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Cartan normal conformal connections from differential equations

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Abstract

We explore and show a natural relationship between all third-order ordinary differential equations that possess a vanishing Wunschmann invariant, with conformal metrics on 3-manifolds and Cartan's normal $O(3, 2)$ conformal connections. The generalization to pairs of second-order PDEs and their relationship to Cartan's normal $O(4, 2)$ conformal connections on four-dimensional manifolds is discussed.

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1. Introduction

The purpose of this work is to show the rich geometric structure that is buried in a large class of differential equations. Some of this structure was known for a long time which will be reviewed here; the other parts of the structure are new. Here we will concentrate on third-order ODEs, first discussing all third-order ODEs and then, via the emerging structure, narrowing the discussion down to a large special subclass. We, however, stress that the discussion could be extended to large classes of PDEs and to lower- and higher-order ODEs.

A generic ordinary differential equation of third order has the form

$$u''' = F(u, u', u'', s), \tag{1}$$

where u is a real function of s and a prime denotes the ordinary derivative with respect to s. The right-hand side is specified by the arbitrary C^∞ function F of four arguments. The equivalence problem, which consists of classifying the functions F which are equivalent under a variety of different transformations (contact, point and fibre preserving) has been extensively studied, most notably by Cartan [1–3] and Chern [4]. It has been revisited more recently by many authors including Tod [5], Frittelli, Kamran, Kozameh and Newman [6, 7] and Nurowski and Godlinski [12]. One remarkable result that follows from the equivalence problem is that the equivalence classes of third-order ODEs split into two major classes; those with a vanishing Wunschmann invariant [8] and those with a non-vanishing invariant. The Wunschmann invariant, $I[F]$, a differential expression involving F and its derivatives in all four variables, was discovered by early workers in the theory of differential equations, extensively used by Chern [4]. An analogue of the Wunschmann invariant in a more general setting (pairs of second-order PDEs as well as single third-order ODEs) was discovered in the context of general relativity [9–11] and is known as the 'metricity condition', a necessary condition for the reformulation of the Einstein equations in terms of null surfaces. (It will be referred to as the generalized Wunschmann invariant.) More specifically, when a third-order ODE satisfies $I[F] = 0$, one can show that the solution space, i.e., the three-dimensional space of constants of integration, $\{x^a\}$, possesses, directly from the differential equation (1), a conformal Lorentzian metric with the level surfaces of the solutions themselves, $u = z(x^a, s)$, forming a one-parameter family of null surfaces. All members of the equivalence class, under contact transformations, yield the same conformal metric. The converse statement is also true; namely, given a three-dimensional conformal Lorentzian spacetime, from any complete solution ($u = U(x^a, s)$) of the eikonal equation, $g^{ab}\partial_a U \partial_b U = 0$, one can obtain a third-order ODE (by differentiating with respect to s three times and eliminating x^a) all belonging to the same equivalence class [6]. This result generalizes to pairs of second-order PDEs that possess a vanishing generalized Wunschmann invariant; the solution space is four dimensional with a conformal Lorentzian 4-metric. All members of the same equivalence class yield the same metric [6].

The purpose of this work is to study further geometric structures arising from third-order ODEs with vanishing Wunschmann invariant and, in particular, to show how a Cartan normal conformal connection arises in this context. Though these structures are invariant under general contact transformations, the discussion of this issue will be presented elsewhere [4, 12].

In section 2, we describe in a very general context (with no apparent relationship to our third-order ODE) certain geometric structures and curvature decompositions associated with an arbitrary Weyl geometry on an n-dimensional spacetime. This will be essentially pedagogical and presented without the proofs, which are simple. This general discussion is associated with the remainder of the paper in the following manner. In the special case of three dimensions, these results describe a simple cross-section of a bundle constructed in the latter sections from the differential equation.

In section 3, we review, and partially relate to the previous section, the basic structures associated with our third-order ODE, a set of three (Pfaffian) 1-forms on a 4-manifold from which one finds an associated torsion-free (partial) connection. It is here that the condition for the vanishing Wunschmann invariant arises; the set of all third-order ODEs splits into two classes. In section 4, the existence and construction of a three-dimensional Lorentzian conformal metric on the solution space is described and some properties of the derivatives arising from the connection are discussed, while in section 5 a few special cases of three arbitrary functions (a Weyl 1-form) arising in the connection are briefly mentioned. In section 6, the curvature 2-forms are first introduced. We show that the ordinary curvature of the Weyl geometry decomposes in a non-standard manner; the Ricci part into a unique 1-form and a new curvature (the first Cartan curvature), which in general is trace-free but in this three-dimensional case vanishes. This leads, in section 7, to the introduction of a further (a second Cartan) curvature 2-form and the realization that we are dealing with a Cartan normal $O(3, 2)$ conformal connection. In other words, third-order ODEs, satisfying the Wunschmann condition, encode an entire $O(3, 2)$ conformal connection on the solution space. Finally in section 9, we discuss how this extends to pairs of second-order PDEs and an $O(4,2)$ conformal connection on the four-dimensional solution space and its relationship to twistor theory.

2. Weyl geometry

We begin, in a completely conventional manner, with an n-dimensional manifold and n arbitrary independent 1-forms θ^i which determine a metric g by

$$g = \eta_{ij}\theta^i\theta^j, \tag{2}$$

with η_{ij} a constant flat metric (signature $(1, n-1)$)[5], which is used to raise and lower indices and a connection 1-form $\omega^i{}_j$ (defining covariant differentiation) that is uniquely determined by

$$\omega_{ij} = \omega_{[ij]} + \eta_{ij}A, \tag{3}$$

(with A a given but arbitrary (Weyl) 1-form) and the torsion-free structure equation acting as an algebraic equation for $\omega_{[ij]}$,

$$\mathrm{d}\theta^i + \omega^i{}_j \wedge \theta^j = 0. \tag{4}$$

The curvature 2-form $\Theta^i{}_j$ is then defined by

$$\Theta^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \tag{5}$$

At this point we break with the conventional treatment and define a new curvature 2-form $\Omega^i{}_j$ (referred to as the first Cartan curvature) by

$$\Omega^i{}_j = \Theta^i{}_j + \theta^i \wedge \Psi_j + \eta^{il}\Psi_l \wedge \theta^m\eta_{jm} - \delta^i_j\Psi_k \wedge \theta^k \tag{6}$$

with Ψ_j determined uniquely by the following condition. Expanding both $\Omega^i{}_j$ and $\Theta^i{}_j$ as

$$\Omega^i{}_j = \tfrac{1}{2}\Omega^i{}_{jlm}\theta^l \wedge \theta^m \qquad \Theta^i{}_j = \tfrac{1}{2}\Theta^i{}_{jlm}\theta^l \wedge \theta^m \tag{7}$$

and requiring that

$$\Xi_{ij} \equiv \Omega^l{}_{ilj} = 0, \tag{8}$$

one finds that Ψ_j is given uniquely in terms of the 'Ricci tensor' of $\Theta^i{}_{jlm}$, i.e., in terms of $R_{ij} \equiv \Theta^l{}_{ilj}$. Explicitly, we find that

$$\Psi_j = K_{jm}\theta^m, \qquad K_{jm} = -\frac{1}{n}R_{[jm]} - \frac{1}{n-2}R^{TF}_{(jm)} - \frac{1}{2n(n-1)}R\eta_{jm}, \tag{9}$$

with $R^{TF}_{(jm)}$ the trace-free part of $R_{(ij)}$ and $R = \eta^{ij}R_{ij}$.

We have shifted the curvature information from $\Theta^i{}_j$ to the traceless $\Omega^i{}_j$ and the 1-form Ψ_i which contains all the 'Ricci tensor' information.

Continuing, we define a second Cartan curvature 2-form by

$$\Omega_i = \mathrm{d}\Psi_i + \Psi_j \wedge \omega^j{}_i \equiv \nabla \wedge \Psi_i. \tag{10}$$

All the quantities, the 1-forms $\omega^i{}_j$ and Ψ_i, and the 2-forms $\Omega^i{}_j$ and Ω_j are unique functions of the given forms θ^i and A. $\Omega^i{}_j$ is independent of A and is the Weyl tensor. In three dimensions $\Omega^i{}_j = 0$. When $A = 0$, we have that $\omega^i{}_j$ is the ordinary metric connection. When $n = 3$ and $A = 0$, $\Omega_i \equiv C_{ijk}\theta^j \wedge \theta^k$ is the Cotton–York tensor. For dimensions $n > 3$, Ω_i is determined by first derivatives of the Weyl tensor and A. All curvature statements can be made in terms of $\Omega^i{}_j$, Ω_i and Ψ_j. As an example, the Bach tensor can be given in terms of a derivative of Ω_i.

[5] This is easily generalized to arbitrary signature, (p, q).

Most of the above statements can be checked by direct calculation. However, the construction of the Bach tensor from Ω_i is apparently of considerable difficulty [13].

In this section, we have constructed a trivial local cross-section of a principal H-bundle P, with a Cartan normal conformal connection [14].

To explain this, we begin with the conformal group $G = Conformal(1, n-1)$, composed of the conformal rotations (dilatation and rotations), i.e., $CO(1, n-1)$, the n translations, T_n, and the n special conformal translations, T_n^*. The principal bundle P is constructed from the group H which is a subgroup of the conformal group G consisting of $CO(1, n-1)$ and the n special conformal translations, T^*. (The translation part of G have been omitted.) The Cartan conformal connection takes values in the Lie algebra of G, i.e., G'. Since $Conformal(1, n-1)$ is isomorphic to $O(2, n)$, we can represent the Lie algebra G' by $(n+2) \times (n+2)$ matrices [14].

The Cartan connection, ω, and curvature, R, which are given respectively by

$$\omega = \left(\theta^i, \omega^j{}_i, \Psi_j\right) \tag{11}$$

$$R = \left(T^j = 0, \Omega^i{}_j, \Omega_i\right) \tag{12}$$

can be represented by the $(n+2, n+2)$ matrices of 1 and 2-forms by

$$\omega_B^A = \begin{bmatrix} -A & \Psi_i & 0 \\ \theta^i & \eta^{ik}\omega_{[kj]} & \eta^{ij}\Psi_j \\ 0 & \eta_{ij}\theta^j & A \end{bmatrix} \tag{13}$$

$$R_B^A = \begin{bmatrix} 0 & \Omega_i - A_j\Omega^j{}_i & 0 \\ 0 & \Omega^i{}_j & \eta^{ij}\Omega_j \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

with the relationship

$$R_B^A = \mathrm{d}\omega_B^A + \omega_C^A \wedge \omega_B^C.$$

In this representation of G', the generic element of H is represented by the matrix

$$b = \begin{bmatrix} \mathrm{e}^{-\phi} & \mathrm{e}^{-\phi}\xi_j & \frac{\mathrm{e}^{-\phi}}{2}\xi_i\xi_j\eta^{ij} \\ 0 & \Lambda^i{}_j & \Lambda^i{}_j\eta^{jk}\xi_k \\ 0 & 0 & \mathrm{e}^{\phi} \end{bmatrix} \tag{15}$$

with

$$\Lambda^i{}_l\Lambda^j{}_m\eta^{jm} = \eta^{ij}.$$

Lifting the forms (13) and (14), via equation (15), to P, by

$$\begin{aligned} \widehat{\omega}_B^A &= (b^{-1})_C^A\omega_D^C(b)_B^D + (b^{-1})_C^A(\mathrm{d}b)_B^C \\ \widehat{R}_B^A &= (b^{-1})_C^A R_D^C(b)_B^D \end{aligned} \tag{16}$$

yields the full Cartan conformal connection and curvature on the principal H-bundle.

In the following sections, we will see how this type of structure, with $n = 3$, arises from the third-order ODE. In this case $G = Conformal\ (1, 2) \approx O(2, 3)$ is a ten-parameter group while the subgroup $H = CO(1, 2) \otimes_s T_3^*$ has seven parameters. Adding in the $n = 3$ for the base space, the bundle P is ten-dimensional. We will obtain directly only a one-parameter subgroup of H (so that the bundle obtained, $\widetilde{P}$, is only four dimensional) and not the full seven-parameter group H. Nevertheless, it is easily seen how the other six parameters can be inserted (via a version of equation (16)) and furthermore one clearly sees their geometric meaning.

3. Basic relations

We begin by reviewing some properties of the general third-order ODE,

$$u''' = F(u, u', u'', s) \tag{17}$$

or equivalently the Pfaffian system;

$$\beta^1 = \mathrm{d}u - u'\,\mathrm{d}s, \qquad \beta^2 = \mathrm{d}u' - u''\,\mathrm{d}s, \qquad \beta^3 = \mathrm{d}u'' - F(u, u', u'', s)\,\mathrm{d}s \tag{18}$$

on the second jet bundle J^2 with local bundle coordinates $x^\alpha = (u, u', u'', s)$. On this space we introduce our basis 1-forms $(\theta^A) = (\theta^0, \theta^i)$ with

$$\theta^0 = \mathrm{d}s, \qquad \theta^1 = \beta^1, \qquad \theta^2 = \beta^2, \qquad \theta^3 = \beta^3 + a\beta^1 + b\beta^2. \tag{19}$$

with (a, b), at the moment, two arbitrary functions of x^α which will, shortly, be determined. The dual vector bases, $e_A = (e_0, e_i)$ are

$$\begin{aligned} e_0 &= \mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}s} = \partial_s + u'\partial_u + u''\partial_{u'} + F\partial_{u''}, \\ e_1 &= \partial_u - a\partial_{u''} \qquad e_2 = \partial_{u'} - b\partial_{u''} \qquad e_3 = \partial_{u''}. \end{aligned} \tag{20}$$

This jet bundle possesses an alternative bundle structure (which will be referred to as $\widetilde{P}$) [6]. If solutions of the third-order ODE are given by $u = z(x^a, s)$, with x^a being three constants of integration (the solution space), then the integral curves of e_0 are the fibres over the solution space. These base space coordinates can be taken to be the initial values of (u, u', u''), i.e., $(u(0), u'(0), u''(0)) = (u_0, u'_0, u''_0)$. Though we do not explicitly use these base space coordinates, the point of view associated with this alternative bundle is basic for what follows.

On this four-dimensional space, $\widetilde{P}$, we *define* a three-dimensional distribution by the three vectors e_i, a subspace of the tangent space. This distribution is tangent to the surfaces of fixed value of s with arbitrary values of (u, u', u''). We construct a 'partial' connection that is associated with this subspace of the tangent bundle over $\widetilde{P}$ in the following manner.

On $\widetilde{P}$ we introduce the tensor field (*a degenerate quadratic form*) by

$$g = \eta_{ij}\theta^i \otimes \theta^j \tag{21}$$

with η_{ij} being the flat Minkowski 3-metric (in null-null coordinates) given by

$$\eta_{ij} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{22}$$

as well as a covariant derivative operator ∇_B such that

$$\nabla_B g_{CD} = 2A_B g_{CD} \tag{23}$$

with $A = A_i\theta^i + A_0\theta^0$, an arbitrary 1-form on $\widetilde{P}$ that we will refer to as a Weyl 1-form.

The 'partial' connection, compatible with equation (23) is

$$\omega^i{}_j = \omega^i{}_{j(A)}\theta^A = \omega^i{}_{j(k)}\theta^k + \omega^i{}_{j(0)}\theta^0, \qquad \omega_{ij} = \eta_{ik}\omega^k{}_j = \omega_{[ij]} + A\eta_{ij},$$

which are a set of four 1-forms obtained as the algebraic solution of the (torsion free) structure equation.

Covariant derivatives are given by

$$\nabla_A e_i \equiv e_j\omega^j{}_{iA} \tag{24}$$

$$\nabla_A\theta^i \equiv -\omega^i{}_{jA}\theta^j. \tag{25}$$

Because of the degeneracy of g, this result does not uniquely follow from equation (23). We require that $\omega^i{}_{jA}$ satisfy the analogue of the first structure equation

$$\mathrm{d}\theta^i + \omega^i{}_j \wedge \theta^j = 0. \tag{26}$$

From the following relations, obtained directly from the exterior derivatives of θ^i,

$$\begin{aligned}
\mathrm{d}\theta^1 &= \mathrm{d}s \wedge \theta^2,\\
\mathrm{d}\theta^2 &= \mathrm{d}s \wedge (\theta^3 - a\theta^1 - b\theta^2),\\
\mathrm{d}\theta^3 &= (F_u - aF_{u''} + \mathrm{D}a - ab)\,\mathrm{d}s \wedge \theta^1 + (F_{u'} - bF_{u''} + a + \mathrm{D}b - b^2)\,\mathrm{d}s \wedge \theta^2\\
&\quad + (F_{u''} + b)\,\mathrm{d}s \wedge \theta^3 + (a_{u'} - ba_{u''} - b_u + ab_{u''})\theta^2 \wedge \theta^1 + a_{u''}\theta^3 \wedge \theta^1 + b_{u''}\theta^3 \wedge \theta^2
\end{aligned} \tag{27}$$

and (using $\omega_{33} = \omega_{11} = 0$, $\omega_{22} = -A$, $\omega_{12} = -\omega_{21}$, $\omega_{23} = -\omega_{32}$, $\omega_{13} = \omega_{[13]} + A$, $\omega_{31} = -\omega_{[13]} + A$) equation (26) written as

$$\begin{aligned}
&\mathrm{d}\theta^1 + (w_{[31]} + A) \wedge \theta^1 + w_{[32]} \wedge \theta^2 = 0,\\
&\mathrm{d}\theta^2 - w_{[21]} \wedge \theta^1 + A \wedge \theta^2 - w_{[23]} \wedge \theta^3 = 0,\\
&\mathrm{d}\theta^3 + w_{[12]} \wedge \theta^2 + (w_{[13]} + A) \wedge \theta^3 = 0,
\end{aligned} \tag{28}$$

we obtain the follwing *three results*.

1. The equations have solutions *only if* the function $F(u, u', u'', s)$ *is* restricted by the condition

$$I[F] = F_u - aF_{u''} + \mathrm{D}a - ab = 0, \tag{29}$$

a relation known as the Wunschmann condition or vanishing of the Wunschmann invariant. Though a severe condition, it nevertheless allows a very large space of solutions; the space of solutions can be put into a one-to-one correspondence with the space of all conformal three-dimensional Lorentzian metrics. For the remainder of this work we will always assume that equation (11) is satisfied. (Note, however, that a non-vanishing Wunschmann invariant could be interpreted as the existence of a non-vanishing torsion tensor associated with the connection.)

2. The two functions (a, b) that were arbitrary to start with are now uniquely determined as

$$a = -\tfrac{1}{2}F_{u'} - \tfrac{1}{9}(F_{u''})^2 + \tfrac{1}{6}(\mathrm{D}F_{u''}) \qquad b = -\tfrac{1}{3}F_{u''}. \tag{30}$$

3. The connection is determined by

$$\begin{aligned}
\omega_{[32]} &= \left(A_2 - \tfrac{1}{2}b_{u''}\right)\theta^1 + A_3\theta^2 - \theta^0\\
\omega_{[31]} &= (A_1 - a_{u''})\theta^1 - \tfrac{1}{2}b_{u''}\theta^2 - A_3\theta^3 - b\theta^0\\
\omega_{[21]} &= -(a_{u'} - ba_{u''} - b_u + ab_{u''})\theta^1 - A_1\theta^2 + \left(\tfrac{1}{2}b_{u''} - A_2\right)\theta^3 - a\theta^0\\
\omega_{22} &= -A = -A_i\theta^i - b\theta^0,
\end{aligned} \tag{31}$$

where only the three components of A, namely $A_i = (A_1, A_2, A_3)$, are completely arbitrary.

Note that the 'partial' connection is *uniquely* given in terms of F, up to the *arbitrary choice of the three coefficients* A_i. Though we will see that there are several natural choices of A_i, there is no unique determination of them. For the most part we leave them completely arbitrary (see equation (3)).

Remark 1. We point out now, and return to it later, that there is a natural generalization of the forms (19) to

$$\begin{aligned}
\widehat{\theta}^0 &= \mathrm{d}s\\
\widehat{\theta}^1 &= \alpha\beta^1,\\
\widehat{\theta}^2 &= \Phi(\beta^2 + \tau\beta^1),\\
\widehat{\theta}^3 &= \alpha^{-1}\Phi^2(\beta^3 + a\beta^1 + b\beta^2),
\end{aligned} \tag{32}$$

where the parameters τ, α and Φ describe respectively, a null rotation around $\widehat{\theta}^1$, a boost transformation in the $(\widehat{\theta}^1, \widehat{\theta}^3)$ plane and a conformal rescaling of the metric, g (equation (21)). Up to scale, they preserve the degenerate metric g. Our manifold $\widetilde{P}$ could be extended from four to ten dimensions by adding in as new coordinates the three (τ, α, Φ) and the three A_i. This enlarged space possesses a natural geometric structure of a Cartan normal conformal connection. (See the discussion at the end of sections 2 and 8.)

4. Construction of conformal metrics

In order to demonstrate the existence of a conformal metric on the solution space, we first note that

$$\pounds_{e_0}\theta^1 = \theta^2, \qquad \pounds_{e_0}\theta^2 = \theta^3 - a\theta^1 - b\theta^2, \qquad \pounds_{e_0}\theta^3 = -a\theta^2 - 2b\theta^3 \tag{33}$$

which follow from the definitions and the Wunschmann condition (equation (29)). From equation (33), one easily sees that

$$\pounds_{e_0} g = \tfrac{2}{3} F_{,u''} g \tag{34}$$

so that a conformal factor U can be found from

$$\pounds_{e_0} U = e_0(U) = -\tfrac{1}{3} F_{,u''} \tag{35}$$

so that

$$\pounds_{e_0} \widetilde{g} = 0 \tag{36}$$

with

$$\widetilde{g} \equiv \mathrm{e}^{2U} g. \tag{37}$$

The residual conformal freedom is given by U_0 with $\pounds_{e_0} U_0 = 0$ or $U_0 = U_0(x^i)$ with x^i the base space coordinates, i.e., an ordinary conformal rescaling of a metric on the solution space.

We have thus shown (again, but by a different argument) [6, 7], that a third-order ODE satisfying the Wunschmann condition defines a Lorentzian three-dimensional conformal metric. We point out that the converse is also true; any Lorentzian three-dimensional conformal metric defines, via a complete solution of the eikonal equation, an equivalence class of third-order ODEs that leads back to the metric by the process just explained.

For completeness, from equations (24) and (25), we display the covariant derivatives of the e_i and the θ^i in the vertical directions via

$$\begin{aligned} &e_0 \nabla e_i = e_j \omega^j{}_{i(0)}, && e_0 \nabla e_1 = -a e_2, \\ &e_0 \nabla e_2 = e_1 - b e_2 - a e_3, && e_0 \nabla e_3 = e_2 - 2b e_3, \\ &e_0 \nabla \theta^i = -\omega^i{}_{j(0)} \theta^j, && e_0 \nabla \theta^1 = -\theta^2, \\ &e_0 \nabla \theta^2 = a\theta^1 + b\theta^2 - \theta^3, && e_0 \nabla \theta^3 = a\theta^2 + 2b\theta^3. \end{aligned}$$

5. Choices for A

Though we will not consider it as an important issue, we simply mention that there is a series of different 'natural choices' for the first three components A_i of the Weyl 1-form A.

One can simply take them as zero or as a gradient field, i.e., $A_i = \nabla_i \Phi$.

Alternatively, it can be considered as some additional covector field leading to a Weyl geometry on the base space.

Cartan [2], in order to give the curvature tensor certain simple properties, chose them as

$$A_1 = \tfrac{1}{3}(F_{u''u'} - \mathrm{D}F_{u''u''}) \qquad A_2 = \tfrac{1}{3}F_{u''u''} \qquad A_3 = 0.$$

However, we simply leave them as three arbitrary functions, whose role in geometry will be clarified later.

6. Curvature

In this section, we introduce two different, but closely related, curvature 2-forms, Θ_{ij} and Ω_{ij} where the 'ordinary' curvature

$$\Theta_{ij} \equiv \tfrac{1}{2}\Theta_{ijlm}\theta^l \wedge \theta^m + \Theta_{ijm0}\theta^m \wedge \theta^0 \tag{38}$$

is defined by

$$\Theta_{ij} = \mathrm{d}\omega_{ij} + \eta^{kl}\omega_{ik} \wedge \omega_{lj} \tag{39}$$

and the 'first Cartan' curvature by

$$\Omega_{ij} = \Theta_{ij} + \eta_{il}\theta^l \wedge \Psi_j + \Psi_i \wedge \theta^l\eta_{jl} - \eta_{ij}\Psi_k \wedge \theta^k, \tag{40}$$

with the expansion

$$\Omega_{ij} = \tfrac{1}{2}\Omega_{ijlm}\theta^l \wedge \theta^m + \Omega_{ijm0}\theta^m \wedge \theta^0. \tag{41}$$

The 1-form Ψ_i, which can be written as

$$\Psi_i = w_i\theta^0 + K_{ij}\theta^j \tag{42}$$

will be chosen so that

- the coefficient of θ^0 vanishes, i.e., $\Omega_{ijm0} = 0$,
- the trace on the first and third indices of Ω_{ijlm} vanishes, i.e.,

$$\eta^{il}\Omega_{ijlm} = 0. \tag{43}$$

Remark 2. As an aside we point out that with a three-dimensional base space, these two conditions imply that $\Omega_{ij} = 0$. That, however, is true only in three dimensions.

We first show that w_i can be chosen so that $\Omega_{ijm0} = 0$.

By pointing out that Θ_{ij} inherits from equation (39) the same symmetries as ω_{ij} and can be written as

$$\Theta_{ij} = \Theta_{[ij]} - \eta_{ij}\Theta_{22}, \tag{44}$$

we have

$$\Theta_{13ij} + \Theta_{31ij} = 2\Theta_{(13)ij} = -2\Theta_{22ij}. \tag{45}$$

It follows that Θ_{ij} has four 2-form components, $\Theta_{[12]}$, $\Theta_{[13]}$, $\Theta_{[23]}$ and Θ_{22}.

By taking the exterior derivative of equation (26) we obtain the first Bianchi identity

$$\Theta_{ij} \wedge \theta^j = 0. \tag{46}$$

Remembering, from equation (38), that Θ_{ij} decomposes into two types of terms

$$\Theta_{ijlm} \qquad \text{and} \qquad \Theta_{ijm0}$$

the Bianchi identities yield the following relations

$$\Theta_{2313} = -\Theta_{3123} \qquad \Theta_{2213} = \Theta_{2312} - \Theta_{1223} \qquad \Theta_{1312} = \Theta_{1213} \tag{47}$$

for the first set. From the second set, we obtain

$$\begin{aligned}\Theta_{[13]l0}\theta^l \wedge \theta^0 &= \chi_1\theta^1 \wedge \theta^0 - \chi_3\theta^3 \wedge \theta^0,\\ \Theta_{[23]l0}\theta^l \wedge \theta^0 &= \chi_2\theta^1 \wedge \theta^0 + \chi_3\theta^2 \wedge \theta^0,\\ \Theta_{[12]l0}\theta^l \wedge \theta^0 &= -\chi_1\theta^2 \wedge \theta^0 - \chi_2\theta^3 \wedge \theta^0,\end{aligned} \tag{48}$$

where, by definition, χ_i are given by

$$\Theta_{22l0}\theta^l \wedge \theta^0 \mathrel{\mathring{=}} \chi_1\theta^1 \wedge \theta^0 + \chi_2\theta^2 \wedge \theta^0 + \chi_3\theta^3 \wedge \theta^0. \tag{49}$$

Thus all the terms of Θ_{ijl0} are expressed in terms of χ_i.

By simply taking

$$w_i = \chi_i, \tag{50}$$

and a direct calculation, using equation (40) and the first term of equation (42), we immediately have that $\Omega_{ijm0} = 0$.

Since, in the transformation, equation (40) there is no interaction between the forms $\theta^m \wedge \theta^0$ and $\theta^l \wedge \theta^m$, we can now concentrate just on the relationship between Ω_{ijlm} and Θ_{ijlm} using

$$\Psi_i = \chi_i\theta^0 + K_{ij}\theta^j. \tag{51}$$

Rewriting equation (40) as

$$\tfrac{1}{2}\Omega_{ijlm}\theta^l \wedge \theta^m = \left(\tfrac{1}{2}\Theta_{ijlm} + \eta_{il}K_{jm} - K_{im}\eta_{jl} - \eta_{ij}K_{ml}\right)\theta^l \wedge \theta^m \tag{52a}$$

or

$$\Omega_{ijlm} = \Theta_{ijlm} + [\eta_{il}K_{jm} - \eta_{im}K_{jl}] - [K_{im}\eta_{jl} - K_{il}\eta_{jm}] - [\eta_{ij}K_{ml} - \eta_{ij}K_{lm}] \tag{53a}$$

and then multiplying by η^{il} and using equation (43), we obtain

$$\begin{aligned}0 &= \eta^{il}\Omega_{ijlm}\\ &= \eta^{il}\Theta_{ijlm} + 2K_{jm} - K_{mj} + K\eta_{jm}.\end{aligned}$$

Simplifying, with $\eta^{il}\Theta_{ijlm} \equiv R_{jm}$,

$$0 = R_{jm} + 2K_{jm} - K_{mj} + K\eta_{jm} \tag{54}$$

we have the algebraic equation to determine K_{jm} from R_{jm}. By taking the skew part we obtain

$$K_{[jm]} = -\tfrac{1}{3}R_{[jm]}. \tag{55}$$

From the symmetric part

$$R_{(jm)} + K_{(jm)} + K\eta_{jm} = 0$$

and its trace

$$R + 4K = 0$$

we find the symmetric part of K_{jm}

$$K_{(jm)} = \tfrac{1}{4}R\eta_{jm} - R_{(jm)} \tag{56}$$

so that

$$K_{jm} = K_{(jm)} + K_{[jm]} = -\tfrac{1}{12}\eta_{jm}R - R^{TF}_{(jm)} - \tfrac{1}{3}R_{[jm]}; \tag{57}$$

the unique determination of K_{jm} to make the trace of Ω_{ijlm} vanish. This is in agreement with equation (9).

We thus have for the 1-form Ψ_i

$$\Psi_i = \chi_i\theta^0 + K_{im}\theta^m, \tag{58}$$

where all terms are unique functions of $F(u, u', u'', s)$ and A_i.

7. A new curvature

In this section, a new curvature 2-form, Ω_i (the Cartan second curvature) is defined by the third structure equation using Ψ_i from equation (58)

$$\Omega_i = \mathrm{d}\Psi_i + \eta^{lk}\Psi_l \wedge \omega_{ki}. \tag{59}$$

Though Ω_i can be decomposed into

$$\Omega_i = \Omega_{ijk}\theta^j \wedge \theta^k + \Omega_{ij0}\theta^j \wedge \theta^0, \tag{60}$$

in fact, it can be shown that

$$\Omega_{ij0} = 0.$$

The proof is lengthy but straightforward. We first take the exterior derivative of equation (63), which leads, after much cancellations, to the second set of Bianchi identities

$$\Omega_k \wedge \theta^k = 0 \qquad \eta_{jm}\theta^m \wedge \Omega_i - \Omega_j \wedge \theta^m \eta_{mi} = 0. \tag{61}$$

(A third set, which we will not use, is obtained by the exterior derivative of equation (59).)

By substituting equation (60) into the Bianchi identities, one obtains an algebraic equation for Ω_{ij0}, whose only solution is $\Omega_{ij0} = 0$.

At this point the general structure that we have been describing is complete; it is clear that everything $(\theta^i, \Omega_i, \Psi_i, \omega_{ij})$ can be expressed explicitly in terms of $F(u, u', u'', s)$, the A_i and their derivatives. For completeness they are given explicitly in the appendix.

Though it is clear that we are dealing with the differential geometry of a conformal 3-manifold, it is not yet a conventional treatment. We can ask where the curvature information is. The Ricci tensor, R_{ij}, of the ordinary curvature Θ_{ij} has been put into the Ψ_i. The first Cartan curvature Ω_{ij} which is the Weyl tensor is known to vanish in three dimensions. In the special case of vanishing A_i, the remaining curvature Ω_i is the Cotton–York tensor and its generalization when $A_i \neq 0$.

8. Summary and unification

Many technical ideas have been introduced, some naturally appearing and other perhaps appearing rather arbitrarily; however, we have followed closely, but from a different starting point, the treatment by Kobayashi [14] of Cartan's normal conformal connections via the three structure equations (26), (39) and (59).

In this section, we will summarize what has been shown and then tie them together into a unified geometric structure. We show that, *essentially*, we have recovered a Cartan normal conformal, $O(3, 2)$, connection on the principal H-bundle over the solution space, with H being a seven-dimensional subgroup of $O(3, 2)$ [14]. More precisely, we have recovered a variety of cross-sections of this bundle.

We began with a third-order ODE satisfying the Wunschmann condition and a set of three associated 1-forms, θ^i, on the four-dimensional space $\widetilde{P}$. We then found a 'partial' connection

$$\omega_{ij} = \omega_{[ij]} + A\eta_{ij}$$

satisfying the first (torsion-free) structure equation

$$\mathrm{d}\theta^i + \omega^i{}_j \wedge \theta^j = 0 \tag{62a}$$

with three arbitrary functions A_i. A vanishing first-Cartan curvature 2-form was found via the second structure equation

$$\Omega_{ij} = 0 = \mathrm{d}\omega_{ij} + \eta^{kl}\omega_{ik} \wedge \omega_{lj} + \eta_{il}\theta^l \wedge \Psi_j + \Psi_i \wedge \theta^l \eta_{jl} - \eta_{ij}\Psi_k \wedge \theta^k \tag{63}$$

with the proper choice of Ψ_i, equation (58). Finally, a last structure equation and second-Cartan curvature 2-form was introduced by

$$\Omega_i = \mathrm{d}\Psi_i + \eta^{jk}\Psi_j \wedge \omega_{ki} \equiv \mathrm{D}\Psi_i \tag{64}$$

with the property that

$$\Omega_i = \Omega_{ijk}\theta^j \wedge \theta^k.$$

The issue is: what is the meaning of this resulting structure?

If we now *try to consider* the three sets of 1-forms, ten in number,

$$\omega = \left(\theta^j, \omega^k{}_i, \Psi_i\right)$$

to be a connection on $\widetilde{P}$, taking values in the Lie algebra of a group, the group turns out to be the ten-dimensional group $G = O(3, 2)$. Its Lie algebra can be graded as

$$O(3, 2)' = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$$

with

$$\theta^j \epsilon \mathfrak{g}_{-1} \qquad \omega^k{}_i \epsilon \mathfrak{g}_0 \qquad \Psi_i \epsilon \mathfrak{g}_1.$$

Unfortunately, this does not work immediately; we have ten 'connection' 1-forms on a four-dimensional manifold there $\widetilde{P}$, where instead the manifold should be the ten-dimensional bundle P. Aside from the shortage of dimensions, *all the conditions* for a Cartan normal, $O(3, 2)$, conformal connection are there [14]; we have the three structure equations (62*a*), (63), (64), with, in addition, zero torsion, trace-free (even vanishing) Ω_{ij} and $\Omega_i = \Omega_{ijk}\theta^j \wedge \theta^k$.

It is clear that we are dealing with a four-dimensional cross-section of the full ten-dimensional bundle. The question is where and what are the six missing coordinates?

They actually have been discussed earlier; first we have the three arbitrary A_i, three components of the Weyl 1-form A. They can be simply added to $\widetilde{P}$ making it immediately a seven-dimensional manifold. The remaining three coordinates (α, τ, Φ), which were introduced in equation (32), describe scale and triad freedom that preserved the conformal metric, g. The three parameters (s, α, τ) parametrize the Lorentz transformation given by $\Lambda^i{}_j$ in equation (15), while Φ describes the conformal rescaling corresponding to e^ϕ in equation (15). The three arbitrary A_i can be created or destroyed by the use of the variable ξ_k in (15).

In the subsequent discussion, (α, τ, Φ) were taken to have definite values, $\alpha = \Phi = 1$ and $\tau = 0$, though all the calculations could have been done with these parameters as arbitrary coordinates on the ten-dimensional manifold.

The picture that emerges is a ten-dimensional space with coordinates, $(u, u', u'', s, A_i, \alpha, \Phi, \tau)$ (or $(x^a, s, A_i, \alpha, \Phi, \tau)$ if we had introduced base space coordinates, x^a, via ($u = u(x^a, s)$, $u' = u'(x^a, s)$, $u'' = u''(x^a, s)$). Using the base space coordinates x^a, we obtain seven fibre coordinates $(s, A_i, \alpha, \Phi, \tau)$. Taking any or all of these as functions on the base space is a choice of cross-section. In our construction, we have taken the cross-section given by $\alpha = \Phi = 1$, $\tau = 0$ and A_i as functions of (u, u', u'', s). Had the calculation been done without this restriction, we would have obtained the full Cartan, $O(3, 2)$, conformal connection over the three-dimensional base space.

9. Generalization

There is an obvious generalization of the material presented here. Though the details are far from complete—the calculations being quite large—the following seems to be clearly true.

If we begin with a pair of overdetermined PDEs in two independent variables and one dependent variable [6, 7], satisfying a generalized Wunschmann condition (or metricity condition) and some weak inequalities, there exists a rich associated geometric structure. From these PDEs, a four-dimensional conformal Lorentzian solution space (a spacetime manifold) can be defined. On that solution space (the base space), a Cartan normal conformal connection with values in the 15-dimensional group $G = O(4, 2)$ is naturally found. The eleven-dimensional fibres of the subgroup $H = CO(1, 3) \otimes_s T^*$ can be coordinatized by the six parameters of the Lorentz group, a conformal factor and the four components of a Weyl 1-form.

It appears almost certainly that local twistor theory [15] is contained in this structure though how, in detail, is still not clear. Furthermore, since the structures can be associated with all conformal Lorentzian 4-spaces, the conformal Einstein equations must be contained as a restriction on the choice of the pair of PDEs. Work has begun on these problems.

Appendix

For completeness, we give the expressions for the basic variables

$$\{\theta^i, \omega_{[ki]}, \Psi_i, \Omega_i\}$$

directly in terms of $F(u, u', u'', s)$ and A_i. All expressions are modulo the vanishing of the Wunschmann invariant, i.e.,

$$I[F] = F_u - aF_{u''} + \mathrm{D}a - ab = 0.$$

(a)

$$\begin{aligned}\theta^0 &= \mathrm{d}s\\ \theta^1 &= \mathrm{d}u - u'\,\mathrm{d}s,\\ \theta^2 &= \mathrm{d}u' - u''\,\mathrm{d}s,\\ \theta^3 &= \mathrm{d}u'' + b\,\mathrm{d}u' + a\,\mathrm{d}u - [F(u, u', u'', s) + au' + bu'']\,\mathrm{d}s,\end{aligned} \tag{65}$$

with

$$\begin{aligned}a &= -\tfrac{1}{2}F_{u'} - \tfrac{1}{9}(F_{u''})^2 + \tfrac{1}{6}(\mathrm{D}F_{u''})\\ b &= -\tfrac{1}{3}F_{u''}.\end{aligned} \tag{66}$$

(b)

$$\begin{aligned}\omega_{[32]} &= \left(A_2 - \tfrac{1}{2}b_{u''}\right)\theta^1 + A_3\theta^2 - \theta^0\\ \omega_{[31]} &= (A_1 - a_{u''})\theta^1 - \tfrac{1}{2}b_{u''}\theta^2 - A_3\theta^3 - b\theta^0\\ \omega_{[21]} &= L\theta^1 - A_1\theta^2 + \left(\tfrac{1}{2}b_{u''} - A_2\right)\theta^3 - a\theta^0\\ \omega_{22} &= -A = -A_i\theta^i - b\theta^0,\end{aligned} \tag{67}$$

with

$$L = -\tfrac{1}{3}(F_{uu''} - aF_{u''u''} + 3a_{u'} + F_{u''}a_{u''}). \tag{68}$$

(c)

$$\begin{aligned}\Psi_1 = [\mathrm{D}A_1 - b_u - aA_2 + ab_{u''}]\theta^0 &\\ + \left[A_2L + \tfrac{1}{3}F_{u''}L_{u''} + L_{u'} - A_1^2 - A_{1u} + a_{u''}A_1 - aA_{1u''}\right]\theta^1 &\\ + \left[A_{1u'} - bA_{1u''} + \tfrac{1}{2}ab_{u''u''} + \tfrac{1}{2}A_1\,b_{u''} - A_1A_2 + L_{u''} - \tfrac{1}{2}b_{uu''}\right]\theta^2 &\\ + \left[\tfrac{1}{2}\,b_{u''}A_2 - \tfrac{1}{8}b_{u''}^2 + A_{1u''} - \tfrac{1}{2}A_2^2 - \tfrac{1}{2}\,a_{u''u''}\right]\theta^3 &\end{aligned}$$

$$\begin{aligned}\Psi_2 = {} & [\mathrm{D}A_2 - aA_3 - bA_2 + A_1 - b_{u'} + bb_{u''}]\theta^0 \\ & + \left[\tfrac{1}{2}\, ab_{u''u''} - aA_{2u''} + A_{2u} - \tfrac{1}{2}b_{uu''} + L_{u''} + LA_3 - A_1A_2 + \tfrac{1}{2}b_{u''}A_1\right]\theta^1 \\ & + \tfrac{1}{2}\left[bb_{u''u''} - 2\,bA_{2u''} + \tfrac{1}{4}b_{u''}^2 - 2A_1A_3 + 2A_{2u'} - A_2^2 - a_{u''u''} - b_{u''u'}\right]\theta^2 \\ & + \left[A_{2u''} - \tfrac{1}{2}b_{u''u''} + \tfrac{1}{2}b_{u''}A_3 - A_2A_3\right]\theta^3\end{aligned}$$

$$\begin{aligned}\Psi_3 = {} & [\mathrm{D}A_3 + A_2 - 2\,bA_3 - b_{u''}]\theta^0 \\ & + \left[\tfrac{1}{2}b_{u''}A_2 - a_{u''}A_3 - \tfrac{1}{8}\,b_{u''}^2 - aA_{3u''} + A_{3u} - \tfrac{1}{2}A_2^2 - \tfrac{1}{2}a_{u''u''}\right]\theta^1 \\ & + \left[A_{3u'} - bA_{3u''} - \tfrac{1}{2}b_{u''u''} - A_2A_3 - b_{u''}A_3\right]\theta^2 + \left[A_{3u''} - A_3^2\right]\theta^3.\end{aligned}$$

(d)

$$\begin{aligned}\Omega_1 &= \tfrac{1}{6}P\theta^1\wedge\theta^2 + \tfrac{1}{2}N_{u''}\theta^1\wedge\theta^3 - \tfrac{1}{2}L_{u''u''}\theta^2\wedge\theta^3 \\ \Omega_2 &= \tfrac{1}{2}N_{u''}\theta^1\wedge\theta^2 - L_{u''u''}\theta^1\wedge\theta^3 + \tfrac{1}{2}a_{u''u''u''}\theta^2\wedge\theta^3, \\ \Omega_3 &= -\tfrac{1}{2}L_{u''u''}\theta^1\wedge\theta^2 + \tfrac{1}{2}a_{u''u''u''}\theta^1\wedge\theta^3 - \tfrac{1}{6}F_{u''u''u''u''}\theta^2\wedge\theta^3,\end{aligned} \tag{69}$$

with

$$\begin{aligned}N &= \tfrac{1}{3}F_{u''u''}L - \tfrac{2}{3}F_{u''}L_{u''} - 2L_{u'} + aa_{u''u''} - a_{uu''} - \tfrac{1}{2}(a_{u''})^2, \\ P &= -3a_{u''u''}L + 3a_{u''}L_{u''} - 3aL_{u''u''} + 3L_{uu''} + 3N_{u'} + F_{u''}N_{u''}.\end{aligned}$$

It is worth noting that

$$F_{u''u''u''u''} = 0 \tag{71}$$

which implies that

$$a_{u''u''u''} = L_{u''u''} = N_{u''} = P = 0. \tag{72}$$

Thus, the vanishing of $F_{u''u''u''u''}$ is necessary and sufficient for the vanishing of the curvature of the Cartan normal conformal connection associated with the equations for which the Wunschmann invariant is zero. All such equations are contact equivalent to

$$u''' = 0. \tag{73}$$

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