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On certain classes of $\mathbf{Sp}(4, \mathbb{R})$ symmetric G_2 structures

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Abstract
We find two different families of $\mathbf{Sp}(4, \mathbb{R})$ symmetric G_2 structures in seven dimensions. These are G_2 structures with G_2 being the split real form of the simple exceptional complex Lie group G_2. The first family has $\tau_2 \equiv 0$, while the second family has $\tau_1 \equiv \tau_2 \equiv 0$, where τ_1, τ_2 are the celebrated G_2-invariant parts of the intrinsic torsion of the G_2 structure. The families are different in the sense that the first one lives on a homogeneous space $\mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$, and the second one lives on a homogeneous space $\mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$. Here $\mathbf{SL}(2, \mathbb{R})_l$ is an $\mathbf{SL}(2, \mathbb{R})$ corresponding to the $\mathfrak{sl}(2, \mathbb{R})$ related to the long roots in the root diagram of $\mathfrak{sp}(4, \mathbb{R})$, and $\mathbf{SL}(2, \mathbb{R})_s$ is an $\mathbf{SL}(2, \mathbb{R})$ corresponding to the $\mathfrak{sl}(2, \mathbb{R})$ related to the short roots in the root diagram of $\mathfrak{sp}(4, \mathbb{R})$.

Keywords Homogeneous G2 structures · Skew symmetric torsion · Split signature metric

1 Introduction: a question of Maciej Dunajski

Recently, together with Hill [5], we uncovered an $\mathbf{Sp}(4, \mathbb{R})$ symmetry of the nonholonomic kinematics of a car. I talked about this at the Abel Symposium in Ålesund, Norway, in June 2019. After my talk Maciej Dunajski, intrigued by the root diagram of $\mathfrak{sp}(4, \mathbb{R})$ which appeared in the talk, asked me if using it I can see a G_2 structure on a 7-dimensional homogeneous space $\mathrm{M} = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})$.

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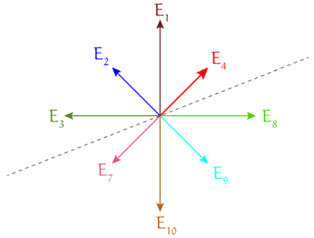

Root diagram for $\mathfrak{sp}(4,\mathbb{R})$

My immediate answer was: 'I can think about it, but I have to know which of the $\mathbf{SL}(2,\mathbb{R})$ subgroups of $\mathbf{Sp}(4,\mathbb{R})$ I shall use to built M.' The reason for the 'but' word in my answer was that there are at least two $\mathbf{SL}(2,\mathbb{R})$ subgroups of $\mathbf{Sp}(4,\mathbb{R})$, which lie quite differently in there. One can see them in the root diagram above: the first $\mathbf{SL}(2,\mathbb{R})$ corresponds to the *long* roots, as, for example, E_1 and E_{10}, whereas the second one corresponds to the *short* roots, as, for example, E_2 and E_9. Since Maciej never told me which $\mathbf{SL}(2,\mathbb{R})$ he wants, I decided to consider both of them and to determine what kind of G_2 structures one can associate with the respective choice of a subgroup.

I emphasize that in the below considerations I will use the *split real form* of the simple exceptional Lie group G_2. Therefore, the corresponding G_2 structure metrics will *not* be Riemannian.[1] They will have signature (3, 4).

2 The Lie algebra $\mathfrak{sp}(4,\mathbb{R})$

The Lie algebra $\mathfrak{sp}(4,\mathbb{R})$ is given by the 4×4 matrices

$$E=(E^\alpha{}_\beta)=\begin{pmatrix} a_5 & a_7 & a_9 & 2a_{10}\\ -a_4 & a_6 & a_8 & a_9\\ a_2 & a_3 & -a_6 & -a_7\\ -2a_1 & a_2 & a_4 & -a_5\end{pmatrix},$$

where the coefficients a_I, $I=1,2,\ldots 10$, are real constants. The Lie bracket in $\mathfrak{sp}(4,\mathbb{R})$ is the usual commutator $[E,E']=E\cdot E'-E'\cdot E$ of two matrices E and E'. We start with the following basis (E_I),

$$E_I=\frac{\partial E}{\partial a_I},\qquad I=1,2,\ldots 10,$$

in $\mathfrak{sp}(4,\mathbb{R})$.

In this basis, modulo the antisymmetry, we have the following nonvanishing commutators: $[E_1,E_5]=2E_1$, $[E_1,E_7]=-2E_2$, $[E_1,E_9]=-2E_4$, $[E_1,E_{10}]=4E_5$, $[E_2,E_4]=E_1$, $[E_2,E_5]=E_2$, $[E_2,E_6]=E_2$, $[E_2,E_7]=2E_3$, $[E_2,E_8]=E_4$, $[E_2,E_9]=-E_5-E_6$, $[E_2,E_{10}]=-2E_7$, $[E_3,E_4]=-E_2$, $[E_3,E_6]=2E_3$, $[E_3,E_8]=-E_6$, $[E_3,E_9]=-E_7$,

[1] For some of the Riemannian counterparts of the structures considered here, see for example, [6].

$[E_4,E_5]=E_4,\ \ [E_4,E_6]=-E_4,\ \ [E_4,E_7]=E_5-E_6,\ \ [E_4,E_9]=-2E_8,\ \ [E_4,E_{10}]=-2E_9,$
$[E_5,E_7]=E_7,[E_5,E_9]=E_9,[E_5,E_{10}]=2E_{10},[E_6,E_7]=-E_7,[E_6,E_8]=2E_8,[E_6,E_9]=E_9,$
$[E_7,E_8]=E_9,[E_7,E_9]=E_{10}.$

We see that there are at least *two* $\mathfrak{sl}(2,\mathbb{R})$ Lie algebras here. The first one is

$$\mathfrak{sl}(2,\mathbb{R})_l=\mathrm{Span}_{\mathbb{R}}(E_1,E_5,E_{10}),$$

and the second is

$$\mathfrak{sl}(2,\mathbb{R})_s=\mathrm{Span}_{\mathbb{R}}(E_2,E_5+E_6,E_9).$$

The reason for distinguishing these two is as follows:

The eight 1-dimensional vector subspaces $\mathfrak{g}_I=\mathrm{Span}(E_I)$, $I=1,2,3,4,7,8,9,10$, of $\mathfrak{sp}(4,\mathbb{R})$ are the *root spaces* of this Lie algebra. They correspond to the Cartan subalgebra of $\mathfrak{sp}(4,\mathbb{R})$ given by $\mathfrak{h}=\mathrm{Span}(E_5,E_6)$. It follows that the pairs (E_I,E_J) of the root vectors, such that $I+J=11$, $I,J\neq 5,6$, correspond to the *opposite roots* of $\mathfrak{sl}(2,\mathbb{R})$. Knowing the Killing form for $\mathfrak{sl}(2,\mathbb{R})$, which in the basis (E_I), and its dual basis (E^I), $E_I \lrcorner E^J=\delta^J{}_I$, is

$$\begin{aligned}K=\tfrac{1}{12}K_{IJ}E^I\odot E^J&=-4E^1\odot E^{10}+2E^2\odot E^9+E^3\odot E^8\\&-2E^4\odot E^7+E^5\odot E^5+E^6\odot E^6,\end{aligned}$$

one can see that the roots corresponding to the root vectors (E_1,E_{10}) and (E_3,E_8) are *long*, and the roots corresponding to the root vectors (E_2,E_9) and (E_4,E_7) are *short*; compare the Euclidian lengths of these roots as drawn on the G_2 root diagram presented at the beginning of this article.[2] Thus, the Lie algebra $\mathfrak{sl}(2,\mathbb{R})_l$ containing root vectors (E_1,E_{10}) corresponding to the *long* roots lies quite different in $\mathfrak{sp}(4,\mathbb{R})$ than the Lie algebra $\mathfrak{sl}(2,\mathbb{R})_s$ containing the root vectors (E_2,E_9) corresponding to the *short* roots.

3 G_2 structures on $\mathrm{Sp}(4,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})_l$

3.1 Compatible pairs (g,ϕ) on M_l

To consider the homogeneous space $M_l=\mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_l$, it is convenient to change the basis (E_I) in $\mathfrak{sp}(4,\mathbb{R})$ to a new one, (e_I), in which the last three vectors span $\mathfrak{sl}(2,\mathbb{R})_l$. Thus, we take:

$$e_1=E_2,\ e_2=E_3,\ e_3=E_4,\ e_4=E_6,\ e_5=E_7,\ e_6=E_8,\ e_7=E_9,\ e_8=E_1,\ e_9=E_5,\ e_{10}=E_{10}.$$

If now, one considers (e_I) as the basis of the Lie algebra of left invariant vector fields on the Lie group $\mathbf{Sp}(4,\mathbb{R})$ then the dual basis (e^I), $e_I \lrcorner e^J=\delta^J{}_I$, of the left invariant forms on $\mathbf{Sp}(4,\mathbb{R})$ satisfies:

[2] We hope that the reader noticed that we use the same symbol E_I for 'root vectors' spanning 1-dimensional 'root spaces' of $\mathfrak{g}_2$, as well as for the 'roots' E_I of $\mathfrak{g}_2$ depicted on the root diagram.

$$
\begin{aligned}
de^1 &= -e^1 \wedge (e^4 + e^9) + e^2 \wedge e^3 - 2e^5 \wedge e^8 \\
de^2 &= -2e^1 \wedge e^5 - 2e^2 \wedge e^4 \\
de^3 &= -e^1 \wedge e^6 + e^3 \wedge (e^4 - e^9) - 2e^7 \wedge e^8 \\
de^4 &= e^1 \wedge e^7 + e^2 \wedge e^6 + e^3 \wedge e^5 \\
de^5 &= 2e^1 \wedge e^{10} + e^2 \wedge e^7 + e^5 \wedge (e^9 - e^4) \\
de^6 &= 2e^3 \wedge e^7 - 2e^4 \wedge e^6 \\
de^7 &= 2e^3 \wedge e^{10} - e^5 \wedge e^6 + e^7 \wedge (e^4 + e^9) \\
de^8 &= -e^1 \wedge e^3 - 2e^8 \wedge e^9 \\
de^9 &= e^1 \wedge e^7 - e^3 \wedge e^5 - 4e^8 \wedge e^{10} \\
de^{10} &= -e^5 \wedge e^7 - 2e^9 \wedge e^{10}.
\end{aligned}
\tag{3.1}
$$

Here we used the usual formula relating the structure constants $c^I{}_{JK}$, from $[e_J, e_K] = c^I{}_{JK} e_I$, to the differentials of the Maurer–Cartan forms (e^I), $de^I = -\frac{1}{2} c^I{}_{JK} e^J \wedge e^K$.

In this basis, the Killing form on $\mathbf{Sp}(4, \mathbb{R})$ is

$$
K = \tfrac{1}{12} c^I{}_{JK} c^K{}_{LI} e^J \odot e^L = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7 + (e^9)^2 - 4e^8 \odot e^{10}.
$$

Here, we have used the notation $e^I \odot e^J = \frac{1}{2}(e^I \otimes e^J + e^J \otimes e^I)$, $(e^I)^2 = e^I \odot e^I$.

One can now use equations (3.1) to see that the homogeneous space $M_l = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$ is the leaf space of a certain integrable rank 3 distribution D_l on $\mathbf{Sp}(4, \mathbb{R})$, establishing explicitly that $\mathbf{Sp}(4, \mathbb{R})$ has, in particular, the structure of the principal $\mathbf{SL}(2, \mathbb{R})$ fiber bundle $\mathbf{SL}(2, \mathbb{R})_l \to \mathbf{Sp}(4, \mathbb{R}) \to M_l = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$.

Indeed, the 3-dimensional distribution D_l, generated by the vector fields X on $\mathbf{Sp}(4, \mathbb{R})$ annihilating the span of the 1-forms $(e^1, e^2, \dots, e^7)$, is integrable, $de^\mu \wedge e^1 \wedge e^2 \cdots \wedge e^7 \equiv 0$, $\mu = 1, 2 \dots, 7$, so that we have a well-defined 7-dimensional leaf space M_l of the corresponding foliation. Moreover, the Maurer–Cartan equations (3.1), restricted to a leaf defined by $(e^1, e^2, \dots, e^7) \equiv 0$, reduce to $de^8 = -2e^8 \wedge e^9$, $de^9 = -4e^8 \wedge e^{10}$, $de^{10} = -2e^9 \wedge e^{10}$, showing that each leaf can be identified with the Lie group $\mathbf{SL}(2, \mathbb{R})_l$. Thus, the projection $\mathbf{Sp}(4, \mathbb{R}) \to M_l$ from the Lie group $\mathbf{Sp}(4, \mathbb{R})$ to the leaf space M_l is the projection to the homogeneous space $M_l = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_l$.

In this section, I will use from now on Greek indices μ, ν, etc., to run from 1 to 7. They number the first seven basis elements in the bases (e_I) and (e^I).

Now, I look for all bilinear symmetric forms $g = g_{\mu\nu} e^\mu \odot e^\nu$ on $\mathbf{Sp}(4, \mathbb{R})$, with constant coefficients $g_{\mu\nu} = g_{\nu\mu}$, which are constant along the leaves of the foliation defined by D_l. Technically, I search for those g whose Lie derivative with respect to any vector field X from D_l vanishes,

$$
\mathcal{L}_X g = 0 \text{ for all } X \text{ in } D_l. \tag{3.2}
$$

I have the following proposition:

Proposition 3.1 *The most general $g = g_{\mu\nu} e^\mu \odot e^\nu$ satisfying condition (3.2) is*

$$
\begin{aligned}
g = {} & g_{22}(e^2)^2 + 2g_{24} e^2 \odot e^4 + g_{44}(e^4)^2 + 2g_{35}(e^3 \odot e^5 - e^1 \odot e^7) \\
& + 2g_{26} e^2 \odot e^6 + 2g_{46} e^4 \odot e^6 + g_{66}(e^6)^2.
\end{aligned}
$$

Thus, I have a 7-parameter family of bilinear forms on $\mathbf{Sp}(4,\mathbb{R})$ that *descend* to well-defined pseudo-Riemannian metrics on the leaf space M_l. Note that the restriction of the Killing form K to the space where $(e^8, e^9, e^{10}) \equiv 0$ is in this family. This corresponds to $g_{22} = g_{24} = g_{46} = 0$ and $g_{44} = 2g_{26} = -g_{35} = 1$.

Since the aim of my note is *not* to be exhaustive, but rather to show how to produce G_2 structures on $\mathbf{Sp}(4,\mathbb{R})$ homogeneous spaces, from now on I will restrict myself to only one $\mathbf{SL}(2,\mathbb{R})_l$ invariant bilinear form g on $\mathbf{Sp}(4,\mathbb{R})$, namely to

$$g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7, \tag{3.3}$$

coming from the restriction of the Killing form. It follows from Proposition 3.1 that this form is a well-defined (3, 4) signature metric on the quotient space $M_l = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_l$

.

I now look for the 3-forms $\phi = \frac{1}{6}\phi_{\mu\nu\rho}e^\mu \wedge e^\nu \wedge e^\rho$ on $\mathbf{Sp}(4,\mathbb{R})$ that are constant along the leaves of the distribution D_l, i.e., such that

$$\mathcal{L}_X\phi = 0 \text{ for all } X \text{ in } D_l. \tag{3.4}$$

Then, I have the following proposition.

Proposition 3.2 *There is a 10-parameter family of 3-forms* $\phi = \frac{1}{6}\phi_{\mu\nu\rho}e^\mu \wedge e^\nu \wedge e^\rho$ *on* $\mathbf{Sp}(4,\mathbb{R})$ *which satisfy condition* (3.4). *The general formula for them is*:

$$\begin{aligned}\phi = {} & fe^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) \\ & + se^{156} + t(e^{356} - e^{167}) + he^{237} + be^{246} + re^{347} + ue^{367}.\end{aligned}$$

Here $e^{\mu\nu\rho} = e^\mu \wedge e^\nu \wedge e^\rho$, *and* $a, b, f, h, p, q, r, s, t$ *and* u *are real constants.*

Thus there is a 10-parameter family of 3-forms ϕ that descends from $\mathbf{Sp}(4,\mathbb{R})$ to the $\mathbf{Sp}(4,\mathbb{R})$ homogeneous space $M_l = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_l$.

Now, I introduce an important notion of *compatibility* of a pair (g, ϕ) where g is a metric and ϕ is a 3-form on a 7-dimensional oriented manifold M. The pair (g, ϕ) on M *is compatible* if and only if

$$(X \lrcorner \phi) \wedge (X \lrcorner \phi) \wedge \phi = 3\, g(X, Y)\, \mathrm{vol}(g), \qquad \forall X, Y \in TM.$$

Here $\mathrm{vol}(g)$ is a *volume form* on M related to the metric g.

Restricting, as I did, to the $\mathbf{Sp}(4,\mathbb{R})$ invariant metric g_K on M_l as in (3.3), I now ask which of the 3-forms ϕ from Proposition 3.2 are compatible with the metric (3.3). In other words, I now look for the constants $a, b, f, h, p, q, r, s, t$ and u such that

$$(e_\mu \lrcorner \phi) \wedge (e_\nu \lrcorner \phi) \wedge \phi = 3\, g_K(e_\mu, e_\nu)\, e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6 \wedge e^7, \tag{3.5}$$

for $g = g_K$ given in (3.3).

I have the following proposition.

Proposition 3.3 *The general solution to the Eq.* (3.5) *is given by*

$$b = \tfrac{1}{2},\ f = \frac{ap}{1-q},\ h = \frac{a(q-1)}{p},\ r = \frac{q^2-1}{p},\ s = \frac{p(1-q)}{4a},\ t = \frac{1-q^2}{4a},\ u = \frac{(q^2-1)(q+1)}{4ap}.$$

This leads to the following corollary.

Corollary 3.4 *The most general pair (g_K, ϕ) on M_l compatible with the $\mathbf{Sp}(4,\mathbb{R})$ invariant metric*

$$g_K = (e^4)^2 - 2e^3 \odot e^5 + e^2 \odot e^6 + 2e^1 \odot e^7,$$

coming from the Killing form in $\mathbf{Sp}(4,\mathbb{R})$, is a 3-parameter family with ϕ given by:

$$\phi = \frac{ap}{1-q}e^{125} + a(e^{235} - e^{127}) + pe^{145} + q(e^{147} + e^{345}) + \frac{p(1-q)}{4a}e^{156} + \frac{1-q^2}{4a}(e^{356} - e^{167}) + \frac{a(q-1)}{p}e^{237} + \tfrac{1}{2}e^{246} + \frac{q^2-1}{p}e^{347} + \frac{(q^2-1)(q+1)}{4ap}e^{367}.$$

Here $a \neq 0$, $p \neq 0$, $q \neq 1$ are free parameters, and $e^{\mu\nu\rho} = e^\mu \wedge e^\nu \wedge e^\rho$ as before.

3.2 G_2 structures in general

Compatible pairs (g, ϕ) on 7-dimensional manifolds are interesting since they give examples of G_2 structures [2]. In general, a G_2 structure consists of a compatible pair (g, ϕ) of a metric g and a 3-form ϕ on a 7-dimensional manifold M, and it is in addition assumed that the 3-form ϕ is *generic*, meaning that at every point of M it lies in one of the two *open* orbits of the natural action of $\mathbf{GL}(7,\mathbb{R})$ on 3-forms in $\mathbb{R}^7$. The simple exceptional Lie group G_2 appears here as the common stabilizer in $\mathbf{GL}(7,\mathbb{R})$ of both g and ϕ.

It follows (from compatibility) that the G_2 structures can have metrics g of only two signatures: the Riemannian ones and (3, 4) signature ones. If the signature of g is Riemannian, the corresponding G_2 structure is related to the compact real form of the simple exceptional complex Lie group G_2, and in the (3, 4) signature case the corresponding G_2 structure is related to the noncompact (split) real form of the complex group G_2. In this sense, our Corollary 3.4 provides a 3-parameter family of split real form G_2 structures on M_l.

G_2 structures can be classified according to the properties of their *intrinsic torsion* [1, 2]. Making a long story short, this torsion is totally determined by finding *four* p-forms τ_p on M, $p = 0, 1, 2, 3$, each belonging to one of four different irreducible representations of G_2. Before telling on how to find these forms given a G_2 structure (g, ϕ), we need some preparation.

We recall that the group G_2 acts in $\mathbb{R}^7$, and this induces its action on spaces $\bigwedge^p$ of p-forms in $\mathbb{R}^7$. Of course the 1-dimensional space $\bigwedge^0$ is G_2 irreducible, as well as is the space of 1-forms $\bigwedge^1 = \bigwedge^1_7$. The G_2 irreducible decompositions of the spaces of 2- and 3-forms look like $\bigwedge^2 = \bigwedge^2_7 \oplus \bigwedge^2_{14}$ and $\bigwedge^3 = \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27}$. Here we use the convention that the lower index i in $\bigwedge^p_i$ denotes the *dimension* of the corresponding representation. It is further convenient to introduce the Hodge dual, $*$, which is defined on p-forms λ by

$$*\lambda(e_{\mu_1},\dots,e_{\mu_{7-p}})\,\mathrm{vol}(g) = \lambda\wedge g(e_{\mu_1})\wedge\dots\wedge g(e_{\mu_{7-p}}), \qquad X\lrcorner g(e_\mu) = g(e_\mu, X).$$

By the Hodge duality, the decomposition of $\bigwedge^4$ into G_2 irreducible components is similar to this for $\bigwedge^3$. We further mention that the 7-dimensional representations $\bigwedge^1_7$, $\bigwedge^2_7$ and $\bigwedge^3_7$ are all G_2 equivalent. Also, one can see that, e.g., $\bigwedge^3_{27} = \{\alpha\in\bigwedge^3 \text{ s.t. } \alpha\wedge\phi = 0 \;\&\; \alpha\wedge*\phi = 0\}$.

The intrinsic torsion components τ_0, τ_1, τ_2 and τ_3 have values in the following G_2 irreducible modules: the 3-form τ_3 has values in the 27-dimensional *irreducible* representation $\bigwedge^3_{27}$, the 2-form τ_2 has values in the 14-dimensional *irreducible* representation $\bigwedge^2_{14}$, the 1-form τ_1 has values in the 7-dimensional *irreducible* representation $\bigwedge^1_7$, and the 0-form τ_0 has values in the trivial representation $\bigwedge^0_1$.

The result of Bryant [1, 2] states that for every G_2 structure (g,ϕ) on M there exist unique forms τ_0, τ_1, τ_2 and τ_3 on M, with values in the above-mentioned representations, such that

$$\begin{aligned} d\phi &= \tau_0 *\phi + 3\tau_1\wedge\phi + *\,\tau_3 \\ d*\phi &= 4\tau_1\wedge*\phi + \tau_2\wedge\phi. \end{aligned} \tag{3.6}$$

Thus, Eq. (3.6) enable to determine all the intrinsic torsion components τ_0, τ_1, τ_2 and τ_3 of a given G_2 structure (g,ϕ). They are called Bryant's [1, 2] equations. It follows that vanishing or not of each of the forms τ_p is a G_2 invariant property of a G_2 structure.

3.3 All $\mathbf{Sp}(4,\mathbb{R})$ symmetric G_2 structures on M_l with the metric coming from the Killing form

The below theorem characterizes the G_2 structures corresponding to compatible pairs (g_K,ϕ) from Corollary 3.4; it summarizes the already obtained results and, in addition, provides formulas for the intrinsic torsion which are needed for the characterization.

Theorem 3.5 *Let g_K be the $(3,4)$ signature metric on $M_l = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_l$ arising as the restriction of the Killing form K from $\mathbf{Sp}(4,\mathbb{R})$ to M_l,*

$$g_K = (e^4)^2 - 2e^3\odot e^5 + e^2\odot e^6 + 2e^1\odot e^7.$$

Then the most general G_2 structure associated with such g_K is a 3-parameter family (g_K,ϕ) with the 3-form

$$\begin{aligned}\phi = {}&\frac{ap}{1-q}e^{125} + a(e^{235}-e^{127}) + pe^{145} + q(e^{147}+e^{345}) + \frac{p(1-q)}{4a}e^{156} \\ &+ \frac{1-q^2}{4a}(e^{356}-e^{167}) + \frac{a(q-1)}{p}e^{237} + \tfrac{1}{2}e^{246} + \frac{q^2-1}{p}e^{347} + \frac{(q^2-1)(q+1)}{4ap}e^{367}.\end{aligned}$$

For this structure, the torsions τ_μ solving the Bryant's equations (3.6) are:

$$
\begin{aligned}
\tau_0 &= \frac{6}{7}\,\frac{(2a-p)^2 q-(2a+p)^2}{ap},\\
\tau_1 &= \frac{1}{4}\,(2a-p)\Big(-e^2+\frac{1}{2}\,\frac{(2a+p)(q-1)}{ap}\,e^4+\frac{1}{2}\,\frac{q^2-1}{ap}\,e^6\Big),\\
\tau_2 &= 0,\\
\tau_3 &= \Big(\tfrac{3}{28}(2a-p)^2+\frac{8ap}{7(q-1)}\Big)e^{125}+\frac{11p^2+16ap-12a^2+3q(2a-p)^2}{28p}e^{127}\\
&\quad-\frac{44a^2+16ap-3p^2+3q(2a-p)^2}{28a}e^{145}\\
&\quad+\frac{(7-4q)(2a+p)^2-3q^2(2a-p)^2}{28ap}e^{147}\\
&\quad+\frac{3p^2(q-1)^2-12ap(q^2-1)+4a^2(31+22q+3q^2)}{112a^2}e^{156}\\
&\quad-\frac{(q^2-1)(44a^2+16ap-3p^2+3q(2a-p)^2)}{112a^2p}e^{167}\\
&\quad+\frac{12a^2-16ap-11p^2-3q(2a-p)^2}{28p}e^{235}\\
&\quad-\frac{12a^2(q-1)^2-12ap(q^2-1)+p^2(31+22q+3q^2)}{28p^2}e^{237}\\
&\quad+\frac{4ap(6-q)+(4a^2+p^2)(q-1)}{14ap}e^{246}\\
&\quad+\frac{(7-4q)(2a+p)^2-3q^2(2a-p)^2}{28ap}e^{345}\\
&\quad+\frac{(q^2-1)(12a^2-16ap-11p^2-3q(2a-p)^2)}{28ap^2}e^{347}\\
&\quad+\frac{(q^2-1)(44a^2+16ap-3p^2+3q(2a-p)^2)}{112a^2p}e^{356}\\
&\quad+\frac{(q^2-1)(q+1)(12a^2-44ap+3p^2-3q(2a-p)^2)}{112a^2p^2}e^{367},
\end{aligned}
$$

where as usual $e^{\mu\nu}=e^\mu\wedge e^\nu$ *and* $e^{\mu\nu\rho}=e^\mu\wedge e^\nu\wedge e^\rho$.

Thus, the 3-parameter family of G_2 structures on M_l described in this theorem have the entire 14-dimensional torsion $\tau_2=0$. This means that *all* these G_2 structures *are integrable* in the terminology of [3, 4], or what is the same, this means that they all have the *totally skew symmetric torsion*.

4 G_2 structures on $\mathrm{Sp}(4,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})_s$

Now we consider the homogeneous space $M_s=\mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_s$. Since $\mathfrak{sl}(2,\mathbb{R})$ is spanned by E_2, E_5+E_6, E_9, it is convenient to put these vectors at the end of the new basis of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. We choose this new basis (f_I) in $\mathfrak{sp}(4,\mathbb{R})$ as:

$$f_1 = E_1,\ f_2 = E_3,\ f_3 = E_4,\ f_4 = E_6 - E_5,\ f_5 = E_7,\ f_6 = E_8,$$
$$f_7 = E_{10},\ f_8 = E_2,\ f_9 = E_5 + E_6,\ f_{10} = E_9.$$

If now, one considers (f_I) as the basis of the Lie algebra of invariant vector fields on the Lie group $\mathbf{Sp}(4,\mathbb{R})$, then the dual basis (f^I), $f_I \lrcorner f^J = \delta^J{}_I$, of the left invariant forms on $\mathbf{Sp}(4,\mathbb{R})$, satisfies:

$$\begin{aligned}
\mathrm{d}f^1 &= 2f^1 \wedge (f^4 - f^9) + f^3 \wedge f^8\\
\mathrm{d}f^2 &= -2f^2 \wedge (f^4 + f^9) + 2f^5 \wedge f^8\\
\mathrm{d}f^3 &= 2f^1 \wedge f^{10} + 2f^3 \wedge f^4 + f^6 \wedge f^8\\
\mathrm{d}f^4 &= 2f^1 \wedge f^7 + \tfrac{1}{2}f^2 \wedge f^6 + f^3 \wedge f^5\\
\mathrm{d}f^5 &= f^2 \wedge f^{10} + 2f^4 \wedge f^5 - 2f^7 \wedge f^8\\
\mathrm{d}f^6 &= 2f^3 \wedge f^{10} - 2(f^4 + f^9) \wedge f^6\\
\mathrm{d}f^7 &= 2(f^4 - f^9) \wedge f^7 - f^5 \wedge f^{10}\\
\mathrm{d}f^8 &= 2f^1 \wedge f^5 + f^2 \wedge f^3 - 2f^8 \wedge f^9\\
\mathrm{d}f^9 &= -2f^1 \wedge f^7 + \tfrac{1}{2}f^2 \wedge f^6 + f^8 \wedge f^{10}\\
\mathrm{d}f^{10} &= 2f^3 \wedge f^7 - f^5 \wedge f^6 - 2f^9 \wedge f^{10}.
\end{aligned} \tag{4.1}$$

In this basis, the Killing form on $\mathbf{Sp}(4,\mathbb{R})$ is

$$\begin{aligned}
&K = \tfrac{1}{12}c^I{}_{JK}c^K{}_{LI}f^J \odot\\
&f^L = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7 + 2(f^9)^2 + 2f^8 \odot f^{10},
\end{aligned}$$

where as usual the structure constants $c^I{}_{JK}$ are defined by $[f_I, f_J] = c^K{}_{IJ}f_K$.

Using the same arguments, as in the case of M_l, we again see that $\mathbf{Sp}(4,\mathbb{R})$ has the structure of the principal $\mathbf{SL}(2,\mathbb{R})$ fiber bundle $\mathbf{SL}(2,\mathbb{R})_s \to \mathbf{Sp}(4,\mathbb{R}) \to M_s = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_s$ over the homogeneous space $M_s = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_s$. In particular, we have a foliation of $\mathbf{Sp}(4,\mathbb{R})$ by integral leaves of an integrable distribution D_s spanned by the annihilator of the forms $(f^1, f^2, \dots, f^7)$. As before, also in this section, we will use Greek indices μ, ν, etc., to run from 1 to 7. They now number the first seven basis elements in the bases (f_I) and (f^I).

Repeating the procedure from the previous sections, I now search for all bilinear symmetric forms $g = g_{\mu\nu}f^\mu \odot f^\nu$ on $\mathbf{Sp}(4,\mathbb{R})$, with constant coefficients $g_{\mu\nu} = g_{\nu\mu}$, whose Lie derivative with respect to any vector field X from D_l vanishes,

$$\mathcal{L}_X g = 0 \text{ for all } X \text{ in } D_s. \tag{4.2}$$

I have the following proposition.

Proposition 4.1 *The most general $g = g_{\mu\nu}f^\mu \odot f^\nu$ satisfying condition (4.2) is*

$$\begin{aligned}
g = g_{33}\big((f^3)^2 - 2f^1 \odot f^6\big) + g_{44}(f^4)^2 + g_{55}\big((f^5)^2 + 2f^2 \odot f^7\big)\\
+ 2g_{26}\big(-2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7\big).
\end{aligned}$$

Thus, this time, I only have a 4-parameter family of bilinear forms on $\mathbf{Sp}(4,\mathbb{R})$ that *descend* to well-defined pseudo-Riemannian metrics on the leaf space M_s. Note that the restriction of the Killing form K to the space where $(f^8, f^9, f^{10}) \equiv 0$ is in this family. This corresponds to $g_{33} = g_{55} = 0$ and $g_{44} = 2$, $g_{26} = 1/2$.

Again for simplicity reasons, I will solve the problem of finding $\mathbf{Sp}(4, \mathbb{R})$ invariant G_2 structures on M_s restricting to only those pairs (g, ϕ) for which $g = g_K$, where

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7, \tag{4.3}$$

which again means that I only will consider one metric, the one coming from the restriction of the Killing form of $\mathbf{Sp}(4, \mathbb{R})$ to M_s. It is a well-defined (3, 4) signature metric on the quotient space $M_s = \mathbf{Sp}(4, \mathbb{R})/\mathbf{SL}(2, \mathbb{R})_s$.

I now look for the 3-forms $\phi = \frac{1}{6}\phi_{\mu\nu\rho} f^\mu \wedge f^\nu \wedge f^\rho$ on $\mathbf{Sp}(4, \mathbb{R})$ which are such that

$$\mathcal{L}_X \phi = 0 \text{ for all } X \text{ in } D_s. \tag{4.4}$$

I have the following proposition.

Proposition 4.2 *There is precisely a 5-parameter family of 3-forms* $\phi = \frac{1}{6}\phi_{\mu\nu\rho} f^\mu \wedge f^\nu \wedge f^\rho$ *on* $\mathbf{Sp}(4, \mathbb{R})$ *which satisfies condition* (4.4). *The general formula for* ϕ *is*:

$$\begin{aligned}\phi = {} & a(4f^{147} + f^{246} + 2f^{345}) + b(2f^{156} + f^{236} - 4f^{137}) + qf^{136} \\ & + h(f^{256} - 4f^{157} - 2f^{237}) + pf^{257}.\end{aligned}$$

Here $f^{\mu\nu\rho} = f^\mu \wedge f^\nu \wedge f^\rho$, *and* a, b, q, h *and* p *are real constants.*

Solving for all 3-forms ϕ from this 5-parameter family that are compatible, as in (3.5), with the metric g_K from (4.3), I arrive at the following proposition.

Proposition 4.3 *The general solution to the equations* (3.5) *is given by*

$$a = \tfrac{1}{2}, \; b = h = 0, \; p = \frac{1}{q}.$$

This leads to the following corollary.

Corollary 4.4 *The most general pair* (g_K, ϕ) *on* M_s *compatible with the* $\mathbf{Sp}(4, \mathbb{R})$ *invariant metric*

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7,$$

coming from the Killing form in $\mathbf{Sp}(4, \mathbb{R})$, *is a 1-parameter family with* ϕ *given by*:

$$\phi = 2f^{147} + \tfrac{1}{2}f^{246} + f^{345} + qf^{136} + \frac{1}{q}f^{257}.$$

Here $q \neq 0$ *is a free parameter, and* $f^{\mu\nu\rho} = f^\mu \wedge f^\nu \wedge f^\rho$ *as before.*

4.1 All $\mathbf{Sp}(4,\mathbb{R})$ symmetric G_2 structures on M_s with the metric coming from the Killing form

Similarly as in Sect. 3.3 we now summarize the already obtained results about the considered $Sp(2,\mathbb{R})$ symmetric G_2 structures on M_s in a theorem; it is given below and has also a new part consisting of the formulas for the intrinsic torsion.

Theorem 4.5 *Let g_K be the* (3, 4) *signature metric on $M_s = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_s$ arising as the restriction of the Killing form K from $\mathbf{Sp}(4,\mathbb{R})$ to M_s,*

$$g_K = 2(f^4)^2 - 2f^3 \odot f^5 + f^2 \odot f^6 - 4f^1 \odot f^7.$$

Then, the most general G_2 structure associated with such g_K is a 1-parameter family (g_K,ϕ) with the 3-form

$$\phi = 2f^{147} + \tfrac{1}{2}f^{246} + f^{345} + qf^{136} + \frac{1}{q}f^{257}.$$

For this structure

$$\mathrm{d} * \phi = 0,$$

i.e., the torsions

$$\tau_1 = \tau_2 = 0.$$

The rest of the torsions solving Bryant's equations (3.6) *are*:

$$\begin{aligned}\tau_0 &= -\frac{18}{7},\\ \tau_3 &= \tfrac{2}{7}\left(4f^{147} + f^{246} + 2f^{345}\right) - \tfrac{3}{7}\left(qf^{136} + \frac{1}{q}f^{257}\right).\end{aligned}$$

where, as usual $f^{\mu\nu\rho} = f^\mu \wedge f^\nu \wedge f^\rho$; $q \neq 0$.

So on $M_s = \mathbf{Sp}(4,\mathbb{R})/\mathbf{SL}(2,\mathbb{R})_s$ there exists a 1-parameter family of the above G_2 structures which is *coclosed*. Therefore, in particular, it is *integrable*

I note that formally I can also obtain coclosed G_2 structures on M_l, using Theorem 3.5. It is enough to take $p = 2a$ in the solutions of this Theorem. The question if in the resulting 2-parameter family of the coclosed G_2 structures there is a 1-parameter subfamily equivalent to the structures I have on M_s via Theorem 4.5 needs further investigation. However, I doubt that the answer to this question is positive, since it is visible from the root diagram for $\mathbf{Sp}(4,\mathbb{R})$ that the spaces M_l and M_s are geometrically quite different. Indeed, apart from the $\mathbf{Sp}(4,\mathbb{R})$ invariant G_2 structures, which I have just introduced in this note, the spaces M_l and M_s have quite different *additional* $\mathbf{Sp}(4,\mathbb{R})$ invariant structures. A short look at the root diagram on page 1 of this note shows that M_l has *two* well-defined $\mathbf{Sp}(4,\mathbb{R})$ invariant rank 3-distributions, corresponding to the pushforwards from $\mathbf{Sp}(4,\mathbb{R})$ to M_l of the vector spaces $D_{l1} = \mathrm{Span}_\mathbb{R}(E_2,E_3,E_7)$ and $D_{l2} = \mathrm{Span}_\mathbb{R}(E_4,E_8,E_9)$. Likewise M_s, in addition to the discussed G_2 structures, has also a well defined pair of $\mathbf{Sp}(4,\mathbb{R})$ invariant rank 3-distributions, corresponding to the pushforwards from $\mathbf{Sp}(4,\mathbb{R})$ to M_s of the vector spaces $D_{s1} = \mathrm{Span}_\mathbb{R}(E_1,E_4,E_8)$ and $D_{s2} = \mathrm{Span}_\mathbb{R}(E_3,E_7,E_{10})$. The problem is that these two sets of pairs of $\mathbf{Sp}(4,\mathbb{R})$ invariant distributions are quite different. The distributions on M_l have constant growth vector (2, 3), while the distributions on M_s are *integrable*. These pairs of distributions constitute an immanent ingredient of the geometry on the corresponding spaces M_l and M_s and, since they are diffeomorphically nonequivalent and they make the G_2 geometries there quite different. I believe that this fact makes the G_2 structures obtained on M_l and M_s really nonequivalent.

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