Seminar Tromsø, AiT, 14.02.2023

I $\mathbb{R}^{15}$ $\quad x^i, y^i \quad i=1,2,3,4$

$x^{ij} \quad 1\le i<j\le 4$

z

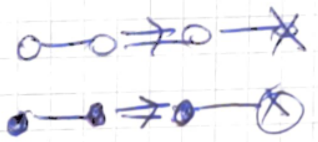
Cartan PhD thesis 1893 (German version)

$$\begin{cases} \lambda^{ij} = dx^{ij} + x^i dx^j + y^k dy^l \\ \lambda = dz + x^1 dy^1 + x^2 dy^2 + x^3 dy^3 + x^4 dy^4 \end{cases}$$

$$X \text{ s.t. } \begin{cases} (\mathcal{L}_X \lambda^{ij}) \wedge \lambda \wedge \lambda^{12} \wedge \lambda^{13} \wedge \lambda^{14} \wedge \lambda^{23} \wedge \lambda^{24} \wedge \lambda^{34} = 0 \\ (\mathcal{L}_X \lambda) \wedge \lambda \wedge \lambda^{12} \cdots \wedge \lambda^{34} = 0 \end{cases}$$

Solution space is 52 dimensional Lie algebra

$\simeq$ split real form of f_4



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II Major Lemma

- $(n_{00}, [\cdot,\cdot]_0)$ – Lie algebra (real, finite dim)
- (ρ, S) – its real finite dim representation $\dim S = s$
- R – real vector space $\dim R = r$
- $\tau : n_0 \xrightarrow{\text{linear}} \mathrm{End}(R)$
- $\omega \in \mathrm{Hom}(\Lambda^2 S, R)$

$A \in n_{00}$

$X, Y \in S$

Assume that (ρ, τ, ω) satisfy

$$\boxed{\omega(\rho(A)X, Y) + \omega(X, \rho(A)Y) = \tau(A)\,\omega(X,Y)}$$

<u>then</u>

① $$\tau([A,B])\,\omega(X,Y) = [\tau(A),\tau(B)]\,\omega(X,Y)$$
$\forall A,B \in n_{00}$
$\forall X,Y \in S$

"almost" a representation of n_{00}

② If τ is a repr. of n_{00} in R then.

Ⓐ $\mathfrak{g}_0 = R \oplus S \oplus n_{00}$ is a Lie algebra with a bracket:

- $[A,B] = [A,B]_0$, $A,B \in n_{00}$
- $[A,X] = \rho(A)X$ $\quad A \in n_{00}, X \in S$
- $[A,X] = \tau(A)X$ $\quad A \in n_{00}, X \in R$
- $[S,R] = [R,R] = 0$
- $[X,Y] = \omega(X,Y)$ $\quad X,Y \in S$

Ⓑ $n_- = R \oplus S$ is a 2-step nilpotent Lie algebra with
$n_{-2} = R$, $n_{-1} = S$.

Ⓒ $$n_{00} \subset n_0 = \{ D : n_- \xrightarrow{\text{linear}} n_- \;\; D[X,Y] = [DX,Y] + [X,DY],\; Dn_{-1} \subset n_{-1},\; Dn_{-2} \subset n_{-2} \}$$
" 0-Tanaka prolongation of n_-

Introduce $\sigma = \tau \oplus \rho$

Action of elements from n_{00} in n_-:

$$A[X,Y] := [A,[X,Y]] = \sigma(A)[X,Y] = \sigma(A)\,\omega(X,Y) = \tau(A)\,\omega(X,Y)$$

Remark note that if you start with proper $(\mathfrak{S}, S)$, $(\mathfrak{R}, R)$ you get $\mathfrak{n}$ whose Tanaka prolongation be graded real simple Lie algebra.

~~III note also that if~~

III Construction

1) first take a basis $\{f_\mu\}_{\mu=1}^{s}$ in S
$\{e_i\}_{i=1}^{r}$ in R

$\{f^\mu\}_{\mu=1}^{s}$ in S^* s.t. (f_μ, f^ν) are dual:

$$f_\mu \lrcorner f^\nu = \delta_\mu{}^\nu,$$

$$\Rightarrow \quad \omega(f_\mu, f_\nu) =: \omega^i{}_{\mu\nu}$$

ASSUME that $\omega: \Lambda^2 S \to R$ is ONTO

$\mathrm{Im}(\omega) = R$

2) Take $S = \mathbb{R}^s$ with coordinates $\{x^\mu\}$
$R = \mathbb{R}^r$ with coordinates $\{u^i\}$

Consider $M = \mathbb{R}^r \times \mathbb{R}^s$ and

$$\lambda^i = du^i + \omega^i{}_{\mu\nu} x^\mu dx^\nu.$$

Now you have $(M, \mathrm{Span}(\lambda^1, \dots, \lambda^r))$

$$\mathcal{D} = \{ X \in TM : \ X \lrcorner \lambda^1 = \dots = X \lrcorner \lambda^r = 0 \}$$

is a rank s distribution on M

s.t.

$$[\mathcal{D}, \mathcal{D}] = TM.$$

Its symmetry algebra is as large as Tanaka prolongation of $\mathfrak{n} = R \oplus S$

IV Examples

1 ~~Cartan Γ_I~~: Bicquard's (para) quaternionic CR's

$$\mathfrak{n}_{00} = \begin{cases} \mathfrak{so}(2,1) & \varepsilon = -1 \\ \mathfrak{so}(0,3) \oplus \mathbb{R} & \varepsilon = 1 \end{cases}$$

ρ - spin representation of $\mathfrak{n}_{00}$ in $S = \mathbb{R}^4$

$$\rho(A_1) = -\frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & \varepsilon \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) = -\frac{1}{2}\sigma_3, \quad \rho(A_2) = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & -\varepsilon & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) = \frac{1}{2}\sigma_2, \quad \rho(A_3) = -\frac{1}{2}\varepsilon\left(\begin{array}{cc|cc} 0 & -1 & & \\ 1 & 0 & & \\ \hline & & 0 & -1 \\ & & 1 & 0 \end{array}\right) = -\frac{1}{2}\varepsilon\sigma_1$$

$$\sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu = 2 g_{\mu\nu} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} -\varepsilon & & \\ & -\varepsilon & \\ & & -1 \end{pmatrix}$$

$$\rho(A_4) = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

τ - vectorial (standard) representation of $\mathfrak{n}_{00}$ in $R = \mathbb{R}^3$

$$\tau(A_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \varepsilon \\ 0 & -1 & 0 \end{pmatrix} \quad \tau(A_2) = \begin{pmatrix} 0 & 0 & -\varepsilon \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \tau(A_3) = \begin{pmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tau(A_4) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$\Rightarrow \omega^1_{\mu\nu} = \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline -1 & & & \\ & 1 & & \end{array}\right) \quad \omega^2_{\mu\nu} = \left(\begin{array}{cc|cc} & & & -1 \\ & & -1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right) \quad \omega^3_{\mu\nu} = \left(\begin{array}{cc|cc} & -\varepsilon & & \\ \varepsilon & & & \\ \hline & & & -1 \\ & & 1 & \end{array}\right)$$

$$\begin{cases} \lambda^1 = du^1 + x^1 dx^3 - x^2 dx^4 \\ \lambda^2 = du^2 - x^1 dx^4 - x^2 dx^3 \\ \lambda^3 = du^3 - \varepsilon x^1 dx^2 - x^3 dx^4 \end{cases}$$

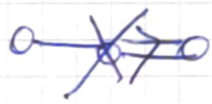
defines $\mathcal{D}$ of rank 4 in dim 7.

On $\mathcal{D}$ there is a natural (para) quaternionic structure.

$\Rightarrow$ flat model for Biquard's quaternionic (para)-CR's with symmetry

$Sp(1,2)$

or $Sp(0,3) \simeq Sp(3,\mathbb{R}) \longleftarrow$ quaternionic



$$\cancel{-8\lambda\left(\theta^{1,19} + \theta^{2,20} + \theta^{3,21} + 2\varepsilon\theta^{4,18} - 2\varepsilon\theta^{5,17} + 2\theta^{6,15} - 2\theta^{7,16} - \theta^2\right)} \quad \cancel{-\theta^2}$$

② Cartan's f_I.

$n_{00} = so(4,3) \oplus \mathbb{R}$.

We need an 8-dim real repr of $so(4,3)$
7-dim real repr of $so(4,3)$.

$\Rightarrow$ (ρ, S) - spin representation in $S = \mathbb{R}^8$
(τ, R) - vectorial representation in $R = \mathbb{R}^7$

$\sigma_z\sigma_x = -\sigma_x\sigma_z = -\varepsilon$

$\varepsilon\sigma_z = -\sigma_z\varepsilon = \sigma_x$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\varepsilon = -i\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\sigma_x^2 = \sigma_z^2 = -\varepsilon^2 = I$

$\sigma_x\varepsilon = -\varepsilon\sigma_x = \sigma_z$

Clifford algebra generators 8-dim Pauli matrices:

$$\sigma_1 = \sigma_x \otimes \sigma_x \otimes \sigma_x$$
$$\sigma_2 = \sigma_x \otimes \sigma_x \otimes \varepsilon$$
$$\sigma_3 = \sigma_x \otimes \sigma_x \otimes \sigma_z$$
$$\sigma_4 = \sigma_x \otimes \varepsilon \otimes I$$
$$\sigma_5 = \sigma_x \otimes \sigma_z \otimes I$$
$$\sigma_6 = \varepsilon \otimes I \otimes I$$
$$\sigma_7 = \sigma_z \otimes I \otimes I$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2g_{ij}\, 1_{8\times 8}$$

$$g_{ij} = \mathrm{diag}(1,-1,1,-1,1,-1,1)$$

$\underline{so(4,3)}$ generated by

$$\boxed{\rho(A_I) = \frac{1}{2}\sigma_i \cdot \sigma_j} \qquad 1 \leq i < j \leq 7$$

$$I = 1 + i + \frac{1}{2}(j-3)j \qquad I(1,2) = 1$$

$I = 11 = I(1,6)$

$i \backslash j$	1	2	3	4	5	6	7
1	X	1	2	4	7	11	16
2		X	3	5	8	12	17
3			X	6	9	13	18
4				X	10	14	19
5					X	15	20
6						X	21
7							X

$\rho(A_{11}) = \frac{1}{2}\sigma_1\sigma_6 =$

$= \frac{1}{2}(\sigma_x \otimes \sigma_x \otimes \sigma_x)(\varepsilon \otimes I \otimes I) =$

$= \frac{1}{2}\sigma_z \otimes \sigma_x \otimes \sigma_x$

etc.

Vectorial representation

$\overset{a}{\rho} = \rho \wedge \rho$ in $S \wedge S = \underset{28}{\Lambda^2 S} = \Lambda_{21} \oplus \Lambda_7$

Apply Casimir $10 K^{IJ} \overset{a}{\rho}(A_I) \overset{a}{\rho}(A_J)$

and look at eigenspaces

$\Rightarrow \quad \tau(A_I)$ in $R = \mathbb{R}^7$.

$\Rightarrow$ Solve magic equation for $\omega^i{}_{\mu\nu}$

$\Rightarrow \quad \Omega^i = du^i + \omega^i{}_{\mu\nu} dx^\mu dx^\nu$

s.t.

$$\lambda^1 = du^1 + x^1 dx^2 - x^7 dx^8$$

$$\lambda^2 = du^2 + x^2 dx^4 - x^6 dx^8$$

$$\lambda^3 = du^3 + x^1 dx^4 - x^5 dx^8$$

$$\lambda^4 = du^4 + \tfrac{1}{2}\left(x^1 dx^6 - x^2 dx^5 - x^3 dx^8 + x^4 dx^7\right)$$

$$\lambda^5 = du^5 + x^2 dx^3 - x^6 dx^7$$

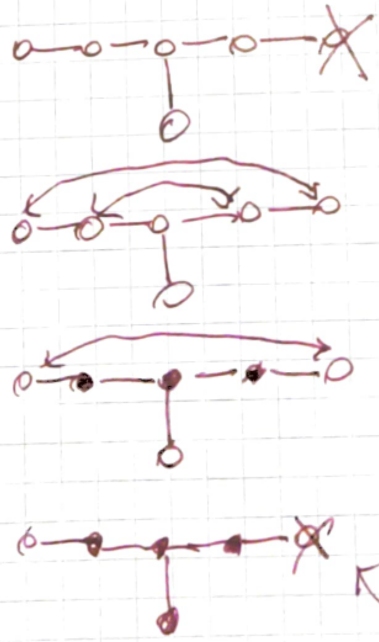
$$\lambda^6 = du^6 + x^1 dx^3 - x^5 dx^7$$

$$\lambda^7 = du^7 + x^3 dx^4 - x^5 dx^6 .$$

$$\mathfrak{f}_2 = \underset{7}{\mathfrak{g}_2} \oplus \underset{8}{\mathfrak{g}_1} \oplus \underset{22}{\mathfrak{g}_0} \oplus \underset{8}{\mathfrak{g}_1} \oplus \underset{7}{\mathfrak{g}_2}$$

vectorial. spin

(3) $\underline{E_6}$ Apart from compact, 4 real forms.



Cartan 1893

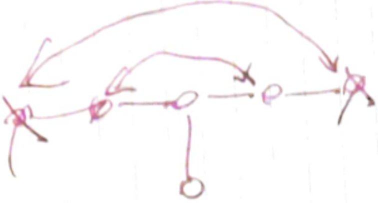
16-dimensional realization

But

$$\underbrace{\mathfrak{g}_{-1}}_{16} \oplus \underset{46}{\mathfrak{g}_0} \oplus \underset{16}{\mathfrak{g}_1}$$

$\mathbb{R}\,\mathrm{Spin}(5,5)$ structure

Also in $\underline{\text{dim } 16}$



realization in dim 24

$$e_{I,IV} = \underbrace{\underset{8}{\mathfrak{g}_{-2}} \oplus \underset{16}{\mathfrak{g}_{-1}}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

integrable J s.t. $J^2 = -id$

$\Downarrow$

CR-structure of CR dimension

the other two are more interesting:

$K^2 = id.$

$$e_6 = \overset{8}{\mathfrak{g}_{-2}} \oplus \overset{16}{\mathfrak{g}_{-1}} \oplus \overset{K}{\mathfrak{g}_0} \oplus \overset{16}{\mathfrak{g}_1} \oplus \overset{8}{\mathfrak{g}_2}$$

$\mathfrak{g}_0 = \mathbb{R} \oplus co(4,4)$

$\mathbb{R} \oplus co(0,8)$.

Clifford $(4,4)$:

$$
\begin{cases}
\gamma_1 = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \\
\gamma_2 = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \varepsilon \\
\gamma_3 = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_z \\
\gamma_4 = \sigma_x \otimes \sigma_x \otimes \varepsilon \otimes I \\
\gamma_5 = \sigma_x \otimes \sigma_x \otimes \sigma_z \otimes I \\
\gamma_6 = \sigma_x \otimes \varepsilon \otimes I \otimes I \\
\gamma_7 = \sigma_x \otimes \sigma_z \otimes I \otimes I \\
\gamma_8 = \varepsilon \otimes I \otimes I \otimes I
\end{cases}
$$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}\, I \otimes I \otimes I \otimes I$$

$$g_{\mu\nu} = \mathrm{diag}(1,-1,1,-1,1,-1,1,-1)$$

$$\Rightarrow \quad \rho(A_I) = \frac{1}{2}\gamma_\mu \gamma_\nu \qquad 1 \le \mu < \nu \le 8$$

$$I = 1 + i + \frac{1}{2}(j-3)j$$

$i \backslash j$	1	2	3	4	5	6	7	8
1		1	2	4	7	11	16	22
2			3	5	8	12	17	23
3				6	9	13	18	24
4					10	14	19	25
5						15	20	26
6							21	27
7								28

How to make Clifford $(8,0)$?

$\cancel{\gamma_1}$

$\cancel{\gamma_2 \to i\gamma_2}$

$$\begin{array}{lll} \gamma_1 \to \gamma_1 & & \bar{\gamma}_1 \\ \gamma_2 \to \gamma_2 = i\gamma_2 & & \bar{\gamma}_2 \\ \gamma_3 \to \gamma_3 & & \bar{\gamma}_3 \\ \gamma_4 \to \gamma_4 = i\gamma_4 & & \bar{\gamma}_4 \\ \gamma_5 \to \gamma_5 & & \bar{\gamma}_5 \\ \gamma_6 \to \gamma_6 = i\gamma_6 & & \bar{\gamma}_6 \\ \gamma_7 \to \gamma_7 & & \bar{\gamma}_7 \\ \gamma_8 \to \gamma_8 = i\gamma_8 & & \bar{\gamma}_8 \end{array}$$

$B \in M_{16\times 16}(\mathbb{R})$ s.t

$$B\gamma_\mu = \bar{\gamma}_\mu B$$

$$B = \left(\begin{array}{c|c} \begin{array}{c|c} 0 & \begin{smallmatrix} & 1 & \\ -1 & & -1 \\ & 1 & \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & & 1 \\ & -1 & \end{smallmatrix} & 0 \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} & \begin{smallmatrix} 1 & -1 \\ & -1 \quad 1 \end{smallmatrix} \\ \hline \begin{smallmatrix} & 1 & \\ -1 & & -1 \\ & 1 & \end{smallmatrix} & \end{array} \end{array}\right) \qquad B^2 = id.$$

$\cancel{\gamma_\mu 2 = \mathrm{Id} + I_2}$

$$\tilde{\gamma}_\mu = (I + iB)\,\gamma_\mu\,(I + iB)^{-1}$$

$$\boxed{B = \sigma_z \otimes \varepsilon \otimes \sigma_z \otimes \varepsilon}$$

$\cancel{\otimes \varepsilon \otimes I \otimes \sigma_z \otimes \varepsilon \otimes \varepsilon}$

$\gamma_1 = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$

$\gamma_2 = -\varepsilon \otimes \sigma_z \otimes \varepsilon \otimes I$

$\gamma_3 = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_z$

$\gamma_4 = \varepsilon \otimes \sigma_z \otimes I \otimes \varepsilon$

$\gamma_5 = \sigma_x \otimes \sigma_x \otimes \sigma_z \otimes I$

$\gamma_6 = \varepsilon \otimes I \otimes \sigma_z \otimes \varepsilon$

$\gamma_7 = \sigma_x \otimes \sigma_z \otimes I \otimes I$

$\gamma_8 = \sigma_x \otimes \varepsilon \otimes \sigma_z \otimes \varepsilon$

$$\gamma_\mu \cdot \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} I_8$$

$$g_{\mu\nu} = \mathrm{diag}(1,1,1,1,1,1,1,1)$$

$$\varsigma(A_I) = \frac{1}{2} \gamma_\mu \gamma_\nu \qquad 1 \le \mu < \nu \le 8$$

$$I = 1 + i + \frac{1}{2}(j-3)j$$

(ς, S) — 16 dimensional Dirac spinors representation of $so(9)$.

It is reducible!

$$\gamma_9 = \gamma_1 \cdot \gamma_2 \cdots \gamma_8 = \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix}$$

$$S = S_+ \oplus S_- \qquad \gamma_9 S_\pm = \pm S_\pm$$

(ς_+, S_+) and (ς_-, S_-)

these two representations in the spaces of Weyl spinors are nonequivalent.

Now I take $\rho_+ \otimes \rho_- = \tau \oplus \rho_{56}$
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τ - gives vectorial representation.

$$
\begin{array}{cccccc}
e_{\mathrm{I}} = & n_{-2} & \oplus\ n_{-1} & \oplus\ n_0 & \oplus\ n_1 & \oplus\ n_2 \\
\mathrm{IV} & \uparrow & \| & & & \\
 & & n_{-1}^+ \oplus n_{-1}^- & \underbrace{\mathbb{R} \oplus co(g)}_{9} & 16 & \| \\
 & 8 & 8 \quad 8 & & \| & 8 \\
 & & & & 8 &
\end{array}
$$

n_0 acts as derivation in

$$n_- = \underbrace{n_{-2}}_{\substack{8 \\ \text{vector}}} \oplus \underbrace{n_{-1}^+}_{\substack{8 \\ \text{spin}_+}} \oplus \underbrace{n_{-1}^-}_{\substack{8 \\ \text{spin}_-}}$$

$\Rightarrow \quad \omega^i{}_{\mu\nu}$

f_{I}

$$\lambda^1 = du^1 + x^1 dx^{10} + x^2 dx^9 + x^7 dx^{16} - x^8 dx^{15}$$
$$\lambda^2 = du^2 + x^3 dx^{12} + x^4 dx^{10} + x^6 dx^{16} - x^8 dx^{14}$$
$$\lambda^3 = du^3 + x^1 dx^{12} + x^4 dx^9 + x^5 dx^{16} - x^8 dx^{13}$$
$$\lambda^4 = du^4 + x^5 dx^{10} + x^6 dx^9 + x^7 dx^{12} - x^8 dx^{11}$$
$$\lambda^5 = du^5 + x^2 dx^{11} + x^3 dx^{10} + x^6 dx^{15} - x^7 dx^{14}$$
$$\lambda^6 = du^6 + x^1 dx^{11} + x^3 dx^9 + x^5 dx^{15} - x^7 dx^{13}$$
$$\lambda^7 = du^7 + x^3 dx^{12} + x^4 dx^{11} + x^5 dx^{14} - x^6 dx^{13}$$
$$\lambda^8 = du^8 + x^1 dx^{14} + x^2 dx^{13} + x^3 dx^{16} - x^4 dx^{15}$$

fin

$$\lambda^1 = x^1 dx^9 + x^2 dx^{10} + x^3 dx^{11} + x^4 dx^{12} + x^5 dx^{13} - x^6 dx^{14} - x^7 dx^{15} - x^8 dx^{16} + du^1$$

$$\lambda^2 = -x^1 dx^{10} + x^2 dx^9 + x^3 dx^{12} - x^4 dx^{11} - x^5 dx^{14} + x^6 dx^{13} + x^7 dx^{16} - x^8 dx^{15} + du^2$$

$$\lambda^3 = -x^1 dx^{11} - x^2 dx^{12} + x^3 dx^9 + x^4 dx^{10} + x^5 dx^{15} + x^6 dx^{16} - x^7 dx^{13} - x^8 dx^{14} + du^3$$

$$\lambda^4 = -x^1 dx^{12} + x^2 dx^{11} - x^3 dx^{10} + x^4 dx^9 + x^5 dx^{16} - x^6 dx^{15} + x^7 dx^{14} - x^8 dx^{13} + du^4$$

$$\lambda^5 = x^1 dx^{13} + x^2 dx^{14} - x^3 dx^{15} - x^4 dx^{16} + x^5 dx^9 + x^6 dx^{10} - x^7 dx^{11} - x^8 dx^{12} + du^5$$

$$\lambda^6 = x^1 dx^{14} - x^2 dx^{13} - x^3 dx^{16} + x^4 dx^{15} - x^5 dx^{10} + x^6 dx^9 + x^7 dx^{12} - x^8 dx^{11} + du^6$$

$$\lambda^7 = x^1 dx^{15} - x^2 dx^{16} + x^3 dx^{13} - x^4 dx^{14} + x^5 dx^{11} - x^6 dx^{12} + x^7 dx^9 - x^8 dx^{10} + du^4$$

$$\lambda^8 = -x^1 dx^{16} - x^2 dx^{15} - x^{13} dx^{14} - x^4 dx^{13} - x^5 dx^{12} - x^6 dx^{11} - x^7 dx^{10} - x^8 dx^9$$