INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **23** (2006) 3123–3127 doi:10.1088/0264-9381/23/9/022

CR structures and asymptotically flat spacetimes

Ezra T Newman[1] **and Pawel Nurowski**[2]

[1] Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh PA, USA
[2] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski,ul Hoza 69, Warszawa, Poland

Received 18 January 2006, in final form 2 March 2006
Published 5 April 2006
Online at stacks.iop.org/CQG/23/3123

Abstract
We discuss the existence, arising by analogy to that in algebraically special spacetimes, of a unique CR structure realized on null infinity for (almost) any asymptotically flat Einstein or Einstein–Maxwell spacetime.

PACS number: 03.50.De

1. Introduction

It has been well known in a large portion of the relativity community that *shear-free null geodesic congruences* play an extremely important role in general relativity [1, 2] and Maxwell theory [3, 4]. Though it is hard to argue from *a priori* knowledge that this should be true, nevertheless from many examples and theoretical discoveries, the importance does, in fact, become easy to see. For example, many of the most important exact solutions of the vacuum Einstein or Einstein–Maxwell equations possess a degenerate principal null vector that is both geodesic and shear free, e.g. the Schwarzschild, the Reissner–Nordstrom, the Kerr and Kerr–Newman metrics. In electrodynamics, the null [3] and the Lienard–Wiechert (as well as the complex Lienard–Wiechert [4]) Maxwell fields have a principal null vector that is also tangent to a shear-free null geodesic congruence. These observations, in turn, led to the more general issue: find all Einstein metrics that possess a principal null vector field that is shear free and geodesic. From this came the discovery of the algebraically special metrics and the beautiful Goldberg–Sachs theorem stating that the degenerate principal null vectors *for Einstein metrics* are always geodesic and shear free. It opened the door to the large subject of studying the properties and integrating the algebraically special metrics. One of the very pretty mathematical discoveries was the automatic existence of a three-dimensional CR structure [5–9] associated with these metrics. In the special case of asymptotically flat algebraically special metrics (or in their flat-space limits), one could choose null infinity, $[\mathfrak{I}^+]$, as the realization space of the CR manifold where a 'portion' of the metric (coming from the congruence itself) defines the CR structure.

From a different perspective, Penrose, in his development of flat-space twistor theory, realized the importance of what is now known as the Kerr theorem, which states that a holomorphic function (of three complex variables) on projective twistor space, CP^3, defines in Minkowski space, a shear-free null geodesic congruence that is, in general, twisting, i.e. is not surface forming. Thus, in some general sense the shear-free null geodesic congruences lie at the centre or origin of twistor theory. The real five-dimensional subspace of projective twistor space, $\mathfrak{N}$, defined by the vanishing of the twistor norm, possesses a five-dimensional CR structure [10, 11]. The intersection of $\mathfrak{N}$ with the subspace of CP^3 that is obtained from the vanishing of a holomorphic function (via the Kerr theorem) is a real three-dimensional CR manifold with the CR structure inherited from that of $\mathfrak{N}$.

The one purpose of this paper is to point out that the shear-free structures associated with these special situations, i.e. flat space or the algebraically special metrics, can be generalized to virtually all asymptotically flat Einstein or Einstein–Maxwell spacetimes. Though shear-free null geodesic congruences cannot be found in arbitrary spacetimes, the idea of *shear-free null geodesic congruences* can be generalized to *asymptotically shear-free null geodesic congruences*. They exist in all asymptotically flat spacetimes.

In several recent articles [12–15] we returned, with a rather unconventional point of view, to the study of asymptotically flat solutions of the Einstein or Einstein–Maxwell equations. The main development in that work was the realization that for *any* given asymptotically flat Einstein or Einstein–Maxwell spacetime with any given Bondi asymptotic shear (with non-vanishing total electric charge), one can find a class of asymptotically shear-free (but in general twisting) null geodesic congruences. Individual members of the class are uniquely given by the choice of an arbitrary complex analytic world lines in the complex four-dimensional space known as H-space. In the case of asymptotically flat vacuum spacetimes, by *mimicking some terms that are found in algebraically special type II metrics*, this complex world line can be chosen uniquely. For the case of the Einstein–Maxwell fields, there is *a pair* of uniquely defined complex world lines in the H-space: one is defined from imposed properties of the Maxwell tensor at null infinity while the other is found again from mimicking terms found in the type II metrics. We concentrated on the special or degenerate case where the *two world lines coincide* or the pure vacuum case. At first it was not at all clear as to what meaning one could assign to this (these) world line(s). Gradually, however, suggestions as to their meaning or physical content did appear. Though it is not the intention here to go into the details of this issue, we remark that the real part of the world line can be identified as a position vector *in some sort* of 'screen' or 'observation' space, with the meaning of a 'centre-of-mass', while the imaginary part can be identified with the asymptotically defined specific spin-angular momentum, i.e. spin per unit mass. The most surprising aspect of this attempt to understand the complex world line was discovering its relationship with the classical Bondi mass–momentum, (M, P^i), and its evolution. Writing the complex world line as $z^a = \xi^a(\tau) = \xi^a_R(\tau) + \mathrm{i}\xi^a_I(\tau)$ and with second-order approximations (essentially around the Reissner–Nordstrom metric), we discovered that we had the relationship [15]

$$P^i = M\frac{\mathrm{d}\xi^i_R}{\mathrm{d}u} - \frac{2q^2}{3c^3}\frac{\mathrm{d}^2\xi^i_R}{\mathrm{d}u^2} - \frac{3}{2c}M\epsilon_{ijk}\frac{\mathrm{d}\xi^j_I}{\mathrm{d}u}\frac{\mathrm{d}\xi^k_R}{\mathrm{d}u} + \cdots. \qquad (1)$$

Then, using the Bondi mass–momentum loss equation, i.e. the equations for $\mathrm{d}P^i/\mathrm{d}u$ and $\mathrm{d}M/\mathrm{d}u$, the equations of motion for both the real and imaginary parts of the world line were determined. One could see immediately that there was, in addition to the standard mass times acceleration, the classical radiation-reaction force and from the mass loss term, there was an additional counterterm that appears to suppress the runaway solutions associated with the radiation-reaction force [15].

2. Asymptotically shear-free null geodesic congruences and CR structures

The main point of this paper is however to point out that there is a simple clear mathematical restatement of our main results [12–15] that generalizes the CR structures associated with the algebraically special spacetimes. What we have shown is that for *any* asymptotically flat vacuum or Einstein–Maxwell spacetime (with a non-vanishing charge), there is (in general) a pair of unique CR structures given on $\mathfrak{I}^+$: one is determined from the Maxwell field, the other from the Weyl tensor. We assumed in that work the special case where the two world lines coincided. Since in the vacuum case there is only one CR structure, we deal in either case with only one CR structure.

The unique CR structure arises in the following manner. We begin with Bondi coordinates $(u, \zeta, \bar{\zeta})$ on $\mathfrak{I}^+$ and with a Bondi one-form basis $(n, l, m, \bar{m})$ and dual vector basis. The one-form n is the covector version of the tangent vector to the generators of $\mathfrak{I}^+$ and l is the covector version of the vector normals to the u = constant slices of $\mathfrak{I}^+$ ($m, \bar{m}$ are the one-form versions of the tangent vectors to the 'slices', u = constant). We then perform a null rotation around n of the form [12]

$$l^* = l + \frac{L}{r}\bar{m} + \frac{\bar{L}}{r}m + O(r^{-2}), \qquad m^* = m + \frac{L}{r}n + O(r^{-2}),$$

where L, at this moment, is an arbitrary complex function on $\mathfrak{I}^+$, i.e. $L = L(u, \zeta, \bar{\zeta})$. The resulting 1-forms on $\mathfrak{I}^+$ are (after a conformal rescaling of m)

$$\begin{aligned} l^* &= \mathrm{d}u - \frac{L}{1+\zeta\bar{\zeta}}\,\mathrm{d}\zeta - \frac{\bar{L}}{1+\zeta\bar{\zeta}}\,\mathrm{d}\bar{\zeta}, \\ m^* &= \frac{\mathrm{d}\bar{\zeta}}{1+\zeta\bar{\zeta}}, \qquad \overline{m}^* = \frac{\mathrm{d}\zeta}{1+\zeta\bar{\zeta}}. \end{aligned} \tag{2}$$

We note that, *for any choice* of the function $L(u, \zeta, \bar{\zeta})$, the three 1-forms from equation (2) are a representative set of 1-forms (up to gauge freedom) that define a CR structure on $\mathfrak{I}^+$. Note also that L can be geometrically interpreted as the complex stereographic angle on the sphere of the past light cones on $\mathfrak{I}^+$, so that geometrically $L(u, \zeta, \bar{\zeta})$ is an angle field on $\mathfrak{I}^+$.

When we required that the new null congruence defined by l^* be asymptotically shear free, we discovered [12] that $L = L(u, \zeta, \bar{\zeta})$ satisfies the nonlinear differential equation

$$\eth L + L L_{,u} = \sigma(u, \zeta, \bar{\zeta}) \tag{3}$$

with $\sigma(u, \zeta, \bar{\zeta})$ the freely chosen radiation data, i.e. the Bondi asymptotic shear.

It was shown that this equation could be transformed [12], so that, surprisingly, one could see immediately that the solutions were given up to the choice of an arbitrary world line in H-space. Specifically, the solutions were constructed in the following manner.

We begin with the transform of (3), the 'good cut equation', i.e.

$$\eth^2 X = \sigma(X, \zeta, \bar{\zeta}).$$

and write the general solution as

$$u = X(z^a, \zeta, \bar{\zeta}), \tag{4}$$

with z^a an arbitrary point in the four complex dimensional H-space. Then choosing an *arbitrary* H-space world line, $z^a = \xi^a(\tau)$, and substituting it into the solution, (4), yields

$$u = X(\xi^a(\tau), \zeta, \bar{\zeta}) \equiv Z(\tau, \zeta, \bar{\zeta}).$$

Finally by the application of ð to $Z(\tau, \zeta, \overline{\zeta})$, we have the solution to equation (3) that is given parametrically by

$$u = Z(\tau, \zeta, \overline{\zeta}) \tag{5}$$

$$L(u, \zeta, \bar{\zeta}) = ðZ(\tau, \zeta, \overline{\zeta}). \tag{6}$$

We thus have the result that each (regular) asymptotically shear-free null geodesic congruence is determined by the arbitrary choice of the H-space world line $z^a = \xi^a(\tau)$. At this stage, we have a CR structure for each choice of the world line.

Finally, by imposing the conditions (mentioned earlier) on the Weyl tensor and Maxwell tensor that mimicked the algebraically special Weyl tensor, we obtained a unique world line and unique solution for $L(u, \zeta, \bar{\zeta})$. Consequently, we have a unique CR structure on $\mathfrak{I}^+$.

An additional point to note is that if the equation $u = Z(\tau, \zeta, \overline{\zeta})$ is inverted so that

$$\tau = T(u, \zeta, \bar{\zeta}), \tag{7}$$

it becomes easy to show that τ is a CR function, i.e. $T(u, \zeta, \bar{\zeta})$ satisfies the CR equation

$$ðT + LT_{,u} = 0, \tag{8}$$

which gives a geometric meaning to the world line parameter τ. The proof of this comes from differentiating the implicit form of τ, i.e. from (5). The u derivative yields

$$1 = \partial_\tau Z(\tau, \zeta, \overline{\zeta}) \cdot T_{,u} \tag{9}$$

$$\partial_\tau Z(\tau, \zeta, \overline{\zeta}) = \frac{1}{T_{,u}}, \tag{10}$$

and the ð derivative gives

$$0 = \partial_\tau Z(\tau, \zeta, \overline{\zeta}) \cdot ðT + ðZ \tag{11}$$

$$0 = \partial_\tau Z(\tau, \zeta, \overline{\zeta}) \cdot ðT + L. \tag{12}$$

Putting (10) into (12) then yields the CR equation, (8).

The local $\mathbf{C}^2$ from which the three-dimensional CR manifold is defined is $(\tau, \bar{\zeta})$, with the parametric form of the embedding given by $(\tau, \bar{\zeta}) = (T(u, \zeta, \bar{\zeta}), \bar{\zeta})$.

3. Discussion

In summary, considering only the special Einstein–Maxwell case where the congruence obtained from the Maxwell tensor coincided with that of the Weyl tensor, we have shown that for any asymptotically flat vacuum Einstein or Einstein–Maxwell spacetime (with non-vanishing total charge), there is a unique asymptotically shear-free null geodesic congruence that is determined by mimicking certain properties of algebraically special Weyl tensors. From the description of this congruence, i.e. from the H-space curve, $z^a = \xi^a(\tau)$, and the related angle field, $L(u, \zeta, \overline{\zeta})$, we found a unique three-dimensional CR structure that was realized on real $\mathfrak{I}^+$. Though we do not see an immediate physical consequence of this observation, there is now considerable unification of the mathematical ideas associated with the algebraically special metrics and general asymptotically flat spacetimes. Perhaps the strange fact that the CR function $\tau = T(u, \zeta, \overline{\zeta})$ parametrizes the motion of the H-space world line that leads to the physical momenta and equations of motion, (1), has physical meaning. For certain flat-space situations, τ can be taken as the proper time along a real world line. In any case, that question needs further study.

We mention that for the Einstein–Maxwell spacetimes, when the two world lines coincide, i.e. when there is a unique CR structure, we found from our physical interpretation of the world line, the surprising result that the gyromagnetic ratio (the ratio of spin to magnetic moment) is the same as that of Dirac, namely $g = 2$. The charged Kerr metric, which possesses a single H-space complex world line and a unique CR structure, is a special case of this result.

Since we are dealing here with the CR structure arising from *asymptotically shear-free* congruences rather than from *shear-free* congruences, a question immediately arises: is there any analogue of the Kerr theorem, which gave the shear-free null geodesic congruences [10] via arbitrary holomorphic functions on twistor space, to our case of finding *asymptotically* shear-free null geodesic congruences. The answer is yes. The full details, which are lengthy, will be presented elsewhere. We simply note the basic structures involved. The role of flat twistor space is played by asymptotic twistor space [11, 16–19]. The analogue of the Kerr theorem, which produces asymptotically shear-free congruences, lies in the arbitrary choice of the complex world line in H-space. The vanishing of the (Kähler) norm [18, 11] of the asymptotic twistor space, which defines the analogue of $\mathfrak{N}$, yields the five-dimensional CR manifold, $\mathfrak{N}^*$, a real subspace of projective asymptotic twistor space. The three-dimensional CR structure given on $\mathfrak{I}^+$, which has been described here, is inherited from $\mathfrak{N}^*$.

Acknowledgments

This material is based upon work (partially) supported by the National Science Foundation under grant PHY-0244513. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science. ETN thanks the NSF for this support. He also thanks both Roger Penrose for certain research suggestions and Jerzy Lewandowski for great help in understanding CR structures. PN was supported by the KBN grant 1 P03B 07529.

References

[1] Robinson I and Trautman A 1962 *Proc. R. Soc.* A **289** 463
[2] Goldberg J N and Sachs R K 1962 *Acta Phys. Pol.* **22** Suppl. 13
[3] Robinson I 1961 *J. Math. Phys.* **2** 290–1
[4] Newman E T 2004 *Class. Quantum Grav.* **21** 1–25
[5] Lewandowski J, Nurowski P and Tafel J 1990 *Class. Quantum Grav.* **7** L241–6
[6] Lewandowski J, Nurowski P and Tafel J 1991 *Class. Quantum. Grav.* **8** 493–501
[7] Lewandowski J and Nurowski P 1990 *Class. Quantum Grav.* **7** 309–28
[8] Tafel J 1985 *Lett. Math. Phys.* **10** 33–9
[9] Trautman A 1984 *J. Geom. Phys.* **1** 85–95
[10] Nurowski P and Trautman A 2002 *Differ. Geom. Appl.* **17** 175–95
[11] Penrose R and Rindler W 1984 *Spinors and Space-Time* vol 2 (Cambridge: Cambridge University Press)
[12] Kozameh C and Newman E T 2005 *Class. Quantum Grav.* **22** 4659–65
[13] Kozameh C and Newman E T 2005 *Class. Quantum Grav.* **22** 4667–78
[14] Kozameh C, Newman E T and Silva-Ortigoza G 2005 *Class. Quantum Grav.* **22** 4679–98
[15] Newman E T and Silva-Ortigoza G 2006 *Class. Quantum Grav.* **23** 91–113
[16] Penrose R and MacCallum M A H 1973 *Phys. Rep.* C 6
[17] Penrose R 1975 Twistor theory, its aims and achievments *Quantum Gravity* ed C J Isham, R Penrose and D Sciama (Oxford: Oxford University Press)
[18] Newman E T and Penrose R 1977 *J. Math. Phys.* **18** 58
[19] Hansen R, Newman E T, Penrose R and Tod K P 1978 *Proc. R. Soc.* A **363** 445