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Obstructions to conformally Einstein metrics in *n* dimensions

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Abstract

We construct polynomial conformal invariants, the vanishing of which is necessary and sufficient for an n-dimensional suitably generic (pseudo-)Riemannian manifold to be conformal to an Einstein manifold. We also construct invariants which give necessary and sufficient conditions for a metric to be conformally related to a metric with vanishing Cotton tensor. One set of invariants we derive generalises the set of invariants in dimension 4 obtained by Kozameh, Newman and Tod. For the conformally Einstein problem, another set of invariants we construct gives necessary and sufficient conditions for a wider class of metrics than covered by the invariants recently presented by Listing. We also show that there is an alternative characterisation of conformally Einstein metrics based on the tractor connection associated with the normal conformal Cartan bundle. This plays a key role in constructing some of the invariants. Also using this we can interpret the previously known invariants geometrically in the tractor setting and relate some of them to the curvature of the Fefferman–Graham ambient metric.

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1. Introduction

The central focus of this article is the problem of finding necessary and sufficient conditions for a Riemannian or pseudo-Riemannian manifold, of any signature and dimension $n \geq 3$, to be locally conformally related to an Einstein metric. In particular we seek invariants, polynomial in the Riemannian curvature and its covariant derivatives, that give a sharp obstruction to conformally Einstein metrics in the sense that they vanish if and only if the metric concerned is conformally related to an Einstein metric. For example in dimension 3 it is well known that this problem is solved by the Cotton tensor, which is a certain tensor part of the first covariant derivative of the Ricci tensor. So 3-manifolds are conformally Einstein if and only if they are conformally flat. The situation is significantly more complicated in higher dimensions. Our main result is that we are able to solve this problem in all dimensions and for metrics of any signature, except that the metrics are required to be non-degenerate in the sense that they are, what we term, weakly generic. This means that, viewed as a bundle map $TM \to \otimes^3 TM$, the Weyl curvature is injective. The results are most striking for Riemannian n-manifolds where we obtain a single trace-free rank two tensor-valued conformal invariant that gives a sharp obstruction. Setting this invariant to zero gives a quasi-linear equation on the metric. Returning to the setting of arbitrary signature, we also show that a manifold is conformally Einstein if and only if a certain vector bundle, the so-called standard tractor bundle, admits a parallel section. This powerful characterisation of conformally Einstein metrics is used to obtain the sharp obstructions for conformally Einstein metrics in the general weakly generic pseudo-Riemannian and Riemannian setting. It also yields a simple geometric derivation, and unifying framework, for all the main theorems in the paper.

The study of conditions for a metric to be conformally Einstein has a long history that dates back to the work of Brinkman [4,5] and Schouten [29]. Substantial progress was made by Szekeres in 1963 [30]. He solved the problem on 4-manifolds, of signature -2, by explicitly describing invariants that provide a sharp obstruction. However his approach is based on a spinor formalism and is difficult to analyse when translated into the equivalent tensorial picture. In the 1980s Kozameh, Newman and Tod (KNT) [19] found a simpler set of conditions. While their construction was based on Lorentzian 4-manifolds the invariants obtained provide obstructions in any signature. However these invariants only give a sharp obstruction to conformally Einstein metrics if a special class of metrics is excluded (see also [20] for the reformulation of the KNT result in terms of the Cartan normal conformal connection). Baston and Mason [3] proposed another pair of conformally invariant obstruction invariants for 4-manifolds. However these give a sharp obstruction for a smaller class of metrics than the KNT system (see [1]).

One of the invariants in the KNT system is the conformally invariant Bach tensor. In higher even dimensions there is an interesting higher order analogue of this trace-free symmetric 2-tensor due to Fefferman and Graham and this is also an obstruction to conformally Einstein metrics [11,17,18]. This tensor arises as an obstruction to their ambient metric construction. It has a close relationship to some of the constructions in this article, but this is described in [17]. Here we focus on invariants which exist in all dimensions. Recently Listing [21] made a substantial advance. He described a trace-free 2-tensor that gives, in dimensions $n \geq 4$, a sharp obstruction for conformally Einstein metrics, subject to the restriction that the metrics are what he terms “non-degenerate”. This means that the Weyl curvature is maximal rank as a map $\Lambda^2 TM \to \Lambda^2 TM$. In this paper metrics satisfying this non-degeneracy condition are instead termed Λ^2-generic.

Following some general background, we show in Sections 2.3 and 2.4 that it is possible to generalise to arbitrary dimension $n \geq 4$ the development of KNT. This culminates in the construction of a pair of (pseudo-)Riemannian invariants F^1_{abc} and F^2_{ab} whose vanishing is necessary and sufficient for the manifold to be conformally Einstein provided we exclude a small class of metrics (but the class is larger than the class failing to be Λ^2-generic) (see Theorem 2.3). These invariants are *natural* in the sense that they are given by a metric partial contraction polynomial in the Riemannian curvature and its covariant derivatives. F^1 is conformally covariant and F^2 is conformally covariant on metrics for which F^2 vanishes. Thus together they form a conformally covariant system.

In Section 2.5 we show that very simple ideas reveal new conformal invariants that are more effective than the system F^1 and F^2 in the sense that they give sharp obstructions to conformal Einstein metrics on a wider class of metrics. Here the broad treatment is based on the assumption that the metrics are weakly generic as defined earlier. This is a strictly weaker restriction than requiring metrics to be Λ^2-generic; any Λ^2-generic metric is weakly generic but in general the converse fails to be true. One of the main results of the paper is Theorem 2.8 which gives a natural conformally invariant trace-free 2-tensor which gives a sharp obstruction for conformally Einstein metrics on weakly generic Riemannian manifolds. Thus in the Riemannian setting this improves Listing’s results. In Riemannian dimension 4 there is an even simpler obstruction, see Theorem 2.9, but an equivalent result is in [21]. In Theorem 2.10 we also recover Listing’s main results for Λ^2-generic metrics as special case of the general setup. In all cases the invariants give quasi-linear equations. The results mentioned are derived from the general result in Proposition 2.7. We should point out that while this proposition does not in general lead to natural obstructions, in many practical situations, for example if a metric is given explicitly in terms of a basis field, this would still provide an effective route to testing whether or not a metric is conformally Einstein, since a choice of tensor $\tilde{D}$ can easily be described. (See the final remark at the end of Section 2.5.)

In Section 2.5 we also pause, in Proposition 2.5 and Theorem 2.6, to observe some sharp obstructions to metrics being conformal to a metric with vanishing Cotton tensor. We believe these should be of independent interest. Since the vanishing of the Cotton tensor is necessary but not sufficient for a metric to be Einstein, it seems that the Cotton tensor could play a role in setting up problems where one seeks metrics suitably “close” to being Einstein or conformally Einstein.

In Section 3, following some background on tractor calculus, we give the characterisation of conformally Einstein metrics as exactly those for which the standard tractor bundle admits a (suitably generic) parallel section. The standard (conformal) tractor bundle is an associated structure to the normal Cartan conformal connection. The derivations in Section 2 are quite simple and use just elementary tensor analysis and Riemannian differential geometry. However they also appear ad hoc. We show in Section 3 that the constructions and invariants of Section 2 have a natural and unifying geometric interpretation in the tractor/Cartan framework. This easily adapts to yield new characterisations of conformally Einstein metrics, see Theorem 3.4. From this we obtain, in Corollary 3.5, obstructions for conformally Einstein metrics that are sharp for weakly generic metrics of any signature. Thus these also improve on the results in [21].

We believe the development in Section 3 should have an important role in suggesting how an analogous programme could be carried out for related conformal problems as well as analogues on, for example, CR structures where the structure and tractor calculus is very similar. We also use this machinery to show that the system F^1, F^2 has a simple interpretation in terms of the curvature of the Fefferman–Graham ambient metric.

Finally in Section 4 we discuss explicit metrics to shed light on the invariants constructed and their applicability. This includes examples of classes metrics which are weakly generic but not Λ^2-generic. Also here, as an example use of the machinery on explicit metrics, we identify the conformally Einstein metrics among a special class of Robinson–Trautman metrics.

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2. Conformal characterisations via tensors

In this section we use standard tensor analysis on (pseudo-)Riemannian manifolds to derive sharp obstructions to conformally Einstein metrics.

2.1. Basic (pseudo-)Riemannian objects

Let M be a smooth manifold, of dimension $n \geq 3$, equipped with a Riemannian or pseudo-Riemannian metric g_{ab}. We employ Penrose's abstract index notation [27] and indices should be assumed abstract unless otherwise indicated. We write $\mathcal{E}^a$ to denote the space of smooth sections of the tangent bundle on M, and $\mathcal{E}_a$ for the space of smooth sections of the cotangent bundle. (In fact we will often use the same symbols for the corresponding bundles, and also in other situations we will often use the same symbol for a given bundle and its space of smooth sections, since the meaning will be clear by context.) We write $\mathcal{E}$ for the space of smooth functions and all tensors considered will be assumed smooth without further comment. An index which appears twice, once raised and once lowered, indicates a contraction. The metric g_{ab} and its inverse g^{ab} enable the identification of $\mathcal{E}^a$ and $\mathcal{E}_a$ and we indicate this by raising and lowering indices in the usual way.

The metric g_{ab} defines the Levi–Civita connection ∇_a with the curvature tensor R^a_{bcd} given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)V^c = R^c_{abd}V^d, \text{ where } V^c \in \mathcal{E}^c.$$

This can be decomposed into the totally trace-free *Weyl curvature* C_{abcd} and the symmetric *Schouten tensor* P_{ab} according to

$$R_{abcd} = C_{abcd} + 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c}.$$

Thus P_{ab} is a trace modification of the Ricci tensor $R_{ab} = R_{ca}{}^{c}{}_{b}$:

$$R_{ab} = (n-2)\mathsf{P}_{ab} + \mathsf{J} g_{ab}, \quad \mathsf{J} := \mathsf{P}_a^a.$$

Note that the Weyl tensor has the symmetries

$$C_{abcd} = C_{[ab][cd]} = C_{cdab}, \quad C_{[abc]d} = 0,$$

where we have used the square brackets to denote the antisymmetrisation of the indices.

We recall that the metric g_{ab} is an Einstein metric if the trace-free part of the Ricci tensor vanishes. This condition, when written in terms of the Schouten tensor, is given by

$$\mathsf{P}_{ab} - \frac{1}{n}\mathsf{J} g_{ab} = 0.$$

In the following we will also need the Cotton tensor A_{abc} and the Bach tensor B_{ab}. These are defined by

$$A_{abc} := 2\nabla_{[b}\mathsf{P}_{c]a} \tag{2.1}$$

and

$$B_{ab} := \nabla^c A_{acb} + \mathsf{P}^{dc} C_{dacb}. \tag{2.2}$$

It is straightforward to verify that the Bach tensor is symmetric. From the contracted Bianchi identity $\nabla^a \mathsf{P}_{ab} = \nabla_b \mathsf{J}$ it follows that the Cotton tensor is totally trace-free. Using this, and that the Weyl tensor is trace-free, it follows that the Bach tensor is also trace-free.

Let us adopt the convention that sequentially labelled indices are implicitly skewed over. For example with this notation the Bianchi symmetry is simply $R_{a_1a_2a_3b} = 0$. Using this symmetry and the definition (2.1) of $A_{ba_1a_2}$ we obtain a useful identity

$$\nabla_{a_1} A_{ba_2a_3} = \mathsf{P}^c_{a_1} C_{a_2a_3bc}. \tag{2.3}$$

Further important identities arise from the Bianchi identity $\nabla_{a_1} R_{a_2a_3de} = 0$:

$$\nabla_{a_1} C_{a_2a_3cd} = g_{ca_1} A_{da_2a_3} - g_{da_1} A_{ca_2a_3}, \tag{2.4}$$

$$(n-3)A_{abc} = \nabla^d C_{dabc}, \tag{2.5}$$

$$\nabla^a \mathsf{P}_{ab} = \nabla_b \mathsf{J}, \tag{2.6}$$

$$\nabla^a A_{abc} = 0. \tag{2.7}$$

2.2. Conformal properties and naturality

Metrics g_{ab} and $\hat{g}_{ab}$ are said to be conformally related if

$$\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}, \quad \Upsilon \in \mathcal{E}, \tag{2.8}$$

and the replacement of g_{ab} with $\hat{g}_{ab}$ is termed a *conformal rescaling*. Conformal rescaling in this way results in a conformal transformation of the Levi–Civita connection. This is given by

$$\widehat{\nabla_a u_b} = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + g_{ab}\Upsilon^c u_c \tag{2.9}$$

for a 1-form u_b. The conformal transformation of the Levi–Civita connection on other tensors is determined by this, the duality between 1-forms and tangent fields, and the Leibniz rule.

A tensor T (with any number of covariant and contravariant indices) is said to be *conformally covariant* (of *weight* w) if, under a conformal rescaling (2.8) of the metric, it transforms according to

$$T \mapsto \hat{T} = \mathrm{e}^{w\Upsilon} T,$$

for some $w \in \mathbb{R}$. We will say T is conformally *invariant* if $w = 0$. We are particularly interested in natural tensors with this property. A tensor T is *natural* if there is an expression for T which is a metric partial contraction, polynomial in the metric, the inverse metric, the Riemannian curvature and its covariant derivatives.

The weight of a conformally covariant depends on the placement of indices. It is well known that the Cotton tensor in dimension $n = 3$ and the Weyl tensor in dimension $n \geq 3$ are conformally invariant with their natural placement of indices, i.e. $\hat{A}_{abc} = A_{abc}$ and $\hat{C}_{ab}{}^c{}_d = C_{ab}{}^c{}_d$. In dimension $n \geq 4$, vanishing of the Weyl tensor is equivalent to the existence of a scale Υ such that the transformed metric $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$ is flat (and so if the Weyl tensor vanishes we say the metric is *conformally flat*). In dimension $n = 3$ the Weyl tensor vanishes identically. In this dimension g_{ab} is conformally flat if and only if the Cotton tensor vanishes.

An example of tensor which fails to be conformally covariant is the Schouten tensor. We have

$$\mathsf{P}_{ab} \to \hat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2}\Upsilon_c \Upsilon^c g_{ab}, \tag{2.10}$$

where

$$\Upsilon_a = \nabla_a \Upsilon.$$

Thus the property of the metric being Einstein is not conformally invariant. A metric g_{ab} is said to be *conformally Einstein* if there exists a conformal scale Υ such that $\hat{g}_{ab} = \mathrm{e}^{2\Upsilon} g_{ab}$ is Einstein.

For natural tensors the property of being conformally covariant or invariant may depend on dimension. For example it is well known that the Bach tensor is conformally covariant in dimension 4. In other dimensions the Bach tensor fails to be conformally covariant.

2.3. Necessary conditions for conformally Einstein metrics

Suppose that g_{ab} is conformally Einstein. As mentioned above this means that there exists a scale Υ such that the Ricci tensor, or equivalently the Schouten tensor for $\hat{g}_{ab} := \mathrm{e}^{2\Upsilon} g_{ab}$, is pure trace. That is

$$\hat{\mathsf{P}}_{ab} - \frac{1}{n}\hat{\mathsf{J}}\hat{g}_{ab} = 0.$$

This equation, when written in terms of Levi–Civita connection ∇ and Schouten tensor P_{ab} associated with g_{ab} reads,

$$\mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{n} T g_{ab} = 0, \tag{2.11}$$

where

$$T = \mathsf{J} - \nabla^a \Upsilon_a + \Upsilon^a \Upsilon_a.$$

Conversely if there is a gradient $\Upsilon_a = \nabla_a \Upsilon$ satisfying (2.11) then $\hat{g}_{ab} := \mathrm{e}^{2\Upsilon} g_{ab}$ is an Einstein metric. Thus, with the understanding that $\Upsilon_a = \nabla_a \Upsilon$, (2.11) will be termed the *conformal Einstein equations*. There exists a smooth function Υ solving these if and only if the metric g is conformally Einstein.

To find consequences of these equations we apply ∇_c to both sides of (2.11) and then antisymmetrise the result over the $\{ca\}$ index pair. Using that the both the Weyl tensor and the Cotton tensor are completely trace-free this leads to the first integrability condition which is

$$A_{abc} + \Upsilon^d C_{dabc} = 0.$$

Now taking ∇^c of this equation, using the definition of the Bach tensor (2.2), the identity (2.5), and again this last displayed equation, we get

$$B_{ab} + \mathsf{P}^{dc} C_{dabc} - (\nabla^c \Upsilon^d - (n-3)\Upsilon^d \Upsilon^c) C_{dabc} = 0.$$

Eliminating $\nabla^c \Upsilon^d$ by means of the Einstein condition (2.11) yields a second integrability condition:

$$B_{ab} + (n-4)\Upsilon^d \Upsilon^c C_{dabc} = 0.$$

Summarising we have the following proposition.

Proposition 2.1. *If g_{ab} is a conformally Einstein metric then the corresponding Cotton tensor A_{abc} and the Bach tensor B_{ab} satisfy the following conditions*

$$A_{abc} + \Upsilon^d C_{dabc} = 0, \tag{2.12}$$

and

$$B_{ab} + (n-4)\Upsilon^d \Upsilon^c C_{dabc} = 0. \tag{2.13}$$

for some gradient

$$\Upsilon_d = \nabla_d \Upsilon.$$

Here Υ is a function which conformally rescales the metric g_{ab} to an Einstein metric $\hat{g}_{ab} = e^{2\Upsilon} g_{ab}$.

Remarks:

- Note that in dimension $n = 3$ the first integrability condition (2.12) reduces to $A_{abc} = 0$ and the Weyl curvature vanishes. Thus, in dimension $n = 3$, if (2.12) holds then (2.13) is automatically satisfied and the conformally Einstein metrics are exactly the conformally flat metrics. The vanishing of the Cotton tensor is the necessary and sufficient condition for a metric to satisfy these equivalent conditions. This well known fact solves the problem in dimension $n = 3$. Therefore, for the remainder of Section 2 we will assume that $n \geq 4$.
- In dimension $n = 4$ the second integrability condition reduces to the conformally invariant Bach equation:

$$B_{ab} = 0. \tag{2.14}$$

2.4. Generalising the KNT characterisation

Here we generalise to dimension $n \geq 4$ the characterisation of conformally Einstein metrics given by Kozameh et al. [19]. Our considerations are local and so we assume, without loss of generality, that M is oriented and write $\boldsymbol{\epsilon}$ for the volume form. Given the Weyl tensor C_{abcd} of the metric g_{ab}, we write $C^*_{b_1 \cdots b_{n-2} cd} := \boldsymbol{\epsilon}_{b_1 \cdots b_{n-2}}{}^{a_1 a_2} C_{a_1 a_2 cd}$. Note that this is completely trace-free due to the Weyl Bianchi symmetry $C_{[a_1 a_2 a_3] b} = 0$. Consider the equations

$$C_{abcd} F^{ab} = 0, \tag{2.15}$$

$$C_{abcd} H^{bd} = 0, \tag{2.16}$$

and

$$C^*_{b_1 \cdots b_{n-2} cd} H^{b_1 d} = 0, \tag{2.17}$$

for a skew symmetric tensor F^{ab} and a symmetric trace-free tensor H^{ab}. We say that the metric g_{ab} is *generic* if and only if the only solutions to Eqs. (2.15)–(2.17) are $F^{ab} = 0$ and $H^{ab} = 0$. Occasionally we will be interested in the superclass of metrics for which (2.15) has only trivial solutions but for which we make no assumptions about (2.16) and (2.17); we will call these Λ^2-*generic* metrics. That is, a metric is Λ^2-generic if and only if the Weyl curvature is injective (equivalently, maximal rank) as a bundle map $\Lambda^2 TM \to \Lambda^2 TM$. Let $\|C\|$ be the natural conformal invariant which is the pointwise determinant of the map

$$C : \Lambda^2 T^*M \to \Lambda^2 T^*M, \tag{2.18}$$

given by $W_{ab} \mapsto C_{ab}^{cd} W_{cd}$ and write $\tilde{C}_{abcd}$ for the tensor field which is the pointwise adjugate (i.e. "matrix of cofactors") of the Weyl curvature tensor, viewed as an endomorphism in this way. Then

$$\tilde{C}^{ab}_{ef} C^{cd}_{ab} = \|C\| \delta^{[c}_{[e} \delta^{d]}_{f]}$$

and if g is a Λ^2-generic metric then $\|C\|$ is non-vanishing and we have

$$\|C\|^{-1} \tilde{C}^{ab}_{ef} C^{cd}_{ab} = \delta^{[c}_{[e} \delta^{d]}_{f]}. \tag{2.19}$$

For later use note that it is easily verified that $\tilde{C}_{abcd}$ is natural (in fact simply polynomial in the Weyl curvature) and conformally covariant.

For the remainder of this subsection we consider only generic metrics, except where otherwise indicated. In this setting, we will prove that the following two conditions are equivalent:

(i) The metric g_{ab} is conformally Einstein.
(ii) There exists a vector field K^a on M such that the following conditions [C] and [B] are satisfied:

$$[\mathrm{C}] \quad A_{abc} + K^d C_{dabc} = 0, \qquad [\mathrm{B}] \quad B_{ab} + (n-4) K^d K^c C_{dabc} = 0.$$

Adapting a tradition from the General Relativity literature (originating in [30]), we call a manifold for which the metric g_{ab} admits K^a such that condition [C] is satisfied a *conformal C-space*. Note that such a metric is *not* necessarily conformal to a metric with vanishing Cotton tensor since in [C] we are not requiring K_a to be a gradient. (Thus some care is necessary when comparing with [30,19] for example where a space with vanishing Cotton tensor is termed a C-space.) However, in the case of a *generic* metric satisfying condition [C] the field K_d must be a gradient. To see this take ∇^a of equation [C]. This gives

$$\nabla^a A_{abc} + C_{dabc} \nabla^a K^d + (n-3) K^a K^d C_{adbc} = 0,$$

where, in the last term, we have used identity (2.5) and eliminated A_{dbc} via [C]. The last term in this expression obviously vanishes identically. On the other hand the first term also vanishes, because of identity (2.7). Thus a simple consequence of equation [C] is $C_{dabc} \nabla^a K^d = 0$. Thus, since the metric is generic (in fact for this result we only need that it is Λ^2-generic), we can conclude that

$$\nabla^{[a} K^{d]} = 0.$$

Therefore, at least locally, there exists a function Υ such that

$$K_d = \nabla_d \Upsilon. \tag{2.20}$$

Thus, we have shown that our conditions [C] and [B] are equivalent to the necessary conditions (2.12) and (2.13) for a metric to be conformally Einstein.

To prove the sufficiency we first take ∇^c of [C]. This, after using the identity (2.5) and the definition of the Bach tensor (2.2), takes the form

$$B_{ab} + \mathsf{P}^{dc}C_{dabc} - C_{dabc}\nabla^c K^d + (n-3)K^d K^c C_{dabc} = 0.$$

Now, subtracting from this equation our second condition [B] we get

$$C_{dabc}(\mathsf{P}^{dc} - \nabla^c K^d + K^d K^c) = 0. \quad (2.21)$$

Next we differentiate equation [C] and skew to obtain

$$\nabla_{a_1} A_{ca_2a_3} - C_{a_2a_3cd}\nabla_{a_1} K^d - K^d \nabla_{a_1} C_{a_2a_3cd} = 0.$$

Then using (2.3), the Weyl Bianchi identity (2.4), and [C] once more we obtain

$$C_{a_2a_3cd}(\mathsf{P}^d_{a_1} - \nabla_{a_1} K^d + K_{a_1} K^d) = 0$$

or equivalently

$$C^*_{b_1\cdots b_{n-2}cd}(\mathsf{P}^{b_1 d} - \nabla^{b_1} K^d + K^{b_1} K^d) = 0. \quad (2.22)$$

But this condition and (2.21) together imply that $\mathsf{P}^{dc} - \nabla^c K^d + K^d K^c$ must be a pure trace, due to (2.16) and (2.17). Thus,

$$\mathsf{P}^{dc} - \nabla^c K^d + K^d K^c = \frac{1}{n} T g^{cd}.$$

This, when compared with our previous result (2.20) on K^a, and with the conformal Einstein equations (2.11), shows that our metric can be scaled to the Einstein metric with the function Υ defined by (2.20). This proves the following theorem.

Theorem 2.2. *A generic metric g_{ab} on an n-manifold M is conformally Einstein if and only if its Cotton tensor A_{abc} and its Bach tensor B_{ab} satisfy*

$$[\mathrm{C}] \quad A_{abc} + K^d C_{dabc} = 0, \qquad [\mathrm{B}] \quad B_{ab} + (n-4)K^d K^c C_{dabc} = 0$$

for some vector field K^a on M.

We will show below, and in the next section that [C] is conformally invariant and that, while [B] is not conformally invariant, the system [C], [B] is. In particular [B] is conformally invariant for metrics satisfying [C], the conformal C-space metrics. Next note that, although we settled dimension 3 earlier, the above theorem also holds in that case since the Weyl tensor vanishes identically and the Bach tensor is just a divergence of the Cotton tensor. In other dimensions we can easily eliminate the *undetermined* vector field K^d from this theorem. Indeed, using the tensor $\|C\|^{-1}\tilde{C}^{bc}_{ed}$ of (2.19) and applying it on the condition [C] we obtain

$$\|C\|^{-1}\tilde{C}^{bc}_{ed} A_{abc} + \tfrac{1}{2}(K_e g_{da} - K_d g_{ea}) = 0.$$

By contracting over the indices $\{ea\}$, this gives

$$K^d = \frac{2}{1-n}\|C\|^{-1}\tilde{C}^{dabc}A_{abc}. \tag{2.23}$$

Inserting (2.23) into the equations [C] and [B] of Theorem 2.2, we may reformulate the theorem as the observation that a generic metric g_{ab} on an n-manifold M (where $n \geq 4$) is conformally Einstein if and only if its Cotton tensor A_{abc} and its Bach tensor B_{ab} satisfy

$$[\mathrm{C}'] \quad (1-n)A_{abc} + 2\|C\|^{-1}C_{dabc}\tilde{C}^{defg}A_{efg} = 0$$

and

$$[\mathrm{B}'] \quad (n-1)^2 B_{ab} + 4(n-4)\|C\|^{-2}\tilde{C}^{defg}C_{dabc}\tilde{C}^{chkl}A_{efg}A_{hkl} = 0.$$

These are equivalent to conditions polynomial in the curvature. Multiplying the left-hand sides of [C′] and [B′] by, respectively, $\|C\|$ and $\|C\|^2$ we obtain natural (pseudo-)Riemannian invariants which are obstructions to a metric being conformally Einstein,

$$F^1_{abc} := (1-n)\|C\|A_{abc} + 2C_{dabc}\tilde{C}^{defg}A_{efg}$$

and

$$F^2_{ab} = (n-1)^2\|C\|^2 B_{ab} + 4(n-4)\tilde{C}^{defg}C_{dabc}\tilde{C}^{chkl}A_{efg}A_{hkl}.$$

By construction the first of these is conformally covariant (see below), the second tensor is conformally covariant for metrics such that $F^1_{abc} = 0$, and we have the following theorem.

Theorem 2.3. *A generic metric g_{ab} on an n-manifold M (where $n \geq 4$) is conformally Einstein if and only if the natural invariants F^1_{abc} and F^2_{ab} both vanish.*

Remarks:

- In dimension $n = 4$ there exist examples of metrics satisfying the Bach equations [B] and not being conformally Einstein (see e.g. [24]). In higher dimensions it is straightforward to write down generic Riemannian metrics which, at least at a formal level, have vanishing Bach tensor but for which the Cotton tensor is non-vanishing. Thus the integrability condition [B] does not suffice to guarantee the conformally Einstein property of the metric. In Section 4 we discuss an example of special Robinson–Trautman metrics, which satisfy the condition [C] and do not satisfy [B]. (These are generic.) Thus condition [C] alone is not sufficient to guarantee the conformal Einstein property.
- The development above parallels and generalises the tensor treatment in [19] which is based in dimension 4. It should be pointed out however that there are some simplifications in dimension 4. Firstly F^2_{ab} simplifies to $9\|C\|^2 B_{ab}$. It is thus sensible to use the conformally invariant Bach tensor B_{ab} as a replacement for F^2 in dimension 4. Also note, from the development in [19], that the conditions that a metric g_{ab} be generic may

be characterised in a particularly simple way in Lorentzian dimension 4. In this case they are equivalent to the non-vanishing of at least one of the following two quantities:

$$C^3 := C_{abcd}C^{cd}_{ef}C^{efab} \text{ or } *C^3 := *C_{abcd} * C^{cd}_{ef} * C^{efab},$$

where $*C_{abcd} = C^*_{abcd} = \epsilon_{abef}C^{ef}_{cd}$.

2.5. *Conformal invariants giving a sharp obstruction*

We will show in the next section that the systems [C] and [B] have a natural and valuable geometric interpretation. However its value, or the equivalent obstructions F^1 and F^2, as a test for conformally Einstein metrics is limited by the requirement that the metric is generic. Many metrics fail to be generic. For example in the setting of dimension 4 Riemannian structures any selfdual metric fails to be generic (and even fails to be Λ^2-generic), since any anti-selfdual two form is a solution of (2.15); at each point the solution space of (2.15) is at least three-dimensional (see Section 4.3 for an explicit Ricci-flat example of this type). In the remainder of this section we show that there are natural conformal invariants that are more effective, for detecting conformally Einstein metrics, than the pair F^1 and F^2.

Let us say that a (pseudo-)Riemannian manifold is *weakly generic* if, at each point $x \in M$, the only solution $V^d \in T_xM$ to

$$C_{abcd}V^d = 0 \text{ at } x \in M \tag{2.24}$$

is $V^d = 0$. From (2.19) it is immediate that all Λ^2-generic spaces are weakly generic and hence all generic spaces are weakly generic. Via elementary arguments we will observe that on weakly generic manifolds there is a (smooth) tensor field $\tilde{D}^{ab}{}_c{}^d$ with the property that

$$\tilde{D}^{ac}{}_d{}^e C_{bc}{}^d{}_e = -\delta^a_b.$$

Of course $\tilde{D}^{ab}{}_c{}^d$ is not uniquely determined by this property. However in many settings there is a canonical choice. For example in the case of Riemannian signature g is weakly generic if and only if $L^a_b := C^{acde}C_{bcde}$ is invertible. Let us write $\tilde{L}^a_b$ for the tensor field which is the pointwise adjugate of L^a_b. $\tilde{L}^a_b$ is given by a formula which is a partial contraction polynomial (and homogeneous of degree $2n-2$) in the Weyl curvature and for any structure we have

$$\tilde{L}^a_b L^b_c = \|L\|\delta^a_c,$$

where $\|L\|$ denotes the determinant of L^a_b. Let us define

$$D^{acde} := -\tilde{L}^a_b C^{bcde}.$$

Then D^{acde} is a natural conformal covariant defined on all structures. On weakly generic Riemannian structures, or pseudo-Riemannian structures where we have $\|L\|$ non-vanishing, there is a canonical choice for $\tilde{D}$, viz.

$$\tilde{D}^{acde} := \|L\|^{-1} D^{acde} = -\|L\|^{-1} \tilde{L}^a_b C^{bcde}. \tag{2.25}$$

In other signatures we may obtain a smooth $\tilde{D}^{ac}{}_d{}^e$ by a similar argument but the construction is no longer canonical. On a manifold M with a metric g of indefinite signature this goes as follows. Instead of defining L as above let $\overline{L}^a_b := \overline{C}^{acde} C_{bcde}$ where $\overline{C}^{acde} := \overline{g}^{af} \overline{g}^{ch} \overline{g}^{di} \overline{g}^{ej} C_{fhij}$ with $\overline{g}^{af}$ the inverse of any fixed *choice* of smooth positive definite metric $\overline{g}$ on M. (Here C_{fhij} is the Weyl curvature for the original metric g.) Then as above we have that the metric g is weakly generic if and only if $\overline{L}^a_b$ is invertible. Thus, with $\tilde{\overline{L}}^a_b$ and $\|\overline{L}\|$ denoting, respectively, the pointwise adjugate and the determinant of $\overline{L}^a_b$, it is clear that by construction $\tilde{D}^{acde} := -\|\overline{L}\|^{-1} \tilde{\overline{L}}^a_b \overline{C}^{bcde}$ is smooth and gives $\tilde{D}^{ac}{}_d{}^e C_{bc}{}^d{}_e = -\delta^a_b$.

The last construction argument proves the existence of a smooth $\tilde{D}$ on indefinite weakly generic manifolds but the construction is not canonical since it depends on the artificial choice of the auxiliary metric $\overline{g}$. The main interest is in canonical constructions. Another such construction arises if (in any signature) g is Λ^2-generic. Then we may take

$$\tilde{D}^{acde} := \frac{2}{1-n} \|C\|^{-1} \tilde{C}^{acde} \tag{2.26}$$

as was done implicitly in the previous section. Recall $\tilde{C}^{ac}{}_d{}^e$ is conformally invariant and natural. The examples (2.25) and (2.26) are particularly important since they are easily described and apply to any dimension (greater than 3). However in a given dimension there are many other possibilities which lead to formulae of lower polynomial order if we know, or are prepared to insist that, certain invariants are non-vanishing (see [10] for a discussion in the context of Λ^2-generic structures). For example in the setting of dimension 4 and Lorentzian signature, Λ^2-generic implies $C^3 = C^{cd}_{ab} C^{ef}_{cd} C^{ab}_{ef}$ is non-vanishing and one may take $\tilde{D}^{acde} = C^{de}_{fg} C^{fgca}/C^3$ cf. [19]. In any case let us fix some choice for $\tilde{D}$. Note that since the Weyl curvature $C_{bc}{}^d{}_e$ for a metric g is the same as the Weyl tensor for a conformally related metric $\hat{g}$, it follows that we can (and will) use the same tensor field $\tilde{D}^{ab}{}_c{}^d$ for all metrics in the conformal class.

For weakly generic manifolds it is straightforward to give a conformally invariant tensor that vanishes if and only if the manifold is conformally Einstein. For the remainder of this section we assume the manifold is weakly generic.

We have observed already that the conformally Einstein manifolds are a subclass of conformal C-spaces. Recall that a conformal C-space is a (pseudo-)Riemannian manifold which admits a 1-form field K_a which solves the equation [C]:

$$A_{abc} + K^d C_{dabc} = 0.$$

If K^d_1 and K^d_2 are both solutions to [C] then, evidently, $(K^d_1 - K^d_2) C_{dabc} = 0$. Thus, if the manifold is weakly generic, $K^d_1 = K^d_2$. In fact if K_d is a solution to [C] then clearly

$$K_d = \tilde{D}^{abc}_d A_{abc}, \tag{2.27}$$

which also shows that at most one vector field K^d solves [C] on weakly generic manifolds. From either result, combined with the observations that the Cotton tensor is preserved by constant conformal metric rescalings and that constant conformal rescalings take Einstein metrics to Einstein metrics, gives the following results.

Proposition 2.4. *On a manifold with a weakly generic metric g, the equation [C] has at most one solution for the vector field K^d.*

Either there are no metrics, conformally related to g, that have vanishing Cotton tensor or the space of such metrics is one-dimensional. Either there are no Einstein metrics, conformally related to g, or the space of such metrics is one-dimensional.

If g is a metric with vanishing Cotton tensor we will say this is a *C-space scale*.

Now, for an alternative view of conformal C-spaces, we may take (2.27) as the *definition* of K_d. Note then that from (2.10), a routine calculation shows that $\hat{A}_{abc} = A_{abc} + \Upsilon^k C_{kabc}$, and so (using the conformal invariance of $\tilde{D}_d^{abc}$) $K_d = \tilde{D}_d^{abc} A_{abc}$ has the conformal transformation

$$\hat{K}_d = K_d - \Upsilon_d,$$

where $\hat{A}_{abc}$ and $\hat{K}_d$ are calculated in terms of the metric $\hat{g} = \mathrm{e}^{2\Upsilon} g$ and $\Upsilon_a = \nabla_a \Upsilon$. Thus $A_{abc} + K^d C_{dabc}$ is conformally invariant. From Proposition 2.4 and (2.27) this tensor is a *sharp obstruction* to conformal C-spaces in the following sense.

Proposition 2.5. *A weakly generic manifold is a conformal C-space if and only if the conformal invariant*

$$A_{abc} + \tilde{D}^{dijk} A_{ijk} C_{dabc}$$

vanishes.

In any case where $\tilde{D}^{dijk}$ is given by a Riemannian invariant formulae rational in the curvature and its covariant derivatives (e.g. g is of Riemannian signature, or that g is Λ^2-generic) we can multiply the invariant here by an appropriate polynomial invariant to obtain a natural conformal invariant. Indeed, in the setting of Λ^2-generic metrics, the invariant F^1_{abc} (from Section 2.4) is an example. Since, on Λ^2-generic manifolds, the vanishing of F^1_{abc} implies that (2.23) is locally a gradient, we have the following theorem.

Theorem 2.6. *For a Λ^2-generic Riemannian or pseudo-Riemannian metric g the conformal covariant F^1_{abc},*

$$(1-n)\|C\| A_{abc} + 2 C_{dabc} \tilde{C}^{defg} A_{efg}$$

vanishes if and only if g is conformally related to a Cotton metric (i.e. a metric $\hat{g}$ such that its Cotton tensor vanishes, $\hat{A}_{abc} = 0$).

In the case of Riemannian signature Λ^2-generic metrics we may replace the conformal invariant F^1_{abc} in the theorem with the conformal invariant,

$$\|L\| A_{abc} - C^{efgh} A_{fgh} \tilde{L}_e^d C_{dabc}, \quad n \geq 4. \tag{2.28}$$

In dimension 4 there is an even simpler invariant. Note that in dimension 4 we have

$$4C^{abcd}C_{abce} = |C|^2\delta_e^d, \tag{2.29}$$

where $|C|^2 := C^{abcd}C_{abcd}$ and so L is a multiple of the identity. Eliminating, from (2.28), the factor of $(|C|^2)^3$ and a numerical scale we obtain the conformal invariant

$$|C|^2 A_{abc} - 4C^{defg}A_{efg}C_{dabc}, \quad n = 4,$$

which again can be used to replace F^1_{abc} in the theorem for dimension 4 Λ^2-generic metrics.

We can also characterise conformally Einstein spaces.

Proposition 2.7. *A weakly generic metric g is conformally Einstein if and only if the conformally invariant tensor*

$$E_{ab} := \textit{Trace-free}[\mathsf{P}_{ab} - \nabla_a(\tilde{D}_{bcde}A^{cde}) + \tilde{D}_{aijk}A^{ijk}\tilde{D}_{bcde}A^{cde}]$$

vanishes.

Proof. The proof that E_{ab} is conformally invariant is a simple calculation using (2.10) and the transformation formula for $K_d = \tilde{D}_d^{abc}A_{abc}$.

If g is conformally Einstein then there is a gradient Υ_a such that

$$\text{Trace-free}[\mathsf{P}_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b] = 0.$$

From Section 2.3 this implies Υ_a solves the C-space equation (see (2.12)) and hence, from (2.27), $\Upsilon_a = \tilde{D}_{aijk}A^{ijk}$, and so $E_{ab} = 0$.

Conversely suppose that $E_{ab} = 0$. Then the skew part of E_{ab} vanishes and since P_{ab} and $\tilde{D}_{aijk}A^{ijk}\tilde{D}_{bcde}A^{cde}$ are symmetric we conclude that $\tilde{D}_{bcde}A^{cde}$ is closed and hence, locally at least, is a gradient. □

Now suppose $\|L\|$ is non-vanishing and take $\tilde{D}_{abcd}$ to be given as in (2.25). Note that since E_{ab} is conformally invariant it follows that $\|L\|^2E_{ab}$ is conformally invariant. This expands to

$$\begin{aligned} G_{ab} := \text{Trace-free}[&\|L\|^2\mathsf{P}_{ab} - \|L\|\nabla_a(D_{bcde}A^{cde}) + (\nabla_a\|L\|)(D_{bcde}A^{cde}) \\ &+ D_{aijk}A^{ijk}D_{bcde}A^{cde}]. \end{aligned}$$

This is natural by construction. Since it is given by a universal polynomial formula which is conformally covariant on structures for which $\|L\|$ is non-vanishing, it follows from an elementary polynomial continuation argument that it is conformally covariant on any structure. Note $\|L\|$ is a conformal covariant of weight $-4n$. Thus we have the following theorem on manifolds of dimension $n \geq 4$.

Theorem 2.8. *The natural invariant G_{ab} is a conformal covariant of weight $-8n$. A manifold with a weakly generic Riemannian metric g is conformally Einstein if and only if G_{ab} vanishes. The same is true on pseudo-Riemannian manifolds where the conformal invariant $\|L\|$ is non-vanishing.*

Recall that in dimension 4 we have the identity (2.29). Thus $\|L\|$ is non-vanishing if and only if $|C|^2$ is non-vanishing and we obtain a considerable simplification. In particular the invariant G_{ab} has an overall factor of $(|C|^2)^6$ that we may divide out and still have a natural conformal invariant. This corresponds to taking $(|C|^2)^2 E_{ab}$ with $\tilde{D}^{abcd} = -\frac{4}{|C|^2} C^{abcd}$. Hence we have a simplified obstruction as follows.

Theorem 2.9. *The natural invariant*

$$\textit{Trace-free}[(|C|^2)^2 \mathsf{P}_{ab} + 4|C|^2 \nabla_a (C_{bcde} A^{cde}) - 4C_{bcde} A^{cde} \nabla_a |C|^2 + 16 C_{aijk} A^{ijk} C_{bcde} A^{cde}]$$

is conformally covariant of weight -8.

A 4-manifold with $|C|^2$ nowhere vanishing is conformally Einstein if and only if this invariant vanishes.

In the case of Riemannian 4-manifolds, requiring $|C|^2$ non-vanishing is the same as requiring the manifold to be weakly generic. In this setting this is a very mild assumption; note that $|C|^2 = 0$ at $p \in M$ if and only if $C_{abcd} = 0$ at p (and so the manifold is conformally flat at p).

Note also that if we denote by F_{ab} the natural invariant in the theorem then on Riemannian 4 manifolds the (conformally covariant) scalar function $F_{ab}F^{ab}$ is an equivalent sharp obstruction to the manifold being conformally Einstein.

Now suppose we are in the setting of Λ^2-generic structures (of any fixed signature). Then E_{ab} is well defined and conformally invariant with $\tilde{D}_{abcd}$ given by (2.26). Thus again by polynomial continuation we can conclude that the natural invariant obtained by expanding $\|C\|^2 E_{ab}$, viz.

$$\bar{G}_{ab} := \text{Trace-free}[(1-n)^2 \|C\|^2 \mathsf{P}_{ab} - 2(1-n)\|C\| \nabla_a (\tilde{C}_{bcde} A^{cde}) + 2(1-n)(\nabla_a \|C\|)(\tilde{C}_{bcde} A^{cde}) + 4\tilde{C}_{aijk} A^{ijk} \tilde{C}_{bcde} A^{cde}]$$

is conformally covariant on any structure (i.e. not necessarily Λ^2-generic). Thus we have the following theorem on manifolds of dimension $n \geq 4$.

Theorem 2.10. *The natural invariant $\bar{G}_{ab}$ is a conformal covariant of weight $2n(1-n)$. A manifold with a Λ^2-generic metric g is conformally Einstein if and only if $\bar{G}_{ab}$ vanishes.*

We should point out that there is further scope, in each specific dimension, to obtain simplifications and improvements to Theorems 2.8 and 2.10 along the lines of Theorem 2.9. For example in dimension 4 the complete contraction $C^3 = C^{cd}_{ab} C^{ef}_{cd} C^{ab}_{ef}$, mentioned earlier, is a conformal covariant which is independent of $|C|^2$ (see e.g. [26]). Thus on pseudo-Riemannian structures this may be non-vanishing when $|C|^2 = 0$. There is the identity

$$4 C^{cd}_{jb} C^{ef}_{cd} C^{ib}_{ef} = \delta^i_j C^{cd}_{ab} C^{ef}_{cd} C^{ab}_{ef}$$

and this may be used to construct a formula for $\tilde{D}$ (and then K_d via (2.23)) alternative to (2.25) and (2.26). (See [19] for this and some other examples.)

Finally note that although generally we need to make some restriction on the class of metrics to obtain a canonical formula for $\tilde{D}_{bcde}$ in terms of the curvature, in other circumstances it is generally easy to make a choice and give a description of a $\tilde{D}$. For example in a non-Riemannian setting one can calculate in a fixed local basis field and artificially nominate a Riemannian signature metric. Using this to contract indices of the Weyl curvature (given in the set basis field) one can then use the formula for L and then D. In this way Proposition 2.7 is an effective and practical means of testing for conformally Einstein metrics, among the class weakly generic metrics, even when it does not lead to a natural invariant.

3. A geometric derivation and new obstructions

The derivation of the system of Theorem 2.2 appears ad hoc. We will show that in fact [C] and [B] are two parts (or components) of a single conformal equation that has a simple and clear geometric interpretation. This construction then easily yields new obstructions. This is based on the observation that conformally Einstein manifolds may be characterised as those admitting a parallel section of a certain vector bundle. The vector bundle concerned is the (standard) conformal tractor bundle. This bundle and its canonical conformally invariant connection are associated structures for the normal conformal Cartan connection of [9]. The initial development of the calculus associated to this bundle dates back to the work of Thomas [31] and was reformulated and further developed in a modern setting in [2]. For a comprehensive treatment exposing the connection to the Cartan bundle and relating the conformal case to the wider setting of parabolic structures see [7,6]. The calculational techniques, conventions and notation used here follow [16,15].

3.1. Conformal geometry and tractor calculus

We first introduce some of the basic objects of conformal tractor calculus. It is useful here to make a slight change of point of view. Rather than take as our basic geometric structure a Riemannian or pseudo-Riemannian structure we will take as our basic geometry only a conformal structure. This simplifies the formulae involved and their conformal transformations. It is also a conceptually sound move since conformally invariant operators, tensors and functions are exactly the (pseudo-)Riemannian objects that descend to be well defined objects on a conformal manifold. A signature (p, q) *conformal structure* $[g]$ on a manifold M, of dimension $n \geq 3$, is an equivalence class of metrics where $\hat{g} \sim g$ if $\hat{g} = e^{2\Upsilon} g$ for some $\Upsilon \in \mathcal{E}$. A conformal structure is equivalent to a ray subbundle $\mathcal{Q}$ of S^2T^*M; points of $\mathcal{Q}$ are pairs (g_x, x) where $x \in M$ and g_x is a metric at x, each section of $\mathcal{Q}$ gives a metric g on M and the metrics from different sections agree up to multiplication by a positive function. The bundle $\mathcal{Q}$ is a principal bundle with group $\mathbb{R}_+$, and we denote by $\mathcal{E}[w]$ the vector bundle induced from the representation of $\mathbb{R}_+$ on $\mathbb{R}$ given by $t \mapsto t^{-w/2}$. Sections of $\mathcal{E}[w]$ are called a *conformal densities of weight* w and may be identified with functions on $\mathcal{Q}$ that are homogeneous of degree w, i.e., $f(s^2 g_x, x) = s^w f(g_x, x)$ for any $s \in \mathbb{R}_+$. We will often use the same notation $\mathcal{E}[w]$ for the space of sections of the bundle. Note that for each choice of a metric g (i.e., section of $\mathcal{Q}$, which we term a *choice of conformal scale*), we may identify a section $f \in \mathcal{E}[w]$ with a function f_g on M by $f_g(x) = f(g_x, x)$. This function is conformally covariant of weight w in the sense of Section 2, since if $\hat{g} = \mathrm{e}^{2\Upsilon} g$, for some $\Upsilon \in \mathcal{E}$, then $f_{\hat{g}}(x) = f(\mathrm{e}^{2\Upsilon_x} g_x, x) = \mathrm{e}^{w\Upsilon_x} f(g_x, x) = \mathrm{e}^{w\Upsilon_x} f_g(x)$. Conversely conformally covariant functions determine homogeneous sections of $\mathcal{Q}$ and so densities. In particular, $\mathcal{E}[0]$ is canonically identified with $\mathcal{E}$.

Note that there is a tautological function $\boldsymbol{g}$ on $\mathcal{Q}$ taking values in S^2T^*M. It is the function which assigns to the point $(g_x, x) \in \mathcal{Q}$ the metric g_x at x. This is homogeneous of degree 2 since $\boldsymbol{g}(s^2 g_x, x) = s^2 g_x$. If ξ is any positive function on $\mathcal{Q}$ homogeneous of degree -2 then $\xi \boldsymbol{g}$ is independent of the action of $\mathbb{R}_+$ on the fibres of $\mathcal{Q}$, and so $\xi \boldsymbol{g}$ descends to give a metric from the conformal class. Thus $\boldsymbol{g}$ determines and is equivalent to a canonical section of $\mathcal{E}_{ab}[2]$ (called the conformal metric) that we also denote $\boldsymbol{g}$ (or $\boldsymbol{g}_{ab}$). This in turn determines a canonical section $\boldsymbol{g}^{ab}$ (or $\boldsymbol{g}^{-1}$) of $\mathcal{E}^{ab}[-2]$ with the property that $\boldsymbol{g}_{ab}\boldsymbol{g}^{bc} = \delta_a^c$ (where δ_a^c is kronecker delta, i.e., the section of $\mathcal{E}_a^c$ corresponding to the identity endomorphism of the tangent bundle). In this section the conformal metric (and its inverse $\boldsymbol{g}^{ab}$) will be used to raise and lower indices. This enables us to work with density valued objects. Conformally covariant tensors as in Section 2 correspond one-one with conformally invariant density valued tensors. Each non-vanishing section σ of $\mathcal{E}[1]$ determines a metric g^σ from the conformal class by

$$g^\sigma := \sigma^{-2} \boldsymbol{g}. \tag{3.1}$$

Conversely if $g \in [g]$ then there is an up-to-sign unique $\sigma \in \mathcal{E}[1]$ which solves $g = \sigma^{-2}\boldsymbol{g}$, and so σ is termed a choice of conformal scale. Given a choice of conformal scale, we write ∇_a for the corresponding Levi–Civita connection. For each choice of metric there is also a canonical connection on $\mathcal{E}[w]$ determined by the identification of $\mathcal{E}[w]$ with $\mathcal{E}$, as described above, and the exterior derivative on functions. We will also call this the Levi–Civita connection and thus for tensors with weight, e.g. $v_a \in \mathcal{E}_a[w]$, there is a connection given by the Leibniz rule. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$.

We next define the standard tractor bundle over $(M, [g])$. It is a vector bundle of rank $n+2$ defined, for each $g \in [g]$, by $[\mathcal{E}^A]_g = \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. If $\hat{g} = \mathrm{e}^{2\Upsilon} g$, we identify $(\alpha, \mu_a, \tau) \in [\mathcal{E}^A]_g$ with $(\hat{\alpha}, \hat{\mu}_a, \hat{\tau}) \in [\mathcal{E}^A]_{\hat{g}}$ by the transformation

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\mu}_a \\ \hat{\tau} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a{}^b & 0 \\ -\frac{1}{2}\Upsilon_c\Upsilon^c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix}. \tag{3.2}$$

It is straightforward to verify that these identifications are consistent upon changing to a third metric from the conformal class, and so taking the quotient by this equivalence relation defines the *standard tractor bundle* $\mathcal{E}^A$ over the conformal manifold. (Alternatively the standard tractor bundle may be constructed as a canonical quotient of a certain 2-jet bundle or as an associated bundle to the normal conformal Cartan bundle [6].) The bundle $\mathcal{E}^A$ admits an invariant metric h_{AB} of signature $(p+1, q+1)$ and an invariant connection, which we shall also denote by ∇_a, preserving h_{AB}. In a conformal scale g, these are given by

$$h_{AB} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \nabla_a \begin{pmatrix} \alpha \\ \mu_b \\ \tau \end{pmatrix} = \begin{pmatrix} \nabla_a \alpha - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\alpha \\ \nabla_a \tau - \mathsf{P}_{ab}\mu^b \end{pmatrix}.$$

It is readily verified that both of these are conformally well defined, i.e., independent of the choice of a metric $g \in [g]$. Note that h_{AB} defines a section of $\mathcal{E}_{AB} = \mathcal{E}_A \otimes \mathcal{E}_B$, where $\mathcal{E}_A$ is the dual bundle of $\mathcal{E}^A$. Hence we may use h_{AB} and its inverse h^{AB} to raise or lower indices of $\mathcal{E}_A$, $\mathcal{E}^A$ and their tensor products.

In computations, it is often useful to introduce the 'projectors' from $\mathcal{E}^A$ to the components $\mathcal{E}[1]$, $\mathcal{E}_a[1]$ and $\mathcal{E}[-1]$ which are determined by a choice of scale. They are respectively denoted by $X_A \in \mathcal{E}_A[1]$, $Z_{Aa} \in \mathcal{E}_{Aa}[1]$ and $Y_A \in \mathcal{E}_A[-1]$, where $\mathcal{E}_{Aa}[w] = \mathcal{E}_A \otimes \mathcal{E}_a \otimes \mathcal{E}[w]$, etc. Using the metrics h_{AB} and $\boldsymbol{g}_{ab}$ to raise indices, we define X^A, Z^{Aa}, Y^A. Then we immediately see that

$$Y_A X^A = 1, \qquad Z_{Ab} Z^A_c = \boldsymbol{g}_{bc}$$

and that all other quadratic combinations that contract the tractor index vanish. This is summarised in Fig. 1.

It is clear from (3.2) that the first component α is independent of the choice of a representative g and hence X^A is conformally invariant. For Z^{Aa} and Y^A, we have the transformation laws:

$$\hat{Z}^{Aa} = Z^{Aa} + \Upsilon^a X^A, \qquad \hat{Y}^A = Y^A - \Upsilon_a Z^{Aa} - \tfrac{1}{2}\Upsilon_a \Upsilon^a X^A. \tag{3.3}$$

Given a choice of conformal scale we have the corresponding Levi–Civita connection on tensor and density bundles. In this setting we can use the coupled Levi–Civita tractor connection to act on sections of the tensor product of a tensor bundle with a tractor bundle. This is defined by the Leibniz rule in the usual way. For example if $u^b V^C \alpha \in \mathcal{E}^b \otimes \mathcal{E}^C \otimes \mathcal{E}[w] =: \mathcal{E}^{bC}[w]$ then $\nabla_a u^b V^C \alpha = (\nabla_a u^b) V^C \alpha + u^b (\nabla_a V^C)\alpha + u^b V^C \nabla_a \alpha$. Here ∇ means the Levi–Civita connection on $u^b \in \mathcal{E}^b$ and $\alpha \in \mathcal{E}[w]$, while it denotes the tractor connection on $V^C \in \mathcal{E}^C$. In particular with this convention we have

$$\nabla_a X_A = Z_{Aa}, \qquad \nabla_a Z_{Ab} = -\mathsf{P}_{ab} X_A - Y_A \boldsymbol{g}_{ab}, \qquad \nabla_a Y_A = \mathsf{P}_{ab} Z^b_A. \tag{3.4}$$

Note that if V is a section of $\mathcal{E}_{A_1 \cdots A_\ell}[w]$, then the coupled Levi–Civita tractor connection on V is not conformally invariant but transforms just as the Levi–Civita connection transforms on densities of the same weight: $\widehat{\nabla}_a V = \nabla_a V + w \Upsilon_a V$.

	Y^A	Z^{Ac}	X^A
Y_A	0	0	1
Z_{Ab}	0	$\delta_b{}^c$	0
X_A	1	0	0

Fig. 1. Tractor inner product.

Given a choice of conformal scale, the *tractor-D operator*

$$D_A : \mathcal{E}_{B\cdots E}[w] \to \mathcal{E}_{AB\cdots E}[w-1]$$

is defined by

$$D_A V := (n+2w-2)wY_A V + (n+2w-2)Z_{Aa}\nabla^a V - X_A \Box V, \tag{3.5}$$

where $\Box V := \Delta V + w\mathsf{P}_b^{\ b} V$. This also turns out to be conformally invariant as can be checked directly using the formulae above (or alternatively there are conformally invariant constructions of D, see e.g. [14]).

The curvature Ω of the tractor connection is defined by

$$[\nabla_a, \nabla_b]V^C = \Omega_{ab}{}^C{}_E V^E \tag{3.6}$$

for $V^C \in \mathcal{E}^C$. Using (3.4) and the usual formulae for the curvature of the Levi–Civita connection we calculate (cf. [2])

$$\Omega_{abCE} = Z_C^c Z_E^e C_{abce} - 2X_{[C} Z_{E]}^e A_{eab}. \tag{3.7}$$

From the tractor curvature we obtain a related higher order conformally invariant curvature quantity by the formula (cf. [14,15])

$$W_{BC}{}^E{}_F := \frac{3}{n-2} D^A X_{[A}\Omega_{BC]}{}^E{}_F.$$

It is straightforward to verify that this can be re-expressed as follows:

$$W_{ABCE} = (n-4)Z_A^a Z_B^b \Omega_{abCE} - 2X_{[A} Z_{B]}{}^b \nabla^p \Omega_{pbCE}. \tag{3.8}$$

This tractor field has an important relationship to the ambient metric of Fefferman and Graham. For a conformal manifold of signature (p, q) the ambient manifold [11] is a signature $(p+1, q+1)$ pseudo-Riemannian manifold with $\mathcal{Q}$ as an embedded submanifold. Suitably homogeneous tensor fields on the ambient manifold upon restriction to $\mathcal{Q}$ determine tractor fields on the underlying conformal manifold [8]. In particular, in dimensions other than 4, W_{ABCD} is the tractor field equivalent to $(n-4)\boldsymbol{R}|_{\mathcal{Q}}$ where $\boldsymbol{R}$ is the curvature of the Fefferman–Graham ambient metric.

3.2. Conformally Einstein manifolds

Recall that we say a Riemannian or pseudo-Riemannian metric g is conformally Einstein if there is a scale Υ such that the Ricci tensor, or equivalently the Schouten tensor, is pure trace. Thus we say that a conformal structure $[g]$ is conformally Einstein if there is a metric $\hat{g}$ in the conformal class (i.e. $\hat{g} \in [g]$) such that the Schouten tensor for $\hat{g}$ is pure trace. We show here that a conformal manifold $(M, [g])$ is conformally Einstein if and only if it admits a parallel standard tractor $\mathbb{I}^A$ which also satisfies the condition that $X_A\mathbb{I}^A$ is nowhere vanishing. Note that in a sense the "main condition" is that $\mathbb{I}$ is parallel since the requirement that $X_A\mathbb{I}^A$ is non-vanishing is an open condition. In more detail we have the following result.

Theorem 3.1. *On a conformal manifold* $(M, [g])$ *there is a 1–1 correspondence between conformal scales* $\sigma \in \mathcal{E}[1]$, *such that* $g^\sigma = \sigma^{-2}\boldsymbol{g}$ *is Einstein, and parallel standard tractors* $\mathbb{I}$ *with the property that* $X_A\mathbb{I}^A$ *is nowhere vanishing. The mapping from Einstein scales to parallel tractors is given by* $\sigma \mapsto \frac{1}{n}D_A\sigma$ *while the inverse is* $\mathbb{I}^A \mapsto X_A\mathbb{I}^A$.

Proof. Suppose that $(M, [g])$ admits a parallel standard tractor $\mathbb{I}^A$ such that $\sigma := X_A\mathbb{I}^A$ is nowhere vanishing. Since $\sigma \in \mathcal{E}[1]$ and is non-vanishing it is a conformal scale. Let g be the metric from the conformal class determined by σ, that is $g = g^\sigma = \sigma^{-2}\boldsymbol{g}$ as in (3.1). In terms of the tractor bundle splitting determined by this metric $\mathbb{I}^A$ is given by some triple with σ as the leading entry, $[\mathbb{I}^A]_g = (\sigma, \mu_a, \tau)$. From the formula for the invariant connection we have

$$0 = [\nabla_a\mathbb{I}^B]_g = \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + \boldsymbol{g}_{ab}\tau + \mathsf{P}_{ab}\sigma \\ \nabla_a\tau - \mathsf{P}_{ab}\mu^b \end{pmatrix}. \tag{3.9}$$

Thus $\mu_a = \nabla_a\sigma$, but $\nabla_a\sigma = 0$ by the definition of ∇ in the scale σ. Thus μ_a vanishes, and the second tensor equation from (3.9) simplifies to

$$\mathsf{P}_{ab}\sigma = -\boldsymbol{g}_{ab}\tau,$$

showing that the metric g is Einstein. Note that tracing the display gives $\tau = -\frac{1}{n}\mathsf{J}\sigma$.

To prove the converse let us now suppose that σ is a conformal scale so that $g = \sigma^{-2}\boldsymbol{g}$ is an Einstein metric. That is, for this metric g, P_{ab} is pure trace. Let us work in this conformal scale. Then we have $\mathsf{P}_{ab} = \frac{1}{n}\boldsymbol{g}_{ab}\mathsf{J}$. Thus $\nabla^a\mathsf{P}_{ab} = (1/n)\nabla_b\mathsf{J}$. On the other hand comparing this to the contracted Bianchi identity $\nabla^a\mathsf{P}_{ab} = \nabla_b\mathsf{J}$ we have that $\nabla_a\mathsf{J} = 0$. Now, we define a tractor field $\mathbb{I}^A$ by $\mathbb{I}^A := \frac{1}{n}D^A\sigma$. Then $[\mathbb{I}]_{g^\sigma} := \left(\sigma, 0, -\frac{1}{n}\mathsf{J}\sigma\right)$. Consider the tractor connection on this. We have

$$[\nabla_a\mathbb{I}^B]_g = \begin{pmatrix} \nabla_a\sigma \\ -\frac{1}{n}\boldsymbol{g}_{ab}\mathsf{J}\sigma + \mathsf{P}_{ab}\sigma \\ -\frac{1}{n}(\sigma\nabla_a\mathsf{J} + \mathsf{J}\nabla_a\sigma) \end{pmatrix}.$$

Once again, by the definition of the Levi–Civita connection ∇ as determined by the scale σ, we have $\nabla\sigma = 0$. Since $\mathsf{P}_{ab} = \frac{1}{n}\boldsymbol{g}_{ab}\mathsf{J}$ the second entry also vanishes. The last component also vanishes from $\nabla\mathsf{J} = 0$ and $\nabla\sigma = 0$. So $\mathbb{I}$ is a parallel standard tractor satisfying that $X_A\mathbb{I}^A = \sigma$ is non-vanishing. □

Remarks:

- Note that $h(\mathbb{I}, \mathbb{I})$ is a conformal invariant of density weight 0. In fact from the formulae above, in the Einstein scale, $h(\mathbb{I}, \mathbb{I}) = -\frac{2}{n}\sigma^2\mathsf{J}$. Recall that in this section $\mathsf{J} = \boldsymbol{g}^{ab}\mathsf{P}_{ab}$ and so has density weight -2 and

$$\sigma^2\mathsf{J} = \sigma^2\boldsymbol{g}^{ab}\mathsf{P}_{ab} = g^{ab}\mathsf{P}_{ab}.$$

That is $-\frac{n}{2}h(\mathbb{I},\mathbb{I})$ is the trace of Schouten tensor using the metric determined by σ. Since ∇ preserves the tractor metric and $\mathbb{I}$ is parallel we recover the (well known) result that P_{ab} (and its trace) is constant for Einstein metrics.

- Suppose we drop the condition that $\sigma := X^A\mathbb{I}^a$ is nowhere vanishing. If $\mathbb{I}^A$ is parallel then from (3.9) it follows that $\mu_a = \nabla_a\sigma$. Furthermore tracing the middle entry on the right-hand side of (3.9) implies that $\tau = -\frac{1}{n}\Box\sigma$. Thus if $\nabla_a\mathbb{I}_B = 0$ at $p \in M$ then at p we have $\mathbb{I}_B = \frac{1}{n}D_B\sigma$. Now clearly $\frac{1}{n}X^B D_B\sigma = \sigma$ vanishes on a neighbourhood if and only if $\frac{1}{n}D_B\sigma$ vanishes on the same neighbourhood. So for parallel $\mathbb{I}^A$, $X_A\mathbb{I}^A$ is non-vanishing on an open dense subset of M. The points where σ vanishes are scale singularities for the metric $g = \sigma^{-2}\boldsymbol{g}$.
- The relationship between parallel tractors and conformally Einstein metrics, while implicit in [2], was probably first observed and treated in some detail by Gauduchon in [13] (and we thank Claude LeBrun for drawing our attention to Gauduchon's results in this area). On dimension 4 spin manifolds it is straightforward to show that the standard tractor bundle is isomorphic to the second exterior power of Penrose's [27] local twistor bundle. Under this isomorphism $\mathbb{I}$ may be identified with the *infinity twistor* (defined for spacetimes). The relationship to conformal Einstein manifolds is well known [22,12] in that setting.
- We should also point out that the theorem above can alternatively be deduced, via some elementary arguments but without any calculation, from the construction of the tractor connection as in [2].

Next we make some elementary observations concerning parallel tractors.

Lemma 3.2. *On a conformal manifold let N be a parallel section of the standard tractor bundle* $\mathbb{T}$. *Then:*

$$\Omega_{bc}{}^D{}_E N^E = 0 \quad \textit{and} \quad W_{BCDE}N^E = 0.$$

Proof. By assumption we have $\nabla_a N^D = 0$. Thus $\Omega_{bc}{}^D{}_E N^E = [\nabla_b, \nabla_c]N^D = 0$ and the first result is established.

Next $W_{A_1A_2}{}^D{}_E N^E = \frac{3}{n-2}(D^{A_0}X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E)N^E$, where, as usual, sequentially labelled indices e.g. A_0, A_1, A_2 are implicitly skewed over. Now the quantity $X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E$ has (density) weight -1, so from the formula (3.5) for D, we have

$$\begin{aligned}(D^{A_0}X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E)N^E &= (4-n)Y^{A_0}X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E N^E \\ &\quad + (n-4)(Z^{A_0a}\nabla_a X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E)N^E \\ &\quad - (X^{A_0}\Delta X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E)N^E \\ &\quad + \mathsf{J}X^{A_0}X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E N^E,\end{aligned}$$

where ∇ and Δ act on everything to their right within the parentheses. The first and last terms on the right-hand side vanish from the previous result. (In fact for last term we could also use that $X^{A_0}X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c=0$.) Next observe that, since $\nabla N=0$, we have

$$(Z^{A_0a}\nabla_aX_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E)N^E=Z^{A_0a}\nabla_a(X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_EN^E)=0,$$

where we have again used the earlier result, $\Omega_{bc}{}^D{}_EN^E=0$. Similarly

$$(X^{A_0}\Delta X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_E)N^E=X^{A_0}\Delta(X_{A_0}Z_{A_1}{}^bZ_{A_2}{}^c\Omega_{bc}{}^D{}_EN^E)=0. \quad \square$$

From the lemma it follows immediately that on conformally Einstein manifolds the parallel tractor $\mathbb{I}$, of Theorem 3.1, satisfies $\Omega_{bc}{}^D{}_E\mathbb{I}^E=0$ and $W_{BCDE}\mathbb{I}^E=0$. In general the converse is also true. More accurately we have the result given in the following theorem. Before we state that, note that since the Weyl curvature is conformally invariant it follows that Eqs. (2.15)–(2.17) are conformally invariant. Thus if any metric from a conformal class is generic then all metrics from the class are generic and we will describe the conformal class as generic.

Theorem 3.3. *A generic conformal manifold of dimension $n\neq 4$ is conformally Einstein if and only if there exists a tractor field $\mathbb{I}^A\in\mathcal{E}^A$ such that $X_A\mathbb{I}^A$ is non-vanishing and*

$$W_{BCDE}\mathbb{I}^E=0.$$

A generic conformal manifold of dimension $n=4$ is conformally Einstein if and only if there exists a tractor field $\mathbb{I}^A\in\mathcal{E}^A$ such that $X_A\mathbb{I}^A$ is non-vanishing,

$$\Omega_{bc}{}^D{}_E\mathbb{I}^E=0 \quad \textit{and} \quad W_{BCDE}\mathbb{I}^E=0.$$

Proof. We have shown that on a conformally Einstein manifold there is a (parallel) standard tractor field satisfying

(i) $X_A\mathbb{I}^A$ nowhere vanishing,
(ii) $\Omega_{bc}{}^D{}_E\mathbb{I}^E=0$,
(iii) $W_{BCDE}\mathbb{I}^E=0$.

It remains to prove the relevant converse statements. First we observe that given (i), (ii) is exactly the conformal C-space equation. From above we have that

$$\Omega_{abCE}=Z^c_CZ^e_EC_{abce}-X_CZ^e_EA_{eab}+X_EZ^e_CA_{eab}.$$

A general tractor $\mathbb{I}^A\in\mathcal{E}^A$ may be expanded to

$$\mathbb{I}^E=Y^E\sigma+Z^{Ed}\mu_d+X^E\tau,$$

where $\sigma=X_A\mathbb{I}^A$ and we assume this is non-vanishing. Hence

$$\Omega_{abCE}\mathbb{I}^E=\sigma Z^c_CA_{cab}+Z^c_C\mu^dC_{abcd}-X_C\mu^dA_{dab}. \tag{3.10}$$

Setting this to zero, as required by (ii), implies that the coefficient of Z_C^c must vanish, i.e., $\sigma A_{cab} + \mu^d C_{abcd} = 0$, or

$$A_{cab} + K^d C_{dcab} = 0, \quad K^d := -\sigma^{-1}\mu^d, \tag{3.11}$$

which is exactly the conformal C-space equation [C] as in Theorem 2.2. Contracting this with μ^c (or K^c) annihilates the second term and so

$$\mu^d A_{dab} = 0,$$

whence the coefficient of X_C in (3.10) vanishes as a consequence of the earlier equation and it is shown that (with (i)) $\Omega_{abCE}\mathbb{I}^E = 0$ is exactly the conformal C-space equation.

Now recall

$$W_{BCDE} = (n-4)Z_B^b Z_C^c \Omega_{bcDE} - 2X_{[B}Z_{C]}^c \nabla^a \Omega_{acDE},$$

and so, in dimensions other 4, $W_{BCDE}\mathbb{I}^E = 0$ implies $\Omega_{bcDE}\mathbb{I}^E = 0$ (and hence the conformal C-space equation). From the display we see that $W_{BCDE}\mathbb{I}^E = 0$ also implies that $\mathbb{I}^E\nabla^a\Omega_{acDE} = 0$ or equivalently $\sigma^{-1}\mathbb{I}^E\nabla^a\Omega_{acDE} = 0$. Once again using the formulae for the tractor connection we obtain

$$\nabla^a \Omega_{acDE} = (n-4)Z_D^d Z_E^e A_{cde} - X_D Z_E^e B_{ec} + X_E Z_D^e B_{ec}, \tag{3.12}$$

where B_{ec} is the Bach tensor. Hence $\sigma^{-1}\mathbb{I}^E\nabla^a\Omega_{acDE} = 0$ expands to

$$-(n-4)Z_D^d K^e A_{cde} + X_D K^e B_{ec} + Z_D^d B_{dc} = 0.$$

From the coefficient of Z_D^d we have

$$B_{dc} - (n-4)K^e A_{cde} = 0$$

which, with the conformal C-space equation (and since B is symmetric), gives

$$B_{cd} + (n-4)K^e K^a C_{acde} = 0 \tag{3.13}$$

which is exactly the second equation [B] of Theorem 2.2. If this holds then it follows at once that $K^c B_{cb} = 0$ and so in the expansion of $\sigma^{-1}\mathbb{I}^E\nabla^a\Omega_{acDE} = 0$ the coefficient of X_D vanishes without further restriction. Thus we have shown that in dimensions other than 4 the single conformally invariant tractor equation $W_{BCDE}\mathbb{I}^E = 0$ is equivalent to the two equations [C] and [B]. In dimension 4 it is clear from (3.8) that $W_{BCDE}\mathbb{I}^E = 0$ is equivalent to $\mathbb{I}^E\nabla^a\Omega_{acDE} = 0$ and this with $\mathbb{I}^E\Omega_{acDE} = 0$ gives the pair of equations [B] and [C]. In either case then the theorem here now follows immediately from Theorem 2.2. □

Remarks:

- Note that conditions (i), (ii) and (iii), as in the theorem, do not imply that $\mathbb{I}$ is parallel. On the other hand the theorem shows that if there exists a standard tractor $\mathbb{I}$ satisfying these conditions then (on generic manifolds) also there exists a parallel standard tractor $\mathbb{I}'$

satisfying these conditions. Calculating in an Einstein scale, it follows from the conformal C-space equation that one has $Z_A^a \mathbb{I}^A = Z_A^a \mathbb{I}^A = 0$. Hence that $\mathbb{I}' = f\mathbb{I} + \rho X$ for some section ρ of $\mathcal{E}[-1]$ and non-vanishing function *f*.

- Recall that in Section 3.1 we pointed out that in dimensions other than 4, W_{ABCD} is the tractor field equivalent [8] to $(n-4)\boldsymbol{R}|_{\mathcal{Q}}$ where $\boldsymbol{R}$ is the curvature of the Fefferman–Graham ambient metric. Thus, in these dimensions, the condition $W_{ABCD}\mathbb{I}^D = 0$ is equivalent to the existence of a suitably homogeneous and generic ambient tangent vector field along $\mathcal{Q}$ in the ambient manifold which annihilates the ambient curvature.
- We had already observed in Section 2.5 that $A_{abc} + K^d C_{dabc}$ is conformally invariant if we assume that K_d has the conformal transformation law $\hat{K}_a = K_a - \Upsilon_a$ (where $\hat{g} = e^{2\Upsilon} g$). From the proof above we see this transformation formula fits naturally into the tractor picture and arises from (3.2) since K_a is a density multiple of the middle component of a tractor field according to (3.11).

3.3. *Sharp obstructions via tractors*

Theorem 3.3 gives a simple interpretation of Theorem 2.2 in terms of tractor bundles. In the proof of this above, this connection was made by recovering the familiar tensor equations from Section 2. Here we first observe that entire derivation of Theorem 2.2 and its proof reduces to a few key lines if we work in the tractor picture. This then leads to a stronger theorem as below.

We summarise the background first. From Theorem 3.1 we know that the existence of a conformal Einstein structure is equivalent to the existence of a parallel tractor $\mathbb{I}$ (at points where $X_A\mathbb{I}^A \neq 0$). This immediately implies that the tractor curvature Ω_{abCD} satisfies

$$[\tilde{\mathrm{C}}] \quad \mathbb{I}^D\Omega_{abCD} = 0, \qquad [\tilde{\mathrm{B}}] \quad \mathbb{I}^D\nabla^a\Omega_{abCD} = 0.$$

We have labelled these [C̃] and [B̃] since (as shown in the proof above) the first equation is equivalent to the earlier [C] and, given this, the second equation is equivalent to the earlier equation [B]. The conformal invariance of the systems [C] and [B] is now immediate in all dimensions from the observation that the conformal transformation of $\nabla^a\Omega_{abCD}$ is

$$\widehat{\nabla^a\Omega_{abCD}} = \nabla^a\Omega_{abCD} + (n-4)\Upsilon^a\Omega_{abCD}, \tag{3.14}$$

and whence the conformal transformation of the left-hand side of equation [B̃] is

$$\mathbb{I}^D\widehat{\nabla^a\Omega_{abCD}} = \mathbb{I}^D\nabla^a\Omega_{abCD} + (n-4)\Upsilon^a\mathbb{I}^D\Omega_{abCD},$$

where $\hat{g} = e^{2\Upsilon} g$; from this it is immediate that [B̃] is invariant on metrics that solve [C̃]. We should point out that in dimension 4 it follows immediately from (3.12) that $\mathbb{I}^D\nabla^a\Omega_{abCD} = 0 \Leftrightarrow \nabla^a\Omega_{abCD} = 0 \Leftrightarrow B_{ab} = 0$.

Now we are interested in the converse. We will show that if the displayed equations [C̃] and [B̃] hold for some tractor $\mathbb{I}$, satisfying that $X_A\mathbb{I}^A$ is non-vanishing, then the structure is conformally Einstein. Here is an alternative proof of Theorem 3.3 (and hence an alternative proof of Theorem 2.2). Equation [C̃] implies that $\nabla_{a_1}(\Omega_{a_2a_3CD}\mathbb{I}^D) = 0$, where as usual sequentially labelled indices are skewed over. From the Bianchi identity for the tractor curvature, $\nabla_{a_1}\Omega_{a_2a_3CD} = 0$, it follows that

$$\Omega_{a_2a_3CD}\nabla_{a_1}\mathbb{I}^D = 0. \tag{3.15}$$

Now equation [$\tilde{\text{C}}$] implies [C], viz. $A_{cab} + K^d C_{dcab} = 0$. As we saw earlier this (using that the metric is Λ^2-generic) implies that K_a is a gradient and that there is a conformal scale such that the Cotton tensor A_{cab} vanishes. In this special C-space scale (see Section 2.5) it is clear that K_a is also zero and (3.15) simplifies (using (3.9) and (3.7)) to $\mathsf{P}_{a_1}{}^d C_{a_2a_3cd} Z_C^c = 0$ or equivalently

$$C^*_{b_1\cdots b_{n-2}cd}\mathsf{P}^{b_1d} = 0. \tag{3.16}$$

Note that if C^* is suitably generic this already implies that the metric that gives the special C-space scale is Einstein.

Using only the weaker assumption that the manifold is generic in the sense of Section 2.4 we must also use [$\tilde{\text{B}}$]. The argument is similar to the above. Equation [$\tilde{\text{C}}$] implies $\nabla^a(\mathbb{I}^D\Omega_{abCD}) = 0$. Thus using [$\tilde{\text{B}}$] we have

$$(\nabla^a\mathbb{I}^D)\Omega_{abCD} = 0.$$

In the special C-space scale this expands to $\mathsf{P}^{ad}C_{abcd}Z_C^c = 0$, which is equivalent to

$$\mathsf{P}^{ad}C_{abcd} = 0. \tag{3.17}$$

Clearly Eqs. (3.17) and (3.16) imply that P is pure trace on generic manifolds and so the theorem is proved. In fact these Eqs. (3.17) and (3.16) are respectively Eqs. (2.21) and (2.22) both written in the C-space scale.

The construction of the systems [$\tilde{\text{B}}$] and [$\tilde{\text{C}}$] immediately suggests alternative systems. In particular we have the following results which only requires the manifold to be weakly generic.

Theorem 3.4. *A weakly generic conformal manifold is conformally Einstein if and only if there exists a non-vanishing tractor field $\mathbb{I}^A \in \mathcal{E}^A$ such that*

$$[\tilde{\text{C}}] \quad \mathbb{I}^E\Omega_{bcDE} = 0, \qquad [\tilde{\text{D}}] \quad \mathbb{I}^E\nabla_a\Omega_{bcDE} = 0.$$

The systems [$\tilde{\text{C}}$] *and* [$\tilde{\text{D}}$] *are conformally invariant.*

Proof. Note that from (2.9), and the invariance of the tractor connection, we have

$$\mathbb{I}^E\widehat{\nabla_a\Omega_{bc}}_{DE} = \mathbb{I}^E\nabla_a\Omega_{bcDE} - 2\Upsilon_a\mathbb{I}^E\Omega_{bcDE} - \Upsilon_b\mathbb{I}^E\Omega_{acDE} - \Upsilon_c\mathbb{I}^E\Omega_{baDE} + \boldsymbol{g}_{ab}\Upsilon^k\mathbb{I}^E\Omega_{kcDE} + \boldsymbol{g}_{ac}\Upsilon^k\mathbb{I}^E\Omega_{bkDE},$$

where $\hat{g} = \mathrm{e}^{2\Upsilon}g$, and so [$\tilde{\text{D}}$] is conformally invariant if the conformally invariant equation [$\tilde{\text{C}}$] is satisfied; the systems [$\tilde{\text{C}}$] and [$\tilde{\text{D}}$] are conformally invariant.

If the manifold is conformally Einstein then there is a parallel tractor $\mathbb{I}^E$. We have observed earlier that this satisfies [$\tilde{\text{C}}$]. Differentiating [$\tilde{\text{C}}$] and then using once again that $\mathbb{I}^E$ is parallel shows that [$\tilde{\text{D}}$] is satisfied.

Now we assume that [$\tilde{\text{C}}$] and [$\tilde{\text{D}}$] hold. If $\mathbb{I}^E = Y^E\sigma + Z^{Ed}\mu_d + X^E\tau$, then $\Omega_{abCE}\mathbb{I}^E$ is given by (3.10). Suppose that $X_A\mathbb{I}^A = \sigma$ vanishes at some point x. Then from (3.10) we have $\mu^d C_{abcd} = 0$ at x (and $\mu^d A_{dab} = 0$ at x) and so, since the conformal class is weakly generic, $\mu^d(x) = 0$. Thus $\mathbb{I}^E = \tau X^E$, at x, and [$\tilde{\text{D}}$] gives $X^E\nabla_a\Omega_{bcDE} = 0$ at x. But, $\nabla_a X^E = Z_a^E$ and from (3.7) $X^E\Omega_{bcDE} = 0$, and so $Z_D^d C_{bcda} - X_D A_{abc} = Z_a^E\Omega_{bcDE} = 0$ at x. But this means $C_{bcda}(x) = 0$ which contradicts the assumption that the conformal class is weakly generic. So $X_A\mathbb{I}^A$ is non-vanishing.

Now, differentiating [$\tilde{\text{C}}$] and then using [$\tilde{\text{D}}$] we obtain

$$\Omega_{bcDE}\nabla_a\mathbb{I}^E = 0.$$

But, since the manifold is weakly generic, Ω_{bcDE} must have rank at least n as a map $\Omega_{bcDE} : \mathcal{E}^{bcD} \to \mathcal{E}_E$. Also, from (3.7) and [$\tilde{\text{C}}$], X^E and $\mathbb{I}^E$ are orthogonal to the range. So the display implies that

$$\nabla_a\mathbb{I}^E = \alpha_a\mathbb{I}^E + \beta_a X^E,$$

for some 1-forms α_a and β_a. (An alternative explanation is to note, as earlier, that if U^E is not a multiple of X^E and $\Omega_{bcDE}U^E = 0$ then from (3.7) it follows that U^E determines a non-trivial solution of the equation [C]. Since $\mathbb{I}^E$ also determines such a solution it follows at once from Proposition 2.4 that $U^E = \alpha\mathbb{I} + \beta X^E$.) Differentiating again and alternating we obtain

$$\Omega_{ba}{}^E{}_D\mathbb{I}^D = 2\mathbb{I}^E\nabla_{[b}\alpha_{a]} + 2\alpha_{[a}\alpha_{b]}\mathbb{I}^E + 2\alpha_{[a}\beta_{b]}X^E + 2X^E\nabla_{[b}\beta_{a]} + 2\beta_{[a}Z_{b]}^E.$$

The left-hand side vanishes by assumption and of course $\alpha_{[a}\alpha_{b]}\mathbb{I}^E = 0$. Contracting X_E into the remaining terms brings us to

$$0 = 2\sigma\nabla_{[a}\alpha_{b]}$$

and so α is closed. Locally then $\alpha_a = \nabla_a f$ for some function f and so $\tilde{\mathbb{I}}^E := \mathrm{e}^{-f}\mathbb{I}^E$ satisfies

$$\nabla_a\tilde{\mathbb{I}}^E = \tilde{\beta}_a X^E \tag{3.18}$$

for some 1-form $\tilde{\beta}_a$. Expanding $\tilde{\mathbb{I}}^E$: $\tilde{\mathbb{I}}^E = Y^E\tilde{\sigma} + Z^{Ed}\tilde{\mu}_d + X^E\tilde{\tau}$ we have $X_E\tilde{\mathbb{I}}^e = \tilde{\sigma}$ (which is non-vanishing) and, from (3.18), the equations

$$\nabla_a\tilde{\sigma} - \tilde{\mu}_a = 0, \qquad \nabla_a\tilde{\mu}_b + \boldsymbol{g}_{ab}\tilde{\tau} + \mathsf{P}_{ab}\tilde{\sigma} = 0$$

cf. (3.9). So for the metric $g := \tilde{\sigma}^{-2}\boldsymbol{g}$ we have $\tilde{\mu}_a = \nabla_a\tilde{\sigma} = 0$ and $\mathsf{P}_{ab} + \boldsymbol{g}_{ab}\tilde{\tau}/\tilde{\sigma} = 0$. That is the metric g is Einstein (and $\frac{1}{n}D_A\tilde{\sigma}$ is parallel). □

We have the following consequence of the theorem above.

Corollary 3.5. *A weakly generic pseudo-Riemannian or Riemannian metric g on an n-manifold is conformally Einstein if and only if the natural invariants*

$$\Omega_{abKD_1}\cdots\Omega_{cdLD_s}\nabla_e\Omega_{fgPD_{s+1}}\cdots\nabla_h\Omega_{k\ell QD_{n+2}},$$

for $s = 0, 1, \ldots, n+1$, *all vanish. Here the sequentially labelled indices* $D_1, \ldots, D_{n+2}$ *are completely skewed over.*

Proof. The theorem can clearly be rephrased to state that g is conformally Einstein if and only if the map

$$(\Omega_{bcDE}, \nabla_a\Omega_{bcDE}) : \mathcal{E}^{bcD} \oplus \mathcal{E}^{abcD} \to \mathcal{E}_E \tag{3.19}$$

given by

$$(V^{bcD}, W^{abcD}) \mapsto V^{bcD}\Omega_{bcDE} + W^{abcD}\nabla_a\Omega_{bcDE}$$

fails to have maximal rank at every point of M. But by elementary linear algebra this happens if and only if the induced alternating multi-linear map to $\Lambda^{n+2}(\mathcal{E}^E)$ vanishes. This is equivalent to the claim in the Corollary, since for any metric the tractor curvature satisfies $\Omega_{bcDE}X^E = 0$. □

If M is oriented (which locally we can assume with no loss of generality) then it is straightforward to show that there is a canonical skew $(n+2)$-tractor consistent with the tractor metric and the orientation. Let us denote this by $\boldsymbol{\epsilon}^{C_1\cdots C_{n+2}}$. Using this, we could equally rephrase the Corollary in terms of the invariants

$$\boldsymbol{\epsilon}^{D_1D_2\cdots D_sD_{s+1}\cdots D_{n+1}D_{n+2}}\Omega_{abKD_1}\cdots\Omega_{cdLD_s}\nabla_e\Omega_{fgPD_{s+1}}\cdots\nabla_h\Omega_{k\ell QD_{n+2}},$$

for $s = 0, 1, \ldots, n+1$. These all vanish if and only if the metric is conformally Einstein.

The natural invariants in the lemma are given by mixed tensor-tractor fields, rather pure tensors. However by expanding Ω_{abCD} and $\nabla_a\Omega_{bcDE}$ using (3.7) and (3.4) it is straightforward to obtain an equivalent set of tensorial obstructions from these. The system of obstructions so obtained is rather unwieldy and could be awkward to apply in practise. Nevertheless this gives a system of invariants, which works equally for all signatures.

As a final remark in this section we note that coming to Proposition 2.7 via the tractor picture is also very easy. If we want to test whether a scale $\sigma \in \mathcal{E}[1]$ is an Einstein scale we define $\mathbb{I}_B := \frac{1}{n}D_B\sigma$ as in Theorem 3.1 and consider $\nabla_a\mathbb{I}_B$. Calculating in terms of an arbitrary metric g from the conformal class we get $\nabla_a\mathbb{I}_B = Z^b_B\sigma E_{ab}$, modulo terms involving X_B, where $E_{ab} = \text{Trace-free}(\mathsf{P}_{ab} - \nabla_aK_b + K_aK_b)$ and $K_a := -\sigma^{-1}\nabla_a\sigma$. Since σ can only be an Einstein scale if $\Omega_{bc}{}^D{}_E\mathbb{I}^E = 0$ we obtain the conformal C-space equation for K_a and we are led to the conclusion that the Riemannian invariant of the proposition is conformally invariant and also the conclusion that it must vanish on conformal Einstein manifolds.

4. Examples

Here we shed light on the various notions of generic metrics, mainly by way of examples. First let us note that each of these is an open condition on the moduli space of possible curvatures. Thus in this sense "almost all" metrics are generic (and hence Λ^2-generic and weakly generic). The many components of the Weyl curvature C_{abcd} arise from a Λ^2-generic metric unless they lie on the closed variety determined by the one condition $\|C\| = 0$ where, recall, $\|C\|$ is the determinant of the map (2.18). The metrics which fail to be weakly generic correspond to a closed subspace contained in the $\|C\| = 0$ variety. In the Riemannian case this subvariety is given by $\|L\| = 0$, where recall $\|L\|$ is the determinant of $C^{acde}C_{bcde}$ and we show below that in dimension 4 the containment is proper.

Another aim in this final section is to establish the independence of the conditions [C] and [B] from Section 2.4. We assume that $n \geq 4$ throughout this section.

4.1. Simple n-dimensional Robinson–Trautman metrics

Let Q be an $(n-2)$-dimensional space of constant curvature κ and denote by x^i, $i = 1, 2, \dots, n-2$, standard stereographic coordinates on Q. We take $M = \mathbf{R}^2 \times Q$, with coordinates (r, u, x^i), where (r, u) are coordinates along the $\mathbf{R}^2$, and equip M with a subclass of Robinson–Trautman [28] metrics g by

$$g = 2\,\mathrm{d}u[\mathrm{d}r + h(r)\,\mathrm{d}u] + r^2 \frac{g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}{\left(1 + \frac{\kappa}{4} g_{kl}x^k x^l\right)^2}. \tag{4.1}$$

Here $g_{ij} = \mathrm{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_{n-2})$, $\epsilon_i = \pm 1$, $\kappa = 1, 0, -1$ and $h = h(r)$ is an arbitrary, sufficiently smooth real function of variable r. In the following we describe conformal properties of the metrics (4.1).

To calculate the Weyl tensor we introduce the null-orthonormal coframe $(\theta^a) = (\theta^+, \theta^-, \theta^i)$ by

$$\theta^+ = \mathrm{d}u, \qquad \theta^- = \mathrm{d}r + h\,\mathrm{d}u, \qquad \theta^i = r\frac{\mathrm{d}x^i}{1 + \frac{\kappa}{4} g_{kl}x^k x^l}. \tag{4.2}$$

In this coframe the metric takes the form $g = g_{ab}\theta^a\theta^b$ where

$$g_{ab} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & g_{ij} \end{pmatrix}. \tag{4.3}$$

We lower and raise the indices by means of the matrix g_{ab} and its inverse g^{ab}. The Levi–Civita connection 1-forms

$$\Gamma_{ab} = \Gamma_{abc}\theta^c$$

are uniquely determined by

$$\mathrm{d}\theta^a + \Gamma^a_{\ b} \wedge \theta^b = 0 \quad \text{and} \quad \mathrm{d}g_{ab} - \Gamma_{ab} - \Gamma_{ba} = 0. \tag{4.4}$$

Explicitly, we find that, the connection 1-forms are

$$\Gamma_{ij} = \frac{\kappa}{2r}(x_i\theta_j - x_j\theta_i), \qquad \Gamma_{-j} = -\frac{1}{r}\theta_j, \qquad \Gamma_{+j} = \frac{h}{r}\theta^j, \qquad \Gamma_{+-} = h'\theta^+, \tag{4.5}$$

where $h' = \frac{\mathrm{d}h}{\mathrm{d}r}$. (Observe that, due to the constancy of the matrix elements of g_{ab}, the matrix Γ_{ab} is skew, $\Gamma_{ab} = -\Gamma_{ba}$.) The curvature 2-forms

$$\Omega_{ab} = \tfrac{1}{2}R_{abcd}\theta^c \wedge \theta^d = \mathrm{d}\Gamma_{ab} + \Gamma^c_{\ a} \wedge \Gamma_{cb}$$

are

$$\Omega_{ij} = \frac{\kappa + 2h}{r^2}\theta_i \wedge \theta_j, \qquad \Omega_{-j} = \frac{h'}{r}\theta^+ \wedge \theta_j, \qquad \Omega_{+j} = \frac{h'}{r}\theta^- \wedge \theta_j,$$
$$\Omega_{+-} = h''\theta^- \wedge \theta^+, \tag{4.6}$$

with the remaining components determined by symmetry. The non-vanishing components of the Ricci tensor

$$R_{ab} = R^c_{\ acb}$$

and the Ricci scalar

$$R = g^{ab}R_{ab}$$

are

$$R_{ij} = \left[(n-3)\frac{\kappa + 2h}{r^2} + \frac{2h'}{r}\right]g_{ij}, \qquad R_{+-} = (n-2)\frac{h'}{r} + h'',$$
$$R = (n-2)\left[(n-3)\frac{\kappa + 2h}{r^2} + \frac{4h'}{r}\right] + 2h''. \tag{4.7}$$

From this we conclude that metrics (4.1) are Einstein,

$$R_{ab} = \Lambda g_{ab},$$

if and only if

$$h(r) = -\frac{\kappa}{2} + \frac{m}{r^{n-3}} + \frac{\Lambda}{2(n-1)}r^2, \tag{4.8}$$

where m and Λ are constants. These metrics form the well known n-dimensional Schwarzschild-(anti-)de Sitter 2-parameter class in which m is interpreted as the mass and Λ as the cosmological constant. (The space is termed de Sitter if $\Lambda > 0$ and anti-de Sitter is $\Lambda < 0$.) Thus, we have the following proposition.

Proposition 4.1. *The only Einstein metrics among the Robinson–Trautman metrics*

$$g = 2\,\mathrm{d}u[\mathrm{d}r + h(r)\,\mathrm{d}u] + r^2 \frac{g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}{\left(1 + \frac{\kappa}{4} g_{kl} x^k x^l\right)^2}$$

are the Schwarzschild-(anti-)de Sitter metrics, for which

$$h(r) = -\frac{\kappa}{2} + \frac{m}{r^{n-3}} + \frac{\Lambda}{2(n-1)} r^2.$$

The Weyl tensor of metrics (4.1) has the following non-vanishing components:

$$C_{ijkl} = 2\Psi(g_{ki}g_{jl} - g_{kj}g_{il}), \qquad C_{-i+k} = (3-n)\Psi g_{ik},$$
$$C_{+-+-} = (3-n)(n-2)\Psi, \tag{4.9}$$

where

$$\Psi = \frac{1}{(n-1)(n-2)}\left[\frac{\kappa + 2h}{r^2} - \frac{2h'}{r} + h''\right],$$

and the further non-vanishing components determined from these by the Weyl symmetries. Now, we consider the equation

$$C_{abcd}F^{cd} = 0 \tag{4.10}$$

for the antisymmetric tensor F_{ab}. We easily find that

$$C_{ijab}F^{ab} = 4\Psi F_{ij}, \qquad C_{i+ab}F^{ab} = (3-n)\Psi g_{ik}F^{k-},$$
$$C_{i-ab}F^{ab} = (3-n)\Psi g_{ik}F^{k+}, \qquad C_{+-ab}F^{ab} = 2(3-n)(n-2)\Psi F^{+-}.$$

Thus, if $\Psi \neq 0$, Eq. (4.10) has unique solution $F_{ab} = 0$. We pass to the equation

$$C_{abcd}H^{bd} = 0 \tag{4.11}$$

for a symmetric and trace-free tensor H_{ab}. In the null-orthonormal coframe (4.2) the trace-free condition reads

$$H + 2H_{+-} = 0, \text{ where } H = g^{ik}H_{ik}. \tag{4.12}$$

Comparing this with

$$C_{ibkd}H^{bd} = 2\Psi[g_{ik}(H + (3-n)H_{-+}) - H_{ik}], \qquad C_{ib-d}H^{bd} = (n-3)\Psi g_{ik}H^{+k},$$
$$C_{ib+d}H^{bd} = (n-3)\Psi g_{ik}H^{-k}, \qquad C_{-b-d}H^{bd} = (n-2)(n-3)\Psi H^{++},$$
$$C_{+b+d}H^{bd} = (n-2)(n-3)\Psi H^{--}$$

proves that the only solution of (4.11) is $H_{ab} = 0$. Thus we have the following proposition.

Proposition 4.2. *If*

$$\Psi = \frac{1}{(n-1)(n-2)}\left[\frac{\kappa + 2h}{r^2} - \frac{2h'}{r} + h''\right] \neq 0$$

the Robinson–Trautman metrics

$$g = 2\,\mathrm{d}u[\mathrm{d}r + h(r)\,\mathrm{d}u] + r^2 \frac{g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}{\left(1 + \frac{\kappa}{4} g_{kl}x^k x^l\right)^2}$$

are generic.

By a straightforward calculation we obtain the following proposition.

Proposition 4.3. *Each Robinson–Trautman metric for which* $\Psi \neq 0$, *satisfies the conformal C-space condition [C] with a vector field* K_a *given by*

$$K_a = \nabla_a \log[r^{(1-n)/(n-3)}\Psi^{1/(3-n)}]. \tag{4.13}$$

From this and Propositions 2.4 and 4.2 it follows that the Robinson–Trautman metrics for which $\Psi \neq 0$ are conformal to Einstein metrics if and only if

$$P_{ab} - \nabla_a K_b + K_a K_b - \frac{1}{n}(P - \nabla^c K_c + K^c K_c)g_{ab} = 0$$

with K_a given by (4.13). (Note that, by the uniqueness asserted in Proposition 2.4, this is equivalent to requiring $E_{ab} = 0$ with E_{ab} as in Proposition 2.7.) Inserting R_{ab} and K_a into this equation one finds that the metric (4.1) is conformal to an Einstein metric if and only if the function $h = h(r)$ is given by

$$h(r) = -\frac{\kappa}{2} + \frac{m}{r^{n-3}} + \frac{\Lambda}{2(n-1)}r^2.$$

This means that among the considered Robinson–Trautman metrics the only metrics which are conformal to Einstein metrics are those belonging to the 2-parameter Schwarzschild-de Sitter family. So we have the following conclusions. The Robinson–Trautman metrics (4.1):

- are all generic,
- all satisfy conformal C-space condition, [C]
- in general do not satisfy the Bach condition, [B].

In fact from the conformal invariance of the systems [C] and [B] (see Section 3.2) and the condition of being generic, the same conclusions hold for all metrics conformally related to Robinson–Trautman metrics.

This, when along with four-dimensional examples of metrics satisfying the Bach conditions [B] and not being conformal to Einstein [1,24], proves independence of the two conditions [C] and [B].

4.2. *n-Dimensional pp-waves*

We noted in Section 2.5 that there are weakly generic metrics that fail to be Λ^2-generic, and hence fail to be generic. Metrics g with non-vanishing Weyl curvature, and such that there are two distinct Einstein metrics in the conformal class of g, fail to be weakly generic. This observation, which dates back to Brinkman [5], follows easily from the C-space equation. Explicit examples of Brinkman's metrics, thus the metrics with non-vanishing Weyl curvature but not weakly generic, are pp-waves. They can be described as follows.

Consider the n-dimensional metric (pp-wave)

$$g = 2\,\mathrm{d}u[\mathrm{d}r + h(x^i, u)\,\mathrm{d}u] + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j,$$

where g_{ij} are the components of a constant non-degenerate $(n-2)\times(n-2)$ matrix. This, in the coframe

$$\theta^+ = \mathrm{d}u, \qquad \theta^- = \mathrm{d}r + h\,\mathrm{d}u, \qquad \theta^i = \mathrm{d}x^i,$$

has curvature forms

$$\Omega_{i+} = -h_{,ik}\theta^k \wedge \theta^+, \qquad \Omega_{ij} = \Omega_{i-} = \Omega_{+-} = 0.$$

So the Ricci scalar vanishes, $R = 0$, and the only non-vanishing components of the Ricci and the Weyl tensors are

$$R_{++} = -2g^{ij}h_{,ij}, \qquad C_{i+j+} = \frac{2}{n-2}[g_{ij}g^{kl}h_{,kl} - (n-2)h_{,ij}],$$

apart from the components determined by these via symmetries. Thus, this metric is Einstein if and only if the function $h = h(x^i, u)$ is harmonic in the x^i variables,

$$g^{ij}h_{,ij} = 0,$$

in which case it is also Ricci flat. Whether this is satisfied or not it is clear that the vector field

$$K = f\partial_r, \tag{4.14}$$

where f is any non-vanishing function, satisfies

$$C_{abcd}K^d = 0. \tag{4.15}$$

Thus, the pp-wave metric is not weakly generic. It is worth noting that if the trace-free part of the matrix $h_{,ij}$ is invertible the vector (4.14) is the most general solution of Eq. (4.15). However, if it is not invertible, there are more vectors K which satisfy (4.15).

4.3. Four-dimensional hyperKähler metrics

Another interesting class of metrics that are weakly generic but not Λ^2-generic or generic can be found in the complex setting. Consider a four-dimensional non-flat hyperKähler manifold. This admits three Kähler structures I, J, K such that they satisfy quaternionic identities, e.g. $IJ + JI = 0$, $K = IJ$ and, as a consequence, is Ricci flat. We claim that all such manifolds are weakly generic, but not Λ^2-generic [23]. To see this, first consider the Riemann tensor viewed as an endomorphism $R(.) : \Lambda^2 T^*M \to \Lambda^2 T^*M$. Since the fundamental forms $\omega_I, \omega_J, \omega_K$, associated with I, J, K, are each parallel we have $R(\omega_I) = R(\omega_J) = R(\omega_K) = 0$. On the other hand from Ricci flatness we have $R(.) = C(.)$, where $C(.)$ is the Weyl tensor, also considered as and endomorphism $C(.) : \Lambda^2 T^*M \to \Lambda^2 T^*M$. Hence also $C(\omega_I) = C(\omega_J) = C(\omega_K) = 0$, which means that the metric is not Λ^2-generic.

On the other hand if there existed a vector field V such that $C_{abcd}V^d = 0$ then, because of the invariance property of C with respect of the structures I, J, K also $C_{abcd}(IV)^d$, $C_{abcd}(JV)^d$ and $C_{abcd}(KV)^d$ would vanish. Since on a hyperKähler 4-manifold a quadruple (V, IV, JV, KV) associated with any non-vanishing vector V constitutes a basis of vectors, at every point, we conclude that in such a case C_{abcd} (and therefore the Riemann tensor) vanishes. Thus, at any point x where the Weyl curvature is not zero we can conclude that $V = 0$ is the only solution to $C_{abcd}V^d = 0$.

Thus we have the following proposition.

Proposition 4.4. *Every non-flat four-dimensional hyperKähler manifold is weakly generic but not Λ^2-generic.*

For a local explicit example of this type see e.g. [25].

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