Twistor space for rolling bodies

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Plan

1. $(2,3,5)$ distributions and G_2
2. Bundles of totally null planes for $(2,2)$ signature metrics
3. Rigid bodies rolling without slipping or twisting

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Both papers give a geometric realisation of the group G_2 as a transformation group of a certain structure on a 5 manifold. The essence of this realisation is as follows:

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- Consider an open set $\mathcal{U}$ of $\mathbb{R}^5$ with coordinates (x, y, p, q, z) and a rank 2-distribution $\mathcal{D}_{q^2} = \mathrm{Span}(X_1, X_2)$ spanned by two vector fields

$$X_1 = \partial_x + p\partial_y + q\partial_p + \frac{1}{2}q^2\partial_z, \qquad X_2 = \partial_q.$$

- The commutator $[X_1, X_2] = -\partial_p - q\partial_z = X_3$.
- Then we have $[X_1, X_3] = \partial_y = X_4$ and $[X_2, X_3] = -\partial_z = X_5$. Modulo antisymmetry all the other commutators vanish.
- Note that $X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge X_5 \neq 0$ at each point of $\mathcal{U}$. The distribution $\mathcal{D}_{q^2}$ is maximally nonintegrable. It is a $(2,3,5)$ distribution - the numbers reflect the growth of the dimension when we take successive commutators.

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Realisation of G_2 by Cartan and Engel (continued)

- Two distributions $\mathcal{D}$ and $\mathcal{D}'$ are (locally) equivalent on $\mathcal{U}$ iff there exists a (local) diffeomorphism $\phi : \mathcal{U} \to \mathcal{U}$ such that $\phi_* \mathcal{D} = \mathcal{D}'$. Selfequivalences for $\mathcal{D}$ are called symmetries of $\mathcal{D}$.
- Locally symmetries are determined by vector fields X on $\mathcal{U}$ such that

$$\mathcal{L}_X \mathcal{D} \subset \mathcal{D},$$

called infinitesimal symmetries.
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- What is the Lie algebra of symmetries of the Cartan-Engel distribution $\mathcal{D}_{q^2}$?
- Answer (Cartan and Engel):
  The Lie algebra $\mathfrak{g}$ of symmetries of $\mathcal{D}_{q^2}$ is a 14-dimensional **simple** real Lie algebra with not-definite Killing form.
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Cartan's invariants of $(2,3,5)$ distribution

- It turns out that generically two $(2,3,5)$ distributions $\mathcal{D}$ and $\mathcal{D}'$ on $\mathcal{U} \subset \mathbb{R}^5$ are not locally equivalent.
- For example, taking a smooth function $f = f(q)$ it is easy to show that the distribution $\mathcal{D}_{2f} = \mathrm{Span}(X_1, X_2)$ with

$$X_1 = \partial_x + p\partial_y + q\partial_p + f(q)\partial_z, \qquad X_2 = \partial_q$$

  is $(2,3,5)$ for all fs such that $f'' \neq 0$. But only very few functions f define $\mathcal{D}_{2f}$ locally equivalent to the Cartan-Engel $\mathcal{D}_{q^2}$.
- In **1910** Cartan gave the full set of local differential invariants which can be used to determine if two $(2,3,5)$ distributions are locally equivalent or not.
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Cartan's quartic

- For this, a certain 4th-rank symmetric tensor $C = C(\mathcal{D})$, built up in terms of quite high derivatives of the functions defining $\mathcal{D}$, must vanish. The tensor C is called Cartan's quartic for $\mathcal{D}$, and there is and explicit formula for calculating it, given $\mathcal{D}$.
- For example the Cartan quartic vanishes for $\mathcal{D}_{2f}$ if and only if f satisfies an ODE:

$$10f^{(6)}f''^3 - 80f''^2 f^{(3)} f^{(5)} - 51 f''^2 {f^{(4)}}^2 + 336 f'' {f^{(3)}}^2 f^{(4)} - 224 {f^{(3)}}^4 = 0.$$

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- What is the fundamental difference between $\mathbb{R}^4$ with a Riemannian metric $x_1^2+x_2^2+x_3^2+x_4^2$ and $\mathbb{R}^4$ with a Lorentzian metric $x_1^2+x_2^2+x_3^2-x_4^2$? ... in Lorentzian case we have **null vectors**, e.g. $n=(0,1,0,1)$.
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$$\big(\ a,\ \mathrm{Span}(n_1, n_2)\ \big) \mapsto \mathrm{Span}(a \cdot n_1, a \cdot n_2).$$

- Since the orthogonal group preserves nullity the resulting space $N^+_a = \mathrm{Span}(a \cdot n_1, a \cdot n_2)$ is also totally null.
- It follows that the **orbit** of N^+_0 w.r.t. this $\mathbf{SO}_0(2,2)$ action forms a cricle

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- Any totally null 2-plane $N = \mathrm{Span}(n_1, n_2)$ in $(\mathbb{R}^4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$ defines a line of a bivector $l(N) = \mathbb{R}n_1 \wedge n_2$.
- It follows that the bivectors $l(N)$ are either selfdual: $*l(N) = l(N)$, or antiselfdual $*l(N) = -l(N)$.
- We say that a totally null plane N is selfdual or antislefdual if its corresponding line $l(N)$ is selfdual or antislefdual, respectively.
- For example planes N_ϕ^+ from the $\mathbf{SO}_0(2,2)$ orbit of N_0^+ are all selfdual.

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- The plane $N_0^- = \mathrm{Span}(n_1, n_3)$ with $n_1 = (1,0,1,0)$ and $n_3 = (0,1,0,-1)$ is antiselfdual.
- The entire $\mathbf{SO}_0(2,2)$ orbit of N_0^-, which is a cricle

$$\mathbb{S}^1_- = \{\ N_\phi^- = \mathrm{Span}\big(n_1(\phi), n_3(\phi)\big) \mid \phi \in [0, 2\pi]\ \}$$

  with $n_1(\phi) = (1, 0, \cos\phi, \sin\phi)$, $n_3(\phi) = (0, 1, \sin\phi, -\cos\phi)$, consists of antiselfdual planes.
- It follows that every totally null plane N in $(\mathbb{R}^4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$ belongs to either $\mathbb{S}^1_+$ or $\mathbb{S}^1_-$.
- The space $\mathcal{Z}(N)$ of all totally null planes in $\mathbb{R}^4$ equipped with the $(2,2)$ signature metric, is a disjoint union of $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$, $\mathcal{Z}(N) = \mathbb{S}^1_+ \cup \mathbb{S}^1_-$.

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$$\mathbb{S}^1_- = \{ N^-_\phi = \mathrm{Span}\big(n_1(\phi), n_3(\phi)\big) \mid \phi \in [0, 2\pi] \}$$

  with $n_1(\phi) = (1, 0, \cos\phi, \sin\phi)$, $n_3(\phi) = (0, 1, \sin\phi, -\cos\phi)$, consists of antiselfdual planes.
- It follows that every totally null plane N in $(\mathbb{R}^4, x_1^2 + x_2^2 - x_3^2 - x_4^2)$ belongs to either $\mathbb{S}^1_+$ or $\mathbb{S}^1_-$.
- The space $\mathcal{Z}(N)$ of all totally null planes in $\mathbb{R}^4$ equipped with the $(2,2)$ signature metric, is a disjoint union of $\mathbb{S}^1_+$ and $\mathbb{S}^1_-$, $\quad \mathcal{Z}(N) = \mathbb{S}^1_+ \cup \mathbb{S}^1_-$.

Circle twistor bundle

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- Then, at every point $y \in M$ we have a circle $\mathbb{S}^1_+(y)$ of totally null selfdual planes $N^+_\phi(y)$ contained in the tangent space $\mathrm{T}_y M$.
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- The horizontal rank 2 distribution $\mathcal{D}$ on $\mathbb{T}(M)$ as defined on the previous slide is called **twistor distribution** on $\mathbb{T}(M)$.
- Note that we found a natural **rank 2** distribution $\mathcal{D}$ on $\mathbb{T}(M)$, which is **five** dimensional.

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Twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ is integrable if and only if the split signature metric g on M has anti-selfdual Weyl tensor. Moreover, if the selfdual Weyl tensor of g is nonvanishing in $\mathcal{U} \subset M$, then in $\pi^{-1}(\mathcal{U})$ there are open sets where the corresponding twistor distribution $\mathcal{D}$ is $(2,3,5)$.

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Twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ is integrable if and only if the split signature metric g on M has anti-selfdual Weyl tensor. Moreover, if the selfdual Weyl tensor of g is nonvanishing in $\mathcal{U} \subset M$, then in $\pi^{-1}(\mathcal{U})$ there are open sets where the corresponding twistor distribution $\mathcal{D}$ is $(2,3,5)$.

Let us assume that the selfdual Weyl tensor of g is not antiselfdual everywhere in M. Then, the key question is: *which such metrics have twistor distributions locally equivalent to the Cartan-Engel distribution $\mathcal{D}_{q^2}$?* (the one with split G_2 symmetry).
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Let (Σ_1, g_1) be a Riemann surface with Gaussian curvature κ, which has a Killing vector, and let (Σ_2, g_2) be a Riemann surface of constant Gaussian curvature λ. Consider a 4-manifold $M = \Sigma_1 \times \Sigma_2$ with a product metric $g = g_1 \oplus (-g_2)$. Then in order for the twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ to have local symmetry G_2, the curvatures must satisfy:

$$(9\kappa - \lambda)(\kappa - 9\lambda)\lambda = 0.$$

Obviously these equations can be satisfied only in two cases:

- the ratios of the curvatures are 1:9 or 9:1, in which case both surfaces has constant curvatures,
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If both surfaces (Σ_1, g_1) and (Σ_2, g_2) have constant Gaussian curvatures, respectively, κ, λ, then the Cartan quartic $C(\mathcal{D})$ of the twistor distribution $\mathcal{D}$ on $\mathbb{T}(M)$ associated with $(M = \Sigma_1 \times \Sigma_2, g = g_1 \oplus (-g_2))$ is

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where $h(\phi)$ is a nowhere vanishing function along the fibers of $\mathbb{T}(M)$

Thus the cases when the ratio of constant curvatures is equal 1:9 or 9:1 correspond to twistor distributions with G_2 symmetry.

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Corollary

The twistor distributions $\mathcal{D}$ associated with the 4-manifold being a product of two spheres $\mathbb{S}^2$, whose radii are in the ratio 1:3 or 3:1 have G_2 symmetry.

The same is true for the product of two hyperboloids.

I will comment on the remaining case $\lambda = 0$ and (Σ_1, g_1) with Gaussian curvature κ and Killing symmetry later.

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We want to describe the space of possible positions for two (smooth) rigid bodies B_1 and B_2 that roll on each other in the 3-space $\mathbb{R}^3$.

- We idealize the surface of body B_1 by a Riemann surface (Σ_1, g_1) and the surface of body B_2 by a Riemann surface (Σ_2, g_2).
- To specify a position of the system, we chose a point x on Σ_1 and a point $\hat{x}$ on Σ_2. These are the points in which the two surfaces kiss each other.
- To fully determine the possition of the system at a given time, we still need to fix the relative angle $\phi \in [0, 2\pi]$ between the tangent spaces $\mathrm{T}_x\Sigma_1$ and $\mathrm{T}_{\hat{x}}\Sigma_2$. This is equivalent to specify a rotation $A(\phi)$ which is an orthogonal transformation $A(\phi) : \mathrm{T}_x\Sigma_1 \to \mathrm{T}_{\hat{x}}\Sigma_2$ identifying the tangent spaces.

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- Thus, to specify the position of the system of rolling bodies at a given time, we need **five** real numbers $(x, \hat{x}, \phi)$ such that:
  - $x \in \Sigma_1$,
  - $\hat{x} \in \Sigma_2$,
  - $A(\phi) \in \{$ orthogonal transformations from the tangent space at x to Σ_1 to the tangent space at $\hat{x}$ to Σ_2 $\}$.
- More formally the configuration space of the system is

$$\mathcal{T}(\Sigma_1, \Sigma_2) = \{ A(\phi) : \mathrm{T}_x\Sigma_1 \to \mathrm{T}_{\hat{x}}\Sigma_2 \},$$

clearly a **circle bundle** over the Cartesian product $M = \Sigma_1 \times \Sigma_2$, with fibers being circles $\mathbb{S}^1$ of orthogonal transformations $A(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

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Identifying configuration space with twistor space

- There is a simple bundle isomorphism between the configuration space $\mathcal{T}(\Sigma_1, \Sigma_2)$ and the twistor circle bundle $\mathbb{T}(\Sigma_1 \times \Sigma_2)$.
- For this we need to show how a point $(x, \hat{x}, A(\phi)) \in \mathcal{T}(\Sigma_1, \Sigma_2)$ defines a point $N^+_\phi(y) \in \mathbb{T}(\Sigma_1 \times \Sigma_2)$.
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- A moment of reflexion yields

$$A(\phi) = \begin{pmatrix} cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \mapsto$$

$$\mathrm{graph}(A(\phi)) = (a, b, a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi)$$

$$= a(1, 0, \cos\phi, \sin\phi) + b(0, 1, -\sin\phi, \cos\phi))$$

$$= \mathrm{Span}(n_1(\phi), n_2(\phi)) = N^+_\phi \subset \mathbb{R}^4.$$

- This identifies bundles $\mathcal{T}(\Sigma_1, \Sigma_2)$ and $\mathbb{T}(\Sigma_1, \Sigma_2)$. This is to say that the positions of the system of rolling bodies are totally null selfdual planes in the semi-Riemanian manifold $M = \Sigma_1 \times \Sigma_2$, $g = g_1 \oplus (-g_2)$.

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Rolling without slipping or twisting

We want to impose nonholonomic constraint of 'rolling without slipping or twisting' on the system of two bodies B_1 and B_2. For this we consider a curve $\gamma(t) = (x(t), \hat{x}(t), A(\phi(t)))$ in the configuration space $\mathcal{T}(\Sigma_1, \Sigma_2)$. It draws two curves: $x = x(t)$ on Σ_1, and $\hat{x} = \hat{x}(t)$ on Σ_2. These curves are just trajectories of the points of contacts.

- $\gamma(t)$ corresponds to the **movement without slipping** iff $A(\phi(t))\dot{x} = \dot{\hat{x}}$,
- $\gamma(t)$ corresponds to the **movement without twisting** iff for every vector field $v(t)$ which is parallel along $x(t)$, the corresponding $A(\phi(t))$ transformed vector field $A(\phi(t))v(t)$ is parallel along $\hat{x}(t)$.

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The nonslipping condition imposes **two** linear constraints at the 5-dimensional space of velocities of the system at each point, and the nontwisting condition adds **one more** linear constraint. Thus the 'nonslipping-nontwisting condition' reduces the velocity space at each point $(x, \hat{x}, \phi)$ from the 5-dimensional tangent space $\mathrm{T}\mathcal{T}(\Sigma_1, \Sigma_2)_{(x,\hat{x},\phi)}$ to its certain 2-dimensional vector space, say $\mathcal{D}_{(x,\hat{x},\phi)}$.

In this way the 'nonslipping-nontwisting condition' defines, point by point, a 2-distribution $\mathcal{D}$ on $\mathcal{T}(\Sigma_1, \Sigma_2)$.

The natural question is: how this distribution is related to the twistor distribution on $\mathbb{T}(\Sigma_1 \times \Sigma_2)$ when both spaces $\mathbb{T}(\Sigma_1 \times \Sigma_2)$ and $\mathcal{T}(\Sigma_1, \Sigma_2)$ are identified.

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No slipping no twisting means horizontality

Theorem

Under the identification $\mathcal{T}(\Sigma_1, \Sigma_2) \equiv \mathbb{T}(\Sigma_1 \times \Sigma_2)$ *obtained via* $(x, \hat{x}, \phi) \mapsto N_\phi^+((x, \hat{x}))$, *the nonslipping-nontwisting rolling distribution* $\mathcal{D}$ *on* $\mathcal{T}(\Sigma_1, \Sigma_2)$ *becomes the twistor distribution* $\mathcal{D}$ *in* $\mathbb{T}(\Sigma_1 \times \Sigma_2)$.

In other words: the nonholonomic 'nonslipping-nontwisting' constraint on the velocity space of two rolling bodies, when viewed on $\mathbb{T}(\Sigma_1 \times \Sigma_2)$ is equivalent to the horizontality of the twistor distribution.

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Reinterpretation of the results about $M = \Sigma_1 \times \Sigma_2$

- Circle twistor space for the manifold $M = \Sigma_1 \times \Sigma_2$ with the metric $g = g_1 \oplus (-g_2)$ is the configuration space of two rolling bodies bounded by the Riemann surfaces (Σ_1, g_1) and (Σ_2, g_2).
- If the bodies roll on each other 'without slipping or twisting' their velocity space is restricted, in such a way that the possible velocities can only be tangent to the twistor distribution.
- If the twistor distribution has G_2 symmetry, then also the restricted velocity space of the rolling system has G_2 symmetry as a $(2,3,5)$ distribution on the configuration space.

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Reinterpretation of the results about $M = \Sigma_1 \times \Sigma_2$

- In paricular two spheres of respective radii with ratio 1:3 or 3:1 when rolling on each other without slipping or twisting have the restricted velocity space with G_2 symmetry. But also two hyperboloids of respective 'hyperbolic radii' with ratio 1:3 or 3:1 have the nonslippng-nontwistuing velocity space with G_2 symmetry.
- Are there other rigid bodies having this property?
- Well...Let us examine the left case $\lambda = 0$, the other surface having Killing symmetry.
- Surprisingly calculation of the Cartan quartic in this case is not only manegable, but also the system of ODE's its vanishing imposes on the metric functions of g_1 can be solved to the very end.

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Surfaces of revolution on the plane with G_2 symmetry

Theorem

Modulo homotheties all metrics corresponding to surfaces with a Killing vector, which when rolling on the **plane** $\mathbb{R}^2$ *'without slipping or twisting', have the velocity distribution* $\mathcal{D}$ *with local symmetry* G_2 *are given by:*

$$\boxed{\begin{aligned} g_{1o} &= \rho^4 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \\ g_{1+} &= (\rho^2+1)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \\ g_{1-} &= (\rho^2-1)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \end{aligned}}$$

Theorem (continued)

Theorem

or, collectively as:

$$\boxed{g_1 = (\rho^2+\epsilon)^2 \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2, \quad \text{where} \quad \epsilon = 0, \pm 1.}$$

Their curvature is given by

$$\kappa = \frac{2}{(\rho^2+\epsilon)^3}.$$

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Theorem

Let $\mathcal{U}$ be a region of one of the Riemann surfaces (Σ_1, g_1) of the previous Theorem, in which the curvature κ is nonnegative. In the case $\epsilon = +1$, such a region can be isometrically embedded in flat $\mathbb{R}^3$ as a surface of revolution. The embedded surface, when written in the Cartesian coordinates (X, Y, Z) in $\mathbb{R}^3$, is algebraic, with the embedding given by

$$\boxed{(X^2 + Y^2 + 2)^3 - 9Z^2 = 0, \qquad \epsilon = +1.}$$

Theorem (continued)

Theorem

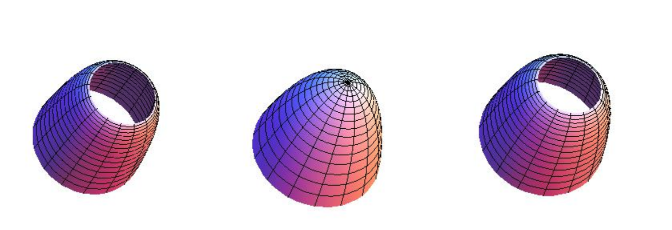
In the case $\epsilon = -1$*, one can find an isometric embedding in* $\mathbb{R}^3$ *of a portion of* $\mathcal{U}$ *given by* $\varphi \in [0, 2\pi[$, $\rho \geq \sqrt{2}$*. This embedding gives another surface of revolution which is also algebraic, and in the Cartesian coordinates* (X, Y, Z)*, given by*

$$\boxed{(X^2 + Y^2 - 2)^3 - 9Z^2 = 0, \qquad \epsilon = -1.}$$

In the case $\epsilon = 0$*, one can embed a portion of* $\mathcal{U}$ *with* $\rho \geq 1$ *in* $\mathbb{R}^3$ *as a surface of revolution*

$$\boxed{Z = f(\sqrt{X^2 + Y^2}), \quad \text{with} \quad f(t) = \int_{\rho=1}^{t} \sqrt{\rho^4 - 1}\, \mathrm{d}\rho.}$$

How do they look?



Rysunek : The Mathematica print of the three surfaces of revolution, whose induced metric from $\mathbb{R}^3$ is given, from left to right, by respective metrics g_{1-}, g_{1+} and g_{1o}. The middle figure embeds all (Σ_1, g_{1+}). In the left figure only the portion of (Σ_1, g_{1-}) with *positive* curvature is embedded, and in the right figure only points of (Σ_1, g_{1o}) with $\rho > 1$ are embedded. It is why the left and right figures have holes on the top. All three surface, when rolling on a plane 'without twisting or slipping' have velocity space $\mathcal{D}_v$ with symmetry G_2.

An update from Robert Bryant

Dear Pawel,
I hope that this finds you well.
Igor Zelenko came to visit me this past week, and we talked a little bit about your G_2 rolling surface example in the context of doing computations for Cartan-type 2-plane fields.
It reminded me of the **left-over question of determining whether there are any other examples besides the constant curvature ones and your rotationally symmetric examples rolling over the plane**, so I took another look at the calculations and at the formula that I worked out for Cartan's C-tensor in this case.

An update from Robert Bryant

It took a little thinking, but, based on this, **I now have a proof** (not too bad) **that, if a pair of Riemannian surfaces has the G_2 rolling distribution, then at least one of the two surfaces has to have constant Gauss curvature.**

An update from Robert Bryant

I still don't know whether, if one fixes a constant Gauss curvature of one surface, the other surface has to have a rotational symmetry (which was your ansatz), but there is a clear line of attack for that, and, when I next have some time to look at this question, I'll see whether or not I can resolve it. **What is clear is that, for each fixed constant Gauss curvature of the one surface, there is at most a finite-dimensional space of isometry classes of germs of metrics that can roll over it with G_2 rolling distribution, and such a metric, if it exists, is completely determined by its 5-jet at one point.**

Yours,

Robert