\begin{aligned} \mathrm{d}\theta^a - \theta^b \wedge \omega^a_b &= k\Theta^a, \\ \mathrm{d}\omega^a_b + \omega^a_c \wedge \omega^c_b &= k\Omega^a_b, \end{aligned} \tag{16}$$

and the ordinary Bianchi identities

$$\mathrm{D}\Theta^a = \theta^b \wedge \Omega^a_b,$$
$$\mathrm{D}\Omega^a_b = 0, \tag{17}$$

are equivalent to the two equations for generalized forms:

$$\mathbf{De}^a = 0, \tag{18}$$
$$\mathbf{F}^a_b = 0. \tag{19}$$

Hence the ordinary Cartan structure equations for an affine connection are satisfied if and only if equations (17) and (18) are satisfied, that is the generalized affine connection is 'flat' and hence

$$\mathbf{e}^a = (\mathbf{b}^{-1})^a_b \mathrm{d}\mathbf{x}^b,$$
$$\Gamma^a_b = (\mathbf{b}^{-1})^a_c \mathrm{d}(\mathbf{b})^c_{b,} \tag{20}$$

where $\mathbf{x}^b$ are generalized 0-forms and $\mathbf{b}^a_b = \beta^a_c(\delta^c_b, B^a_b)$ is a generalized 0-form with values in the Lie group $\mathbf{G}$. Here β^a_b has values in GL(n) (or the appropriate sub-group) and B^a_b has values in the corresponding Lie algebra.

Generalized gauge transformations are determined by generalized 0-forms on M with values in the Lie group $\mathbf{G}$. The gauge transformations determined by an element of $\mathbf{G}$, $\mathbf{a}^a_b = \alpha^a_c(\delta^c_b, A^c_b)$, are given, for the generalized forms, by

$$\mathbf{e}^a \to (\mathbf{a}^{-1})^a_b \mathbf{e}^b,$$
$$\Gamma^a_b \to (\mathbf{a}^{-1})^a_c \mathrm{d}\mathbf{a}^c_b + (\mathbf{a}^{-1})^a_c \Gamma^c_d \mathbf{a}^d_b,$$
$$\mathbf{De}^a \to (\mathbf{a}^{-1})^a_b \mathbf{De}^b,$$
$$\mathbf{F}^a_b \to (\mathbf{a}^{-1})^a_c \mathbf{F}^c_d \mathbf{a}^d_b \tag{21}$$

These are equivalent to the following transformations of the ordinary forms

$$\theta^a \to (\alpha^{-1})^a_b \theta^b,$$
$$\omega^a_b \to (\alpha^{-1})^a_c \mathrm{d}\alpha^c_b + (\alpha^{-1})^a_c \omega^c_d \alpha^d_b - kA^a_b \equiv \varpi^a_b - kA^a_b,$$
$$\Theta^a \to (\alpha^{-1})^a_b [\Theta^b - \alpha^b_c A^c_d (\alpha^{-1})^d_e \wedge \theta^e],$$
$$\Omega^a_b \to (\alpha^{-1})^a_c \Omega^c_d \alpha^d_b - \mathrm{D}_\varpi A^a_b + kA^a_c \wedge A^c_b, \tag{22}$$

where D_ϖ denotes the covariant exterior derivative with respect to $\varpi^a_b = (\alpha^{-1})^a_c \mathrm{d}\alpha^c_b + (\alpha^{-1})^a_c \omega^c_\mathrm{d} \alpha^d_b$.

These formulae show how the affine structure is encoded and the formalism provides a unifying framework for different affine connections. A simple application is provided by the following result.

Proposition 1. *Let a metric on M have line element*

$$\mathrm{d}s^2 = \eta_{ab} \theta^a \otimes \theta^b, \tag{23}$$

where $\eta_{ab}=\eta_{ba}$ *are the components of the metric with respect to the co-frame* θ^a*, are constant. Let* $\omega^a_b = \omega^a_{bc}\theta^c$ *be a general connection with torsion* $\Theta^a = \frac{1}{2}\Theta^a_{bc}\theta^b \wedge \theta^c$, $\Theta^a_{bc} = -\Theta^a_{cb}$, *and let* $A_{ab} = A_{abc}\theta^c$*. Then, if in the above generalized gauge transformations* $\alpha^a_b = \delta^a_b$*, and if* $A_{(ab)} = k^{-1}\omega_{(ab)}$*, the transformed ordinary connection is metric. If, in addition,* $A_{[ab]} = \{1/2[\Theta_{cab} - \Theta_{bca} - \Theta_{abc}] + k^{-1}\omega_{(ac)b} - k^{-1}\omega_{(bc)a}\}\theta^c$*, the transformed ordinary connection is also torsion free, and hence is the Levi-Civita connection of the metric.*

It is also straightforward to define generalized connections on principal and associated bundles. Rather than pursue that line in detail here, an illustration will be given in which generalized forms are used to provide a simple formulation of the (possibly complex) Einstein vacuum field equations in four dimensions. Here it will be assumed that k is not only non-zero *but also not equal to one*. Upper case Latin indices sum and range over 0-1 and are two-component spinor indices, [5]. Any 4-metric on a four dimensional manifold M can be written, locally, in the form

$$ds^2 = \alpha_A \otimes \beta^A + \beta^A \otimes \alpha_A, \tag{24}$$

where, α^A and β^A are spinor-valued 1-forms on M. Define the two generalized spinor valued 1-forms

$$\begin{aligned} \mathbf{r}^A &\equiv (\alpha^A, -k^{-1}\mu^A), \\ \mathbf{s}^A &\equiv (\beta^A, k^{-1}\upsilon^A), \end{aligned} \tag{25}$$

and the generalized sl(2,C)-valued connection 1-form

$$\Gamma^A_B = (\omega^A_B, -\Omega^A_B), \tag{26}$$

where

$$\Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B, \tag{27}$$

and ω^A_B is an ordinary sl(2,C)-valued connection 1-form. The sl(2,C)-valued curvature 2-form of Γ^A_B is given by

$$\mathbf{F}^A_B = d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B = ([1-k]\Omega^A_B, 0). \tag{28}$$

Then it can be seen directly, or from [6], (see also [7]), that the metric is Ricci flat if and only if the generalized spinor valued 1-forms $\mathbf{r}^A$ and $\mathbf{s}^A$ have vanishing generalized exterior covariant derivatives, and the symmetric part of their generalized exterior product is an ordinary 2-form. That is, the metric is Ricci flat if and only if $\mathbf{r}^{(A} \wedge \mathbf{s}^{B)}$ are ordinary 2-forms and

$$\begin{aligned} \mathbf{Dr}^A &\equiv d\mathbf{r}^A - \mathbf{r}^B \wedge \Gamma^A_B = 0, \\ \mathbf{Ds}^A &\equiv d\mathbf{s}^A - \mathbf{s}^B \wedge \Gamma^A_B = 0. \end{aligned} \tag{29}$$

Here $\mathbf{D}$ is the generalized covariant exterior derivative determined by Γ^A_B. These conditions encode the Ricci flatness of the metric and ensure that ω^A_B is the anti-self dual part of the Levi-Civita spin connection. Imposition of the usual reality conditions leads to the real Einstein equations. (Equations (29) are formally similar to the first order equations used by Plebanski [8], in his analysis of half-flat geometries.)

In conclusion it should be noted that the concept of a generalized p-form discussed above is a special case of a broader generalization in which generalized p-forms are represented by n + 1-tuples of ordinary forms. Using the type of notation introduced by Sparling and mentioned above, on an n dimensional manifold assume that n minus one-forms ς_a exist and satisfy the conditions,

$$\begin{aligned} &\varsigma_1 \wedge \varsigma_2 \ldots \wedge \varsigma_n \neq 0, \\ &d\varsigma_a = k_a, \\ &k_a \text{constants}, a = 1 \ldots n. \end{aligned} \tag{30}$$

Now define a generalized p-form to be

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}{}^{a_1} \wedge \varsigma_{a_1} + \ldots \overset{n}{\alpha}{}^{a_1 a_2 \ldots a_{n-p}} \wedge \varsigma_{a_1} \wedge \varsigma_{a_2} \ldots \wedge \varsigma_{a_{n-p}} \tag{31}$$

or, including zeros, the equivalent (n+1) tuple. Here $\overset{k}{\alpha}{}^{a_1\ldots a_{k-p}} = \overset{k}{\alpha}{}^{[a_1\ldots a_{k-p}]}$, $k = p$ to n, are ordinary k-forms (all sub-scripted indices ranging and summing over 1 to n). The rules for exterior multiplication and exterior derivative can be computed immediately from the last two equations. Extensions to include super-symmetry and the infinite dimensional case appear to pose no major formal problems. Further developments, both real and complex, of the above formalism, including the definitions of Lie derivatives, other generalized connection and metric geometries, generalized Hodge duality, co-differentials or adjoints, inner products, Laplacians, co-homology, and physical applications, will be presented elsewhere.

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