

# Generalized forms and their applications

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## Abstract

The calculus of generalized  $p$ -forms,  $-1 \leq p \leq n$ , is developed further. On an  $n$ -dimensional manifold with metric,  $M$ , a (Hodge) star operator, inner product, co-differential and Laplacian operators are introduced. The inner product and Lie derivative with respect to vector fields on  $M$  are defined. Applications are made to Hamiltonian mechanics, field theories and Einstein's vacuum equations.

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## 1. Introduction

On an  $n$ -dimensional manifold  $M$ , a generalized  $p$ -form  $\mathbf{a} \in \Lambda_G^p$  is an ordered pair of an ordinary  $p$ -form  $\alpha^p$  and an ordinary  $(p+1)$ -form  $\alpha^{p+1}$ ,  $\mathbf{a} = (\alpha^p, \alpha^{p+1})$ ,  $-1 \leq p \leq n$ , where  $\alpha^{-1} = 0$ . The algebra and calculus of these generalized differential forms, originally introduced by Sparling [1–4], were developed in [5] and that paper included a generalized form version of Cartan's structure equations for an affine connection. In this paper the calculus of generalized forms is extended further. The investigations here centre on the generalized forms on a manifold  $M$  equipped with a metric<sup>3</sup>. The notation of [5] is generally followed; in particular generalized forms are denoted by bold letters to distinguish them from ordinary forms. A Hodge star operation and the notion of duality for generalized forms are defined first. The star operation is then used in the definition of an inner product of, and co-differential and Laplacian operators for, generalized  $p$ -forms. These definitions are similar to those for ordinary forms but there are some significant differences. For example, the notions of self-dual or anti-self-dual generalized forms apply only on odd-dimensional manifolds and the Laplacian operator on generalized forms contains an additional 'mass-like' term. Furthermore, one definition of an inner product of generalized  $p$ -forms, used here in the Lagrangian formulation of Einstein's

<sup>3</sup> Here semi-Riemannian manifolds with symmetric metrics of all possible signatures are explicitly considered. These ideas can be easily extended, for example, to symplectic manifolds.

vacuum equations, admits the inclusion of boundary terms. Next vectors and vector fields and generalized forms are considered. Here tangent vector fields to  $M$  are mainly considered although more general possibilities are mentioned. The inner product and Lie derivative of generalized forms with respect to tangent vectors are defined.

Thirdly, a number of applications of the results above, to mechanical and physical field theories, are exhibited. These include generalized form analogues of Hamiltonian systems, scalar fields, Maxwell and Yang–Mills fields, and Einstein’s vacuum field equations. It is seen that generalized forms provide a natural formalism within which to consider potentials and gauge conditions. The extension of zero rest mass systems to systems with non-zero rest mass occurs naturally in this framework. This recalls the association made in [5] between the generalized form version of Cartan’s structure equations, torsion and translations.

## 2. Manifolds with metrics and differential forms

In this section we first recall some of the definitions and theorems which apply to ordinary differential forms on an oriented  $n$ -dimensional manifold  $M$  with metric  $g$  [6]. Analogous results, applicable to generalized forms are then formulated.

Let  $\Lambda^p$  denote the module of (ordinary)  $p$ -forms on  $M$ . Recall that the Hodge star operator maps a  $p$ -form  $\alpha$  to the dual  $(n-p)$ -form

$$* : \Lambda^p \rightarrow \Lambda^{n-p} \quad \text{by} \quad \alpha \mapsto *\alpha. \quad (1)$$

Furthermore

$$**\alpha = \overset{p}{s} \alpha, \quad (2)$$

where,

$$\overset{p}{s} = (-1)^{p(n+1)} \operatorname{sgn}(\det g). \quad (3)$$

The symmetric inner product of two  $p$ -forms  $\alpha$  and  $\beta$  is given by

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta, \quad (4)$$

when the integral is defined. In the Riemannian case it is positive definite. The co-differential, or adjoint derivative operator,  $\delta : \Lambda^p \rightarrow \Lambda^{p-1}$ , is defined by

$$\delta\alpha = \overset{p}{\sigma} * d*\alpha, \quad (5)$$

where

$$\overset{p}{\sigma} = (-1)^{np+n+1} \operatorname{sgn}(\det g). \quad (6)$$

Recall that a  $p$ -form,  $\alpha$ , is co-closed if  $\delta\alpha = 0$ , and co-exact if  $\alpha = \delta\beta$ , for some  $(p+1)$ -form  $\beta$ . The Laplacian  $\Delta$  is the linear operator mapping  $p$ -forms to  $p$ -forms given by

$$\Delta = (d + \delta)^2 = d\delta + \delta d, \quad (7)$$

and a  $p$ -form  $\alpha$  is harmonic if  $\Delta\alpha = 0$ . On a compact orientable Riemannian manifold  $\alpha$  is harmonic if and only if  $\alpha$  is both closed and co-closed, and the Hodge theorems, see e.g. [6], apply.

Consider next the generalized  $p$ -forms. Recall from [5] that the (left) exterior product of a generalized  $p$ -form  $\overset{p}{\mathbf{a}} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$  and a generalized  $q$ -form  $\overset{q}{\mathbf{b}} = (\overset{q}{\beta}, \overset{q+1}{\beta})$  is given by

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} \equiv (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}), \quad (8)$$

and the exterior derivative of  $\mathbf{a}^p$  is given by

$$d\mathbf{a}^p \equiv (d\alpha^p + (-1)^{p+1}k\alpha^{p+1}, d\alpha^{p+1}). \tag{9}$$

Here  $k$  is constant, a necessary and sufficient condition for  $d^2 = 0$ . In this paper we again follow the convention that generalized forms are denoted by bold lowercase Latin letters and ordinary forms are denoted by Greek letters. Furthermore, ordinary forms such as  $\alpha^p$ , with  $p$  negative, are (conventionally) zero. Unless it is explicitly stated otherwise, the constant  $k$  will be assumed to be non-zero, and  $-1 \leq p \leq n$ .

The Poincaré lemma for generalized forms is simple because generalized forms are closed if and only if they are exact.

**Theorem.** Let  $\mathbf{a}^p = (\alpha^p, \alpha^{p+1})$  be closed so that  $d\mathbf{a}^p = 0$ . Then  $\mathbf{a}^p$  is exact and

- (a) if  $p \geq 0$ ,  $\mathbf{a}^p = d\mathbf{b}^{p-1}$  where  $\mathbf{b}^{p-1} = (\beta^{p-1}, (-1)^p k^{-1}[\alpha^p - d\beta^{p-1}])$  for any ordinary  $(p-1)$ -form  $\beta^{p-1}$ ;
- (b) if  $p = -1$ , any closed minus one-form is zero.

In the following a Hodge star operator, co-differential and Laplacian, etc for generalized forms are defined. These are analogues rather than direct extensions of the results given above for ordinary forms. The definitions involve, amongst other things, particular choices of signs. Other choices of signs would lead to slightly different definitions.

First the (Hodge) dual and star operator,  $\star$ , for generalized forms are defined by

$$\star : \Lambda_G^p \rightarrow \Lambda_G^{n-p-1} \quad \text{by} \quad \mathbf{a}^p \mapsto \star\mathbf{a} \equiv ([-1]^{n-p-1} * \alpha^{p+1}, * \alpha^p). \tag{10}$$

This definition gives, as the dual to a generalized  $p$ -form, a  $(n-p-1)$ -form rather than a  $(n-p)$ -form as is the case for ordinary  $p$ -forms. For example, the dual of a  $-1$ -form,  $\star\mathbf{a}^{-1}$ , is a generalized  $n$ -form.

Since

$$\star\star\mathbf{a}^p = s^p (-1)^p \mathbf{a}^p \equiv \lambda^2 \mathbf{a}^p, \tag{11}$$

where  $s^p$  is defined in equation (3), the possible eigenvalues of  $\star$  are  $\pm\lambda$ , where in fact  $\lambda = 1$  or  $i$ . With the above sign conventions, a generalized  $p$ -form is said to be self-dual if  $\star\mathbf{a}^p = \lambda \mathbf{a}^p$ , and anti self-dual if  $\star\mathbf{a}^p = -\lambda \mathbf{a}^p$ , where the choice of  $\lambda$  depends, as usual, on the signature of the metric. It is straightforward to see that necessary conditions for a generalized  $p$ -form to be either self-dual or anti self-dual are that  $n$ , the dimension of  $M$  be odd, and that  $p = \frac{n-1}{2} \geq 0$ .

In fact  $\star\mathbf{a}^p = \pm\lambda \mathbf{a}^p$  if and only if  $\mathbf{a}^p = (\alpha^{\frac{n-1}{2}}, \pm\lambda^{-1} * \alpha^{\frac{n-1}{2}})$ .

For the two generalized forms  $\mathbf{a}^p$  and  $\mathbf{b}^q$

$$\mathbf{a}^p \wedge \mathbf{b}^q = ([-1]^{n-q-1} \alpha^p \wedge * \beta^{q+1}, \alpha^p \wedge * \beta^q + \alpha^{p+1} \wedge * \beta^q). \tag{12}$$

The expression for  $\frac{1}{2}(\mathbf{a}^p \wedge \star\mathbf{b}^p + \mathbf{b}^p \wedge \star\mathbf{a}^p)$  and the definition of an inner product for ordinary forms suggests that a symmetric inner product of two generalized  $p$ -forms  $\mathbf{a}^p$  and  $\mathbf{b}^p$  be defined as

$$(\mathbf{a}^p, \mathbf{b}^p) = \int_M (\alpha^p \wedge * \beta^p + \alpha^{p+1} \wedge * \beta^{p+1}) \tag{13}$$

when the integrals are defined. This inner product is necessarily positive definite for a Riemannian manifold and it reduces to the usual definition for ordinary  $p$ -forms. In particular

$$(\mathbf{a}, \mathbf{a}) = \int_M (\alpha^p \wedge * \alpha^p + \alpha^{p+1} \wedge * \alpha^{p+1}), \tag{14}$$

and this expression will be used in section 4 in the construction of Lagrangians.

A co-differential operator  $\delta : \Lambda_G^p \rightarrow \Lambda_G^{p-1}$ , by  $\mathbf{a} \mapsto \delta \mathbf{a}$  is defined by the equation

$$\delta \mathbf{a} = (-1)^{p+1} \sigma \star d \star \mathbf{a} \tag{15}$$

where  $\sigma$  is as in the definition of the co-differential of ordinary forms above. This definition is equivalent to

$$\delta \mathbf{a} = (\delta \alpha^p, \delta \alpha^{p+1} + (-1)^p k \alpha^p). \tag{16}$$

From these definitions it follows that

- (a)  $\delta^2 = 0$ ,
- (b)  $\delta \mathbf{a}^{-1} = 0$ ,
- (c)  $\delta \mathbf{a}^0 = (0, \delta \alpha^1 + k \alpha^0)$ ,
- (d) if  $\mathbf{a}$  is co-closed, that is  $\delta \mathbf{a}^p = 0$ , then in a result analogous to the Poincaré lemma above, it is co-exact, that is it is the co-differential of a generalized  $(p+1)$ -form. More specifically, if  $\delta \mathbf{a}^p = 0$ , then if  $-1 \leq p \leq n-1$ ,

$$\mathbf{a}^p = \delta \mathbf{b}^{p+1}, \quad \mathbf{b}^{p+1} = ((-1)^{p+1} k^{-1} [\alpha^{p+1} - \delta \beta^{p+2}], \beta^{p+2}) \tag{17}$$

for any ordinary  $(p+2)$ -form  $\beta^{p+2}$ . Furthermore,  $\delta \mathbf{a}^n = 0$  if and only if  $\mathbf{a}^n = 0$ .

Note that any  $-1$ -form is both co-closed and co-exact. It should also be noted that when  $M$  has no boundary, the condition,  $(d\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \delta \mathbf{b})$ , for this co-differential operator on generalized forms to be the adjoint of  $d$ , holds.

A Laplacian for generalized forms,  $\Delta : \Lambda_G^p \rightarrow \Lambda_G^p$ , is defined to be  $\Delta = d\delta + \delta d$ . Computation, with the choice of signs made in this paper, gives the simple expression

$$\Delta \mathbf{a}^p = (\Delta \alpha^p + k^2 \alpha^p, \Delta \alpha^{p+1} + k^2 \alpha^{p+1}). \tag{18}$$

Hence  $\mathbf{a}^p$  is a harmonic generalized form, that is

$$\Delta \mathbf{a}^p = 0, \tag{19}$$

if and only if

$$\Delta \alpha^p + k^2 \alpha^p = 0 \quad \text{and} \quad \Delta \alpha^{p+1} + k^2 \alpha^{p+1} = 0. \tag{20}$$

A generalized form is harmonic only when its constituent ordinary forms satisfy a Klein-Gordon type of equation with a ‘mass-squared’ term given by  $k^2$ . The extension from ordinary forms to generalized forms provides a framework within which translations and torsion appear naturally within the framework of Cartan’s structure equations [5] and mass terms appear naturally in the context of the Laplacian operators. This provides a physically interesting interpretation of the constant  $k$ .

In concluding this section it should be noted that the choices of signs in the above definitions have been made in order to make generalized forms eigenforms of the operator  $\star \star$ , to give a definition of  $\delta$  which was simply related to the definition for ordinary forms and to ensure that the Laplacian on generalized forms was computable in terms of the Laplacian, not some other second-order differential operator, acting on ordinary forms.

### 3. Vectors and generalized forms

A discussion of vectors could start by considering the ring of generalized 0-forms,  $\mathcal{F}$  and the module  $V^*$  of generalized 1-forms over  $\mathcal{F}$ . Then generalized vectors could be defined to be the elements of  $V$  the module over  $\mathcal{F}$  dual to  $V^*$ . That is, if  $\mathbf{a} = \binom{-1}{\alpha, \alpha} \in V^*$  and  $\mathbf{X} = \binom{1}{X, X} \in V$ , then by definition

$$\langle \mathbf{X}, \mathbf{a} \rangle = \binom{1}{X} \lrcorner \binom{1}{\alpha} + \binom{2}{X} \lrcorner \binom{2}{\alpha}, \tag{21}$$

where  $\binom{1}{X}$  is an ordinary vector field tangent to  $M$ ,  $\binom{2}{X}$  is an ordinary bi-vector field on  $M$  and  $\lrcorner$  denotes the ordinary hook or inner product operation. The tensor algebra (and subsequently calculus) over  $V, V^*$  and  $\mathcal{F}$  could then be constructed. However, the Leibniz rule does not then hold in the usual way. For example in the special case where  $\mathbf{a} = \binom{0}{d\mathbf{b}}$ ,  $\langle \mathbf{X}, \mathbf{a} \rangle$  cannot be identified with a derivation  $\mathbf{X}(\mathbf{b})$  because the Leibniz rule does not always hold. Another possibility would be to consider only modules over the ordinary functions and define

$$\langle \mathbf{X}, \mathbf{a} \rangle = \binom{1}{X} \lrcorner \binom{1}{\alpha} + \binom{2}{X} \lrcorner \binom{2}{\alpha}, \tag{22}$$

but this is not of immediate use, for example, in the forthcoming application to Hamilton's equations. Hence attention here will be confined to the inner product of ordinary vector fields and generalized forms, and definitions will be formulated so that the usual Leibniz formulae for products hold.

Let  $X$  be an ordinary vector field tangent to  $M$  and let the generalized  $p$ -form  $\mathbf{a}$  and  $q$ -form  $\mathbf{b}$  be given, respectively, by the pairs of ordinary forms  $(\binom{p}{\alpha}, \binom{p+1}{\alpha})$  and  $(\binom{q}{\beta}, \binom{q+1}{\beta})$ . The hook operator (contraction or inner product) on generalized forms is defined for  $-1 \leq p \leq n$  by

$$\lrcorner : \Lambda_G^p \rightarrow \Lambda_G^{p-1} \quad X \lrcorner \mathbf{a} = (X \lrcorner \binom{p}{\alpha}, X \lrcorner \binom{p+1}{\alpha}), \tag{23}$$

where as always by definition,  $\binom{-1}{\alpha} = 0$ , and as usual  $X \lrcorner \binom{0}{\alpha} = 0$ . Hence  $X \lrcorner \mathbf{a} = 0$  and  $X \lrcorner \mathbf{b} = (0, X \lrcorner \binom{1}{\beta})$ . It is a straightforward matter to show that this definition implies that

$$X \lrcorner (\mathbf{a} \wedge \mathbf{b}) = (X \lrcorner \mathbf{a}) \wedge \mathbf{b} + (-1)^p \mathbf{a} \wedge (X \lrcorner \mathbf{b}), \tag{24}$$

that is

$$X \lrcorner (\mathbf{a} \wedge \mathbf{b}) = (X \lrcorner (\binom{p}{\alpha} \wedge \binom{q}{\beta}), X \lrcorner (\binom{p}{\alpha} \wedge \binom{q+1}{\beta}) + (-1)^q X \lrcorner (\binom{p+1}{\alpha} \wedge \binom{q}{\beta})). \tag{25}$$

Next the Lie derivative with respect to  $X$ ,  $\mathfrak{L}_X$  is defined by the usual expression

$$\mathfrak{L}_X \mathbf{a} = X \lrcorner d \mathbf{a} + d(X \lrcorner \mathbf{a}), \tag{26}$$

or equivalently

$$\mathfrak{L}_X \mathbf{a} = (\mathfrak{L}_X \binom{p}{\alpha}, \mathfrak{L}_X \binom{p+1}{\alpha}). \tag{27}$$

It follows that

$$\mathfrak{L}_X (\mathbf{a} \wedge \mathbf{b}) = (\mathfrak{L}_X \mathbf{a}) \wedge \mathbf{b} + \mathbf{a} \wedge \mathfrak{L}_X (\mathbf{b}), \tag{28}$$

and all the Cartan formulae [9] are satisfied; in particular,

$$\mathfrak{L}_X \circ d = d \circ \mathfrak{L}_X. \tag{29}$$

#### 4. Applications of generalized forms

In this section a number of examples, providing applications of generalized forms will be presented. One particular aim is to highlight similarities and differences with some standard field theories formulated in terms of ordinary forms. First, the use of generalized forms in a version of Hamilton's equations will be considered. Secondly, massive and massless scalar field equations will be discussed using generalized forms. Thirdly, Maxwell type equations for generalized forms will be presented and their relation to the Proca equation exhibited. Next, a generalized form version of the Yang–Mills equations will be presented. Then, in a short illustration, the conditions for  $G_2$  structures to be parallel or nearly parallel will be reformulated in terms of generalized forms. Finally, a brief discussion of integrals and generalized forms will be presented and illustrated by a Lagrangian formulation of the complex Einstein vacuum field equations.

**Example 1** (Hamilton's equations and generalized forms). First recall the standard results for ordinary forms. Let  $M$  be a manifold of dimension  $2m$ , with symplectic 2-form  $\omega$  with  $d\omega = 0$ . If  $H$  is a function on  $M$  the corresponding Hamiltonian vector field  $X$  is determined by the equation

$$X \lrcorner \omega = -dH. \quad (30)$$

If  $(p_i, q^i)$  are (local) symplectic coordinates so that

$$\omega = dp_i \wedge dq^i, \quad (31)$$

then

$$X = \partial H / \partial p_i \partial / \partial q^i - \partial H / \partial q^i \partial / \partial p_i, \quad (32)$$

and the integral curves of  $X$  satisfy Hamilton's equations

$$dq^i / dt = \partial H / \partial p_i, \quad dp_i / dt = -\partial H / \partial q^i. \quad (33)$$

It is straightforward to write similar equations for generalized forms by replacing  $\omega$  and  $H$  by a generalized closed 2-form and a generalized 0-form, respectively. If however  $\omega$  is replaced by a closed generalized 1-form and  $H$  by a  $-1$ -form results directly related to the standard results are obtained as follows.

Let  $\mathbf{a} = (\alpha, \overset{2}{\alpha})$  be a closed generalized 1-form on  $M$  so that

$$d\overset{1}{\alpha} = -k\overset{2}{\alpha}, \quad d\overset{2}{\alpha} = 0. \quad (34)$$

It will be assumed that  $\overset{2}{\alpha}$  is of maximal rank. Let  $\overset{-1}{\mathbf{E}} = (0, \overset{0}{\varepsilon})$  be a generalized  $-1$ -form. Now define the Hamiltonian vector field determined by  $\overset{-1}{\mathbf{E}}$  to be the vector field  $X$ , tangent to  $M$ , which satisfies the equation

$$X \lrcorner \mathbf{a} = -d\overset{-1}{\mathbf{E}}. \quad (35)$$

These equations are more restrictive than the equations for ordinary forms because they are equivalent to the ordinary equation for a Hamiltonian vector field

$$X \lrcorner \overset{2}{\alpha} = -d\overset{0}{\varepsilon} \quad (36)$$

together with the additional equation

$$X \lrcorner \alpha = -k\overset{0}{\varepsilon}. \quad (37)$$

Equation (36) determines, as usual, the Hamiltonian vector field corresponding to the Hamiltonian function  $\varepsilon^0$  with integral curves corresponding to solutions of Hamilton's equations. Equation (34) determines a symplectic potential,  $-k^{-1}\alpha^1$ , for the symplectic 2-form  $\alpha^2$ . This symplectic potential must satisfy equation (37). It follows then that the symplectic 2-form and the symplectic potential are invariant; they both have vanishing Lie derivatives with respect to the Hamiltonian vector field  $X$ .

In symplectic coordinates  $(p_i, q^i), i = 1 \dots m$ , these equations state that

$$\alpha^2 = dp_i \wedge dq^i, \tag{38}$$

$$X = (\partial \varepsilon^0 / \partial p_i) \partial / \partial q^i - \partial \varepsilon^0 / \partial q^i \partial / \partial p_i, \tag{39}$$

and

$$-k^{-1}\alpha^1 = p_i dq^i + df, \tag{40}$$

where the function  $f$  satisfies equation (37), that is

$$\{f, \varepsilon^0\} + p_i \partial \varepsilon^0 / \partial p_i - \varepsilon^0 = 0, \tag{41}$$

and the first bracket is the Poisson bracket. Hence this generalized form approach automatically incorporates into the standard Hamiltonian formalism symplectic potentials which must satisfy the condition given by equations (37) or (41) and are consequently invariant potentials.

**Example 2** (Scalar field equations on an  $n$ -dimensional Lorentzian manifold  $M$ ). First consider the  $-1$ -form

$$\mathbf{a}^{-1} = (0, \alpha^0). \tag{42}$$

The dual generalized  $n$ -form is  $\star \mathbf{a}^{-1} = ((-1)^n \star \alpha^0, 0)$ . Consider the action

$$\begin{aligned} A &= (d \mathbf{a}^{-1}, d \mathbf{a}^{-1}) \\ &= \int_M (k^2 \alpha^0 \wedge \star \alpha^0 + d\alpha^0 \wedge \star d\alpha^0). \end{aligned} \tag{43}$$

The Euler-Lagrange equation is

$$d \star d \mathbf{a}^{-1} = 0 \quad \text{or} \quad \Delta \mathbf{a}^{-1} = 0. \tag{44}$$

This is in fact the Klein-Gordon equation

$$\Delta \alpha^0 + k^2 \alpha^0 = 0, \tag{45}$$

with mass term  $m^2 = k^2$ .

On the other hand, starting again, consider the generalized 0-form

$$\mathbf{a}^0 = (\alpha^0, 0). \tag{46}$$

For such a form, the dual is given by the generalized  $(n-1)$ -form  $\star \mathbf{a}^0 = (0, \star \alpha^0)$ . Using this type of generalized 0-form consider now the action

$$\begin{aligned} A &= (d \mathbf{a}^0, d \mathbf{a}^0) \\ &= \int_M d\alpha^0 \wedge \star d\alpha^0. \end{aligned} \tag{47}$$

The Euler–Lagrange equation which follows from this action is

$$\Delta^0 \mathbf{a} - k^2 \mathbf{a} = 0, \quad (48)$$

or equivalently the standard zero rest-mass scalar field equation

$$\Delta^0 \alpha = 0. \quad (49)$$

These results prompt the speculative observation that scalar fields with zero and non-zero rest-mass might be viewed as being related by a type of ‘internal gauge transformations in the space of generalized forms’, that is by a mapping between  $(\alpha, 0)$  and  $(0, \alpha)$ .

**Example 3** (Maxwell-type equations for generalized forms on a four-dimensional Lorentzian manifold). In this example it will be assumed that  $M$  is a four-dimensional Lorentzian manifold and the equations

$$d^1 \mathbf{F} = 0 \quad (50)$$

and

$$d \star^1 \mathbf{F} = 4\pi \star^0 \mathbf{J}, \quad (51)$$

for a generalized 1-form and 0-form,  $\mathbf{F}$  and  $\mathbf{J}$  will be studied. The consideration of these equations is motivated by the structure of the Maxwell equations expressed in terms of ordinary forms

$$dF = 0 \quad \text{and} \quad d \star F = 4\pi \star J, \quad (52)$$

where  $F$  is the Maxwell field 2-form and  $J$  is the current density 1-form. Both sets of equations admit conservation laws

$$d \star^0 \mathbf{J} = 0, \quad (53)$$

and

$$d \star J = 0. \quad (54)$$

Writing  $\mathbf{J} = (\rho, \rho)$  it follows from equation (53) that  $\rho = -k^{-1} \delta^1 \rho$ . Applying the Poincaré lemma to the closed generalized 1-form  $\mathbf{F}$ , it follows that a (non-unique) generalized 0-form potential  $\mathbf{a} = (\alpha, \alpha)$  can be introduced such that  $\mathbf{F} = d\mathbf{a}$ . Here it will prove convenient to rewrite the potential in terms of the ordinary form  $A = \alpha - k^{-1} d\alpha$ , so that

$$\mathbf{a} = (\alpha, A + k^{-1} d\alpha), \quad (55)$$

and hence

$$\mathbf{F} = (-kA, dA), \quad (56)$$

and the (gauge) freedom determined by  $\alpha$  is more clearly exhibited. It now follows from equation (51) that  $A$  satisfies the equations

$$\delta dA + k^2 A = -4\pi \rho. \quad (57)$$

In four-dimensional Lorentzian spacetime equation (57) is the Proca equation for a massive spin one field with source, [7], when the mass squared term is equal to  $k^2$ , as in the previous example. It also follows from the above, and indeed directly from equation (57) that

$$\delta(A + 4\pi k^{-2} \rho) = 0, \quad (58)$$

which, in the source-free case, is a Lorentz gauge condition on the potential 1-form  $A$ .



Conversely, let  $A$  be a 1-form which satisfies equation (57). Let  $\overset{1}{\alpha} = A + k^{-1} d\overset{0}{\alpha}$ , where  $\overset{0}{\alpha}$  is any (gauge) function, and let  $\overset{0}{\mathbf{a}} = (\overset{0}{\alpha}, \overset{1}{\alpha})$ , and  $\overset{1}{\mathbf{F}} = d\overset{0}{\mathbf{a}}$ . Then  $\overset{1}{\mathbf{F}}$  satisfies equations (50) and (51) with  $\overset{0}{\mathbf{J}} = (-k^{-1}\delta^1 \rho, \overset{1}{\rho})$ . These Maxwell-like generalized form equations representing the Proca equation (and when  $k$  is taken to be zero, the Maxwell equations) automatically incorporate both field and potential equations.

In the source-free case, an action for equations (50) and (51) is given by the inner product for generalized forms, as in equation (14), that is

$$(\overset{1}{\mathbf{F}}, \overset{1}{\mathbf{F}}) = \int_M (kA \wedge *A + dA \wedge *dA). \tag{59}$$

The (formal) analogues of these results with  $p$  greater than 1 are clearly similar.

**Example 4** (Generalized Yang–Mills equations). Here a generalization of the Yang–Mills field equations for a generalized 1-form, on an  $n$ -dimensional manifold with metric, will be presented.

Let  $\overset{1}{\mathbf{a}} = (\overset{1}{\alpha}, \overset{2}{\alpha})$  be a generalized 1-form (generalized Yang–Mills potential) with values in the Lie algebra of a Lie group  $G$ . Let its generalized curvature 2-form (generalized Yang–Mills field) be given by

$$\overset{2}{\mathbf{F}} = d\overset{1}{\mathbf{a}} + \frac{1}{2}[\overset{1}{\mathbf{a}}, \overset{1}{\mathbf{a}}], \tag{60}$$

so that

$$\overset{2}{\mathbf{F}} = (F + k\overset{2}{\alpha}, D\overset{2}{\alpha}), \tag{61}$$

where

$$F = d\overset{1}{\alpha} + \frac{1}{2}[\overset{1}{\alpha}, \overset{1}{\alpha}] \tag{62}$$

and  $D$  denotes the covariant exterior derivative with respect to the ordinary Yang–Mills potential  $\overset{1}{\alpha}$ , that is for a  $p$ -form

$$\begin{aligned} D\overset{p}{\beta} &= d\overset{p}{\beta} + (-1)^{p+1}\overset{p}{\beta} \wedge \overset{1}{\alpha} + \overset{1}{\alpha} \wedge \overset{p}{\beta}, \\ &\equiv d\overset{p}{\beta} + (-1)^{p+1}[\overset{p}{\beta}, \overset{1}{\alpha}]. \end{aligned} \tag{63}$$

The dual of the generalized Yang–Mills field is an  $(n-3)$ -form

$$\star\overset{2}{\mathbf{F}} = ((-1)^{n-1} *D\overset{2}{\alpha}, *F + k*\overset{2}{\alpha}). \tag{64}$$

The generalized Yang–Mills field equation is defined to be

$$D\star\overset{2}{\mathbf{F}} = d\star\overset{2}{\mathbf{F}} + (-1)^n[\star\overset{2}{\mathbf{F}}, \overset{1}{\mathbf{a}}] = 0, \tag{65}$$

where  $D$  is the generalized covariant exterior derivative corresponding to the generalized Yang–Mills potential  $\overset{1}{\mathbf{a}}$ , and is given by the obvious analogue of equation (63) above. The equations can be written in terms of ordinary forms by using the expression

$$D\star\overset{2}{\mathbf{F}} = (\{-1\}^{n-1}\{D*D\overset{2}{\alpha} - k*F - k^2*\overset{2}{\alpha}\}, D*F + kD*\overset{2}{\alpha} - [*D\overset{2}{\alpha}, \overset{2}{\alpha}]). \tag{66}$$

If  $\overset{2}{\alpha} = cF$  where  $c = -k^{-1}$  then the equations are automatically satisfied. When this is not the case, and  $k$  is non-zero, it follows that the generalized Yang–Mills equation is equivalent to the equations

$$[*D\overset{2}{\alpha}, F + k\overset{2}{\alpha}] = 0, \quad kF + k^2\overset{2}{\alpha} = s *D*D\overset{2}{\alpha}. \tag{67}$$

**Example 5** (Parallel and nearly parallel  $G_2$ -structures). Recall that a seven-dimensional oriented Riemannian manifold  $(M, g)$  equipped with a 3-form  $\omega$  given by

$$\begin{aligned} \theta^1 \wedge \theta^2 \wedge \theta^7 + \theta^1 \wedge \theta^3 \wedge \theta^5 - \theta^1 \wedge \theta^4 \wedge \theta^6 - \theta^2 \wedge \theta^3 \wedge \theta^6 \\ - \theta^2 \wedge \theta^4 \wedge \theta^5 + \theta^3 \wedge \theta^4 \wedge \theta^7 + \theta^5 \wedge \theta^6 \wedge \theta^7, \end{aligned} \tag{68}$$

where  $\{\theta^i\}$  is an orthonormal basis of 1-forms, is called a  $G_2$ -manifold (or  $G_2$ -structure). Special cases of  $G_2$ -structures are distinguished by means of the natural splitting of the covariant derivative of  $\omega$  into four components. This splitting can be also described in terms of analysis of  $d\omega$  and  $d*\omega$ . Special cases of particular interest include (see, for example, [10, 11]):

- *parallel*  $G_2$ -structures, which are defined by conditions  $d\omega = 0$  and  $d*\omega = 0$ ,
- *nearly parallel*  $G_2$ -structures, which have  $d\omega = c*\omega$  and  $d*\omega = 0$ , with  $c$  being a real constant.

The conditions for these special structures can be naturally reformulated in terms of generalized forms by introducing the generalized 3-form

$$\mathbf{w} = (\omega, *\omega). \tag{69}$$

Now

$$d\mathbf{w} = (d\omega + k*\omega, d(*\omega)), \quad \delta\mathbf{w} = (\delta\omega, *d\omega - k\omega). \tag{70}$$

Hence it follows that the  $G_2$ -structure is *parallel* if and only if the generalized 3-form  $\mathbf{w}$  is both closed and co-closed (and hence  $k = 0$ ) and it is *nearly parallel* if and only if  $\mathbf{w}$  is either closed ( $c = -k$ ) or co-closed ( $c = k$ ).

**Example 6** (Integrals of forms and an action for the complex Einstein vacuum field equations).

In this example a possible definition of the integral of a generalized form is presented and illustrated by an action for Einstein’s vacuum equations, in the form presented in [5], in four dimensions. Let  $U \subseteq M$ , be a  $q$ -dimensional sub-manifold,  $0 \leq q \leq n$ , with a (possibly but not necessarily non-empty) compatibly oriented boundary  $\partial U$ . Let  $\mathbf{a}^{q-1} = (\alpha^{q-1}, \alpha^q)$  be a generalized  $(q-1)$ -form and define the integral of this form over  $U$  to be

$$\int_U \mathbf{a}^{q-1} \equiv \int_{\partial U} \alpha^{q-1} + \int_U \alpha^q, \tag{71}$$

$$= \int_U (d\alpha^{q-1} + \alpha^q). \tag{72}$$

With this definition<sup>4</sup> Stoke’s theorem, re-formulated in terms of generalized forms, is

$$\int_U d\mathbf{a}^{q-2} = (1 - (-1)^q k) \int_{\partial U} \mathbf{a}^{q-2} \tag{73}$$

<sup>4</sup> If, motivated by equation (14) and using this notation, an inner product,  $(\cdot)_1$  is defined by

$$(\mathbf{a}, \mathbf{b})_1 = \frac{1}{2} \int_M (\mathbf{a} \wedge \star \mathbf{b} + \mathbf{b} \wedge \star \mathbf{a}),$$

then a different inner product from the previously introduced one is obtained. For this new inner product

$$\begin{aligned} (\mathbf{a}, \mathbf{a})_1 &= \int_M \mathbf{a} \wedge \star \mathbf{a} \\ &= (-1)^{n-p-1} \int_{\partial M} \alpha^p \wedge \star \alpha^{p+1} + \int_M (\alpha^p \wedge \star \alpha^p + \alpha^{p+1} \wedge \star \alpha^{p+1}) \\ &= \int_M (-1)^{n-p-1} d(\alpha^p \wedge \star \alpha^{p+1}) + \alpha^p \wedge \star \alpha^p + \alpha^{p+1} \wedge \star \alpha^{p+1}, \end{aligned}$$

the condition,  $(da, b)_1 = (a, \delta b)_1$ , holds when the integral over  $\partial M$  vanishes, but when that integral is not zero the condition will generally not hold.

Using this definition an action for Einstein's (complex) vacuum equations in four dimensions can be written, when it is assumed that  $k$  is not equal to 1, in the simple form

$$A = \int \mathbf{r}_A \wedge \mathbf{D}\mathbf{s}^A, \quad (74)$$

where, using the notation of [5], upper case Latin indices are two component spinor indices which range and sum over 1 and 2,  $\mathbf{r}_A$  and  $\mathbf{s}_A$  are generalized spinor-valued 1-forms.  $\mathbf{D}$  is an  $\mathfrak{sl}(2, \mathbb{C})$ -valued generalized covariant exterior derivative determined by a generalized  $\mathfrak{sl}(2, \mathbb{C})$ -valued connection 1-form

$$\Gamma_B^A = (\omega_B^A, -\Omega_B^A), \quad \text{where } \Omega_B^A = d\omega_B^A + \omega_C^A \wedge \omega_B^C, \quad (75)$$

and  $\omega_B^A$  is an ordinary  $\mathfrak{sl}(2, \mathbb{C})$ -valued connection 1-form, [5]. Variation of the action with respect to the dynamical variables,  $\mathbf{r}_A$ ,  $\mathbf{s}_A$  and  $\Gamma_B^A$  gives as the Euler–Lagrange equations the equations

$$\begin{aligned} \mathbf{D}\mathbf{s}^A &\equiv d\mathbf{s}^A - \mathbf{s}^B \wedge \Gamma_B^A = 0, \\ \mathbf{D}\mathbf{r}^A &\equiv d\mathbf{r}^A - \mathbf{r}^B \wedge \Gamma_B^A = 0, \end{aligned} \quad (76)$$

and the condition that  $\mathbf{r}^{(A} \wedge \mathbf{s}^{B)}$  are ordinary 2-forms. The latter ensures that  $\omega_B^A$  is the anti self-dual part of the Levi-Civita connection of a Ricci-flat metric. Again, when using the notation of [5], the dynamical variables  $\mathbf{r}^A$  and  $\mathbf{s}^A$  are written in terms of ordinary forms as

$$\begin{aligned} \mathbf{r}^A &\equiv (\alpha^A, -k^{-1}\mu^A), \\ \mathbf{s}^A &\equiv (\beta^A, k^{-1}\nu^A), \end{aligned} \quad (77)$$

and this metric is

$$ds^2 = \alpha_A \otimes \beta^A + \beta^A \otimes \alpha_A. \quad (78)$$

## 5. Concluding comments

Different differential calculi can be introduced on manifolds, an issue that has been explored, for example, in the context of the development of non-commutative differential geometry, [9]. The algebra and calculus discussed in this paper and in [5] have a number of interesting features. These include natural unifications of potentials and fields, and the equations they satisfy and the natural incorporation of mass-like terms in generalized Laplacians. In addition, equations containing connections, such as the Cartan structure equations and the Einstein equations take particularly simple ‘flat’ forms when written in terms of generalized forms. As was noted in [5], further generalizations along the lines presented here are possible, with it being a simple matter to formulate, as was done there, an exterior algebra and calculus of  $p$ -forms where  $-n \leq p \leq n$ . The usefulness of these different calculi, and their relation to other calculi [9], remain interesting matters for investigation.

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