LOCALLY SASAKIAN MANIFOLDS *

Michał Godliński, Wojciech Kopczyński[†], Paweł Nurowski[‡]
Instytut Fizyki Teoretycznej
Uniwersytet Warszawski, ul. Hoża 69, Warszawa, Poland

May 30, 2000

Abstract

We show that every Sasakian manifold in dimension $2k+1$ is locally generated by a free real function of $2k$ variables. This function is a Sasakian analogue of the Kähler potential for Kähler geometry. It is also shown that every locally Sasakian-Einstein manifold in $2k+1$ dimensions is generated by a locally Kähler-Einstein manifold in dimension $2k$.

*Research supported by Komitet Badań Naukowych (Grant nr 2 P03B 017 12)
[†]`e-mail: kopcz@fuw.edu.pl`
[‡]`e-mail: nurowski@fuw.edu.pl`

1 Introduction

The Sasakian structure, wich is defined on an odd dimensional manifold is, in a sense, the closest possible analog of the Kähler geometry of even dimension. It was introduced by S. Sasaki [12] in 1960, who considered it as a special kind of contact geometries. Sasakian structure consists, in paraticular, of the contact 1-form η and the Riemannian metric g. The differential of η defines a 2-form, which constitutes an analog of the fundamental form of Kähler geometry.
Sasakian geometry was primarily studied as a substructure within the catagory of contact structures. A review of this approach can be found in [1, 14]. In this letter we exploit the analogy between Sasakian and Kähler geometry. We show that a well known fact that a Kähler geometry can be locally generated by a Kähler potential has its Sasakian counterpart. This result may be of some use in constructing a vast family of examples of Sasakian and Sasakian-Einstein structures.
The Sasakian and Sasakian-Einstein structures appear in physics in the context of string theory. It turns out that a metric cone $(C(\mathcal{S})=\mathbf{R}_+\times\mathcal{S},\bar{g}=\mathrm{d}r^2+r^2g)$ over a Sasakian-Einstein manifold $(\mathcal{S},g)$ is Kähler and Ricci flat, i.e. it constitutes a Calabi-Yau manifold. Moreover, the Sasaki-Einstein manifolds in dimensions $2k+1$ and Sasakian manifolds with three Sasakian structures in dimesnion $4k+3$ are related to the Maldacena conjecture [3, 4, 6, 13]. It turns out that they are one of very few structures which can serve as a compact factor $\mathcal{S}$ in (anti-de-Sitter)$\times\mathcal{S}$ background for classical field theories which, via the Maldacena conjecture, correspond to the large N limit of certain quantum conformal field theories.
A formal definition of a Sasaki manifold is as follows.
Definition 1
Let $\mathcal{S}$ be a $(2k+1)$-dimensional manifold equipped with a structure (ϕ,ξ,η,g) such that:
(i) ϕ is a (1,1) tensor field,
(ii) ξ is a vector field,
(iii) η is a field of an 1-form,
(iv) g is a Riemannian metric.
Assume, in addition, that for any vector fields X and Y on $\mathcal{S}$, (ϕ,ξ,η,g) satisfy the following algebraic conditions:
(1) $\phi^2X=-X+\eta(X)\xi$,
(2) $\eta(\xi)=1$,
(3) $g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$,
(4) $g(\xi,X)=\eta(X)$,
and the folllowing differential conditions:
(5) $N_\phi+\mathrm{d}\eta\otimes\xi=0$,
where $N_\phi(X,Y)=[\phi X,\phi Y]+\phi^2[X,Y]-\phi[\phi X,Y]-\phi[X,\phi Y]$ is the Nijenhuis tensor for ϕ,
(6) $\mathrm{d}\eta(X,Y)=g(\phi X,Y)$.
Then $\mathcal{S}$ is called a Sasakian manifold.
Example 1
A standard example of a Sasaki manifold is an odd dimensional sphere

$$\mathbf{S}^{2k+1}=\{\ \mathbf{C}^{k+1}\ni(z^1,...,z^{k+1})\ :\ |z^1|^2+...+|z^{k+1}|^2=1\ \}\subset\mathbf{C}^{k+1},$$

viewed as a submanifold of $\mathbf{C}^{k+1}$. Let J be the standard complex structure on $\mathbf{C}^{k+1}$, $\tilde{g}$ the standard flat metric on $\mathbf{C}^{k+1}\equiv\mathbf{R}^{2k+2}$, and n be the unit normal to the sphere. The vector field ξ on $\mathbf{S}^{2k+1}$ is defined by $\xi=-Jn$. If X is tangent vector to the sphere then JX uniquely decomposes onto the part paralell to n and the part tangent to the sphere. Denote this decomposition by $JX=\eta(X)n+\phi X$. This defines 1-form η and tensor field ϕ on $\mathbf{S}^{2k+1}$. Denoting the restriction of $\tilde{g}$ to $\mathbf{S}^{2k+1}$ by g we obtain (ϕ,ξ,η,g) structure on $\mathbf{S}^{2k+1}$. It is a matter of checking that this structure equips $\mathbf{S}^{2k+1}$ with a structure of a Sasakian-Einstein manifold. This construction is, in a certain sense, a Sasakian counterpart of the Fubini-Study Kähler structure on $\mathbf{CP}^k$.

Notation
We adapt the following notation:
$K_{,l}$ denotes the partial derivative of a function K with respect to the coordinate z^l.
Complex conjugate of an indexed quantity, e.g. $a^i{}_j$, is usually denoted by a bar over it, i.e. $\overline{a^i{}_j}$. Our notation is: $\overline{a^i{}_j} = \bar{a}^{\bar{i}}{}_{\bar{j}}$.
Symmetrized tensor products of 1-forms η and λ is denoted by $\eta\lambda = \frac{1}{2}(\eta \otimes \lambda + \lambda \otimes \eta)$.
The main result
The pouropuse of this letter is to prove the following theorem, which locally characterizes all Sasakian and Sasakian-Einstein manifolds.
Theorem.
Let $\mathcal{U}$ be an open set of $\mathbf{C}^k \times \mathbf{R}$ and let $(z^1, z^2, ..., z^k, x)$ be Cartesian coordinates in $\mathcal{U}$. Consider:
a vector field $\xi = \partial_x$
a real-valued function K on $\mathcal{U}$ such that $\xi(K) = 0$
an 1-form

$$\eta = \mathrm{d}x + i \sum_{m=1}^{k} (K_{,m} \mathrm{d}z^m) - i \sum_{\bar{m}=1}^{k} (K_{,\bar{m}} \mathrm{d}\bar{z}^{\bar{m}})$$

a bilinear form

$$g = \eta^2 + 2 \sum_{m,\bar{k}=1}^{k} K_{,m\bar{k}} \mathrm{d}z^m \mathrm{d}\bar{z}^{\bar{k}}$$

a tensor field

$$\phi = -i \sum_{m=1}^{k} [(\partial_m - iK_{,m}\partial_x) \otimes \mathrm{d}z^m] + i \sum_{\bar{m}=1}^{k} (\partial_{\bar{m}} + iK_{,\bar{m}}\partial_x) \otimes \mathrm{d}\bar{z}^{\bar{m}}].$$

I) If the function K is choosen in such a way that the bilinear form g has positive definite signature then $\mathcal{U}$ equipped with the structure (ϕ, ξ, η, g) is a Sasakian manifold. Moreover, every Sasakian manifold can locally be generated by such a function K.
II) The above Sasakian structure satisfies Einstein equation $Ric(g) = \lambda g$ if and only if $\lambda = 2k$ and the function K satsifies

$$-[\log \det(K_{,i\bar{j}})]_{,m\bar{n}} = 2(k+1)K_{,m\bar{n}} \, .$$

2 Almost conatact versus Sasakian manifolds

Definition 2
Consider $(2k+1)$-dimensional manifold $\mathcal{S}$ equipped with a structure consisting of a (1,1) tensor field ϕ, a vector field ξ and a field of an 1-form η. Assume, in addition, that for every vector field X on $\mathcal{S}$ (ξ, η, ϕ) satisfy the following algebraic conditions:
(1) $\phi^2 X = -X + \eta(X)\xi$,
(2) $\eta(\xi) = 1$.
Then $\mathcal{S}$ is called almost contact manifold. If, in addition, an almost contact manifold $(\mathcal{S}, (\xi, \eta, \phi))$ is equipped with Riemannian metric g such that for every veector fields X and Y on $\mathcal{S}$ we have
(3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,
(4) $g(\xi, X) = \eta(X)$,
then the almost contact manifold is called almost contact metric manifold.
Note that every Sasakian manifold is an almost contact metric manifold.

Let $\mathrm{T}^{\mathbf{C}}\mathcal{S}$ be the complexification of the tangent bundle of an almost contact manifold $\mathcal{S}$. The almost contact structure (ξ, η, ϕ) on $\mathcal{S}$ defines the decomposition

$$\mathrm{T}^{\mathbf{C}}\mathcal{S} = \mathbf{C} \otimes \xi \oplus N \oplus \bar{N},$$

where $\mathbf{C} \otimes \xi$, N and $\bar{N}$ are eigenspaces of ϕ with eigenvalues 0, $-i$ and i, respectively.
We say that a vector subbundle Z of $\mathrm{T}^{\mathbf{C}}\mathcal{S}$ is involutive if and only if $[\Gamma(Z), \Gamma(Z)] \subset \Gamma(Z)$, where $\Gamma(Z)$ denotes the set of all sections of Z.

Lemma

For an almost contact structure the condition $N_\phi + \mathrm{d}\eta \otimes \xi = 0$ is satisfied if and only if the bundle N is involutive, $[\Gamma(N), \Gamma(N)] \subset \Gamma(N)$, and $[\xi, \Gamma(N)] \subset \Gamma(N)$.

Proof.

Let $X, Y \in \Gamma(N)$. Making use of the eigenvalue property of ϕ and of property (1) of Definition 2, we get the following expressions for the Nijenhuis tensor of ϕ:

$$N_\phi(X, Y) = -2[X, Y] + 2i\phi([X, Y]) + \eta([X, Y])\xi,$$

$$N_\phi(X, \bar{Y}) = \eta([X, \bar{Y}])\xi,$$

$$N_\phi(X, \xi) = -[X, \xi] + i\phi([X, \xi]) + \eta([X, \xi])\xi.$$

Observe that the last component of the above formulae is the action of $-\mathrm{d}\eta \otimes \xi$ on (X, Y), $(X, \bar{Y})$ and (X, ξ), respectively. Therefore $N_\phi + \mathrm{d}\eta \otimes \xi = 0$ if and only if

$$\phi([X, Y]) = -i[X, Y]$$

and

$$\phi([X, \xi]) = -i[X, \xi].$$

This finishes the proof.

Corollary

For an almost contact structure satisfying $N_\phi + \mathrm{d}\eta \otimes \xi = 0$ (in particular for a Sasaki structure) the bundle $\mathbf{C} \otimes \xi \oplus N$ is involutive.

3 Sasakian geometry in a null frame

Let $(\mathcal{S}, (\xi, \eta, \phi, g))$ be a Sasakian manifold of dimension $2k + 1$. The algebraic conditions (1)-(4) of Definition 1 imply an existence of a local basis $(\xi, m_i, \bar{m}_{\bar{i}})$, $i, \bar{i} = 1, 2, ..., k$, of complex-valued vector fields on $\mathcal{S}$, with a cobasis $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$, such that

$$g = \eta^2 + 2 \sum_{l=\bar{l}, l=1}^{k} \mu^l \bar{\mu}^{\bar{l}}, \tag{1}$$

$$\phi = i \sum_{\bar{j}=1}^{k} (\bar{m}_{\bar{j}} \otimes \bar{\mu}^{\bar{j}}) - i \sum_{j=1}^{k} (m_j \otimes \mu^j). \tag{2}$$

Since $(\mathcal{S}, (\xi, \eta, \phi, g))$ is Sasakian then its bundle $\mathbf{C} \otimes \xi \oplus N$ is involutive. This is equivalent to the condition that the forms $\mu^1, \mu^2, ..., \mu^k$ generate a closed differential ideal i.e.

$$\mathrm{d}\mu^i \wedge \mu^1 \wedge \mu^2 \wedge ... \wedge \mu^k = 0 \qquad \forall i = 1, 2, ..., k. \tag{3}$$

Condition (6) of Definition 1 of a Sasakian manifold in this basis reads

$$\mathrm{d}\eta = -2i \sum_{l=\bar{l},l=1}^{k} \mu^l \wedge \bar{\mu}^{\bar{l}}. \tag{4}$$

Thus, the fact that the manifold is Sasakian necessarily implies the existence of a local basis $(\xi, m_i, \bar{m}_{\bar{i}})$ with a dual basis $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$ such that (1)-(4) holds. The converse is also true: if a real vector field ξ on a manifold $\mathcal{S}$ can be suplemented by k complex-valued vector fields m_i such that $(\xi, m_i, \bar{m}_{\bar{i}})$ and $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$ form a mutually dual basis for $\mathrm{T}^{\mathbf{C}}\mathcal{S}$ and $\mathrm{T}^{*\mathbf{C}}\mathcal{S}$, respectively, satisfying (3)-(4), then the structure (ξ, η, ϕ, g) defined by ξ, η and g, ϕ of (1)-(2) is a Sasakian manifold. This fact can be seen by noting that condition (3) is equivalent to the existence of complex-valued functions a_{ijk}, $b_{i\bar{j}k}$ and c_{ij} such that

$$\mathrm{d}\mu^i = \sum_{j,n=1}^{k} a_{ijn}\mu^j \wedge \mu^k + \sum_{\bar{j},n=1}^{k} b_{i\bar{j}n}\bar{\mu}^{\bar{j}} \wedge \mu^k + \sum_{j}^{k} c^j_{ij} \wedge \eta. \tag{5}$$

The dual conditions to conditions (4)-(5) imply that N is involutive and that $[\xi, \Gamma(N)] \subset \Gamma(N)$. These, when compared with Lemma of Corollary 1, imply condition (7), which is the only condition from Definition 1 which, a'priori, was not assumed for (ξ, η, ϕ, g). This proves the following Proposition.

Proposition 1

(I) (Local version)

Let (ξ, η, ϕ, g) be a Sasakian structure on a manifold $\mathcal{S}$ of dimension $2k+1$. Then there exisists a local basis $(\xi, m_i, \bar{m}_{\bar{i}})$, $i, \bar{i} = 1, 2, ...k$ of $\mathrm{T}^{\mathbf{C}}\mathcal{S}$ with dual basis $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$ such that

$$g = \eta^2 + 2 \sum_{l=\bar{l},l=1}^{k} \mu^l \bar{\mu}^{\bar{l}}$$

$$\phi = i\sum_{\bar{j}=1}^{k}(\bar{m}_{\bar{j}} \otimes \bar{\mu}^{\bar{j}}) - i\sum_{j=1}^{k}(m_j \otimes \mu^j),$$

$$\mathrm{d}\mu^i \wedge \mu^1 \wedge \mu^2 \wedge ... \wedge \mu^k = 0 \qquad \forall i = 1, 2, ..., k,$$

$$\mathrm{d}\eta = -2i \sum_{l=\bar{l},l=1}^{k} \mu^l \wedge \bar{\mu}^{\bar{l}}.$$

(II) (Global version)

Every almost contact metric structure which satisfies condition (6) of Definition 1 is Sasakian if and only if its canonical decomposition $\mathrm{T}^{\mathbf{C}}\mathcal{S} = \mathbf{C} \otimes \xi \oplus N \oplus \bar{N}$, consists of involutive $\mathbf{C} \otimes \xi \oplus N$ part.

We close this section with a quick application of part (I) of Proposition 1. It is well known that a vector field ξ on a Sasakian manifold $(\mathcal{S}, (\xi, \eta, \phi, g))$ is a Killing vector field. This in particular means that the Lie derivatives $\mathcal{L}_\xi$ of g and η vanish. The second of these facts is an immediate consequence of (4). To calculate $\mathcal{L}_\xi g$ one uses (4) and (5). After some work one shows that vanishing of $\mathcal{L}_\xi g$ is equivalent to $c_{ij} + \overline{c_{ji}} = 0$ where c_{ij} are functions appearing in (5). On the other hand, these equations are automatically implied by application of d on both sides of equation (4).

4 Analogue of the Kähler potential

We pass to a construction of local coordinates on a Sasakian manifold $(\mathcal{S},(\xi,\eta,\phi,g))$. We assume that all the fields defining the Sasakian structure are smooth on $\mathcal{S}$.
In a considered region of $\mathcal{S}$, we chose a local frame $(\xi, m_i, \bar{m}_{\bar{i}})$ of Proposition 1. Now, the fact that ξ is a Killing vector field on $\mathcal{S}$ together with the complex version of the Fröbenius theorem, assures that condition (3) is equivalent to an existence of complex-valued functions ${f^i}_j$ and z^i, $i,j=1,2,...k$ such that

$$\mu^i = {f^i}_j \mathrm{d}z^j. \tag{6}$$

Since the forms $(\mu^i, \bar{\mu}^{\bar{i}})$ form a part of the basis on the considered region of $\mathcal{S}$ then we also have

$$\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge ... \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2 \wedge ... \wedge \mathrm{d}\bar{z}^k \neq 0. \tag{7}$$

For the basis $(\xi, \partial_{z^1}, ..., \partial_{z^k}, \partial_{\bar{z}^1}, ..., \partial_{\bar{z}^k})$ and its dual $(\eta, \mathrm{d}z^1, ..., \mathrm{d}z^k, \mathrm{d}\bar{z}^1, ..., \mathrm{d}\bar{z}^k)$ the Maurer-Cartan relations for $\mathrm{d}(\mathrm{d}z^i) = 0 = \mathrm{d}(\mathrm{d}\bar{z}^{\bar{i}})$ readilly show that $[\xi, \partial/\partial z^i] = 0 = [\xi, \partial/\partial \bar{z}^{\bar{i}}]$. Therefore, there exists a real coordinate x complementary to $z^1, ..., z^k, \bar{z}^1, ..., \bar{z}^k$ such that

$$\xi = \partial_x \tag{8}$$

and the form η reads

$$\eta = \mathrm{d}x + p_j \mathrm{d}z^j + \bar{p}_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}. \tag{9}$$

Comparing this with the fact that ξ preserves η leads to the conclusion that the functions p_i are independent of coordinate x, $\partial p_i/\partial x = 0$.
Condition (4) is now equivalent to the follwing two conditions for the differentails of functions p_i:

$$p_{i,j} - p_{j,i} = 0 \tag{10}$$

and

$$p_{j,\bar{i}} - \bar{p}_{\bar{i},j} = 2i \sum_{l=\bar{l},l=1}^{k} {f^l}_j {\bar{f}^{\bar{l}}}_{\bar{i}}. \tag{11}$$

In a simply connected region of $\mathcal{S}$ equation (10) guarantees an existence of a complex-valued function V such that

$$p_i = \frac{\partial V}{\partial z^i}. \tag{12}$$

Since p_i is independent of x it is enough to consider functions V such that $\partial V/\partial x = 0$. Inserting so determined p_i to equation (11) we show that now (11) is equivalent to

$$K_{,j\bar{i}} = \sum_{l=\bar{l},l=1}^{k} {f^l}_j {\bar{f}^{\bar{l}}}_{\bar{i}}, \tag{13}$$

where we have introduce $\mathrm{Im}V = K$ and $\mathrm{Re}V = L$. Finally we note that now

$$\eta = \mathrm{d}(x+L) + i\sum_{j=1}^{k} K_{,j}\mathrm{d}z^j - i\sum_{\bar{j}=1}^{k} K_{,\bar{j}}\mathrm{d}\bar{z}^{\bar{j}},$$

so redefining the x coordinate by $x \to x + L$ we simplify η to the form $\eta = \mathrm{d}x + i\sum_{j=1}^{k} K_j \mathrm{d}z^j - i\sum_{\bar{j}=1}^{k} K_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}$. Using (13) we can eliminate functions ${f^i}_j$ from formulae defining our Sasakian structure. Indeed,

$$g = \eta^2 + 2\sum_{l=\bar{l},l=1}^{k} \mu^l \bar{\mu}^{\bar{l}} =$$

$$\eta^2+2\sum_{l=\bar{l},l=1}^{k}\sum_{j,\bar{i}=1}^{k} f^l{}_j\bar{f}^{\bar{l}}{}_{\bar{i}}\mathrm{d}z^j\mathrm{d}\bar{z}^{\bar{i}}=$$
$$\eta^2+2\sum_{j,\bar{i}=1}^{k}K_{,j\bar{i}}\mathrm{d}z^j\mathrm{d}\bar{z}^{\bar{i}}.$$

In this way we obtain the following theorem.
Theorem 1.
Let $\mathcal{U}$ be an open set of $\mathbf{C}^k\times\mathbf{R}$ and let $(z^1,z^2,...,z^k,x)$ be Cartesian coordinates in $\mathcal{U}$. Consider:
(i) a vector field $\xi=\partial_x$
(ii) a real-valued function K on $\mathcal{U}$ such that $\xi(K)=0$
(iii) an 1-form

$$\eta=\mathrm{d}x+i\sum_{m=1}^{k}(K_{,m}\mathrm{d}z^m)-i\sum_{\bar{m}=1}^{k}(K_{,\bar{m}}\mathrm{d}\bar{z}^{\bar{m}})$$

(iv) a bilinear form

$$g=\eta^2+2\sum_{m,\bar{k}=1}^{k}K_{,m\bar{k}}\mathrm{d}z^m\mathrm{d}\bar{z}^{\bar{k}}$$

(v) a tensor field

$$\phi=-i\sum_{m=1}^{k}[(\partial_m-iK_{,m}\partial_x)\otimes\mathrm{d}z^m]+i\sum_{\bar{m}=1}^{k}(\partial_{\bar{m}}+iK_{,\bar{m}}\partial_x)\otimes\mathrm{d}\bar{z}^{\bar{m}}].$$

If the function K is choosen in such a way that the bilinear form g has positive definite signature then $\mathcal{U}$ equipped with the structure (ϕ,ξ,η,g) is a Sasakian manifold. Moreover, every Sasakian manifold can locally be generated by such a function K.

The function K appearing in the above theorem is a Sasakian analogue of the Kähler potential generating Kähler geometries. We call it a Sasakian potential.

We close this section with a remark that several Sasakian potentials may generate the same Sasakian structure. This is evident if one notes that the following transformations

$$K\to K+f(z^j)+\bar{f}(\bar{z}^{\bar{j}})\qquad x\to x+i\bar{f}(\bar{z}^{\bar{j}})-if(z^j),\tag{14}$$

with f beeing a holomorphic function of z^js, do not change the Sasakian structure of Theorem 1. Thus, transformations (14) are the gauge transformations for the Sasakian potential.

5 Locally Sasakian-Einstein structures

In this section we calculate the Ricci tensor for the Sasakian metric g generated in a region $\mathcal{U}$ by the Sasakian potential K of Theorem 1. We also derive the equation which the Sasakian potential has to obey for Sasakian metric to satisfy Einstein equations $Ric(g)=\lambda g$. In this section we use the Einstein summation convention.
Let $\mathcal{U}$ be a simply connected region of of $\mathbf{C}^k\times\mathbf{R}$ as in Theorem 1. Consider a Sasakian structure defined in this Theorem by the Sasakian potential K. In the holonomic cobasis

$$(\mathrm{d}y^\mu)=(\mathrm{d}x,\mathrm{d}z^j,\mathrm{d}\bar{z}^{\bar{j}})$$

the covariant components of the Sasakian metric read

$$g_{\mu\nu} = \left(\begin{array}{ccc} 1 & iK_{,j} & -iK_{,\bar{j}} \\ iK_{,i} & -K_{,i}K_{,j} & K_{,i\bar{j}} + K_{,i}K_{,\bar{j}} \\ -iK_{,\bar{i}} & K_{,\bar{i}j} + K_{,\bar{i}}K_{,j} & -K_{,\bar{i}}K_{,\bar{j}} \end{array} \right). \tag{15}$$

The contravariant components of the metric read

$$g^{\mu\nu} = \left(\begin{array}{ccc} 1 + 2K_{,i}K_{,\bar{j}}\kappa^{i\bar{j}} & iK_{,\bar{j}}\kappa^{i\bar{j}} & -iK_{,j}\kappa^{j\bar{i}} \\ iK_{,\bar{i}}\kappa^{j\bar{i}} & 0 & \kappa^{j\bar{i}} \\ -iK_{,i}\kappa^{i\bar{j}} & \kappa^{i\bar{j}} & 0 \end{array} \right), \tag{16}$$

where

$$\kappa^{j\bar{l}}K_{,i\bar{l}} = \delta^{j}{}_{i} \qquad \kappa^{l\bar{j}}K_{,l\bar{i}} = \delta^{\bar{j}}{}_{\bar{i}} \qquad \overline{\kappa^{l\bar{j}}} = \kappa^{j\bar{l}} = \kappa^{\bar{l}j}. \tag{17}$$

The connection 1-forms $\Gamma_{\mu\nu} = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu})\mathrm{d}y^{\rho}$ read

$$\Gamma_{xx} = 0 \qquad \Gamma_{xi} = iK_{,ij}\mathrm{d}z^{j} \qquad \Gamma_{x\bar{i}} = \overline{\Gamma_{xi}} \qquad \Gamma_{ix} = iK_{,i\bar{j}}\mathrm{d}\bar{z}^{\bar{j}} \qquad \Gamma_{\bar{i}x} = \overline{\Gamma_{ix}}$$

$$\Gamma_{ij} = -K_{,i}K_{,jl}\mathrm{d}z^{l} - K_{,j}K_{,i\bar{l}}\mathrm{d}\bar{z}^{\bar{l}} \qquad \Gamma_{\bar{i}\bar{j}} = \overline{\Gamma_{ij}} \tag{18}$$

$$\Gamma_{i\bar{j}} = iK_{,i\bar{j}}\mathrm{d}x - K_{,l}K_{,i\bar{j}}\mathrm{d}z^{l} + (K_{,i\bar{j}\bar{l}} + K_{,i}K_{,\bar{j}\bar{l}} + K_{,\bar{j}}K_{,i\bar{l}} + K_{,\bar{l}}K_{,i\bar{j}})\mathrm{d}\bar{z}^{\bar{l}} \qquad \Gamma_{,\bar{i}j} = \overline{\Gamma_{i\bar{j}}}.$$

It is convenient to introduce the following functions:

$$C^{i}{}_{jm} = \kappa^{i\bar{l}}K_{,\bar{l}jm} \qquad B^{i}{}_{jm} = C^{i}{}_{jm} + \delta^{i}{}_{m}K_{,j} + \delta^{i}{}_{j}K_{,m} \qquad A_{jm} = C^{l}{}_{jm}K_{,l} + 2K_{,j}K_{,m} - K_{,jm}$$

$$C^{\bar{i}}{}_{\bar{j}\bar{m}} = \overline{C^{i}{}_{jm}} \qquad B^{\bar{i}}{}_{\bar{j}\bar{m}} = \overline{B^{i}{}_{jm}} \qquad A_{\ \bar{j}\bar{m}} = \overline{A_{\ jm}}.$$

Then the connection 1-forms $\Gamma^{\mu}{}_{\nu}$ read

$$\Gamma^{x}{}_{x} = -\mathrm{d}K \qquad \Gamma^{x}{}_{j} = -K_{,j}\mathrm{d}x - iA_{jm}\mathrm{d}z^{m} \qquad \Gamma^{i}{}_{x} = -i\mathrm{d}z^{i} \qquad \Gamma^{x}{}_{\bar{j}} = \overline{\Gamma^{x}{}_{j}} \qquad \Gamma^{\bar{i}}{}_{x} = \overline{\Gamma^{i}{}_{x}}$$

$$\Gamma^{i}{}_{j} = -i\delta^{i}{}_{j}\mathrm{d}x - \delta^{i}{}_{j}K_{,\bar{l}}\mathrm{d}\bar{z}^{\bar{l}} + B^{i}{}_{jl}\mathrm{d}z^{l} \qquad \Gamma^{i}{}_{\bar{j}} = -K_{,\bar{j}}\mathrm{d}z^{i} \qquad \Gamma^{\bar{i}}{}_{\bar{j}} = \overline{\Gamma^{i}{}_{j}} \qquad \Gamma^{\bar{i}}{}_{j} = \overline{\Gamma^{i}{}_{\bar{j}}}.$$

The curvature 2-forms $\Omega^{\mu}{}_{\nu} = \frac{1}{2}R^{\mu}{}_{\nu\rho\sigma}\mathrm{d}y^{\rho} \wedge \mathrm{d}y^{\sigma} = \mathrm{d}\Gamma^{\mu}{}_{\nu} + \Gamma^{\mu}{}_{\rho} \wedge \Gamma^{\rho}{}_{\nu}$ read

$$\Omega^{x}{}_{x} = iK_{,l}\mathrm{d}x \wedge \mathrm{d}z^{l} - iK_{,\bar{l}}\mathrm{d}x \wedge \mathrm{d}\bar{z}^{\bar{l}}$$

$$\Omega^{x}{}_{j} = -K_{,j}K_{,l}\mathrm{d}x \wedge \mathrm{d}z^{l} + (K_{,j\bar{l}} + K_{,j}K_{,\bar{l}})\mathrm{d}x \wedge \mathrm{d}\bar{z}^{\bar{l}} + iA_{jl,\bar{m}}\mathrm{d}z^{l} \wedge \mathrm{d}\bar{z}^{\bar{m}}$$

$$\Omega^{i}{}_{x} = -\delta^{i}{}_{j}\mathrm{d}x \wedge \mathrm{d}z^{j} + i\delta^{i}{}_{j}K_{,l}\mathrm{d}z^{j} \wedge \mathrm{d}z^{l} + iK_{,\bar{l}}\mathrm{d}\bar{z}^{\bar{l}} \wedge \mathrm{d}z^{i}$$

$$\Omega^{i}{}_{j} = -i\delta^{i}{}_{l}K_{,j}\mathrm{d}x \wedge \mathrm{d}z^{l} + (B^{i}{}_{mn}B^{m}{}_{jl} - B^{i}{}_{jn,l} + A_{jn}\delta^{i}{}_{l})\mathrm{d}z^{n} \wedge \mathrm{d}z^{l} + (K_{,j}K_{,\bar{l}}\delta^{i}{}_{n} - \delta^{i}{}_{j}K_{,n\bar{l}} - B^{i}{}_{jn,\bar{l}})\mathrm{d}z^{n} \wedge \mathrm{d}\bar{z}^{\bar{l}}$$

$$\Omega^{i}{}_{\bar{j}} = i\delta^{i}{}_{l}K_{,\bar{j}}\mathrm{d}x \wedge \mathrm{d}z^{l} + (K_{,\bar{j}l} + K_{,\bar{j}}K_{,l})\delta^{i}{}_{n}\mathrm{d}z^{n} \wedge \mathrm{d}z^{l} - \delta^{i}{}_{n}K_{,\bar{j}}K_{\bar{l}}\mathrm{d}z^{n} \wedge \mathrm{d}\bar{z}^{\bar{l}}$$

$$\Omega^{x}{}_{\bar{j}} = \overline{\Omega^{x}{}_{j}} \quad \Omega^{\bar{i}}{}_{x} = \overline{\Omega^{i}{}_{x}} \quad \Omega^{\bar{i}}{}_{\bar{j}} = \overline{\Omega^{i}{}_{j}} \quad \Omega^{\bar{i}}{}_{j} = \overline{\Omega^{i}{}_{\bar{j}}}.$$

The Ricci tensor $R_{\nu\sigma} = R^{\mu}{}_{\nu\mu\sigma}$ componenets read

$$R_{xx} = 2k \qquad R_{xj} = 2ikK_{,j} \qquad R_{ij} = -2kK_{,i}K_{,j} \qquad R_{\bar{i}j} = 2kK_{,\bar{i}}K_{,j} - 2K_{,\bar{i}j} - C^{\bar{m}}{}_{\bar{m}\bar{i},j}$$

$$R_{x\bar{j}} = \overline{R_{xj}} \qquad R_{\bar{i}\bar{j}} = \overline{R_{ij}}.$$

Now, the Einstein equations $Ric(g) = \lambda g$, which are nontrivial only for the components R_{xx} and $R_{\bar{i}j}$ become

$$\lambda = 2k \qquad -(\kappa^{\bar{m}l}K_{,\bar{m}l\bar{i}})_{,j} = 2(k+1)K_{,\bar{i}j}.$$

Since the matrix $(\kappa^{\bar{m}l})$ is the inverse of $(K_{,\bar{i}j})$ then the left hand side of the second equations above is

$$-(\kappa^{\bar{m}l}K_{,\bar{m}l\bar{i}})_{,j} = -\log(\det(K_{,m\bar{n}}))_{,\bar{i}j},$$

see e.g. [7].
Thus we arrive to the following theorem.
Theorem 2
Any Sasakian manifold of dimension $(2k+1)$ can be locally represented by the Sasakian potential K of Theorem 1. In the region where the potential is well defined the manifold satisfies Einstein equations $Ric(g) = \lambda g$ if and only if the cosmological constant

$$\lambda = 2k$$

and the potential satisfies

$$-\log(\det(K_{,m\bar{n}}))_{,\bar{i}j} = 2(k+1)K_{,\bar{i}j}. \tag{19}$$

Surprisingly equation (19) is the same as the Einstein condition $Ric(h) = 2(k+1)h$ for the Kähler metric $h = 2K_{,\bar{i}j}\mathrm{d}\bar{z}^{\bar{i}}\mathrm{d}z^j$ in dimension $2k$. Thus we have the following Corollary.
Corollary
Every Sasakian-Einstein manifold in dimension $(2k+1)$ is locally in one to one correspondence with a Kähler-Einstein manifold in dimension $2k$ whose cosmological constant $\lambda = 2(k+1)$. The correspondence is obtained by identifying the Kähler potential for the Kähler-Einstein manifold with the Sasaki potential for the Sasaki-Einstein manifold.

Examples
1). Sasakian potential for the sphere $\mathbf{S}^{2k+1}$.
Consider a function

$$\mathcal{K} = \frac{1}{2}\log(z^1\bar{z}^1 + ... + z^{k+1}\bar{z}^{k+1})$$

defined on $\mathbf{C}^{k+1} - \{0\}$. Let

$$N = i(\mathcal{K}_{,j}\mathrm{d}z^j - \mathcal{K}_{,\bar{j}}\mathrm{d}\bar{z}^{\bar{j}}),$$

$$H = 2\mathcal{K}_{,i\bar{j}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}}$$

and

$$G = N^2 + H.$$

The tensor fields N and G restrict to a sphere

$$\mathbf{S}^{2k+1} = \{(z^1, ..., z^{k+1}) \in \mathbf{C}^{k+1} - \{0\} \mid z^1\bar{z}^1 + ... + z^{k+1}\bar{z}^{k+1} = 1\}.$$

Denote these restrictions by η and g, respectively. Then the 1-form η and the Riemannian metric g define a Sasakian-Einstein structure on $\mathbf{S}^{2k+1}$. This structure coincides with the one defined in Example 1 of Section 1.
To see this, recal the Hopf fibration $\mathbf{U}(1) \to \mathbf{S}^{2k+1} \to \mathbf{CP}^k$ with the action of $\mathrm{e}^{i\phi} \in \mathbf{U}(1)$ on $(z^1, ..., z^{k+1}) \in \mathbf{S}^{2k+1}$ defined by $\mathrm{e}^{i\phi}(z^1, ..., z^{k+1}) = (\mathrm{e}^{i\phi}z^1, ..., \mathrm{e}^{i\phi}z^{k+1})$. The canonical projection is given by $\mathbf{S}^{2k+1} \ni (z^1, ..., z^{k+1}) \to \mathrm{dir}(z^1, ..., z^{k+1}) \in \mathbf{CP}^k$. The sphere is covered by $k+1$ charts

$$U_j = V_j \times \mathbf{U}(1),$$

where

$$V_j = \{\mathrm{dir}(z^1, ..., z^{k+1}) \mid (z^1, ..., z^{k+1}) \in \mathbf{S}^{2k+1} \text{ and } z^j \neq 0\}.$$

The local coordinates on each U_j are

$$(\xi^{ij} = \frac{z^i}{z^j},\ \phi_j = \frac{i}{2}\log\frac{z^j}{\bar{z}^j}), \quad i = 1, ..., k+1,\ i \neq j.$$

Then on each chart U_j the form $\eta_j = \eta|_{U_j}$ reads

$$\eta_j = \mathrm{d}\phi_j + \frac{i}{2}\frac{\sum_{i=1, i\neq j}^{k+1}(\bar{\xi}^{ij}\mathrm{d}\xi^{ij} - \xi^{ij}\mathrm{d}\bar{\xi}^{ij})}{1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2}.$$

The metric g restricted to U_j is

$$g_j = (\eta_j)^2 + \frac{(1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2)(\sum_{i=1,i\neq j}^{k+1}|\mathrm{d}\xi^{ij}|^2) - |\sum_{i=1,i\neq j}^{k+1}(\xi^{ij}\mathrm{d}\bar{\xi}^{ij})|^2}{(1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2)^2}.$$

Now, observe that on each U_j the structure (g_j, η_j) may be obtained by means of Theorems 1 and 2 choosing a Sasakian potential

$$K^j = \frac{1}{2}\log(1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2)$$

on the corresponding V_j. It is easy to check that K^j satisfies equation (19) on V_j. Thus, Theorem 2 assures that the Sasakian structure generated by (g_j, η_j) is Sasakian-Einstein. Easy, but lenghty, calculation shows that the Weyl tensor of g_j vanishes identically on U_j. This proves that (U_j, g_j) is locally isometric to a standard Riemannian structure on $\mathbf{S}^{2k+1}$. Since (g_j, η_j) originate from the global structure (g, η) then this global Sasakian structure must coincide with the standard Sasakian structure of Example 1. Note also that $h_j = g_j - (\eta_j)^2$ projects to V_j and patched together defines the Fubini-Study metric on $\mathbf{CP}^k$. In this sense the standard Sasakian structure on $\mathbf{S}^{2k+1}$ described in Example 1 is the analogue of the Fubini-Study Kähler structure on $\mathbf{CP}^k$.
2.) Sasakian-Einstein structure on $\mathbf{C}^q \times \mathbf{C}^n \times \mathbf{R}$.
Consider a function

$$K = \frac{1}{q+n+1}[\sum_{i=1}^{q}\log(1+|v^i|^2)] + \frac{n+1}{2(q+n+1)}\log(1+\sum_{I=1}^{n}|w^I|^2)$$

defined on $\mathbf{C}^q \times \mathbf{C}^n$, with coordinates $(z^\mu) = (v^i, w^I)$. It is easy to check that

$$[\log\det(K_{,\mu\bar{\nu}})]_{,\rho\bar{\sigma}} = -2(q+n+1)K_{,\rho\bar{\sigma}}.$$

Thus, via Theorems 1 and 2, such K generates a Sasakian-Einstein structure on $\mathbf{C}^q \times \mathbf{C}^n \times \mathbf{R}^1$.
3) Locally Sasakian-Einstein structures in dimension 5.
If $k = 2$ then, modulo the gauge (14), equation (19) may be integrated to the form

$$K_{,1\bar{1}}K_{,2\bar{2}} - K_{,1\bar{2}}K_{,2\bar{1}} = \mathrm{e}^{-6K}.$$

This is a well known equation describing the gravitational instantons in four dimensions [2, 5, 9, 10]. Examples of the Kähler-Einstein metrics generated by its solutions can be found e.g. in [5, 8, 11]. Via Theorems 1 and 2, each of these Kähler-Einstein structures defines a nontrivial Sasakian-Einstein manifold in dimension 5.

References

[1] Blair D E 1976 "Contact manifolds in Riemannian geometry" *Lecture Notes in Mathematics* **509**, (New York: Springer)

[2] Boyer C P, Finley J D III and Plebański J F 1980 "Complex general relativity, H and HH spaces - a survey of one approach" in *General relativity and gravitation (Einstein's memorial volume)* vol 2, ed. A Held, (New York: Plenum) pp. 241-81

[3] Boyer C P, Galicki K 1998 "On Sasakian-Einstein geometry", preprint, math.DG/9811098

[4] Figueroa-O'Farrill J M 1998, "Near-horizon geometries of supersymmetric branes", preprint, hep-th/9807149

[5] Gibbons G W, Pope C N 1979 "The positive action conjecture and asymptotically Euclidean metrics in quantum gravity" *Comm. Math. Phys.* **66**, 267-90

[6] Maldacena J 1998 "The large N limit of superconformal field theories and supergravity", *Adv. Theor. Math. Phys.* **2** 231-252; preprint: hep-th/9711200

[7] Landau L D, Lifshitz 1959 "The classical theory of fields" (London, Paris: Pergamon) p. 266

[8] Pedersen H, Poon Y S 1991 "Hamiltonian constructions of Kaehler-Einstein metrics and Kaehler metrics of constant scalar curvature" *Commun. Math. Phys.* **136** 309-326

[9] Przanowski M, Broda B 1983, "Locally Kähler gravitational instantons" *Acta Phys. Polon.* **B 14** 637-61

[10] Przanowski M, Baka B 1984, "One-sided type D gravitational instantons" *Gen. Rel. Grav.* **16** 797-803

[11] Plebański J F, Przanowski M 1998 "Hermite-Einstein four-dimensional manifolds with symmetry" *Class. Quantum Grav.* **15** 1721-35

[12] Sasaki S 1960 "On differentiable manifolds with certain structures which are closely related to almost contact structure" *Tohoku Math. J.* **12** 459-76

[13] Witten E 1998 "Anti-de Sitter space and holography", *Adv. Theor. Math. Phys.* **2** 253-291; preprint: hep-th/9802150

[14] Yano K, Kon M "Structures on manifolds" (World Scientific)