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Projective connections associated with second-order ODEs

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Abstract
We show that every second-order ODE defines a four-parameter family of projective connections on its two-dimensional solution space. In a special case of ODEs, for which a certain point transformation invariant vanishes, we find that this family of connections always has a preferred representative. This preferred representative turns out to be identical to the projective connection described in Cartan's classic paper (Cartan E 1924 *Bull. Soc. Math. France* **52** 205–41, 1955 *Oeuvres III* **1** 825–62).

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1. Introduction

In recent years there has been a return of interest in the two related classical issues associated with differential equations: (1) the equivalence problem (under a variety of transformation types) for the equations and (2) the natural geometric structures induced by the equations on their solution spaces. The original studies began, among others, with the works of Lie [10] and his student, Tresse [11, 12]. This was soon followed by Wünschmann's contribution [13] and reached its peak with the work of Cartan [2] and Chern [3]. Cartan devised an extremely powerful but difficult scheme for the analysis of the equivalence problem under the three classes of transformation: fibre preserving, point and contact. Though equivalence relations were established for a variety of equations and transformation classes, the calculations were extraordinarily complicated and long and, as a consequence, many problems were only partially completed. (The modern advent of algebraic computers has allowed the completion of many of these problems and opened the door to a variety of new problems [4, 5, 7, 8].) Early on in these studies—then confined to general second- and third-order ODEs—it was realized that the equations themselves defined certain geometric structures on their (finite-dimensional) solution spaces. For example, Wünschmann discovered that a (large) class of third-order ODEs define a conformal (Lorentzian) metric on the three-dimensional solution space. This class was defined by the vanishing of a certain function of the third-order equation. Later, in the context of Cartan and Chern's work, this function was understood as a (relative) invariant of the equation under contact transformations and became known as the Wünschmann invariant. (As an aside we mention that in the modern context of general relativity, this work was generalized to pairs of second-order ODEs whose solution space is four dimensional. The vanishing of a generalized Wünschmann invariant for these equations leads to a conformal Lorentzian metric on the solution space. All four-dimensional Lorentzian metrics are obtainable in this manner [4, 5].)

Cartan, following Lie and Tresse, using his scheme for the analysis of second-order ODEs under point transformations realized [1] that a large class of second-order ODEs induced a natural projective structure on their two-dimensional solution space. This class was defined (analogously to the third-order ODE case) by the vanishing of a certain Wünschmann-like function of the second-order equation.

In the present work we return to the problem of the geometry associated with any second-order ODE. Without recourse to Cartan's equivalence technique, we find that any second-order ODE defines, via the torsion-free first Cartan structure equation, a four-parameter family of projective connections on the solution space.

In the second section we review the general theory of normal projective connections on n-manifolds from the point of view of Cartan connections. We also define projective structures as equivalence classes of certain sets of 1-forms on these manifolds.

As an example of projective connections, in the third section, we consider the geometry associated with a second-order ODE. We find a natural four-parameter family of projective connections living on its two-dimensional space of solutions. In general, these connections are quite complicated. They are parametrized by the solutions of a certain *linear* ODE of fourth order, which is naturally associated with our ODE. We find that among all the ODEs $y'' = Q(x, y, y')$, there is a large class for which the associated fourth-order ODE is *homogeneous*. This class of equations is characterized in terms of the vanishing of a certain function constructed solely from Q and its derivatives, which is directly analogous to the Wünschman function. It turns out that the trivial solution of the homogeneous fourth-order ODE singles out a preferred connection from the four-parameter family. Then, this class of second-order ODEs together with this preferred connection turns out to be identical to the class that Cartan obtained from a study of the equivalence problem. In the last section we discuss the relationship between our and Cartan's method of obtaining this class.

The work described here is part of a larger project, namely the study of natural geometric structures induced on the finite-dimensional solution spaces of both ODEs and certain overdetermined PDEs. In an earlier work [6] we saw how all four-dimensional conformal metrics and Cartan normal conformal connections were contained in the space of pairs of PDEs satisfying generalized Wünschmann equations. (Similar results hold for all third-order ODEs satisfying the Wünschmann equation.) In the present work we have extended these results to unique Cartan normal projective connections associated with second-order ODEs satisfying a Wünschmann-like equation.

2. Projective connection

2.1. Cartan connection

In this subsection we will first define a *Cartan connection* and then specialize it to a Cartan *projective* connection (see [9] for more details).

Consider a structure (P, H, M, G) such that

- (P, H, M) is the principal fibre bundle, over an n-dimensional manifold, with a structure Lie group H;
- G is a Lie group, of dimension $\dim G = \dim P$, for which H is a closed subgroup.

Denote by B^* the fundamental vector field associated with an element B of the Lie algebra H' of H. Let ω be a G'-valued 1-form on P such that

- $\omega(B^*) = B$ for each $B \in H'$;
- $R_b^*\omega = b^{-1}\omega b$ for each $b \in H$;
- $\omega(X) = 0$ if and only if the vector field X vanishes identically on P.

Then ω is called *Cartan's connection on* (P, H, M, G).

The Cartan *projective connection* is a Cartan connection for which

$$G = \mathbf{SL}(n+1, \mathbf{R})/(\text{centre}),$$
$$H = \left\{ \begin{pmatrix} \mathbf{A} & 0 \\ A^T & (\det \mathbf{A})^{-1} \end{pmatrix}, \mathbf{A} \in \mathbf{GL}(n, \mathbf{R}), A \in \mathbf{R}^n \right\} \Big/ (\text{centre})$$

In the next two subsections, we present a convenient way of defining a projective connection on a local trivialization $U \times H$ of the bundle P.

2.2. *Normal projective connection on $U \in M$*

Here, working on the base space M we define a *normal projective connection* on $U \subset M$.

Consider a coframe (ω^i), $i = 1, 2, \ldots, n$, on an open neighbourhood U of M. Suppose that in addition you have n^2, 1-forms $\omega^i{}_j$, $i, j = 1, 2, \ldots, n$, on M such that

$$\mathrm{d}\omega^i + \omega^i{}_j \wedge \omega^j = 0, \qquad \forall\, i = 1, 2, \ldots, n. \tag{1}$$

Then, the system of forms $(\omega^i, \omega^i{}_j)$ defines a torsion-free connection on U.

Take n *arbitrary* 1-forms (ω_i), $i = 1, 2, \ldots, n$, on U. The forms $(\omega^i, \omega^i{}_j, \omega_j)$ define the n^2 2-forms $\Omega^i{}_j$ and n 2-forms Ψ_j on U by

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j + \omega^i \wedge \omega_j + \delta^i{}_j \omega^k \wedge \omega_k, \tag{2}$$

$$\Psi_i = \mathrm{d}\omega_i + \omega_k \wedge \omega^k{}_j. \tag{3}$$

Decompose $\Omega^i{}_j$ onto the basis (ω^i),

$$\Omega^i{}_j = \tfrac{1}{2}\Omega^i{}_{jkl}\omega^k \wedge \omega^l.$$

Find all (ω_i) for which the so-called *normal* condition

$$\Omega^i{}_{jil} = 0, \qquad \forall\, j, l = 1, 2, \ldots, n, \tag{4}$$

is satisfied. It turns out that if $n \geqslant 2$, the forms ω_i are determined *uniquely* by equations (4). Indeed, by using the Riemann 2-forms

$$R^i{}_j = \tfrac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j \tag{5}$$

and the Ricci tensor

$$R_{jl} = R^i_{jil} \tag{6}$$

of the connection $\omega^i{}_j$, one finds that

$$\omega_i = \left[\frac{1}{1-n}R_{(ij)} - \frac{1}{1+n}R_{[ij]}\right]\omega^j. \tag{7}$$

Having determined the forms ω_i, collect the system of 1-forms $\left(\omega^i, {\omega^i}_j, \omega_j\right)$ into a matrix

$$\omega_u = \begin{pmatrix} {\omega^i}_k - \frac{1}{n+1}{\omega^l}_l {\delta^i}_k & \omega^i \\ \omega_k & -\frac{1}{n+1}{\omega^l}_l \end{pmatrix}. \tag{8}$$

Note that ω_u is a 1-form on U which has values in the Lie algebra $G' = \mathbf{SL}'(n+1, \mathbf{R})$. It is called a *normal projective connection* on U.

2.3. Normal projective connection on $U \times H$

Earlier, we defined a Cartan projective connection on the principal H-bundle (P, M, H, G). Here we show how the normal projective connection on $U \subset M$ can be lifted to (P, M, H, G).

Choose a generic element of H in the form

$$b = \begin{pmatrix} {A^i}_k & 0 \\ A_k & a^{-1} \end{pmatrix}, \tag{9}$$

where $\left({A^i}_j\right)$ is a real-valued $n \times n$ matrix with nonvanishing determinant $a = \det\left({A^i}_j\right)$, and (A_i) is a real row n-vector.

Define a G'-valued 1-form ω on $U \times H$ by

$$\omega = b^{-1}\omega_u b + b^{-1}\,\mathrm{d}b. \tag{10}$$

The 1-form ω defines a *projective connection* on $U \times H$. This projective connection on $U \times H$ is called the *normal* projective connection. The term *normal* refers to condition (4), which this connection satisfies.

The explicit formulae for the normal projective connection (10) are written below:

$$\omega = \begin{pmatrix} {\omega'^i}_k - \frac{1}{n+1}{\omega'^l}_l {\delta^i}_k & \omega'^i \\ \omega'_k & -\frac{1}{n+1}{\omega'^l}_l \end{pmatrix}, \tag{11}$$

where

$$\omega'^i = a^{-1}{A^{-1i}}_j\omega^j, \tag{12}$$

$${\omega'^i}_j = {A^{-1i}}_k{\omega^k}_l{A^l}_j + {A^{-1i}}_k\omega^k A_j + {\delta^i}_j A_l {A^{-1l}}_k\omega^k + {A^{-1i}}_k\,\mathrm{d}{A^k}_j + {\delta^i}_j a^{-1}\,\mathrm{d}a, \tag{13}$$

$$\omega'_i = a\left(\omega_k {A^k}_i - A_l{A^{-1l}}_j{\omega^j}_k{A^k}_i - A_l{A^{-1l}}_j\omega^j A_i + \mathrm{d}A_i - A_l{A^{-1l}}_j\,\mathrm{d}{A^j}_i\right), \tag{14}$$

and we have used the fact that

$$\mathrm{d}a = a{A^{-1l}}_k\,\mathrm{d}{A^k}_l. \tag{15}$$

The curvature

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega \tag{16}$$

of ω has the form

$$\Omega = b^{-1}\Omega_u b, \qquad \text{where} \qquad \Omega_u = \mathrm{d}\omega_u + \omega_u \wedge \omega_u = \begin{pmatrix} {\Omega^i}_j - \frac{1}{n+1}{\delta^i}_j{\Omega^l}_l & 0 \\ \Psi_j & -\frac{1}{n+1}{\Omega^l}_l \end{pmatrix}. \tag{17}$$

It is worthwhile to note that if $n \geqslant 3$, then the vanishing of ${\Omega^i}_j$ implies the vanishing of Ψ_i. This follows from the Bianchi identity $\mathrm{d}\Omega - \Omega\wedge\omega + \omega\wedge\Omega = 0$. It is known that in dimension $n = 2$, the forms ${\Omega^i}_j$ are identically equal to zero. In this dimension, all the information about the curvature of the normal projective connection is encoded in the forms Ψ_i.

Remark. To globalize the local trivialization construction of the normal projective connection described above, one needs assumptions about the topology of M. In the local treatment we use in this paper, these assumptions are not necessary.

2.4. Projective structure on M

An alternative view of formulae (12)–(14) is to consider them as an equivalence class of connections on U. This motivates the following definition.

A *projective structure* on an n-dimensional manifold M is an equivalence class $\left[\left(\omega^i, \omega^i{}_j\right)\right]$ of sets of 1-forms $\left(\omega^i, \omega^i{}_j\right)$ on M such that

- (ω^i), $i = 1, 2, \ldots, n$, is a coframe on M such that
$$\mathrm{d}\omega^i + \omega^i{}_j \wedge \omega^j = 0, \qquad \forall\, i = 1, 2, \ldots, n$$
- two sets $\left(\omega^i, \omega^i{}_j\right)$ and $\left(\omega'^i, \omega'^i{}_j\right)$ are in the same equivalence class iff there exist functions $A^i{}_j$ and A_i on M such that
$$\omega'^i = a^{-1} A^{-1i}{}_j \omega^j \tag{18}$$
and
$$\omega'^i{}_j = A^{-1i}{}_k \omega^k{}_l A^l{}_j + A^{-1i}{}_k \omega^k A_j + \delta^i{}_j A_l A^{-1l}{}_k \omega^k + A^{-1i}{}_k \,\mathrm{d}A^k{}_j + \delta^i{}_j a^{-1}\,\mathrm{d}a, \tag{19}$$
with $a = \det\left(A^i{}_j\right) \neq 0$ at every point of M.

It turns out that all the torsion-free connections from the equivalence class of a given projective structure have the same set of geodesics on M. To see this, consider a representative $\left(\omega^i, \omega^i{}_j\right)$ of a projective structure on M. Let (e_i) be the set of n-vector fields dual to the coframe (ω^i), i.e. $\omega^i(e_j) = \delta^i{}_j$. Let $\gamma(t)$ be a geodesic curve, for the connection 1-forms $\omega^i{}_j = \omega^i{}_{jk}\omega^k$. This means that if $V = \frac{\mathrm{d}}{\mathrm{d}t} = V^i e_i$ is a vector tangent to this curve, then
$$\frac{\mathrm{d}V^i}{\mathrm{d}t} + \omega^i{}_{jk} V^j V^k = f V^i, \tag{20}$$
with a certain function f on M. If $\left(\omega'^i, \omega'^i{}_j\right)$ belongs to the same projective structure as $\left(\omega^i, \omega^i{}_j\right)$, then equation (20) for V^i and the relations between $\left(\omega^i, \omega^i{}_j\right)$ and $\left(\omega'^i, \omega'^i{}_j\right)$ imply that in the coframe (ω'^i), the V'^i component of the vector $V = V'^i e'_i$ satisfies geodesic equation
$$\frac{\mathrm{d}V'^i}{\mathrm{d}t} + \omega'^i{}_{jk} V'^j V'^k = f' V'^i, \tag{21}$$
with merely new function $f' = f + 2aA_j V^{(\prime j)}$. Thus, the curve $\gamma(t)$ is also a geodesic in connection $\omega'^i{}_j$.

Note that if $A^i{}_j = \delta^i{}_j$, then
$$\omega'^i = \omega^i \tag{22}$$
and
$$\omega'^i{}_j = \omega^i{}_j + \omega^i A_j + \delta^i{}_j A, \tag{23}$$
with $A = A_i\omega^i$. Thus, for a given projective structure $\left(\omega^i, \omega^i{}_j\right)$, fixing the coframe does not fix the gauge in the choice of $\omega'^i{}_j$. There exists an entire class (23) of connections that, together with the fixed coframe (ω^i), represents the same projective structure.

2.5. Equivalence of projective structures

We say that two projective structures $\left(\omega^i, \omega^i{}_j\right)$ and $\left(\bar\omega^i, \bar\omega^i{}_j\right)$ on two respective n-dimensional manifolds M and $\bar M$ are (locally) *equivalent* iff there exists a (local) diffeomorphism $\phi : M \to \bar M$ and functions $A^i{}_j$ and A_j on M such that
$$\phi^*(\bar\omega^i) = a^{-1} A^{-1i}{}_j \omega^j$$

and

$$\phi^*(\bar{\omega})^i{}_j = A^{-1i}{}_k\omega^k{}_l A^l{}_j + A^{-1i}{}_k\omega^k A_j + \delta^i{}_j A_l A^{-1l}{}_k\omega^k + A^{-1i}{}_k\,\mathrm{d}A^k{}_j + \delta^i{}_j a^{-1}\,\mathrm{d}a,$$

with $a = \det\left(A^i{}_j\right) \neq 0$.

If, given a projective structure $\left(\omega^i, \omega^i{}_j\right)$ on M, we have a diffeomorphism $\phi : M \to M$ with $A^i{}_j$ and A_j as above, such that

$$\phi^*(\omega^i) = a^{-1}A^{-1i}{}_j\omega^j \tag{24}$$

and

$$\phi^*(\omega)^i{}_j = A^{-1i}{}_k\omega^k{}_l A^l{}_j + A^{-1i}{}_k\omega^k A_j + \delta^i{}_j A_l A^{-1l}{}_k\omega^k + A^{-1i}{}_k\,\mathrm{d}A^k{}_j + \delta^i{}_j a^{-1}\,\mathrm{d}a, \tag{25}$$

then we call ϕ a symmetry of $\left(\omega^i, \omega^i{}_j\right)$. Locally, a one-parameter group of symmetries $\phi_t : M \to M$ of $\left(\omega^i, \omega^i{}_j\right)$ is expressible in terms of the corresponding vector field X, called an *infinitesimal symmetry*. Taking the Lie derivative with respect to X of equations (24) and (25), one obtains the following characterization of infinitesimal symmetries.

A vector field X is an infinitesimal symmetry of a projective structure $\left(\omega^i, \omega^i{}_j\right)$ iff there exist functions $B^i{}_j$ and B_j on M such that

$$\mathcal{L}_X\omega^i = -\left(B^i{}_j + B^k{}_k\delta^i{}_j\right)\omega^j, \tag{26}$$

$$\mathcal{L}_X\omega^i{}_j = \omega^i{}_j B^l{}_j - B^i{}_l\omega^l{}_j + \omega^i B_j + \delta^i{}_j B_l\omega^l + \mathrm{d}B^i{}_j + \delta^i{}_j\,\mathrm{d}B^k{}_k. \tag{27}$$

It is easy to check that a Lie bracket $[X_1, X_2]$ of two infinitesimal symmetries is an infinitesimal symmetry, hence the infinitesimal symmetries generate a Lie algebra. This is the Lie algebra of infinitesimal symmetries of the structure $\left(\omega^i, \omega^i{}_j\right)$.

3. Projective structures of second-order ODEs

3.1. Contact forms associated with a second-order ODE

We now show that a second-order ODE defines a projective structure on the space of its solutions.

A second-order ODE

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Q\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}\right) \tag{28}$$

for a function $\mathbf{R} \ni x \to y = y(x) \in \mathbf{R}$ can be alternatively written as a system of the two first-order ODEs

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p, \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = Q(x, y, p) \tag{29}$$

for two functions $\mathbf{R} \ni x \to y = y(x) \in \mathbf{R}$ and $\mathbf{R} \ni x \to p = p(x) \in \mathbf{R}$. This system defines two (contact) 1-forms

$$\omega^1 = \mathrm{d}y - p\,\mathrm{d}x, \qquad \omega^2 = \mathrm{d}p - Q\,\mathrm{d}x, \tag{30}$$

which live on a three-dimensional manifold J^1, the *first jet space*, parametrized by coordinates (x, y, p). All the information about the ODE (28) is encoded in these two forms. For example, any solution to (28) is a curve $\gamma(x) = (x, y(x), p(x)) \subset J^1$ on which forms (30) vanish.

Given an ODE (28), we look for a set $\left(\omega^1{}_1, \omega^1{}_2, \omega^2{}_1, \omega^2{}_2\right)$ of 1-forms on J^1 such that

$$\mathrm{d}\omega^1 + \omega^1{}_1 \wedge \omega^1 + \omega^1{}_2 \wedge \omega^2 = 0, \qquad \mathrm{d}\omega^2 + \omega^2{}_1 \wedge \omega^1 + \omega^2{}_2 \wedge \omega^2 = 0. \tag{31}$$

Introducing the third 1-form

$$\omega^3 = \mathrm{d}x, \tag{32}$$

which together with ω^1 and ω^2 constitutes a basis of 1-forms on J^1, we find that

$$\mathrm{d}\omega^1 = -\omega^2 \wedge \omega^3, \qquad \mathrm{d}\omega^2 = -(Q_y\omega^1 + Q_p\omega^2) \wedge \omega^3, \tag{33}$$

and that the general solution to 'vanishing torsion' equations (31) is

$$\omega^1{}_1 = \omega^1{}_{11}\omega^1 + \omega^1{}_{12}\omega^2, \qquad \omega^1{}_2 = \omega^1{}_{12}\omega^1 + \omega^1{}_{22}\omega^2 - \omega^3, \tag{34}$$

$$\omega^2{}_1 = \omega^2{}_{11}\omega^1 + \omega^2{}_{12}\omega^2 - Q_y\omega^3, \qquad \omega^2{}_2 = \omega^2{}_{12}\omega^1 + \omega^2{}_{22}\omega^2 - Q_p\omega^3, \tag{35}$$

with some unspecified functions $\left(\omega^1{}_{11}, \omega^1{}_{12}, \omega^1{}_{22}, \omega^2{}_{11}, \omega^2{}_{12}, \omega^2{}_{22}\right)$ on J^1. Here, and in the following, we denote the partial derivatives with respect to a variable, as a subscript on the function whose partial derivative is evaluated, e.g. $Q_y := \frac{\partial Q}{\partial y}$.

The annihilator of the contact forms ω^1 and ω^2 is spanned by the vector field

$$D = \partial_x + p\partial_y + Q\partial_p, \tag{36}$$

which is defined up to a multiplicative factor. Its integral curves, which coincide with the solutions $\gamma(x)$ of the original equation, are intrinsically defined. Also, the notion of surfaces S, *transversal* to D is unambiguous.

Any choice of 1-forms $\left(\omega^1{}_1, \omega^1{}_2, \omega^2{}_1, \omega^2{}_2\right)$ of the form given by equations (34) and (35) on the jet space J^1 determines projective structures $\left[\left(\omega^k; \omega^i{}_j\right)_{|S}\right]$ on each two-dimensional surface S transversal to D. These projective structures are defined on each S by transformations (18) and (19) applied to the 1-forms $\left(\omega^k; \omega^i{}_j\right)_{|S}$. They, in turn, were defined as the restrictions of the 1-forms $\left(\omega^1, \omega^2; \omega^1{}_1, \omega^1{}_2, \omega^2{}_1, \omega^2{}_2\right)$ from J^1 to S. Given a particular choice of functions ω^i_{jk} in (34) and (35) and a pair of transversal to D surfaces S and S', the projective structures $\left[\left(\omega^k; \omega^i{}_j\right)_{|S}\right]$ and $\left[\left(\omega^k; \omega^i{}_j\right)_{|S'}\right]$ will be, in general, inequivalent. It is therefore interesting to ask whether there exists a choice of forms (34) and (35) which, on *all* transversal surfaces S, defines the same (modulo equivalence) projective structure. Locally, this requirement is equivalent to the existence of a choice of forms (34) and (35) on J^1 such that the Lie derivative of the forms $\left(\omega^i; \omega^k{}_j\right)$ along D is simply the infinitesimal version of the transformations (24) and (25). Explicitly, we ask for the existence of $\omega^i{}_{jk}$ of (34) and (35) and the existence of functions $B^i{}_j$ and B_k on J^1 such that

$$\mathcal{L}_D\omega^i = -\left(B^i{}_j + B^k{}_k\delta^i{}_j\right)\omega^j, \tag{37}$$

$$\mathcal{L}_D\omega^i{}_j = \omega^i{}_jB^l{}_j - B^i{}_l\omega^l{}_j + \omega^i B_j + \delta^i{}_jB_l\omega^l + \mathrm{d}B^i{}_j + \delta^i{}_j\,\mathrm{d}B^k{}_k \qquad i, j = 1, 2. \tag{38}$$

If we were able to find a solution $\omega^i{}_{jk}$ to the above equations, it would generate the same projective structure on all surfaces transversal to D. This structure would therefore descend to the two-dimensional space of integral lines of D endowing it, or what is the same, endowing the parameter space of solutions to the original ODE, with a projective structure.

To solve equations (37) and (38), we take the most general forms $\left(\omega^1{}_1, \omega^1{}_2, \omega^2{}_1, \omega^2{}_2\right)$ (from (34) and (35)) that are associated with the ODE. We then use the gauge freedom (22) and (23) preserving

$$\omega^1 = \mathrm{d}y - p\,\mathrm{d}x, \qquad \omega^2 = \mathrm{d}p - Q\,\mathrm{d}x$$

to achieve

$$\omega^1{}_1 = 0$$

everywhere on J^1. The forms $\left(\omega^1, \omega^2; \omega^1{}_1, \omega^1{}_2, \omega^2{}_1, \omega^2{}_2\right)$ with $\omega^1{}_1 = 0$, when restricted to each S, will therefore represent the same projective structure on S as the original general forms we started with. Thus, without loss of generality, we solve equations (37) and (38) for forms

$$\omega^1 = \mathrm{d}y - p\,\mathrm{d}x, \qquad \omega^2 = \mathrm{d}p - Q\,\mathrm{d}x \tag{39}$$

and

$$\begin{aligned}
&\omega^1{}_1 = 0, \qquad \omega^1{}_2 = \omega^1{}_{22}\omega^2 - \omega^3,\\
&\omega^2{}_1 = \omega^2{}_{11}\omega^1 + \omega^2{}_{12}\omega^2 - Q_y\omega^3, \qquad \omega^2{}_2 = \omega^2{}_{12}\omega^1 + \omega^2{}_{22}\omega^2 - Q_p\omega^3.
\end{aligned} \tag{40}$$

It is a matter of straightforward calculation to achieve the following proposition.

Proposition 1. *The forms (39) and (40) satisfy equations (37) and (38) if and only if*

$$\begin{aligned}
\omega^2{}_{22} &= D\omega^1{}_{22} + 2Q_p\omega^1{}_{22},\\
\omega^2{}_{12} &= \tfrac{1}{4}\left[-D^2\omega^1{}_{22} - 3Q_pD\omega^1{}_{22} + \left(3Q_y - 2Q_p^2 - 2DQ_p\right)\omega^1{}_{22} - Q_{pp}\right]\\
\omega^2{}_{11} &= \tfrac{1}{6}\left[D^3\omega^1{}_{22} + 3D^2\omega^1{}_{22} + \left(5DQ_p + 2Q_p^2 - 7Q_y\right)D\omega^1{}_{22}\right.\\
&\qquad \left. + (2D^2Q_p - 3DQ_y + 4Q_pDQ_p - 8Q_pQ_y)\omega^1{}_{22} + DQ_{pp} - 4Q_{py}\right]
\end{aligned} \tag{41}$$

and $\omega^1{}_{22}$ satisfies the differential equation

$$D^4\omega^1{}_{22} + a_4D^3\omega^1{}_{22} + a_3D^2\omega^1{}_{22} + a_2D\omega^1{}_{22} + a_1\omega^1{}_{22} + a_0 = 0 \tag{42}$$

with coefficients a_0, a_1, a_2, a_3, a_4 given by

$$\begin{aligned}
a_4 &= 2Q_p,\\
a_3 &= \left(8DQ_p - Q_p^2 - 10Q_y\right),\\
a_2 &= \left(7D^2Q_p - 10DQ_y + 3Q_pDQ_p - 2Q_p^3 - 10Q_pQ_y\right),\\
a_1 &= \left(2D^3Q_p - 3D^2Q_y + 4(DQ_p)^2 + 2Q_pD^2Q_p - 5Q_pDQ_y\right.\\
&\qquad \left. - 4Q_p^2DQ_p - 14Q_yDQ_p + 2Q_p^2Q_y + 9Q_y^2\right),\\
a_0 &= D^2Q_{pp} - 4DQ_{py} - Q_pDQ_{pp} + 4Q_pQ_{py} - 3Q_{pp}Q_y + 6Q_{yy}.
\end{aligned} \tag{43}$$

Thus, modulo equivalence, the only forms (30)–(35) that generate the same projective structure on all surfaces transversal to D are given by (40) and (41) with the coefficient $\omega^1{}_{22}$ satisfying differential equations (42) and (43). Now, recalling that the space of solutions of the second-order ODE can be identified with the two-dimensional space of integral lines of D in J^1, we obtain the following theorem.

Theorem 1. *Every solution $\omega^1{}_{22}$ to the fourth-order differential equations (42) and (43) defines a natural projective structure on the space of solutions J^1/D of the second-order ODE $y'' = Q(x, y, y')$. The structure is given by the projection from J^1 to J^1/D of forms*

$$\omega^1 = \mathrm{d}y - p\,\mathrm{d}x, \qquad \omega^2 = \mathrm{d}p - Q\,\mathrm{d}x$$

with

$$\begin{aligned}
\omega^1{}_1 &= 0, \qquad \omega^1{}_2 = \omega^1{}_{22}\omega^2 - \omega^3, \qquad \omega^3 = \mathrm{d}x,\\
\omega^2{}_1 &= \tfrac{1}{6}\left[D^3\omega^1{}_{22} + 3D^2\omega^1{}_{22} + \left(5DQ_p + 2Q_p^2 - 7Q_y\right)D\omega^1{}_{22} + (2D^2Q_p - 3DQ_y\right.\\
&\qquad \left. + 4Q_pDQ_p - 8Q_pQ_y)\omega^1{}_{22} + DQ_{pp} - 4Q_{py}\right]\omega^1 + \tfrac{1}{4}\left[-D^2\omega^1{}_{22}\right.\\
&\qquad \left. - 3Q_pD\omega^1{}_{22} + \left(3Q_y - 2Q_p^2 - 2DQ_p\right)\omega^1{}_{22} - Q_{pp}\right]\omega^2 - Q_y\omega^3,\\
\omega^2{}_2 &= \tfrac{1}{4}\left[-D^2\omega^1{}_{22} - 3Q_pD\omega^1{}_{22} + \left(3Q_y - 2Q_p^2 - 2DQ_p\right)\omega^1{}_{22} - Q_{pp}\right]\omega^1\\
&\qquad + \left[D\omega^1{}_{22} + 2Q_p\omega^1{}_{22}\right]\omega^2 - Q_p\omega^3.
\end{aligned}$$

Since equations (42) and (43) are of fourth order, they have four independent solutions. Thus, all the corresponding projective structures on J^1/D should be treated on equal footing. However, in the case of second-order ODEs satisfying some additional conditions, some of these structures may be more distinguished. In particular, Lie [10] and Cartan [1] considered second-order ODEs satisfying the additional condition

$$a_0 = D^2 Q_{pp} - 4DQ_{py} - Q_p DQ_{pp} + 4Q_p Q_{py} - 3Q_{pp}Q_y + 6Q_{yy} \equiv 0. \tag{44}$$

For such ODEs, equations (42) and (43) are homogeneous and as such have a preferred solution ${\omega^1}_{22} = 0$. Thus, for this class of second-order ODEs, there exists a distinguished, natural projective structure on J^1/D associated with the solution ${\omega^1}_{22} = 0$ of (42) and (43). Explicitly, for any second-order ODE satisfying $a_0 \equiv 0$, this structure is given by

$$\omega^1 = \mathrm{d}y - p\,\mathrm{d}x, \qquad \omega^2 = \mathrm{d}p - Q\,\mathrm{d}x \tag{45}$$

with

$$ {\omega^1}_1 = 0, \qquad {\omega^1}_2 = -\omega^3, \qquad \omega^3 = \mathrm{d}x, \tag{46}$$

$$ {\omega^2}_1 = \tfrac{1}{6}(DQ_{pp} - 4Q_{py})\omega^1 - \tfrac{1}{4}Q_{pp}\omega^2 - Q_y\omega^3, \tag{47}$$

$$ {\omega^2}_2 = -\tfrac{1}{4}Q_{pp}\omega^1 - Q_p\omega^3. \tag{48}$$

In general, any projective structure described by theorem 1 leads to a projective $\mathbf{SL}(3,\mathbf{R})$ connection on an eight-dimensional bundle $P \to J^1/D$. One of the features of the projective structures, which via ${\omega^1}_{22} = 0$ are associated with $a_0 \equiv 0$, is that each of them leads to a *normal* projective $\mathbf{SL}(3,\mathbf{R})$ connection on P. Using the local parameters $(x, y, p, \alpha, \beta, \gamma, \nu, \mu)$ for P and equations (11) and (14), (45)–(48), we find that this $\mathbf{SL}(3,\mathbf{R})$ connection reads

$$\omega = \begin{pmatrix} \frac{1}{3}(\Omega_2 - 2\Omega_1) & -\theta^3 & \theta^1 \\ -\Omega_3 & \frac{1}{3}(\Omega_1 - 2\Omega_2) & \theta^2 \\ \Omega_5 & -\Omega_4 & \frac{1}{3}(\Omega_1 + \Omega_2) \end{pmatrix}, \tag{49}$$

where $(\theta^1, \theta^2, \theta^3, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5)$ are given by

$$\theta^1 = \alpha\omega^1, \qquad \theta^2 = \beta(\omega^2 + \gamma\omega^1), \qquad \theta^3 = \frac{\alpha}{\beta}(\omega^3 + \nu\omega^1),$$

$$\Omega_1 = \mathrm{d}\log\alpha - \mu\theta^1 + \frac{\nu}{\beta}\theta^2 - \frac{\beta\gamma}{\alpha}\theta^3,$$

$$\Omega_2 = \mathrm{d}\log\beta - \frac{1}{4\alpha}[6\gamma\nu + 4\nu Q_p - Q_{pp} + 2\alpha\mu]\theta^1 + 2\frac{\nu}{\beta}\theta^2 + \frac{\beta}{\alpha}[\gamma + Q_p]\theta^3,$$

$$\Omega_3 = \frac{\beta}{\alpha}\,\mathrm{d}\gamma - \frac{\beta}{6\alpha^2}[DQ_{pp} - 6\gamma^2\nu - 6\gamma\nu Q_p + 3\gamma Q_{pp} - 4Q_{py} + 6\nu Q_y]\theta^1$$
$$- \frac{1}{4\alpha}[2\gamma\nu - Q_{pp} + 2\alpha\mu]\theta^2 - \frac{\beta^2}{\alpha^2}[\gamma^2 + \gamma Q_p - Q_y]\theta^3,$$

$$\Omega_4 = \frac{1}{\beta}\,\mathrm{d}\nu - \frac{1}{6\alpha\beta}[6\gamma\nu^2 + 6\nu^2 Q_p - 3\nu Q_{pp} + Q_{ppp}]\theta^1 + \frac{\nu^2}{\beta^2}\theta^2$$
$$- \frac{1}{4\alpha}[-2\gamma\nu - 4\nu Q_p + Q_{pp} + 2\alpha\mu]\theta^3,$$

$$2\Omega_5 = \mathrm{d}\mu + \mu \mathrm{d}\log\alpha - \frac{\nu}{\alpha}\mathrm{d}\gamma + \frac{\gamma}{\alpha}\mathrm{d}\nu - \frac{1}{24\alpha^2}\big[12\alpha^2\mu^2 + 48\nu Q_{py} - 48\nu^2 Q_y - 12\nu D Q_{pp} + 36\gamma^2\nu^2 + 48\gamma\nu^2 Q_p - 36\gamma\nu Q_{pp} + 12\gamma Q_{ppp} + 8DQ_{ppp} + 8Q_pQ_{ppp} - 12Q_{ppy} - 3Q_{pp}^2\big]\theta^1 + \frac{1}{6\alpha\beta}[6\gamma\nu^2 - 3\nu Q_{pp} + Q_{ppp} + 6\alpha\nu\mu]\theta^2 - \frac{\beta}{6\alpha^2}[DQ_{pp} - 6\gamma^2\nu - 12\gamma\nu Q_p + 3\gamma Q_{pp} - 4Q_{py} + 12\nu Q_y + 6\alpha\gamma\mu]\theta^3.$$

The curvature of this connection reads

$$\Omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{6\alpha^2\beta} b_{01}\,\theta^1\wedge\theta^2 & -\frac{1}{6\alpha\beta^2} b_0\,\theta^1\wedge\theta^2 & 0 \end{pmatrix},$$

where we have introduced

$$b_0 = Q_{pppp} \qquad \text{and} \qquad b_{01} = Db_0 + (\gamma + 2Q_p)b_0.$$

The relatively simple form of this curvature agrees with the general theory of normal projective connections for $n = 2$ (compare with the note at the end of section 2.3).

The following section is devoted to explaining the Lie/Cartan motivation for considering the class of ODEs leading to the structure defined above.

3.2. Equivalence classes of second-order ODEs modulo point transformations

A point transformation of variables

$$(x, y) = (x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y})) \tag{50}$$

applied to the second-order ODE

$$y'' = Q(x, y, y') \tag{51}$$

changes it to the new form

$$\bar{y}'' = \bar{Q}(\bar{x}, \bar{y}, \bar{y}'). \tag{52}$$

The function $Q = Q(x, y, y')$ transforms in a rather complicated way into a new function $\bar{Q} = \bar{Q}(\bar{x}, \bar{y}, \bar{y}')$. But, using appropriate derivatives of Q, one can construct functions which have nice transformation properties under transformations (50). In particular, the *relative invariants* of equation (51) are such functions which, under transformations (50), scale by a factor. Their vanishing is the point invariant property of the equation. One such relative invariant is

$$a_0 = D^2Q_{pp} - 4DQ_{py} - Q_pDQ_{pp} + 4Q_pQ_{py} - 3Q_{pp}Q_y + 6Q_{yy},$$

the same function that appears in equations (43). This fact was already known to Lie [10]. Cartan [1] considered the problem of finding *all* point invariants of (51). He used his *equivalence method*, which enabled him to determine another relative invariant

$$b_0 = Q_{pppp}.$$

Both a_0 and b_0 are of the same order and it follows from the Cartan analysis that equation (51) has no more point invariants of order less than or equal to 4. Thus, according to Cartan, the second-order ODEs modulo point transformations split into four major classes which are

(i) $a_0 = b_0 = 0$;
(ii) $a_0 = 0$ and $b_0 \neq 0$;

(iii) $a_0 \neq 0$ and $b_0 = 0$;
(iv) $a_0 \neq 0$ and $b_0 \neq 0$.

Cases (i) and (ii) were analysed by Cartan completely. In particular, he showed that if $a_0 = 0$, then with each point equivalence class of second-order ODEs is associated a natural normal projective connection, whose curvature provides all the point invariants of the class. This connection equips the space of solutions of each of the equations from the equivalence class with a projective structure. It follows that the projective structures originating in this way from different equations from the same point equivalence class are equivalent. This distinguished projective structure associated with the class of equation $y'' = Q(x, y, y')$ coincides with the structure (47) defined in the previous section.

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