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**LETTER TO THE EDITOR**

# Locally Sasakian manifolds*

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**Abstract.** We show that every Sasakian manifold in dimensions $2k + 1$ is locally generated by a free real function of $2k$ variables. This function is a Sasakian analogue of the Kähler potential for the Kähler geometry. It is also shown that every locally Sasakian–Einstein manifold in $2k + 1$ dimensions is generated by a locally Kähler–Einstein manifold in dimension $2k$.

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## 1. Introduction

The Sasakian structure, which is defined on an odd dimensional manifold is, in a sense, the closest possible analogue of the Kähler geometry of even dimension. It was introduced by Sasaki [12] in 1960, who considered it as a special kind of contact geometry. Sasakian structure consists, in particular, of the contact 1-form $\eta$ and the Riemannian metric $g$. The differential of $\eta$ defines a 2-form, which constitutes an analogue of the fundamental form of Kähler geometry.

Sasakian geometry was primarily studied as a substructure within the category of contact structures. A review of this approach can be found in [1, 14]. In this Letter we exploit the analogy between Sasakian and Kähler geometry. We show that the well known fact that a Kähler geometry can be locally generated by a Kähler potential has its Sasakian counterpart. This result may be of some use in constructing a vast family of examples of Sasakian and Sasakian–Einstein structures.

The Sasakian and Sasakian–Einstein structures appear in physics in the context of the string theory. It turns out that a metric cone ($C(\mathcal{S}) = \boldsymbol{R}_+ \times \mathcal{S}, \bar{g} = \mathrm{d}r^2 + r^2 g$) over a Sasakian–Einstein manifold $(\mathcal{S}, g)$ is Kähler and Ricci flat, i.e. it constitutes a Calabi–Yau manifold. Moreover, the Sasakian–Einstein manifolds in dimensions $2k + 1$ and Sasakian manifolds with three Sasakian structures in dimension $4k + 3$ are related to the Maldacena conjecture [3, 4, 6, 13]. It turns out that they are one of very few structures which can serve as a compact factor $\mathcal{S}$ in the (anti-de Sitter) $\times\mathcal{S}$ background for classical field theories which, via the Maldacena conjecture, correspond to the large $N$ limit of certain quantum conformal field theories.

A formal definition of a Sasakian manifold is as follows.

**Definition 1**
Let $\mathcal{S}$ be a $(2k + 1)$-dimensional manifold equipped with a structure $(\phi, \xi, \eta, g)$ such that:

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(i) $\phi$ is a (1,1) tensor field,
(ii) $\xi$ is a vector field,
(iii) $\eta$ is a field of a 1-form,
(iv) $g$ is a Riemannian metric.

Assume, in addition, that for any vector fields $X$ and $Y$ on $\mathcal{S}$, $(\phi, \xi, \eta, g)$ satisfy the following algebraic conditions:

(1) $\phi^2 X = -X + \eta(X)\xi$,
(2) $\eta(\xi) = 1$,
(3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,
(4) $g(\xi, X) = \eta(X)$,

and the following differential conditions:

(5) $N_\phi + \mathrm{d}\eta \otimes \xi = 0$, where $N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ is the Nijenhuis tensor for $\phi$,
(6) $\mathrm{d}\eta(X, Y) = g(\phi X, Y)$.

Then $\mathcal{S}$ is called a Sasakian manifold.

**Example 1**
A standard example of a Sasakian manifold is the odd dimensional sphere

$$\boldsymbol{S}^{2k+1} = \{\, \boldsymbol{C}^{k+1} \ni (z^1, \ldots, z^{k+1}) \ :\ |z^1|^2 + \ldots + |z^{k+1}|^2 = 1 \,\} \subset \boldsymbol{C}^{k+1},$$

viewed as a submanifold of $\boldsymbol{C}^{k+1}$. Let $J$ be the standard complex structure on $\boldsymbol{C}^{k+1}$, $\tilde{g}$ the standard flat metric on $\boldsymbol{C}^{k+1} \equiv \boldsymbol{R}^{2k+2}$, and $n$ be the unit normal to the sphere. The vector field $\xi$ on $\boldsymbol{S}^{2k+1}$ is defined by $\xi = -Jn$. If $X$ is a tangent vector to the sphere then $JX$ uniquely decomposes onto the part parallel to $n$ and the part tangent to the sphere. Denote this decomposition by $JX = \eta(X)n + \phi X$. This defines the 1-form $\eta$ and the tensor field $\phi$ on $\boldsymbol{S}^{2k+1}$. Denoting the restriction of $\tilde{g}$ to $\boldsymbol{S}^{2k+1}$ by $g$ we obtain $(\phi, \xi, \eta, g)$ structure on $\boldsymbol{S}^{2k+1}$. It is a matter of checking that this structure equips $\boldsymbol{S}^{2k+1}$ with a structure of a Sasakian–Einstein manifold. This construction is, in a certain sense, a Sasakian counterpart of the Fubini–Study Kähler structure on $\boldsymbol{CP}^k$.

**Notation**
We adapt the following notation: $K_{,l}$ denotes the partial derivative of a function $K$ with respect to the coordinate $z^l$. The complex conjugate of an indexed quantity, e.g. of $a^i{}_j$, is usually denoted by a bar over it, i.e. $\overline{a^i{}_j}$. Our notation is: $\overline{a^i{}_j} = \bar{a}^{\bar{i}}{}_{\bar{j}}$. The symmetrized tensor products of 1-forms $\eta$ and $\lambda$ is denoted by $\eta\lambda = \frac{1}{2}(\eta \otimes \lambda + \lambda \otimes \eta)$.

**The main result**
The purpose of this letter is to prove the following theorem, which locally characterizes all Sasakian and Sasakian–Einstein manifolds.

**Theorem**
Let $\mathcal{U}$ be an open set of $\boldsymbol{C}^k \times \boldsymbol{R}$ and let $(z^1, z^2, \ldots, z^k, x)$ be Cartesian coordinates in $\mathcal{U}$. Consider:

a vector field $\xi = \partial_x$
a real-valued function $K$ on $\mathcal{U}$ such that $\xi(K) = 0$

an 1-form

$$\eta = \mathrm{d}x + i\sum_{m=1}^{k}(K_{,m}\mathrm{d}z^m) - i\sum_{\bar m=1}^{k}(K_{,\bar m}\mathrm{d}\bar z^{\bar m})$$

a bilinear form

$$g = \eta^2 + 2\sum_{m,\bar k=1}^{k} K_{,m\bar k}\mathrm{d}z^m\mathrm{d}\bar z^{\bar k}$$

a tensor field

$$\phi = -i\sum_{m=1}^{k}[(\partial_m - iK_{,m}\partial_x)\otimes \mathrm{d}z^m] + i\sum_{\bar m=1}^{k}[(\partial_{\bar m} + iK_{,\bar m}\partial_x)\otimes \mathrm{d}\bar z^{\bar m}].$$

(I) If the function $K$ is chosen in such a way that the bilinear form $g$ has the positive definite signature, then $\mathcal{U}$ equipped with the structure $(\phi, \xi, \eta, g)$ is a Sasakian manifold. Moreover, every Sasakian manifold can locally be generated by such a function $K$.

(II) The above Sasakian structure satisfies Einstein equation $Ric(g) = \lambda g$ if and only if $\lambda = 2k$ and the function $K$ satisfies

$$-[\log\det(K_{,i\bar j})]_{,m\bar n} = 2(k+1)K_{,m\bar n}\,.$$

## 2. Almost contact *versus* Sasakian manifolds

**Definition 2**

Consider a $(2k+1)$-dimensional manifold $\mathcal{S}$ equipped with a structure consisting of a (1,1) tensor field $\phi$, a vector field $\xi$ and a field of an 1-form $\eta$. Assume, in addition, that for every vector field $X$ on $\mathcal{S}$ the triple $(\xi, \eta, \phi)$ satisfies the following algebraic conditions:

(1) $\phi^2 X = -X + \eta(X)\xi$,

(2) $\eta(\xi) = 1$.

Then $\mathcal{S}$ is called an almost contact manifold. If, in addition, an almost contact manifold $(\mathcal{S}, (\xi, \eta, \phi))$ is equipped with a Riemannian metric $g$ such that for every vector fields $X$ and $Y$ on $\mathcal{S}$ we have

(3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,

(4) $g(\xi, X) = \eta(X)$,

then the almost contact manifold is called an almost contact metric manifold. Note that every Sasakian manifold is an almost contact metric manifold.

Let $\mathrm{T}^{\boldsymbol{C}}\mathcal{S}$ be the complexification of the tangent bundle of an almost contact manifold $\mathcal{S}$. The almost contact structure $(\xi, \eta, \phi)$ on $\mathcal{S}$ defines the decomposition

$$\mathrm{T}^{\boldsymbol{C}}\mathcal{S} = \boldsymbol{C}\otimes\xi \oplus N \oplus \bar N,$$

where $\boldsymbol{C}\otimes\xi$, $N$ and $\bar N$ are eigenspaces of $\phi$ with eigenvalues 0, $-i$ and $i$, respectively.

We say that a vector subbundle $Z$ of $\mathrm{T}^{\boldsymbol{C}}\mathcal{S}$ is involutive if and only if $[\Gamma(Z), \Gamma(Z)] \subset \Gamma(Z)$, where $\Gamma(Z)$ denotes the set of all sections of $Z$.

**Lemma**

For an almost contact structure the condition $N_\phi + \mathrm{d}\eta\otimes\xi = 0$ is satisfied if and only if the bundle $N$ is involutive, $[\Gamma(N), \Gamma(N)] \subset \Gamma(N)$, and $[\xi, \Gamma(N)] \subset \Gamma(N)$.

**Proof**

Let $X, Y \in \Gamma(N)$. Making use of the eigenvalue property of $\phi$ and of property (1) of definition 2, we get the following expressions for the Nijenhuis tensor of $\phi$:

$$N_\phi(X, Y) = -2[X, Y] + 2i\phi([X, Y]) + \eta([X, Y])\xi,$$

$$N_\phi(X, \bar{Y}) = \eta([X, \bar{Y}])\xi,$$

$$N_\phi(X, \xi) = -[X, \xi] + i\phi([X, \xi]) + \eta([X, \xi])\xi.$$

Observe that the last component of the above formulae is the action of $-\mathrm{d}\eta \otimes \xi$ on $(X, Y)$, $(X, \bar{Y})$ and $(X, \xi)$, respectively. Therefore $N_\phi + \mathrm{d}\eta \otimes \xi = 0$ if and only if

$$\phi([X, Y]) = -i[X, Y]$$

and

$$\phi([X, \xi]) = -i[X, \xi].$$

This finishes the proof.

**Corollary**

For an almost contact structure satisfying $N_\phi + \mathrm{d}\eta \otimes \xi = 0$ (in particular for a Sasakian structure) the bundle $\boldsymbol{C} \otimes \xi \oplus N$ is involutive.

## 3. Sasakian geometry in a null frame

Let $(\mathcal{S}, (\xi, \eta, \phi, g))$ be a Sasakian manifold of dimension $2k + 1$. The algebraic conditions (1)–(4) of definition 1 imply an existence of a local basis $(\xi, m_i, \bar{m}_{\bar{i}})$, $i, \bar{i} = 1, 2, \ldots, k$, of complex-valued vector fields on $\mathcal{S}$, with a cobasis $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$, such that

$$g = \eta^2 + 2\sum_{l=\bar{l},l=1}^{k} \mu^l \bar{\mu}^{\bar{l}}, \tag{1}$$

$$\phi = i\sum_{\bar{j}=1}^{k}(\bar{m}_{\bar{j}} \otimes \bar{\mu}^{\bar{j}}) - i\sum_{j=1}^{k}(m_j \otimes \mu^j). \tag{2}$$

Since $(\mathcal{S}, (\xi, \eta, \phi, g))$ is Sasakian then its bundle $\boldsymbol{C} \otimes \xi \oplus N$ is involutive. This is equivalent to the condition that the forms $\mu^1, \mu^2, \ldots, \mu^k$ generate a closed differential ideal i.e.

$$\mathrm{d}\mu^i \wedge \mu^1 \wedge \mu^2 \wedge \ldots \wedge \mu^k = 0 \qquad \forall i = 1, 2, \ldots, k. \tag{3}$$

Condition (6) of definition 1 of a Sasakian manifold in this basis reads

$$\mathrm{d}\eta = -2i\sum_{l=\bar{l},l=1}^{k} \mu^l \wedge \bar{\mu}^{\bar{l}}. \tag{4}$$

Thus, the fact that the manifold is Sasakian necessarily implies the existence of a local basis $(\xi, m_i, \bar{m}_{\bar{i}})$ with a dual basis $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$ such that (1)–(4) holds. The converse is also true: if a real vector field $\xi$ on a manifold $\mathcal{S}$ can be supplemented by $k$ complex-valued vector fields $m_i$ such that $(\xi, m_i, \bar{m}_{\bar{i}})$ and $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$ form a mutually dual basis for $\mathrm{T}^C\mathcal{S}$ and $\mathrm{T}^{*C}\mathcal{S}$, respectively, satisfying (3)–(4), then the structure $(\xi, \eta, \phi, g)$ defined by $\xi$, $\eta$ and $g$, $\phi$ of (1)–(2) is a Sasakian manifold. This fact can be seen by noting that condition (3) is equivalent to the existence of complex-valued functions $a_{ijk}$, $b_{i\bar{j}k}$ and $c_{ij}$ such that

$$\mathrm{d}\mu^i = \sum_{j,n=1}^{k} a_{ijn}\mu^j \wedge \mu^n + \sum_{\bar{j},n=1}^{k} b_{i\bar{j}n}\bar{\mu}^{\bar{j}} \wedge \mu^n + \sum_{j=1}^{k} c_{ij}\mu^j \wedge \eta. \tag{5}$$

The dual conditions to conditions (4)–(5) imply that $N$ is involutive and that $[\xi, \Gamma(N)] \subset \Gamma(N)$. These, when compared with lemma of corollary 1, imply condition (7), which is the only condition from definition 1 which, *a priori*, was not assumed for $(\xi, \eta, \phi, g)$. This proves the following proposition.

**Proposition 1**
(I) *Local version*
Let $(\xi, \eta, \phi, g)$ be a Sasakian structure on a manifold $\mathcal{S}$ of dimension $2k+1$. Then there exists a local basis $(\xi, m_i, \bar{m}_{\bar{i}})$, $i, \bar{i} = 1, 2, \ldots k$ of $\mathrm{T}^{\mathbf{C}}\mathcal{S}$ with dual basis $(\eta, \mu^i, \bar{\mu}^{\bar{i}})$ such that

$$g = \eta^2 + 2\sum_{l=\bar{l},l=1}^{k} \mu^l \bar{\mu}^{\bar{l}}$$

$$\phi = i\sum_{\bar{j}=1}^{k}(\bar{m}_{\bar{j}} \otimes \bar{\mu}^{\bar{j}}) - i\sum_{j=1}^{k}(m_j \otimes \mu^j),$$

$$\mathrm{d}\mu^i \wedge \mu^1 \wedge \mu^2 \wedge \ldots \wedge \mu^k = 0 \qquad \forall i = 1, 2, \ldots, k,$$

$$\mathrm{d}\eta = -2i\sum_{l=\bar{l},l=1}^{k} \mu^l \wedge \bar{\mu}^{\bar{l}}.$$

(II) *Global version*
Every almost contact metric structure which satisfies condition (6) of definition 1 is Sasakian if and only if its canonical decomposition $\mathrm{T}^{\mathbf{C}}\mathcal{S} = \mathbf{C} \otimes \xi \oplus N \oplus \bar{N}$, consists of involutive $\mathbf{C} \otimes \xi \oplus N$ part.

We close this section with a quick application of part (I) of proposition 1. It is well known that a vector field $\xi$ on a Sasakian manifold $(\mathcal{S}, (\xi, \eta, \phi, g))$ is a Killing vector field. This in particular means that the Lie derivatives $\mathcal{L}_\xi$ of $g$ and $\eta$ vanish. The second of these facts is an immediate consequence of (4). To calculate $\mathcal{L}_\xi g$ one uses (4) and (5). After some work one shows that vanishing of $\mathcal{L}_\xi g$ is equivalent to $c_{ij} + \overline{c_{ji}} = 0$ where $c_{ij}$ are functions appearing in (5). On the other hand, these equations are automatically implied by application of d on both sides of equation (4).

## 4. Analogue of the Kähler potential

We pass to a construction of local coordinates on a Sasakian manifold $(\mathcal{S}, (\xi, \eta, \phi, g))$. We assume that all the fields defining the Sasakian structure are smooth on $\mathcal{S}$.

In a considered region of $\mathcal{S}$, we chose a local frame $(\xi, m_i, \bar{m}_{\bar{i}})$ of proposition 1. Now, the fact that $\xi$ is a Killing vector field on $\mathcal{S}$ together with the complex version of the Frobenius theorem, assures that condition (3) is equivalent to the existence of complex-valued functions $f^i{}_j$ and $z^i$, $i, j = 1, 2, \ldots k$ such that

$$\mu^i = \sum_{j=1}^{k} f^i{}_j \mathrm{d}z^j. \tag{6}$$

Since the forms $(\mu^i, \bar{\mu}^{\bar{i}})$ form a part of the basis on the considered region of $\mathcal{S}$ then we also have

$$\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \ldots \mathrm{d}z^k \wedge \mathrm{d}\bar{z}^1 \wedge \mathrm{d}\bar{z}^2 \wedge \ldots \wedge \mathrm{d}\bar{z}^k \neq 0. \tag{7}$$

For the basis $(\xi, \partial_{z^1}, \ldots, \partial_{z^k}, \partial_{\bar{z}^1}, \ldots, \partial_{\bar{z}^k})$ and its dual $(\eta, \mathrm{d}z^1, \ldots, \mathrm{d}z^k, \mathrm{d}\bar{z}^1, \ldots, \mathrm{d}\bar{z}^k)$ the Maurer–Cartan relations readily show that $[\xi, \partial/\partial z^i] = 0 = [\xi, \partial/\partial \bar{z}^{\bar{i}}]$. Therefore, there exists a real coordinate $x$ complementary to $z^1, \ldots, z^k, \bar{z}^1, \ldots, \bar{z}^k$ such that

$$\xi = \partial_x \tag{8}$$

and the form $\eta$ reads

$$\eta = \mathrm{d}x + p_j \mathrm{d}z^j + \bar{p}_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}. \tag{9}$$

Comparing this with the fact that $\xi$ preserves $\eta$ leads to the conclusion that the functions $p_i$ are independent of the coordinate $x$, $\partial p_i / \partial x = 0$.

Condition (4) is now equivalent to the following two conditions for the differentials of functions $p_i$:

$$p_{i,j} - p_{j,i} = 0 \tag{10}$$

and

$$p_{j,\bar{i}} - \bar{p}_{\bar{i},j} = 2i \sum_{l=\bar{l},l=1}^{k} f^l{}_j \bar{f}^{\bar{l}}{}_{\bar{i}}. \tag{11}$$

In a simply connected region of $\mathcal{S}$ equation (10) guarantees an existence of a complex-valued function $V$ such that

$$p_i = \frac{\partial V}{\partial z^i}. \tag{12}$$

Since $p_i$ is independent of $x$ it is enough to consider functions $V$ such that $\partial V / \partial x = 0$. Inserting a so determined $p_i$ in equation (11) we show that now (11) is equivalent to

$$K_{,j\bar{i}} = \sum_{l=\bar{l},l=1}^{k} f^l{}_j \bar{f}^{\bar{l}}{}_{\bar{i}}, \tag{13}$$

where we have introduced $\mathrm{Im}V = K$ and $\mathrm{Re}V = L$. Finally we note that now

$$\eta = \mathrm{d}(x + L) + i \sum_{j=1}^{k} K_{,j} \mathrm{d}z^j - i \sum_{\bar{j}=1}^{k} K_{,\bar{j}} \mathrm{d}\bar{z}^{\bar{j}},$$

so redefining the $x$ coordinate by $x \to x + L$ we simplify $\eta$ to the form $\eta = \mathrm{d}x + i \sum_{j=1}^{k} K_j \mathrm{d}z^j - i \sum_{\bar{j}=1}^{k} K_{\bar{j}} \mathrm{d}\bar{z}^{\bar{j}}$. Using (13) we can eliminate functions $f^i{}_j$ from formulae defining our Sasakian structure. Indeed,

$$g = \eta^2 + 2 \sum_{l=\bar{l},l=1}^{k} \mu^l \bar{\mu}^{\bar{l}} = \eta^2 + 2 \sum_{l=\bar{l},l=1}^{k} \sum_{j,\bar{i}=1}^{k} f^l{}_j \bar{f}^{\bar{l}}{}_{\bar{i}} \mathrm{d}z^j \mathrm{d}\bar{z}^{\bar{i}} = \eta^2 + 2 \sum_{j,\bar{i}=1}^{k} K_{,j\bar{i}} \mathrm{d}z^j \mathrm{d}\bar{z}^{\bar{i}}.$$

In this way we obtain the following theorem.

**Theorem 1**

Let $\mathcal{U}$ be an open set of $\boldsymbol{C}^k \times \boldsymbol{R}$ and let $(z^1, z^2, \ldots, z^k, x)$ be Cartesian coordinates in $\mathcal{U}$. Consider:

(i) a vector field $\xi = \partial_x$

(ii) a real-valued function $K$ on $\mathcal{U}$ such that $\xi(K) = 0$

(iii) a 1-form

$$\eta = \mathrm{d}x + i \sum_{m=1}^{k} (K_{,m} \mathrm{d}z^m) - i \sum_{\bar{m}=1}^{k} (K_{,\bar{m}} \mathrm{d}\bar{z}^{\bar{m}})$$

(iv) a bilinear form

$$g = \eta^2 + 2 \sum_{m,\bar{k}=1}^{k} K_{,m\bar{k}} \mathrm{d}z^m \mathrm{d}\bar{z}^{\bar{k}}$$

(v) a tensor field

$$\phi = -i \sum_{m=1}^{k} [(\partial_m - i K_{,m} \partial_x) \otimes \mathrm{d}z^m] + i \sum_{\bar{m}=1}^{k} [(\partial_{\bar{m}} + i K_{,\bar{m}} \partial_x) \otimes \mathrm{d}\bar{z}^{\bar{m}}].$$

If the function $K$ is chosen in such a way that the bilinear form $g$ has positive definite signature then $\mathcal{U}$ equipped with the structure $(\phi, \xi, \eta, g)$ is a Sasakian manifold. Moreover, every Sasakian manifold can locally be generated by such a function $K$.

Since the characteristic vector field $\xi$ is non-vanishing everywhere, it defines a one-dimensional foliation on the Sasakian manifold $\mathcal{S}$. Each such foliation can be locally viewed as a principal $\boldsymbol{U}(1)$-bundle, i.e. more precisely, each point of $\mathcal{S}$ has a neighbourhood isomorphic to an open subset of a principal $\boldsymbol{U}(1)$-bundle. With such identification, the vector field $\xi$ is a fundamental vector field of the bundle and the 1-form $\eta$ is a connection form. Our coordinates $(x, z^i)$ are consistent with a local trivialization of the bundle. The relation between the Sasakian metric on $\mathcal{S}$, the Kähler metric on the base space and the connection form can be understood as a special case of the construction known in the Kaluza–Klein unification scheme. In the context of the Sasakian structure, our result is to some extent known (see theorem 2.8 of [3]), however under some global assumptions concerning the manifold $S$ and the Sasakian structure on it (quasi-regularity assumption). We underline that the form of the Sasakian metric given above is valid without any topological restriction, however it is valid locally.

The function $K$ appearing in the above theorem is a Sasakian analogue of the Kähler potential generating Kähler geometries. We call it a Sasakian potential.

We close this section with a remark that several Sasakian potentials may generate the same Sasakian structure. This is evident if one notes that the following transformations

$$K \to K + f(z^j) + \bar{f}(\bar{z}^{\bar{j}}) \qquad x \to x + i\bar{f}(\bar{z}^{\bar{j}}) - if(z^j), \tag{14}$$

with $f$ being a holomorphic function of $z^j$s, do not change the Sasakian structure of theorem 1. Thus, transformations (14) are the gauge transformations for the Sasakian potential.

## 5. Locally Sasakian–Einstein structures

In this section we calculate the Ricci tensor for the Sasakian metric $g$ generated in a region $\mathcal{U}$ by the Sasakian potential $K$ of theorem 1. We also derive the equation which the Sasakian potential has to obey for the Sasakian metric to satisfy the Einstein equations $Ric(g) = \lambda g$. In this section we use the Einstein summation convention.

Let $\mathcal{U}$ be a simply connected region of of $\boldsymbol{C}^k \times \boldsymbol{R}$ as in theorem 1. Consider a Sasakian structure defined in this theorem by the Sasakian potential $K$. In the holonomic cobasis

$$(\mathrm{d}y^\mu) = (\mathrm{d}x, \mathrm{d}z^j, \mathrm{d}\bar{z}^{\bar{j}})$$

the covariant components of the Sasakian metric read

$$g_{\mu\nu} = \begin{pmatrix} 1 & iK_{,j} & -iK_{,\bar{j}} \\ iK_{,i} & -K_{,i}K_{,j} & K_{,i\bar{j}} + K_{,i}K_{,\bar{j}} \\ -iK_{,\bar{i}} & K_{,\bar{i}j} + K_{,\bar{i}}K_{,j} & -K_{,\bar{i}}K_{,\bar{j}} \end{pmatrix}. \tag{15}$$

The contravariant components of the metric read

$$g^{\mu\nu} = \begin{pmatrix} 1 + 2K_{,i}K_{,\bar{j}}\kappa^{i\bar{j}} & iK_{,\bar{j}}\kappa^{i\bar{j}} & -iK_{,j}\kappa^{j\bar{i}} \\ iK_{,\bar{i}}\kappa^{j\bar{i}} & 0 & \kappa^{j\bar{i}} \\ -iK_{,i}\kappa^{i\bar{j}} & \kappa^{i\bar{j}} & 0 \end{pmatrix}, \tag{16}$$

where

$$\kappa^{j\bar{l}}K_{,i\bar{l}} = \delta^j_i \qquad \kappa^{l\bar{j}}K_{,l\bar{i}} = \delta^{\bar{j}}_{\bar{i}} \qquad \overline{\kappa^{l\bar{j}}} = \kappa^{j\bar{l}} = \kappa^{\bar{l}j}. \tag{17}$$

The connection 1-forms $\Gamma_{\mu\nu} = \frac{1}{2}(g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu})\mathrm{d}y^\rho$ read

$$\Gamma_{xx} = 0 \qquad \Gamma_{xi} = iK_{,ij}\mathrm{d}z^j \qquad \Gamma_{x\bar{i}} = \overline{\Gamma_{xi}} \qquad \Gamma_{ix} = iK_{,i\bar{j}}\mathrm{d}\bar{z}^{\bar{j}} \qquad \Gamma_{\bar{i}x} = \overline{\Gamma_{ix}}$$

$$\Gamma_{ij} = -K_{,i}K_{,jl}\mathrm{d}z^l - K_{,j}K_{,i\bar{l}}\mathrm{d}\bar{z}^{\bar{l}} \qquad \Gamma_{\bar{i}\bar{j}} = \overline{\Gamma_{ij}} \tag{18}$$

$$\Gamma_{i\bar{j}} = iK_{,i\bar{j}}\mathrm{d}x - K_{,l}K_{,i\bar{j}}\mathrm{d}z^l + (K_{,i\bar{j}\bar{l}} + K_{,i}K_{,\bar{j}\bar{l}} + K_{,\bar{j}}K_{,i\bar{l}} + K_{,\bar{l}}K_{,i\bar{j}})\mathrm{d}\bar{z}^{\bar{l}} \qquad \Gamma_{,\bar{i}j} = \overline{\Gamma_{i\bar{j}}}.$$

It is convenient to introduce the following functions:

$$C^i{}_{jm} = \kappa^{i\bar{l}}K_{,\bar{l}jm} \qquad B^i{}_{jm} = C^i{}_{jm} + \delta^i{}_m K_{,j} + \delta^i{}_j K_{,m} \qquad A_{jm} = C^l{}_{jm}K_{,l} + 2K_{,j}K_{,m} - K_{,jm}$$

$$C^{\bar{i}}{}_{\bar{j}\bar{m}} = \overline{C^i{}_{jm}} \qquad B^{\bar{i}}{}_{\bar{j}\bar{m}} = \overline{B^i{}_{jm}} \qquad A_{\ \bar{j}\bar{m}} = \overline{A_{\ jm}}.$$

Then the connection 1-forms $\Gamma^\mu_{\ \nu}$ read

$$\Gamma^x{}_x = -\mathrm{d}K \qquad \Gamma^x{}_j = -K_{,j}\mathrm{d}x - iA_{jm}\mathrm{d}z^m \qquad \Gamma^i{}_x = -i\mathrm{d}z^i \qquad \Gamma^x{}_{\bar{j}} = \overline{\Gamma^x{}_j} \qquad \Gamma^{\bar{i}}{}_x = \overline{\Gamma^i{}_x}$$

$$\Gamma^i{}_j = -i\delta^i{}_j\mathrm{d}x - \delta^i{}_jK_{,\bar{l}}\mathrm{d}\bar{z}^{\bar{l}} + B^i{}_{jl}\mathrm{d}z^l \qquad \Gamma^i{}_{\bar{j}} = -K_{,\bar{j}}\mathrm{d}z^i \qquad \Gamma^{\bar{i}}{}_{\bar{j}} = \overline{\Gamma^i{}_j} \qquad \Gamma^{\bar{i}}{}_j = \overline{\Gamma^i{}_{\bar{j}}}.$$

The curvature 2-forms $\Omega^\mu_{\ \nu} = \frac{1}{2}R^\mu_{\ \nu\rho\sigma}\mathrm{d}y^\rho \wedge \mathrm{d}y^\sigma = \mathrm{d}\Gamma^\mu_{\ \nu} + \Gamma^\mu_{\ \rho} \wedge \Gamma^\rho_{\ \nu}$ read

$$\Omega^x{}_x = iK_{,l}\mathrm{d}x \wedge \mathrm{d}z^l - iK_{,\bar{l}}\mathrm{d}x \wedge \mathrm{d}\bar{z}^{\bar{l}}$$

$$\Omega^x{}_j = -K_{,j}K_{,l}\mathrm{d}x \wedge \mathrm{d}z^l + (K_{,j\bar{l}} + K_{,j}K_{,\bar{l}})\mathrm{d}x \wedge \mathrm{d}\bar{z}^{\bar{l}} + iA_{jl,\bar{m}}\mathrm{d}z^l \wedge \mathrm{d}\bar{z}^{\bar{m}}$$

$$\Omega^i{}_x = -\delta^i{}_j\mathrm{d}x \wedge \mathrm{d}z^j + i\delta^i{}_jK_{,l}\mathrm{d}z^j \wedge \mathrm{d}z^l + iK_{,\bar{l}}\mathrm{d}\bar{z}^{\bar{l}} \wedge \mathrm{d}z^i$$

$$\begin{aligned}\Omega^i{}_j = {} & -i\delta^i{}_lK_{,j}\mathrm{d}x \wedge \mathrm{d}z^l + (B^i{}_{mn}B^m_{jl} - B^i{}_{jn,l} + A_{jn}\delta^i{}_l)\mathrm{d}z^n \wedge \mathrm{d}z^l \\ & +(K_{,j}K_{,\bar{l}}\delta^i{}_n - \delta^i{}_jK_{,n\bar{l}} - B^i{}_{jn,\bar{l}})\mathrm{d}z^n \wedge \mathrm{d}\bar{z}^{\bar{l}}\end{aligned}$$

$$\Omega^i{}_{\bar{j}} = i\delta^i{}_lK_{,\bar{j}}\mathrm{d}x \wedge \mathrm{d}z^l + (K_{,\bar{j}l} + K_{,\bar{j}}K_{,l})\delta^i{}_n\mathrm{d}z^n \wedge \mathrm{d}z^l - \delta^i{}_nK_{,\bar{j}}K_{\bar{l}}\mathrm{d}z^n \wedge \mathrm{d}\bar{z}^{\bar{l}}$$

$$\Omega^x{}_{\bar{j}} = \overline{\Omega^x{}_j} \quad \Omega^{\bar{i}}{}_x = \overline{\Omega^i{}_x} \quad \Omega^{\bar{i}}{}_{\bar{j}} = \overline{\Omega^i{}_j} \quad \Omega^{\bar{i}}{}_j = \overline{\Omega^i{}_{\bar{j}}}.$$

The Ricci tensor $R_{\nu\sigma} = R^\mu_{\ \nu\mu\sigma}$ components read

$$R_{xx} = 2k \qquad R_{xj} = 2ikK_{,j} \qquad R_{ij} = -2kK_{,i}K_{,j} \qquad R_{\bar{i}j} = 2kK_{,\bar{i}}K_{,j} - 2K_{,\bar{i}j} - C^{\bar{m}}_{\ \bar{m}\bar{i},j}$$

$$R_{x\bar{j}} = \overline{R_{xj}} \qquad R_{\bar{i}\bar{j}} = \overline{R_{ij}}.$$

Now, the Einstein equations $Ric(g) = \lambda g$, which are nontrivial only for the components $R_{xx}$ and $R_{\bar{i}j}$ become

$$\lambda = 2k \qquad -(\kappa^{\bar{m}l}K_{,\bar{m}l\bar{i}})_{,j} = 2(k+1)K_{,\bar{i}j}.$$

Since the matrix $(\kappa^{\bar{m}l})$ is the inverse of $(K_{,\bar{i}j})$ then the left hand side of the second equations above is

$$-(\kappa^{\bar{m}l}K_{,\bar{m}l\bar{i}})_{,j} = -[\log(\det(K_{,m\bar{n}}))]_{,\bar{i}j},$$

see e.g. [7].

Thus we arrive to the following theorem.

**Theorem 2**

Any Sasakian manifold of dimension $(2k+1)$ can be locally represented by the Sasakian potential $K$ of theorem 1. In the region where the potential is well defined the manifold satisfies Einstein equations $Ric(g) = \lambda g$ if and only if the cosmological constant

$$\lambda = 2k$$

and the potential satisfies

$$-\log[(\det(K_{,m\bar{n}}))]_{,\bar{i}j} = 2(k+1)K_{,\bar{i}j}. \tag{19}$$

Surprisingly equation (19) is the same as the Einstein condition $Ric(h) = 2(k+1)h$ for the Kähler metric $h = 2K_{,\bar{i}j}\mathrm{d}\bar{z}^{\bar{i}}\mathrm{d}z^j$ in dimension $2k$. Thus we have the following corollary.

**Corollary**

Every Sasakian–Einstein manifold in dimension $(2k+1)$ is locally in one to one correspondence with a Kähler–Einstein manifold in dimension $2k$ whose cosmological constant $\lambda = 2(k+1)$. The correspondence is obtained by identifying the Kähler potential for the Kähler–Einstein manifold with the Sasakian potential for the Sasakian–Einstein manifold.

**Examples**

(1) Sasakian potential for the sphere $\boldsymbol{S}^{2k+1}$.

Consider a function

$$\mathcal{K} = \tfrac{1}{2}\log(z^1\bar{z}^1 + \ldots + z^{k+1}\bar{z}^{k+1})$$

defined on $\boldsymbol{C}^{k+1} - \{0\}$. Let

$$N = i(\mathcal{K}_{,j}\mathrm{d}z^j - \mathcal{K}_{,\bar{j}}\mathrm{d}\bar{z}^{\bar{j}}),$$

$$H = 2\mathcal{K}_{,i\bar{j}}\mathrm{d}z^i\mathrm{d}\bar{z}^{\bar{j}}$$

and

$$G = N^2 + H.$$

The tensor fields $N$ and $G$ restrict to a sphere

$$\boldsymbol{S}^{2k+1} = \{(z^1, \ldots, z^{k+1}) \in \boldsymbol{C}^{k+1} - \{0\} \mid z^1\bar{z}^1 + \ldots + z^{k+1}\bar{z}^{k+1} = 1\}.$$

Denote these restrictions by $\eta$ and $g$, respectively. Then the 1-form $\eta$ and the Riemannian metric $g$ define a Sasakian–Einstein structure on $\boldsymbol{S}^{2k+1}$. This structure coincides with the one defined in example 1 of section 1.

To see this, recall the Hopf fibration $\boldsymbol{U}(1) \to \boldsymbol{S}^{2k+1} \to \boldsymbol{CP}^k$ with the action of $\mathrm{e}^{\mathrm{i}\phi} \in \boldsymbol{U}(1)$ on $(z^1, \ldots, z^{k+1}) \in \boldsymbol{S}^{2k+1}$ defined by $\mathrm{e}^{\mathrm{i}\phi}(z^1, \ldots, z^{k+1}) = (\mathrm{e}^{\mathrm{i}\phi}z^1, \ldots, \mathrm{e}^{i\phi}z^{k+1})$. The canonical projection is given by $\boldsymbol{S}^{2k+1} \ni (z^1, \ldots, z^{k+1}) \to \mathrm{dir}(z^1, \ldots, z^{k+1}) \in \boldsymbol{CP}^k$. The sphere is covered by $k+1$ charts

$$U_j = V_j \times \boldsymbol{U}(1),$$

where

$$V_j = \{\mathrm{dir}(z^1, \ldots, z^{k+1}) \mid (z^1, \ldots, z^{k+1}) \in \boldsymbol{S}^{2k+1} \text{ and } z^j \neq 0\}.$$

The local coordinates on each $U_j$ are

$$\left(\xi^{ij} = \frac{z^i}{z^j},\ \phi_j = \frac{i}{2}\log\frac{z^j}{\bar{z}^j}\right), \qquad i = 1, \ldots, k+1,\ i \neq j.$$

Then on each chart $U_j$ the form $\eta_j = \eta|_{U_j}$ reads

$$\eta_j = \mathrm{d}\phi_j + \frac{i}{2}\frac{\sum_{i=1,i\neq j}^{k+1}(\bar{\xi}^{ij}\mathrm{d}\xi^{ij} - \xi^{ij}\mathrm{d}\bar{\xi}^{ij})}{1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2}.$$

The metric $g$ restricted to $U_j$ is

$$g_j = (\eta_j)^2 + \frac{(1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2)(\sum_{i=1,i\neq j}^{k+1}|\mathrm{d}\xi^{ij}|^2) - |\sum_{i=1,i\neq j}^{k+1}(\xi^{ij}\mathrm{d}\bar{\xi}^{ij})|^2}{(1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2)^2}.$$

Now, observe that on each $U_j$ the structure $(g_j, \eta_j)$ may be obtained by means of theorems 1 and 2 choosing a Sasakian potential

$$K^j = \frac{1}{2}\log\left(1 + \sum_{i=1,i\neq j}^{k+1}|\xi^{ij}|^2\right)$$

on the corresponding $V_j$. It is easy to check that $K^j$ satisfies equation (19) on $V_j$. Thus, theorem 2 assures that the Sasakian structure generated by $(g_j, \eta_j)$ is Sasakian–Einstein. Easy, but lengthy, calculation shows that the Weyl tensor of $g_j$ vanishes identically on $U_j$. This proves that $(U_j, g_j)$ is locally isometric to a standard Riemannian structure on $\boldsymbol{S}^{2k+1}$. Since $(g_j, \eta_j)$ originate from the global structure $(g, \eta)$ then this global Sasakian structure must coincide with the standard Sasakian structure of example 1. Note also that $h_j = g_j - (\eta_j)^2$ projects to $V_j$ and patched together defines the Fubini–Study metric on $\boldsymbol{C}P^k$. In this sense the standard Sasakian structure on $\boldsymbol{S}^{2k+1}$ described in example 1 is the analogue of the Fubini–Study Kähler structure on $\boldsymbol{C}P^k$.

(2) Sasakian–Einstein structure on $\boldsymbol{C}^q \times \boldsymbol{C}^n \times \boldsymbol{R}$.

Consider a function

$$K = \frac{1}{q+n+1}\left[\sum_{i=1}^{q}\log(1 + |v^i|^2)\right] + \frac{n+1}{2(q+n+1)}\log\left(1 + \sum_{I=1}^{n}|w^I|^2\right)$$

defined on $\boldsymbol{C}^q \times \boldsymbol{C}^n$, with coordinates $(z^\mu) = (v^i, w^I)$. It is easy to check that

$$[\log\det(K_{,\mu\bar{\nu}})]_{,\rho\bar{\sigma}} = -2(q+n+1)K_{,\rho\bar{\sigma}}.$$

Thus, via theorems 1 and 2, such $K$ generates a Sasakian–Einstein structure on $\boldsymbol{C}^q \times \boldsymbol{C}^n \times \boldsymbol{R}^1$.

(3) Locally Sasakian–Einstein structures in dimension 5.

If $k = 2$ then, modulo the gauge (14), equation (19) may be integrated to the form

$$K_{,1\bar{1}}K_{,2\bar{2}} - K_{,1\bar{2}}K_{,2\bar{1}} = \mathrm{e}^{-6K}.$$

This is a well known equation describing the gravitational instantons in four dimensions [2, 5, 9, 10]. Examples of the Kähler–Einstein metrics generated by its solutions can be found e.g. in [5, 8, 11]. Via theorems 1 and 2, each of these Kähler–Einstein structures defines a nontrivial Sasakian–Einstein manifold in dimension 5.

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