Differential equations and conformal structures

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History

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asking what one has to assume about $F = F(x, y, y', y'')$ to be able to define a *null* distance between the solutions.

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    asking what one has to assume about $F = F(x, y, y', y'')$ to be able to define a *null* distance between the solutions.
  - ⋆ Denoting by $\mathcal{D}$ the total differential, $\mathcal{D} = \partial_x + p\partial_y + q\partial_p + F\partial_q$, where $p = y'$, $q = y''$, he found that the solution space of $(*)$ is naturally equipped with a *conformal Lorentzian* metric iff

$$F_y + (\mathcal{D} - \tfrac{2}{3}F_q)\underbrace{(\tfrac{1}{6}\mathcal{D}F_q - \tfrac{1}{9}F_q^2 - \tfrac{1}{2}F_p)}_{K} \equiv 0. \qquad (W)$$

⋆ The metric reads:

$$g = [\mathrm{d}y - p\mathrm{d}x][\mathrm{d}q - \tfrac{1}{3}F_q\mathrm{d}p + K\mathrm{d}y + (\tfrac{1}{3}qF_q - F - pK)\mathrm{d}x] - [\mathrm{d}p - q\mathrm{d}x]^2.$$

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- ⋆ Wünschman: There is a *one-to-one correspondence* between *equivalence classes of 3rd order ODEs satisfying* (W) *considered modulo contact transformations of variables* and *3-dimensional Lorentzian conformal geometries*.
- ⋆ In particular: all contact invariants of such classes of equations are expressible in terms of the conformal invariants of the associated conformal Lorentzian metrics.

- Chern S S (1940) "The geometry of the differential equations $y''' = F(x, y, y', y'')$" *Sci. Rep. Nat. Tsing Hua Univ.* **4** 97-111:

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  - ⋆ This may be identified with the *Cartan normal conformal connection* associated with the conformal class $[g]$.

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  - ⋆ If in addition to the Wünschmann condition (W) equation $(*)$ satisfies another *point* invariant condition

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  - ⋆ There is a *one-to-one* correspondence between *3-dimensional Lorentzian Einstein-Weyl geometries* and *3rd order ODEs considered modulo point transformations and satisfying conditions* (W) *and* (C).

- Cartan E (1932) “Sur la geometrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes” Part I *Ann. Math. Pura Appl.* **11** 17-90; Part II *Ann. Sc. Norm. Pisa* **1** 333-54:

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- Nurowski P, Sparling GAJ (2003) "3-dimensional Cauchy-Riemann structures and 2nd order ODEs" *Class. Q. Grav.* **20** 4995-5016:
  - ⋆ What are the analogs of the Fefferman metrics for 2nd order ODEs modulo point transformations?

Conformal geometry of $y'' = Q(x, y, y')$

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- on $J^1 \times \mathbb{R}$ consider a metric

$$g = 2[(\mathrm{d}p - Q\mathrm{d}x)\mathrm{d}x - (\mathrm{d}y - p\mathrm{d}x)(\mathrm{d}r + \tfrac{2}{3}Q_p\mathrm{d}x + \tfrac{1}{6}Q_{pp}(\mathrm{d}y - p\mathrm{d}x))], \qquad (F)$$

where r is a coordinate along $\mathbb{R}$ in $J^1 \times \mathbb{R}$.

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- The point invariants of a point equivalence class of ODEs $y'' = Q(x, y, y')$ are expressible in terms of the conformal invariants of the associated conformal class of metrics (F).

- The metrics (F) are very special among all the split signature metrics on 4-manifolds. Their Weyl tensor C has algebraic type (N, N) in the Cartan-Petrov-Penrose classification. Both, the selfdual C^+ and the antiselfdual C^-, parts of C are expressible in terms of only one component.

- C^+ is proportional to

$$w_1 = D^2 Q_{pp} - 4DQ_{py} - DQ_{pp}Q_p + 4Q_pQ_{py} - 3Q_{pp}Q_y + 6Q_{yy}$$

and C^- is proportional to

$$w_2 = Q_{pppp},$$

where

$$D = \partial_x + p\partial_y + Q\partial_p.$$

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- Cartan normal conformal connection associated with any conformal class $[g]$ of metrics (F) is reducible to a certain $\mathbf{SL}(2+1,\mathbf{R})$ connection naturally defined on an 8-dimensional bundle over J^1. This is uniquely associated with the point equivalence class of corresponding ODEs via Cartan's equivalence method.

- The curvature of this connection has a very simple form

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- If $w_1 = 0$ or $w_2 = 0$ this connection can be further understood as a Cartan $normal$ projective connection over a certain two dimensional space S equipped with a projective structure. S can be identified either with the solution space of the ODE in the $w_1 = 0$ case, or with the solution space of its $dual$ in the $w_2 = 0$ case.

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  - ⋆ considered equations of the form $z' = F(x, y, y', y'', z)$ for two real functions $y = y(x)$ and $z = z(x)$.
  - ⋆ He observed that, contrary to the equation $z' = y''F(x, y, y', z) + G(x, y, y', z)$, the general solution to the equation $z' = y''^2$ can not be written in *integral-free* form:

$$x = x(t, w(t), w'(t), ...w^{(k)}(t)),$$

$$y = y(t, w(t), w'(t), ...w^{(k)}(t)),$$

$$z = z(t, w(t), w'(t), ...w^{(k)}(t)).$$

- Cartan E (1910) "Les systemes de Pfaff a cinq variables et les equations aux derivees partielles du second ordre" *Ann. Sc. Norm. Sup.* **27** 109-192:

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  ⋆ solved an equivalence problem for equations

$$z' = F(x, y, y', y'', z) \qquad \text{with} \qquad F_{y''y''} \neq 0, \qquad\qquad (H)$$

  by constructing a 14-dimensional Cartan bundle $P \to J$ over the 5-dimensional space J parametrized by (x, y, y', y'', z).

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- PN (2003) "Differentail equations and conformal structures" *J. Geom. Phys* **55** 19-49:

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  - ⋆ Since G_2 naturally seats in $\mathbf{SO}(3,4)$, that is in a conformal group for $(3,2)$-signature conformal metrics, is it possible to understand Cartan's invariants in terms of inavraints of some conformal structure in 5 dimensions?

Cartan's construction

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- Each equation (H) may be represented by forms

$$\omega^1 = \mathrm{d}z - F(x, y, p, q, z)\mathrm{d}x$$
$$\omega^2 = \mathrm{d}y - p\mathrm{d}x$$
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- every solution to the equation is a curve $\gamma(t) = (x(t), y(t), p(t), q(t), z(t))$ in J on which the forms $(\omega^1, \omega^2, \omega^3)$ simultaneously vanish.

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- Transformation that transforms solutions to solution may mix the forms $(\omega^1, \omega^2, \omega^3)$ among themselves, thus:

Definition

Two equations $z' = F(x, y, y', y'', z)$ and $\bar{z}' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'', \bar{z})$ represented by the respective forms

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$$\phi^* \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \lambda \\ \kappa & \mu & \nu \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix}$$

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- There are two main branches of nonequivalent equations $z' = F(x, y, y', y'', z)$. They are distinguished by vanishing or not of the relative invariant F_{qq}, $q = y''$.
- If $F_{qq} \equiv 0$ then such equations have integral-free solutions.
- There are nonequivalent equations among the equations having $F_{qq} \neq 0$. All these equations are beyond the class of equations with integral-free solutions.

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$$
\begin{aligned}
\mathrm{d}\theta^1 &= \theta^1 \wedge (2\Omega_1 + \Omega_4) + \theta^2 \wedge \Omega_2 + \theta^3 \wedge \theta^4 \\
\mathrm{d}\theta^2 &= \theta^1 \wedge \Omega_3 + \theta^2 \wedge (\Omega_1 + 2\Omega_4) + \theta^3 \wedge \theta^5 \\
\mathrm{d}\theta^3 &= \theta^1 \wedge \Omega_5 + \theta^2 \wedge \Omega_6 + \theta^3 \wedge (\Omega_1 + \Omega_4) + \theta^4 \wedge \theta^5 \\
\mathrm{d}\theta^4 &= \theta^1 \wedge \Omega_7 + \tfrac{4}{3}\theta^3 \wedge \Omega_6 + \theta^4 \wedge \Omega_1 + \theta^5 \wedge \Omega_2 \\
\mathrm{d}\theta^5 &= \theta^2 \wedge \Omega_7 - \tfrac{4}{3}\theta^3 \wedge \Omega_5 + \theta^4 \wedge \Omega_3 + \theta^5 \wedge \Omega_4.
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\end{aligned}$$

We also have formulae for the differentials of the forms Ω_μ, $\mu = 1, 2, ..., 9$.

For example:

$$d\Omega_1 = \Omega_3 \wedge \Omega_2 + \frac{1}{3}\theta^3 \wedge \Omega_7 - \frac{2}{3}\theta^4 \wedge \Omega_5 +$$
$$\frac{1}{3}\theta^5 \wedge \Omega_6 + \theta^1 \wedge \Omega_8 + \frac{3}{8}c_2\theta^1 \wedge \theta^2 +$$
$$b_2\theta^1 \wedge \theta^3 + b_3\theta^2 \wedge \theta^3 +$$
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The other differentials, when decomposed on the basis θ^i, Ω_μ, define more functions, which Cartan denoted by a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, c_1, c_2, c_3, δ_1, δ_2, e, h_1, h_2, h_3, h_4, h_5, h_6, k_1, k_2, k_3.

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We pass to the interpretetion in terms of Cartan connection:

P is a principal fibre bundle over J with the 9-dimensional parabolic subgroup H of G_2 as its structure group.

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On this fibre bundle the following matrix of 1-forms:

$$\omega = \begin{pmatrix} -\Omega_1-\Omega_4 & -\Omega_8 & -\Omega_9 & -\frac{1}{\sqrt{3}}\Omega_7 & \frac{1}{3}\Omega_5 & \frac{1}{3}\Omega_6 & 0 \\ \theta^1 & \Omega_1 & \Omega_2 & \frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{3}\theta^3 & 0 & \frac{1}{3}\Omega_6 \\ \theta^2 & \Omega_3 & \Omega_4 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{3}\theta^3 & -\frac{1}{3}\Omega_5 \\ \frac{2}{\sqrt{3}}\theta^3 & \frac{2}{\sqrt{3}}\Omega_5 & \frac{2}{\sqrt{3}}\Omega_6 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{\sqrt{3}}\Omega_7 \\ \theta^4 & \Omega_7 & 0 & \frac{2}{\sqrt{3}}\Omega_6 & -\Omega_4 & \Omega_2 & \Omega_9 \\ \theta^5 & 0 & \Omega_7 & -\frac{2}{\sqrt{3}}\Omega_5 & \Omega_3 & -\Omega_1 & -\Omega_8 \\ 0 & \theta^5 & -\theta^4 & \frac{2}{\sqrt{3}}\theta^3 & -\theta^2 & \theta^1 & \Omega_1+\Omega_4 \end{pmatrix},$$

is a Cartan connection with values in the Lie algebra of G_2.

The curvature of this connection $R = \mathrm{d}\omega + \omega \wedge \omega$ 'measures' how much a given equivalence class of equations is 'distorted' from the flat Hilbert case corresponding to $F = q^2$.

$(3,2)$-signature conformal metric

Given an equivalence class of equation $z' = F(x, y, y', y'', z)$ consider its corresponding bundle P with the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \Omega_9)$.

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Define a *bilinear form*

$$\tilde{g} = 2\theta^1\theta^5 - 2\theta^2\theta^4 + \frac{4}{3}\theta^3\theta^3$$

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This form is *degenerate* on P and has signature $(3, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

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The 9 degenerate directions generate the vertical space of P.

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- The *Cartan normal conformal connection* associated with this conformal *metric* yields *all the invariant information* about the equivalence class of the equation.
- This $\mathfrak{so}(4,3)$-valued connection *is reducible* and, after reduction, can be identified with the $\mathfrak{g}_2$ Cartan connection ω on P.

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The equations with 7-dimensional group of transitive symmetries are among those equivalent to $z'=F(y'')$ with $F_{y''y''}\neq 0.$

For such F's the $(3,2)$-signature conformal metric reads:

$$
\begin{aligned}
g = {} & 30(F'')^4\,[\,\mathrm{d}q\mathrm{d}y - p\mathrm{d}q\mathrm{d}x\,] \;+\; [\,4F^{(3)2} - 3F''F^{(4)}\,]\,\mathrm{d}z^2 + \\
& 2\,[-5(F'')^2F^{(3)} - 4F'F^{(3)2} + 3F'F''F^{(4)}\,]\,\mathrm{d}p\mathrm{d}z + \\
& 2\,[15(F'')^3 + 5q(F'')^2F^{(3)} - 4FF^{(3)2} + 4qF'F^{(3)2} + \\
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& 2\,[-15F'(F'')^3 + 20q(F'')^4 + 5F(F'')^2F^{(3)} - 10qF'(F'')^2F^{(3)} + \\
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It is always conformal to an Einstein metric $\hat{g} = \mathrm{e}^{2\Upsilon}g$ with the conformal factor $\Upsilon = \Upsilon(q)$ satisfying

$$10(F'')^2\,[\,\Upsilon'' - (\Upsilon')^2\,] - 40F''F^{(3)}\Upsilon' + 17F''F^{(4)} - 56F^{(3)2} = 0.$$

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Any conformal metric originated from our construction, has *special conformal holonomy* $H_C \subseteq G_2$.

It is therefore interesting to look for the ambient metrics for them. These, in turn, may have *special pseudo-riemanian holonomy* $H_{\psi R} \subseteq G_2$.

In the present example it is particularly easy, since g is conformal to Einstein. We find that the Fefferman-Graham ambient metric for this example is:

$$\bar{g} = t^2 g + 2\mathrm{d}r\mathrm{d}t + \frac{2rt}{10F''^2}(56F^{(3)3} - 17F''F^{(4)})\mathrm{d}q^2.$$

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By construction it is *Ricci flat* and has pseudo-riemannian holonomy $H_{\psi R} \subseteq G_2$. Since the metric g is conformal to Einstein, the equality $H_{\psi R} = G_2$ is excluded. We therefore have $H_{\psi R} \subsetneq G_2$.

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By construction it is *Ricci flat* and has pseudo-riemannian holonomy $H_{\psi R} \subseteq G_2$. Since the metric g is conformal to Einstein, the equality $H_{\psi R} = G_2$ is excluded. We therefore have $H_{\psi R} \subsetneq G_2$.

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Conformal metrics from our construction are rarely conformal to Einstein.

Thus, evaluation of the ambient metrics for them should lead to quite nontrivial $(4,3)$-signature metrics which might have strict noncompact G_2 pseudo-riemannian holonomy.

Cartan classified various types of nonequivalent equations $z' = F(x, y, y', y'', z)$ according to the *roots* of $\Psi(z) = a_1 z^4 + 4a_2 z^3 + 6a_3 z^2 + 4a_4 z + a_5$, where $(a_1, a_2, a_3, a_4, a_5)$ are the scalar invariants of the equation.

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This polynomial encodes partial information of the *Weyl tensor* of the associated conformal $(3, 2)$-signature metric. In particular, the well known invariant $I_\Psi = 6a_3^2 - 8a_2 a_4 + 2a_1 a_5$ of this polynomial is, modulo a numerical factor, proportional to the *square of the Weyl tensor* $C^2 = C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}$ of the conformal metric.

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Our example above corresponds to the situation when this multiplicity is equal to 4. According to Cartan, all nonequivalent equations for which Ψ has quartic root are covered by this example.

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Equations $z' = F(x, y, y', y'', z)$ are in relations with 2-plane fields on manifolds of dimension 5. Bryant found description of certain 3-plane fields in dimension 6 in terms of conformal $(3, 3)$-signature geometries.

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  i) $\Upsilon_{ijk} = \Upsilon_{(ijk)},$ (totally *symmetric*)

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- A 5-dimensional Riemannian manifold (M^5, g) equipped with a tensor field Υ satisfying conditions i)-iii) and admitting a unique decomposition $\overset{LC}{\Gamma} = \Gamma + \frac{1}{2}T$, with $T \in \bigwedge^3 \mathbb{R}^5$ and $\Gamma \in \mathfrak{so}(3) \otimes \mathbb{R}^5$ is called *nearly integrable* irreducible $\mathbf{SO}(3)$ structure.

- Such structures are classified according to the decomposition of the totally skew symmetric torsion T onto the irreducibles w.r.t. the $\mathbf{SO}(3)$ action: $\bigwedge^3 \mathbb{R}^5 = \mathbb{R}^3 \bigoplus \mathbb{R}^7$

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- Some of these examples satisfy Strominger equations of type IIB string theory.

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Then the solution space of the equation is naturally equipped with a class of pairs $[(g, \Upsilon)]$ with representatives satisfying our conditions i)-iii).The metric g of signature $(+, +, -, -, -)$ and the tensor Υ are determined by the contact equivalence class of the ODE up to $g \to \mathrm{e}^{2\phi}g$, $\Upsilon \to \mathrm{e}^{3\phi}\Upsilon$.

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