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Conformal Lorentzian metrics on the spaces of curves and 2-surfaces

S Frittelli[1], E T Newman[2] and P Nurowski[3]

[1] Department of Physics, Duquesne University Pittsburgh, PA 15282, USA
[2] Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh PA, USA
[3] Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, ul Hoza 69, Warszawa, Poland

E-mail: simo@mayu.physics.duq.edu, newman@pitt.edu and nurowski@fuw.edu.pl

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Abstract

We show how, beginning with the space of curves in R^2 and the space of 2-surfaces in R^3, one can define conformal Lorentzian three- and four-dimensional metrics on certain special subspaces. It is conjectured that for the case of curves in R^2 this is how Wunschmann obtained his well-known equation. Generalizing the argument to the case of 2-surfaces, leads to a generalized Wunschmann equation. We point out that all conformal Lorentzian three- and four-dimensional metrics (thus including Einstein metrics) can be obtained in this manner.

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1. Introduction

In the past several years there have been a series of papers [1–5] relating the conformal geometries of three and four-dimensional Lorentzian spacetimes, via the solutions of the (conformally invariant) eikonal equation, with certain sets of differential equations. One showed that all three-dimensional conformal Lorentzian geometries were encoded in equivalence classes of third-order ordinary differential equations (ODEs) with the vanishing of a certain function, defined from the differential equation itself, that is referred to as the Wunschmann invariant. Likewise, all four-dimensional conformal Lorentzian geometries were encoded in equivalence classes of pairs of second-order partial differential equations (PDEs) with the vanishing of an analogous generalized Wunschmann invariant. The four-dimensional case is obviously of greater interest to physicists since among all four-dimensional Lorentzian geometries the conformal Einstein equations are included. This, however, means that to code or obtain the conformal Einstein equations into the pairs of second-order PDEs further restrictions (in addition to that of the vanishing of the generalized Wunschmann invariant) must be imposed on the pairs. This is the basic point of view [6] of what has been called the *null surface formulation* of general relativity (GR). Though at present these further conditions are not explicitly known, recent work [7, 8] has suggested how those conditions could be found—and calculations to this end have begun. When we know how to restrict the pairs of second-order PDEs to those containing the conformal Einstein equations we will have another tool to investigate general relativity.

The purpose of the present work is to go back and understand the origin of the Wunschmann invariant. Though it only codifies the conformally invariant four-dimensional spacetimes and not specifically the Einstein equations, understanding the Wunschmann condition is essential. Specifically, one would like to see geometrically how the Wunschmann condition is expressed from the spacetime point of view.

We begin with a brief historical review.

In the first years of the twentieth century a great deal of effort in mathematics was devoted to the study and classification of ordinary differential equations according to their equivalence classes under a variety of transformations and the resulting induced geometries on the solution spaces. As an example, Lie and Tresse, and later Cartan [9–11] studied the classification of general second-order equations under point transformations and related them to induced projective structures in the two-dimensional solution space.

Among the many discoveries concerning the study of the geometry associated with ODEs there is one that appeared to us to be particularly worth revisiting. Wunschmann [12] discovered that in the set of all third-order equations there was a particular subclass of equations (referred to as the Wunschmann class) with the rather remarkable property of inducing on the (three-dimensional) solution space, an indefinite conformal metric. (In fact all indefinite conformal three-metrics can be obtained in this fashion.) An interesting fact about Wunschmann's work is that, even though the defining equation for the Wunschmann class is often quoted [1, 3, 13–17] and referred to as the Wunschmann equation, we, however, have never seen a description of how Wunschmann obtained this class nor how he defined the conformal three-metric from the ODE.

The purpose of this paper is twofold; (1) we will show how Wunschmann *might* have discovered his class of equations and their associated conformal three-metrics and (2) show how this (presumed) method of Wunschmann can be generalized to pairs of overdetermined second-order PDEs, so that one obtains on the four-dimensional solution space, conformal Lorentzian four-metrics. (Again we remark that all conformal Lorentzian four-metrics can be obtained in this fashion.)

In section 2, we will begin with the general third-order ODE (prime denotes s derivatives)

$$u''' = F(u, u', u'', s) = F(u, p, q, s) \tag{1}$$

whose solution space consists of three independent parameters, $x^i = (t, x^1, x^2)$. Then by studying the three-parameter family of curves in the (u, s) space defined as solutions to equation (1), namely

$$u = z(s, x^i) \tag{2}$$

and their generalized Jacobi fields, $(\delta u, \delta u', \delta u'')$ obtained by the independent variation of the three constants of integration x^i and two s derivatives of equation (2), we will show how a simple condition on neighbouring curves that touch (i.e., are tangent) leads to the Wunschmann equation and the associated conformal metric.

In section 3, we generalize our considerations and, instead of equation (1), consider the pair of PDEs (the subscripts s and t denote partial derivatives)

$$u_{ss} = S(u, u_s, u_t, u_{st}, s, t) \qquad u_{tt} = S^*(u, u_s, u_t, u_{st}, s, t) \tag{3}$$

which possess a four-parameter [3] (or dimensional) solution space, x^a. The solutions

$$u = Z(s, t, x^a) \tag{4}$$

define a four-parameter family of 2-surfaces in the three-space, (u, s, t). By varying the parameters x^a we get variations in the u and its s and t derivatives that we denote by $(\delta u, \delta u_s, \delta u_t, \delta u_{st},)$ and refer to as a generalized Jacobi field. Again by imposing a simple condition on the touching of two neighbouring 2-surfaces we show how a generalization of Wunschmann's equation arises and with it a conformal Lorentzian four-metric determined uniquely in terms of the functions S and S^*.

The two problems addressed here, i.e., the geometry associated with third-order ODEs and pairs of second-order PDEs, are not new; there already exists a large literature dealing with them. In this work we are simply showing an alternate approach that appears to us to be both simple and clear and thus complements the earlier work.

2. Third-order ODEs

Beginning with the general third-order ODE

$$u''' = F(u, u', u'', s) \tag{5}$$

and a three-parameter solution

$$u = z(s, x^i) \tag{6}$$

we define the varied solution by

$$\delta u = \partial_i z \,\mathrm{d}x^i \equiv z_i(s, x^j)\,\mathrm{d}x^i \tag{7}$$

which, in turn, satisfies the deviation equation (or linearized equation (5))

$$\delta u''' = \delta F(u, u', u'', s) \equiv F_u \delta u + F_{u'}\delta u' + F_{u''}\delta u''. \tag{8}$$

In the two-dimensional (u, s) space, for fixed x^j, equation (7) defines, via the three $\mathrm{d}x^i$, a three-parameter family of neighbouring curves. Rather than treating the variation to be associated with the independent $\mathrm{d}x^i$, we will treat the three $(\delta u(s), \delta u'(s), \delta u''(s))$ as independent variations (the s behaviour, of course, subject to the deviation equation). On this three-space of independent curves we *will attempt to define an infinitesimal quadratic distance* (i.e., metric distance) between nearby curves. At the beginning of the discussion this *distance* will clearly depend on the value of s, i.e., will depend on a point on the first curve for the comparison with the neighbouring curve. As conditions are imposed, this *distance* becomes a conformal *distance*; i.e., for two neighbouring curves there will be a *distance* between them that is unique up to an s-dependent conformal factor.

To begin with we *define* a metric *distance* (the g_{AB} to be determined) between nearby curves, at the point s, by

$$\begin{aligned} g(s, x^i) &= g_{AB}\delta u^A \delta u^B \\ &= g_{00}\delta u\delta u + 2g_{0+}\delta u\delta u' + 2g_{01}\delta u\delta u'' + g_{++}\delta u'\delta u' + 2g_{+1}\delta u'\delta u'' + g_{11}\delta u''\delta u'' \end{aligned} \tag{9}$$

where our notation associates $(0, +, 1)$ with $(\delta u, \delta u', \delta u'') = (\delta u^0, \delta u^+, \delta u^1)$. The following two conditions are imposed on the metric.

- The first condition is that the *distance* between neighbouring curves that are tangent to each other, *at some point* s_0, should vanish at s_0. Tangency means that both δu and $\delta u'$ vanish at s_0. This then implies, via equation (9), that we must have $g_{11} = 0$.

- The second condition imposed is that when two curves are tangent at some point s_0, the *distance* between them should *vanish at all points* along either curve. This means that if $g(s, x^i)$ vanishes at s_0, its derivative must also vanish there. This implies that the basic condition imposed on the metric should be

$$g' = \Lambda g. \tag{10}$$

Remark 1. The notation g' means that the s derivatives are applied to both the g_{AB} and δu^A.

Remark 2. When equation (10) is satisfied, then for $\tilde{g} = \Omega g$, with $\Omega = \Omega_0\, \mathrm{e}^{-\int^s \Lambda \mathrm{d}s}$ and $\Omega_0' = 0$, we find that

$$\tilde{g}' = 0,$$

i.e. a conformal metric exists on the three-space of curves. We thus see that the condition (10), if it can be imposed on the function $F(u, u', u'', s)$, is a very powerful constraint. We now prove the following theorem:

Theorem 1. *Given a third-order ODE (equation (5)) and its deviation equation, if the metric defined by equation (9) with $g_{11} = 0$ satisfies equation (10), then the metric, up to an arbitrary conformal factor, is uniquely determined as a function of $F(u, u', u'', s)$ and furthermore, $F(u, u', u'', s)$ satisfies the differential equation (the vanishing of the Wunschmann invariant),*

$$D^2 F_{u''} - 2F_{u''} D F_{u''} - 3DF_{u'} + 6F_u + \tfrac{4}{9}F_{u''}^3 + 2F_{u''}F_{u'} = 0 \tag{11}$$

with the total s derivative D defined by

$$DJ(u, u', u'', s) \equiv J_s + J_u u' + J_{u'} u'' + J_{u''} F. \tag{12}$$

Definition 2. *We will use a D to denote s derivatives of functions of (u, u', u'', s) and simply a prime to denote s derivatives of functions only of s.*

Before proving the theorem, we first prove the lemma;

Lemma 3. *Equation (9), with $g_{11} = 0$, and equation (10) imply that $g_{1+} = 0$.*

Proof. This is easily seen by first writing equation (9), with $g_{11} = 0$, as

$$\begin{aligned} g(s, x^i) &= g_{00}(\delta u)^2 + 2g_{0+}\delta u\delta u' + 2g_{01}\delta u\delta u'' + g_{++}\delta u'\delta u' + 2g_{+1}\delta u'\delta u'' \\ &\equiv \hat{g}(s, x^i) + 2g_{+1}\delta u'\delta u'' \end{aligned} \tag{13}$$

and then noting that, since g has no $\delta u''\delta u''$ term, equation (10)

$$Dg = D\hat{g} + 2Dg_{+1}\delta u'\delta u'' + 2g_{+1}\delta u''\delta u'' + 2g_{+1}\delta u'\delta u''' = \Lambda g$$

implies that

$$g_{+1} = 0. \qquad \square$$

Proof of theorem 1. By choosing a conformal factor (done for simplicity but not needed) so that

$$g_{01} = 1,$$

the metric has the form

$$g(s, x^i) = 2\alpha\delta u\delta u + 2\beta\delta u\delta u' + \gamma\delta u'\delta u' + 2\delta u\delta u''$$

with three unknown functions, α, β and γ. Explicitly differentiating g and writing $Dg = \Lambda g$ we obtain

$$2D\alpha\delta u\delta u + 2(2\alpha + D\beta)\delta u\delta u' + (2\beta + D\gamma)\delta u'\delta u' + 2\beta\delta u\delta u'' + 2(\gamma + 1)\delta u'\delta u'' + 2\delta u\delta u''' = \Lambda\{2\alpha\delta u\delta u + 2\beta\delta u\delta u' + \gamma\delta u'\delta u' + 2\delta u\delta u''\}.$$

Replacing $\delta u'''$ by the deviation equation

$$\delta u''' = F_u\delta u + F_{u'}\delta u' + F_{u''}\delta u'' \tag{14}$$

we obtain

$$(2D\alpha + 2F_u)\delta u\delta u + 2(2\alpha + D\beta + F_{u'})\delta u\delta u' + (2\beta + D\gamma)\delta u'\delta u' + 2(\beta + F_{u''})\delta u\delta u'' + 2(\gamma + 1)\delta u'\delta u'' = \Lambda\{2\alpha\delta u\delta u + 2\beta\delta u\delta u' + \gamma\delta u'\delta u' + 2\delta u\delta u''\}.$$

Equating the coefficients of the quadratic deviation terms, we immediately obtain

$$D\alpha + F_u = \Lambda\alpha$$
$$2\alpha + D\beta + F_{u'} = \Lambda\beta$$
$$2\beta + D\gamma = \Lambda\gamma$$
$$(\beta + F_{u''}) = \Lambda$$
$$\gamma + 1 = 0$$

and thus finally have

$$\gamma = -1 \tag{15}$$

$$\Lambda = -2\beta \tag{16}$$

$$\beta = -\tfrac{1}{3}F_{u''} \tag{17}$$

$$\alpha = -\beta^2 - \tfrac{1}{2}D\beta - \tfrac{1}{2}F_{u'} \tag{18}$$

$$D\alpha + F_u + 2\beta\alpha = 0 \tag{19}$$

where the first four relations define the unknown functions in terms of F, while the last equation, using the two previous ones, yields the Wunschmann equation:

$$D^2F_{u''} - 3DF_{u'} - 2F_{u''}DF_{u''} + 6F_u + \tfrac{4}{9}F_{u''}^3 + 2F_{u''}F_{u'} = 0. \tag{20}$$

□

We thus see that by requiring that the vanishing of the *distance* function between two touching curves be preserved as one moves along the curves, we obtain a conformal (2, 1) metric on the solution space of the ODE. From a vague comment or clue in Cartan's work [13] we believe that the argument just given is essentially that of Wunschmann. Without discussion, we point out that it has been shown elsewhere [3] that all conformal Lorentzian three-metrics can be obtained from third-order ODEs that satisfy the Wunschmann equation.

3. Pairs of second-order PDEs

Here we show that the (presumed) Wunschmann argument for the third-order ODEs is easily extended to pairs of second-order PDEs and leads to a generalized Wunschmann equation and a four-dimensional conformal Lorentzian metric on the solution space of the PDEs.

To begin we establish the following notation; our independent variable is z as a function of the two dependent variables (s and t), i.e., $u = z(s, t)$. For functions of (s, t) we denote their derivatives by $(z_s, z_t, z_{st}, z_{ss}, z_{tt})$ while for functions of $(z, z_s, z_t, z_{st}, s, t)$ we will denote the total (s and t) derivatives by D_s and D_t. For example,

$$D_s J(z, z_s, z_t, z_{st}, s, t) = J_z z_s + J_{z_s} z_{ss} + J_{z_t} z_{ts} + J_{z_{ts}} z_{sst} + J_s$$

where z_{ss} and z_{sst} are expressed in terms of the differential equations.

We begin with the general pair of equations

$$G_1(z, z_s, z_t, z_{st}, z_{ss}, z_{tt}) = 0 \qquad G_2(z, z_s, z_t, z_{st}, z_{ss}, z_{tt}) = 0$$

and assume that they can be solved for (z_{ss}, z_{tt}) yielding the pair

$$z_{ss} = S(z, z_s, z_t, z_{st}, s, t) \qquad z_{tt} = T(z, z_s, z_t, z_{st}, s, t) \tag{21}$$

that satisfy the integrability condition

$$D_s^2 T = D_t^2 S \tag{22}$$

and the weak inequality

$$1 - S_{z_{st}} T_{z_{st}} > 0. \tag{23}$$

With no loss of generality (but with a simplification of the calculations), we take both (s, t) and (S, T) to be complex conjugate pairs and use instead the notation (s, s^*) and (S, S^*). The PDEs are then

$$z_{ss} = S(z, z_s, z_{s^*}, z_{ss^*}, s, s^*) \qquad z_{s^*s^*} = S^*(z, z_s, z_{s^*}, z_{ss^*}, s, s^*). \tag{24}$$

It has been shown [3] that, for this pair of equations satisfying equations (22) and (23), the solution space is four dimensional so that the solutions can be written as

$$u = z(s, s^*, x^a) \tag{25}$$

and interpreted as a four-parameter family, x^a, of 2-surfaces in the three-dimensional space (u, s, s^*). (We make an assumption of genericity, namely, that the four functions, $z, z_s, z_{s^*}, z_{ss^*}$ are independent functions of the x^a and hence can be inverted.) For a surface with fixed value of x^a there is a four-parameter family of neighbouring surfaces with coordinates $x^a + \mathrm{d}x^a$. Rather than describing the neighbouring surfaces by the independent $\mathrm{d}x^a$ we will do so by the independent 'connecting vectors',

$$(\delta z, \delta z_s, \delta z_{s^*}, \delta z_{ss^*}) \tag{26}$$

i.e., $\delta z = z_a\, \mathrm{d}x^a$, $\delta z_s = z_{sa}\, \mathrm{d}x^a$, etc.

The variation of the higher derivative terms (e.g., $\delta z_{ss}, \delta z_{sss^*}$, etc) is obtained from the deviation equations obtained by the variation of equations (24) (where for typographic reasons we use $S_0 = S_z$, $S_+ = S_{z_s}$, $S_- = S_{z_{s^*}}$, $S_1 = S_{z_{ss^*}}$, etc), i.e.,

$$\begin{aligned} \delta z_{ss} &= S_0 \delta z + S_+ \delta z_s + S_- \delta z_{s^*} + S_1 \delta z_{ss^*} \\ \delta z_{s^*s^*} &= S_0^* \delta z + S_+^* \delta z_s + S_-^* \delta z_{s^*} + S_1^* \delta z_{ss^*} \end{aligned} \tag{27}$$

and their derivatives

$$\begin{aligned} \delta z_{sss^*} &\equiv K_0 \delta z + K_+ \delta z_s + K_- \delta z_{s^*} + K_1 \delta z_{ss^*} \\ \delta z_{s^*s^*s} &\equiv K_0^* \delta z + K_+^* \delta z_s + K_-^* \delta z_{s^*} + K_1^* \delta z_{ss^*} \end{aligned} \tag{28}$$

with the K (given in the appendix) being further derivatives of S and S^*.

The presumed Wunschmann-like argument begins, as in the previous section, with a quadratic conformal metric defining, initially, the *distance* between two neighbouring surfaces at the point (s, s^*) in terms of the deviation vector, i.e.,

$$g = g_{AB} \delta z^A \delta z^B \tag{29}$$

with

$$\delta z^A = (\delta z, \delta z_s, \delta z_{s^*}, \delta z_{ss^*}) \equiv (\delta z^0, \delta z^+, \delta z^-, \delta z^1).$$

- In analogy with the argument for the third-order ODE, the first condition we impose is that the *distance* between neighbouring 2-surfaces, that are tangent to each other *at some point* (s_0, s_0^*), should vanish at *that* point. Tangency means that $(\delta z, \delta z_s, \delta z_{s^*}) \equiv (\delta z^0, \delta z^+, \delta z^-)$ must vanish at (s_0, s_0^*). This in turn implies that the coefficient of $\delta z^1 \delta z^1$ must vanish, i.e.,

$$g_{11} = 0.$$

- The second condition imposed is that when two 2-surfaces are tangent and hence have a vanishing distance at some point (s_0, s_0^*), the *distance* between them should *vanish at all points*. This means that if $g(s_0, s_0^*, x^i)$ vanishes at (s_0, s_0^*), its derivative must also vanish there. This implies our basic conditions to be imposed on the metric, namely

$$D_s g = \Lambda g, \tag{30}$$

$$D_{s^*} g = \Lambda^* g. \tag{31}$$

In a similar manner to the proof of the lemma in section 2, one can easily see that equations (30) and (31) imply that the coefficients of $\delta z^+ \delta z^1$ and $\delta z^- \delta z^1$ must vanish, i.e.,

$$g_{1+} = g_{1-} = 0.$$

With the choice of conformal factor so that

$$g_{10} = 1$$

the metric, equations (29), with six arbitrary functions, can be written as

$$g = 2\{\theta^{(0}\theta^{1)} - \theta^{(+}\theta^{-)}\}$$

with

$$\begin{aligned} \theta^0 &= \delta z, \\ \theta^1 &= \delta z_{ss^*} + a\delta z_s + \bar{a}\delta z_{s^*} + c\delta z, \\ \theta^+ &= \alpha(\delta z_s + b\delta z_{s^*}), \\ \theta^- &= \alpha(\delta z_{s^*} + \bar{b}\delta z_s) \end{aligned}$$

or

$$\tfrac{1}{2} g = c\delta z^2 + \delta z \delta z_{ss^*} + a\delta z \delta z_s + a^* \delta z \delta z_{s^*} - \alpha^2 \left\{(1 + bb^*)\delta z_s \delta z_{s^*} + b^* \delta z_s^2 + b\delta z_{s^*}^2\right\}. \tag{32}$$

By a simple calculation from the nine non-trivial components of $D_s g = \Lambda g$ (and its conjugate equation) we obtain nine (plus the conjugate) equations that uniquely determine the coefficients $(c, a, a^*, \alpha, b, b^*, \Lambda, \Lambda^*)$ plus a differential condition on the functions S and S^* referred to as the generalized Wunschmann equation. Explicitly, we have the unknown functions expressed completely in terms of the derivatives of S and S^* in the following recursive manner,

$$b = \frac{-1 + \sqrt{1 - S_1 S_1^*}}{S_1^*} \tag{33}$$

$$\bar{b} = \frac{-1 + \sqrt{1 - S_1 S_1^*}}{S_1} \tag{34}$$

$$\alpha^2 = \frac{1 + b\bar{b}}{(1 - b\bar{b})^2} \tag{35}$$

$$a = \alpha^2 b^{-1}[\bar{b}_s b - \bar{b} b_s + 2\bar{b} b S_+ + (1 + b\bar{b})\bar{b} S_-] \tag{36}$$

$$\bar{a} = \alpha^2 \bar{b}^{-1}[b_{s^*}\bar{b} - b\bar{b}_{s^*} + 2\bar{b} b S_-^* + (1 + b\bar{b}) b S_+^*] \tag{37}$$

$$\Lambda = \bar{a} + K_1 - \frac{2ab}{(1 + b\bar{b})} \tag{38}$$

$$2c = (\Lambda - S_+)a - a_s - K_+ + 2\alpha^2 \bar{b} S_0 \tag{39}$$

and

$$b_s + S_- - bS_+ + \frac{(1 - \bar{b}b)}{(1 + \bar{b}b)} b(\bar{a} - ab) = 0 \tag{40}$$

for the (complex) generalized Wunschmann equation. The details for this calculation are given in appendices A and B.

We thus have the theorem:

Theorem 4. *Given the pair of overdetermined PDEs, equations (24), satisfying equations (22) and (23) and the generalized Wunschmann equation, there is induced on the four-dimensional solution space a unique conformal four-dimensional Lorentzian metric.*

By conformally rescaling the metric, equation (32), so that $\tilde{g} = \Omega g$, one can find Ω so that

$$D_s \tilde{g} = D_{s^*} \tilde{g} = 0 \tag{41}$$

showing that a conformal metric does exist on the solution space x^a. Demonstrating this requires showing the integrability of the pair $D_s g = \Lambda g$ and $D_{s^*} g = \Lambda^* g$, a fairly lengthy calculation.

Remark 3. As mentioned earlier, we stress that it has been shown [3, 5] that all conformal Lorentzian four-metrics can be constructed in this manner, i.e., any four-dimensional Lorentzian metric can be constructed from some complex conjugate pair S and S^* satisfying the generalized Wunschmann equation.

4. Conclusions

There has been much work and many publications on the differential geometry associated with both ODEs and PDEs over the years. Much of it was based on the problems arising from the study of equivalence classes for fibre preserving, point and contact transformations. Others arose from the study of the eikonal equation [4] in three- and four-dimensional Lorentzian manifolds and some were even specialized to Einstein manifolds. The rich geometries arising from these considerations can lead to a variety of structures, e.g., Cartan normal conformal or projective connections [3, 18–20].

In this paper we have shown how it could have been possible for Wunschmann, in one of the earliest and simplest efforts, to have approached these issues from the study of a distance function between neighbouring curves in a three-parameter family of curves in two-space and its generalization to a four-parameter family of 2-surfaces in three-space.

As a final comment, we describe how these curves of Wunschmann and their generalization to 2-surfaces have a simple and pretty interpretation in terms of three- and four-dimensional spacetimes. These 'spacetimes' are simply defined as the three- and four-dimensional solutions spaces of the differential equations.

We consider first an arbitrary but general 2-surface $\mathfrak{S}$ embedded in the three-dimensional solution space of the third-order ODE. This surface is to be a realization of the (u, s) space of section 2. The (three-dimensional) family of light cones emanating from (the three-dimensional set) of spacetime points x^i will, in general, intersect $\mathfrak{S}$ in a three-parameter family of curves that can be identified with the Wunschmann curves in the (u, s) space. The point of tangency of two neighbouring curves is interpreted as the intersection with $\mathfrak{S}$ of a common null geodesic on two different light cones so that their neighbouring apexes have a null metric separation. The light cones will be tangent to each other on the geodesic and are thus tangent at $\mathfrak{S}$. This agrees with and gives our choice of conformal metric on the space of curves an alternative meaning; namely that of a conformal metric on the solution space.

This immediately generalizes to the four-dimensional case where the (four-dimensional) family of light cones from spacetime points intersect an arbitrary 3-surfaces, $\mathfrak{I}$ (e.g., the Scri of an asymptotically flat spacetime is often used) in a four-parameter family of 2-surfaces, the cuts of $\mathfrak{I}$. $\mathfrak{I}$ is interpreted as being the (u, s, s^*) space. The conformal metric on the space of 2-surfaces is then reinterpreted as a conformal metric on the four-space of solutions, i.e., on spacetime.

Attempts are now being made to find further restrictions on the pairs of second-order PDEs to those yielding the conformal Einstein metrics.

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Appendix A

By taking the s and s^* derivatives of the deviation equations

$$\begin{aligned}\delta z_{ss} &= S_0\delta z + S_+\delta z_s + S_-\delta z_{s^*} + S_1\delta z_{ss^*}\\ \delta z_{s^*s^*} &= S_0^*\delta z + S_+^*\delta z_s + S_-^*\delta z_{s^*} + S_1^*\delta z_{ss^*}\end{aligned}\tag{A.1}$$

and after some manipulation we obtain

$$\delta z_{sss^*} \equiv K_0\delta z + K_+\delta z_s + K_-\delta z_{s^*} + K_1\delta z_{ss^*}\tag{A.2}$$

and its complex conjugate with

$$\begin{aligned}K_0 &= (1 - S_1S_1^*)^{-1}\{S_-S_0^* + S_{0t} + S_1(S_+^*S_0 + S_{0s}^*)\}\\ K_+ &= (1 - S_1S_1^*)^{-1}\{S_-S_+^* + S_{+t} + S_1(S_0^* + S_+^*S_+ + S_{+s}^*)\}\\ K_- &= (1 - S_1S_1^*)^{-1}\{S_0 + S_-S_-^* + S_{-t} + S_1(S_+^*S_- + S_{-s}^*)\}\\ K_1 &= (1 - S_1S_1^*)^{-1}\{S_-S_1^* + S_{1t} + S_+ + S_1(S_+^*S_1 + S_{1s}^* + S_-^*)\}.\end{aligned}\tag{A.3}$$

Appendix B

Using

$$D_sg \equiv K_{AB}\delta z^A\delta z^B = \Lambda g_{AB}\delta z^A\delta z^B$$

with the deviation equations and their derivatives, we obtain after a lengthy but straightforward calculation

$$
\begin{aligned}
&K_{1+} \doteq 1-\alpha^2(1+b\bar{b})-2\alpha^2\bar{b}S_1=0\\
&K_{1-} \doteq -\alpha^2[(1+b\bar{b})S_1+2b]=0\\
&K_{00} \doteq 2(c_s+S_0a+K_0)=2\Lambda c\\
&K_{0+} \doteq a_s+2c+S_+a+K_+-2S_0\alpha^2\bar{b}=\Lambda a\\
&K_{0-} \doteq \bar{a}_s+S_-a+K_--\alpha^2[1+b\bar{b}]S_0=\Lambda\bar{a}\\
&K_{01} \doteq \bar{a}+K_1-\frac{2ab}{(1+b\bar{b})}=\Lambda\\
&K_{++} \doteq 2\left[a-\alpha^2\left\{\frac{\bar{b}^2(3+b\bar{b})}{(1-b\bar{b})(1+b\bar{b})}b_s+\frac{1+3\bar{b}b}{(1+b\bar{b})(1-b\bar{b})}\bar{b}_s+2\bar{b}S_+\right\}\right]=-2\Lambda\bar{b}\alpha^2\\
&K_{-+} \doteq \bar{a}-\alpha^2\left[\frac{4}{(1-b\bar{b})}(b\bar{b})_s+2S_-\bar{b}+[1+b\bar{b}]S_+\right]=-\Lambda[1+b\bar{b}]\alpha^2\\
&K_{--} \doteq -2\alpha^2\left\{b^2\frac{3+b\bar{b}}{(1-b\bar{b})(1+b\bar{b})}\bar{b}_s+\frac{1+3b\bar{b}}{(1-b\bar{b})(1+b\bar{b})}b_s+S_-(1+\bar{b}b)\right\}=-2\Lambda b\alpha^2.
\end{aligned}
\tag{B.1}
$$

From the analysis of these equations we obtained all the unknown functions of equation (33) etc and the Wunschmann equation

$$b_s+S_--bS_++\frac{(1-bb^*)}{(1+bb^*)}b(a^*-ab)=0.$$

For example, from the $K_{1+}=0$ and $K_{1-}=0$ equations and their conjugates, we obtain (b,b^*,α^2).

Many of the equations satisfy identities among themselves and their conjugates and even contain first and second derivatives of the Wunschmann equation. Their analysis and sorting out the details were fairly involved.

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