# Decaphonic piano
# Its mathematics, physics and sound

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GRIEG meets Chopin
Warszawa, July 11, 2023

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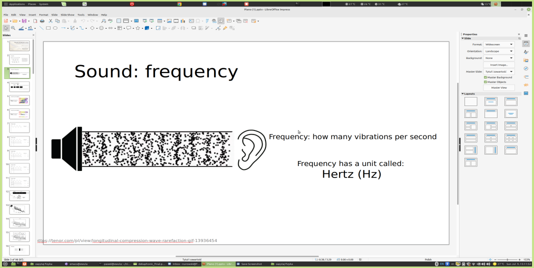
# Part I: MATHEMATICS

# What is sound?




- Sound is a **vibration of air preasure** that **we sense with our ears**[1].
- **The rate** at which these vibration heat our eardrums is called **the frequency of the sound.** This is **measured in Hertzs**, which is the **number of vibrations per second**.

[1]Non decaphonic part of this talk uses a lot from the youtube video of **Yuval Nov**. One can consult his video at `https://www.youtube.com/watch?v=nK2jYk37Rlg&ab_channel=Formant` in case one is lost in this part

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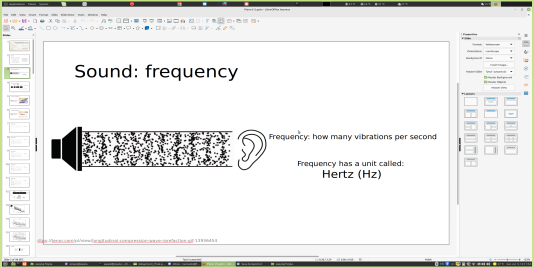


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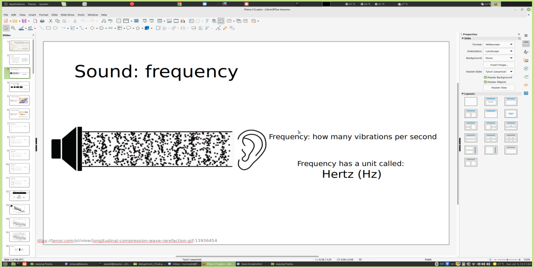


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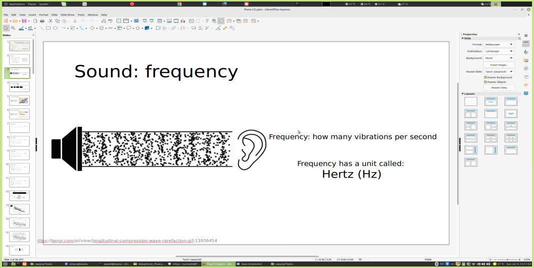


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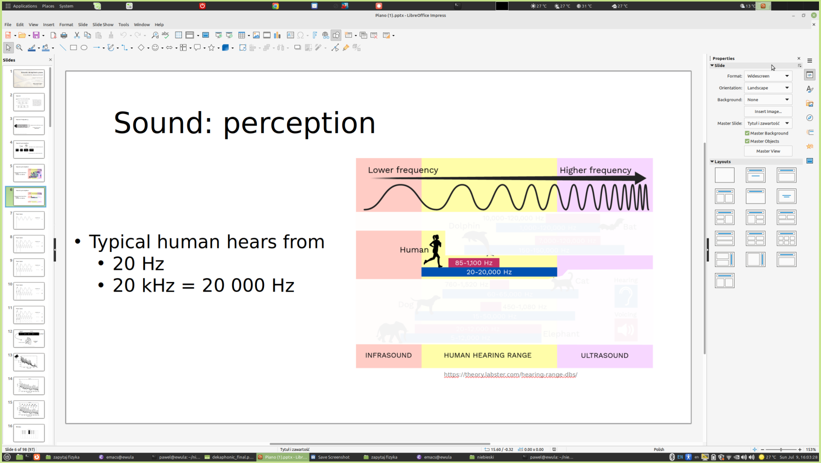
# Frequency range

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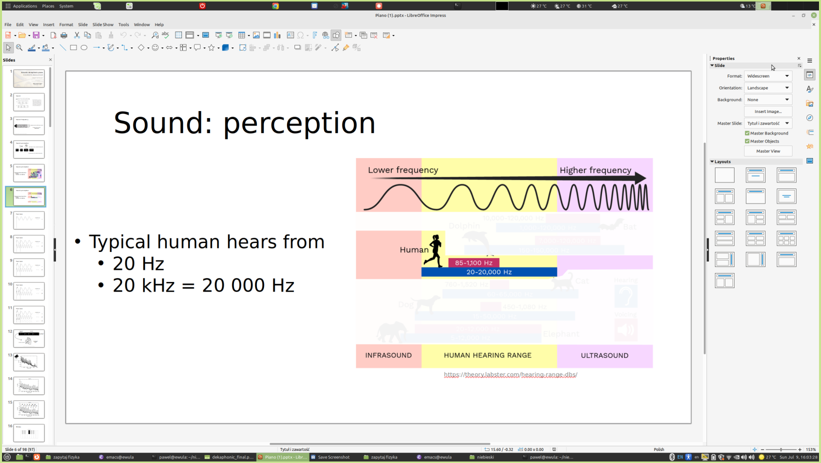


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- Our brain percives higher frequencies detected by our ears as higher pitch
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# Pure tones

- All sounds are made, in a specific way, by building blocks, namely by the **pure tones**.
- We model a pure tone with sound frequency $f$, by a function *sinus*, which changes with time $t$ as $\sin(2\pi f\, t)$.

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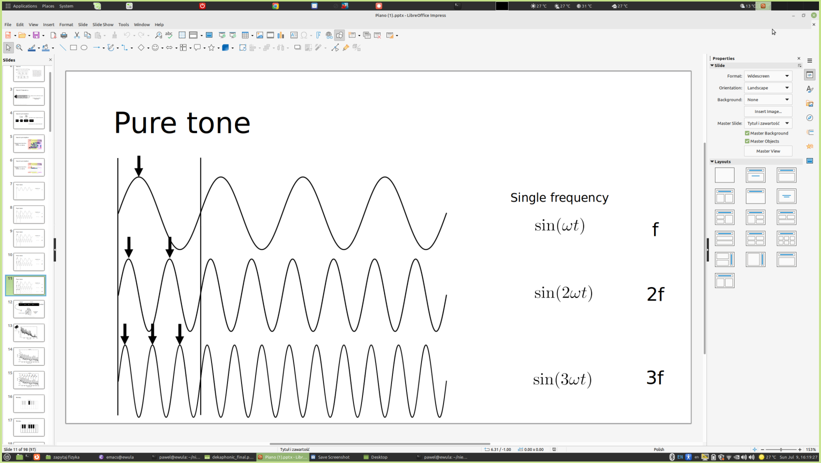


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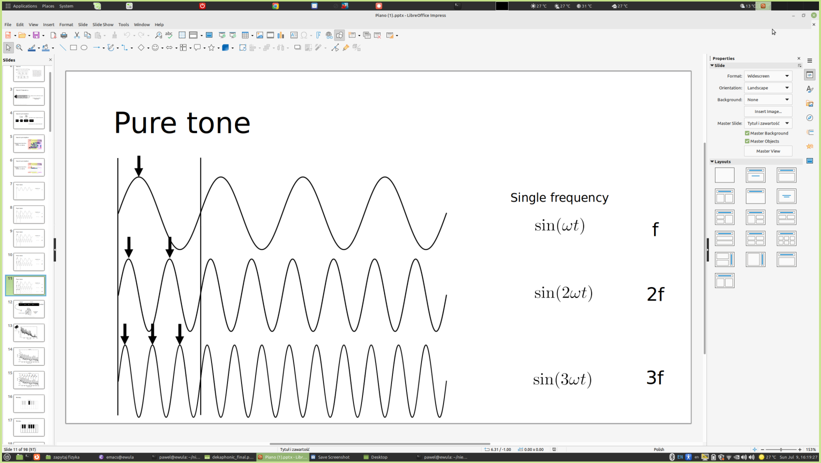


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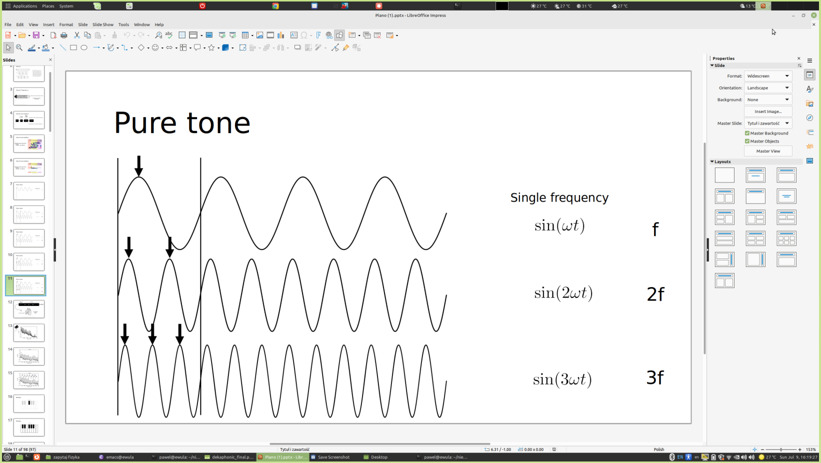


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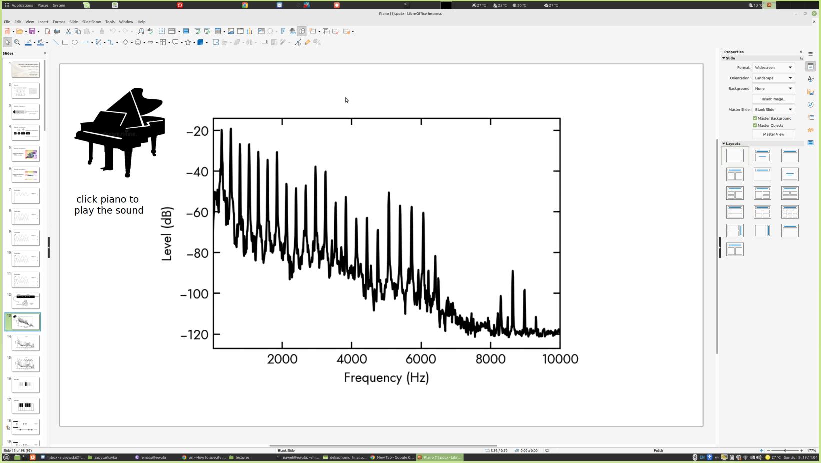
# Tones and overtones for a piano



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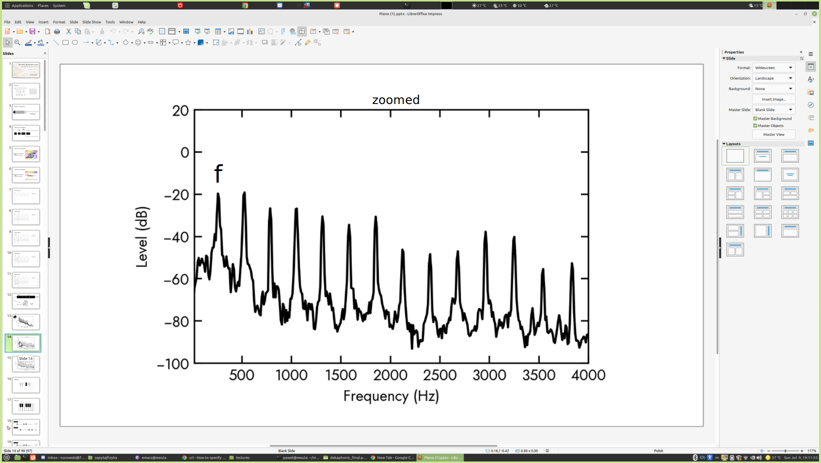

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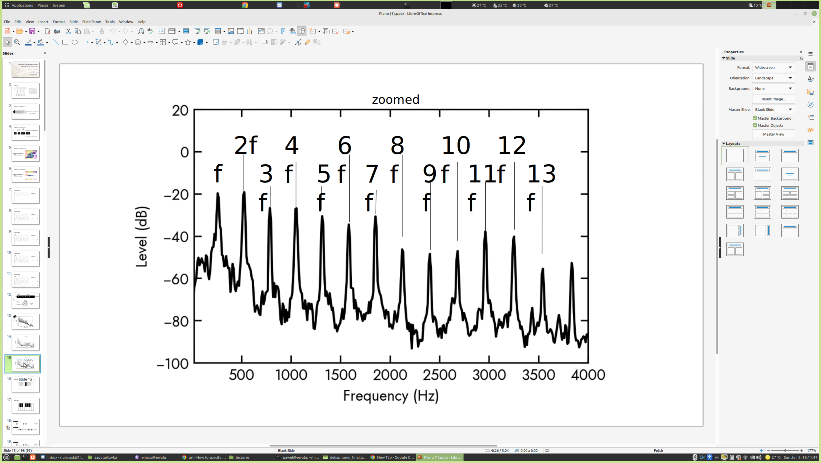
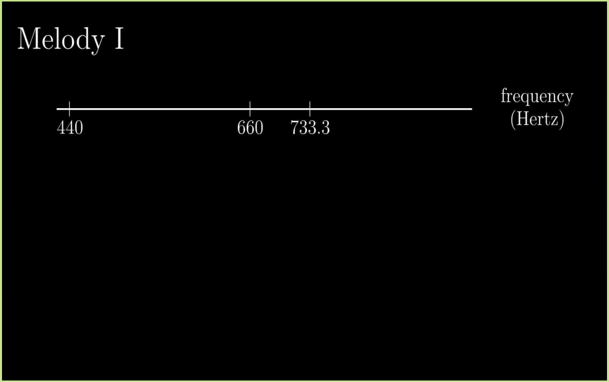


- For a melody, we need more than one tone.



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- What makes these into melodies? Answer: the change in pitch.
- Are these melodies the same? Strictly speaking NO, since none of the notes of the first melody is the same as a note of the second melody.
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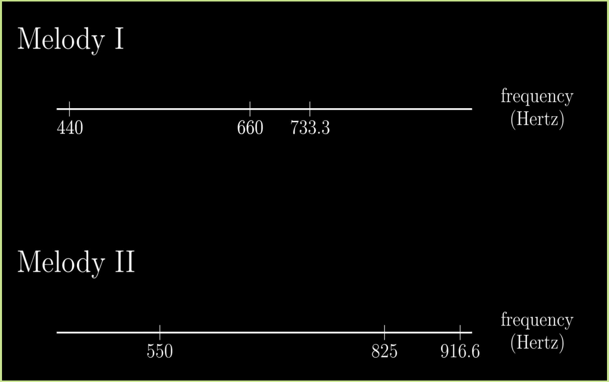


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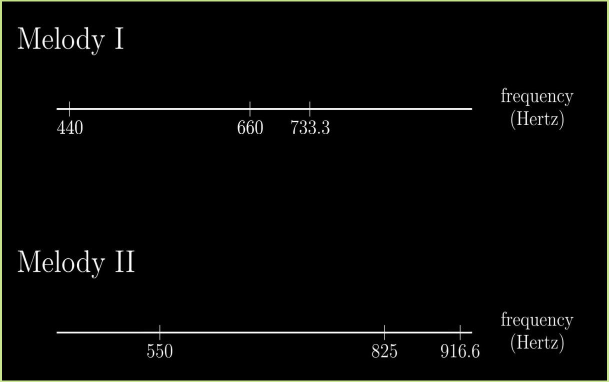


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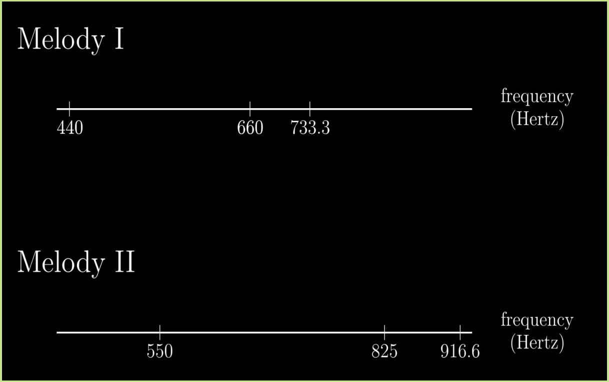


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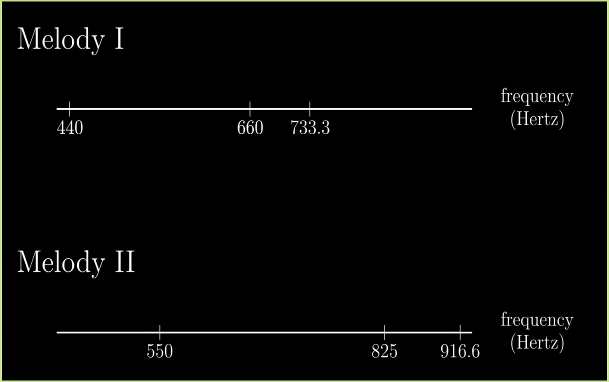


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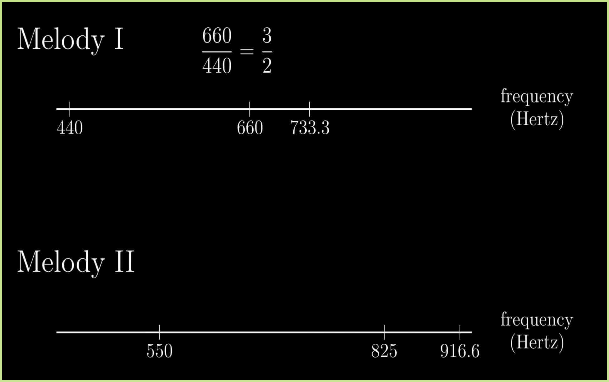
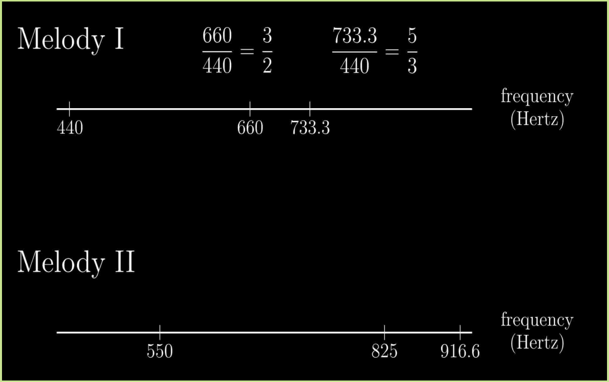

Melody I
$\frac{660}{440} = \frac{3}{2}$
frequency (Hertz)
440
660
733.3
Melody II
frequency (Hertz)
550
825
916.6

# What is a melody?



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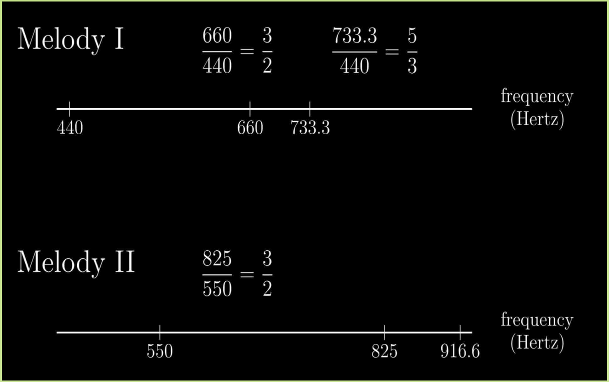

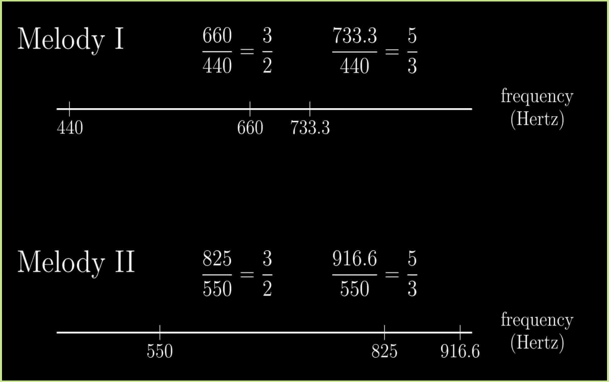
Melody I
660/440 = 3/2
733.3/440 = 5/3
440
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733.3
frequency (Hertz)
Melody II
825/550 = 3/2
916.6/550 = 5/3
550
825
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# What is a melody?

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# Same frequency $\Leftrightarrow$ same melody!

- These are the **ratios of neighbouring frequences** which **are the same!**
- Conclusion:

**the same melodies** $\Longleftrightarrow$ **same ratios between the keys**

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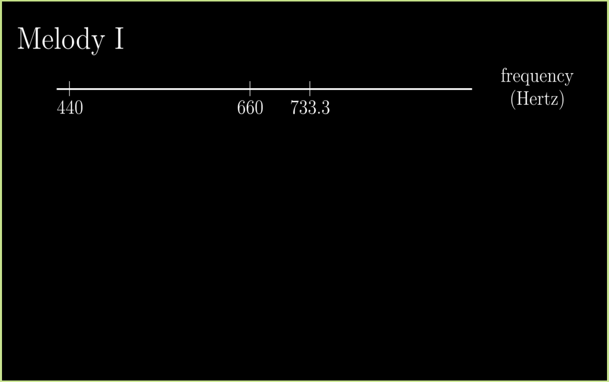
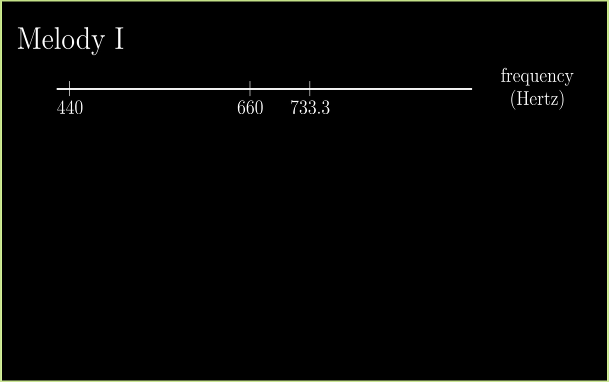



The act of changing of all the frequencies of a given melody by the same factor - so that the ratios between the frequencies stay the same - is called **transposition.**

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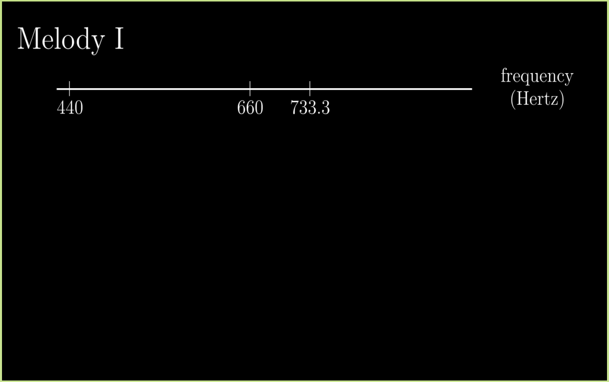


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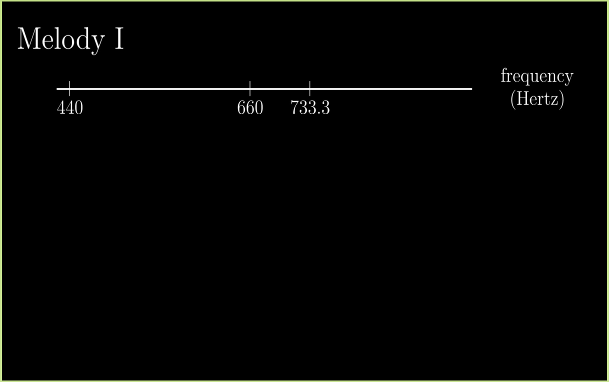


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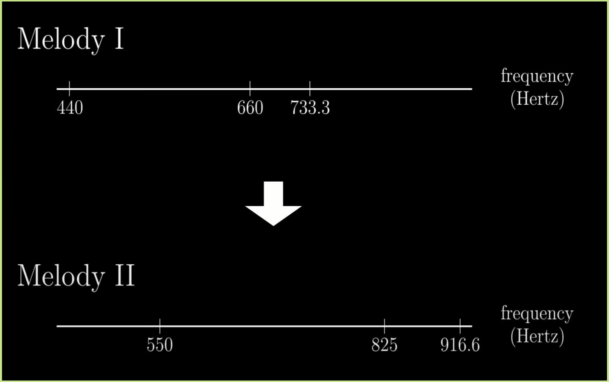


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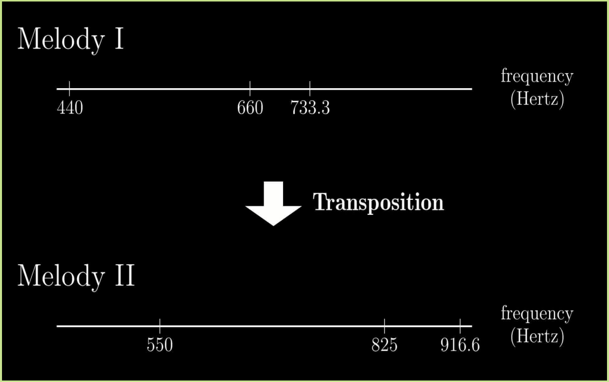


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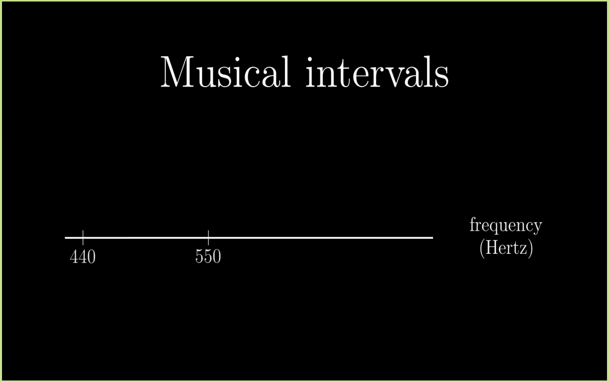
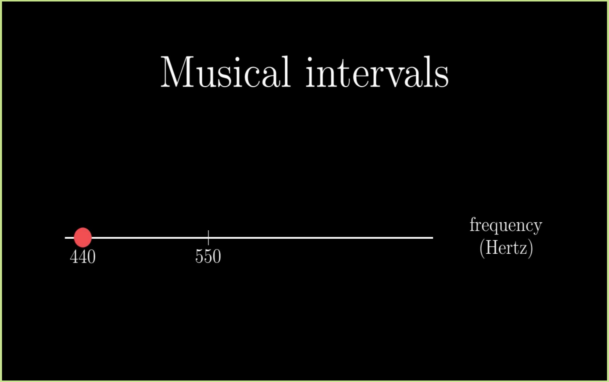



- Between any two tones there is a **musical interval**

# Musical intervals




- musical interval = musical distance between two tones

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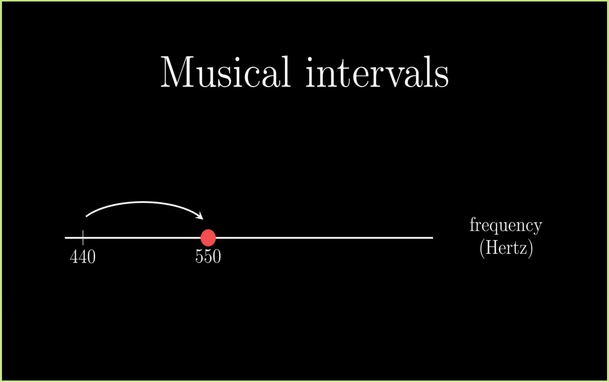


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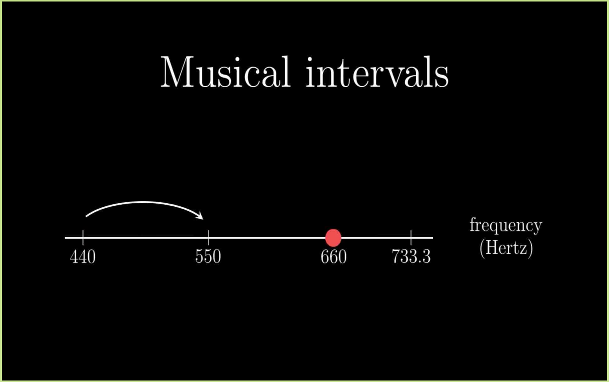


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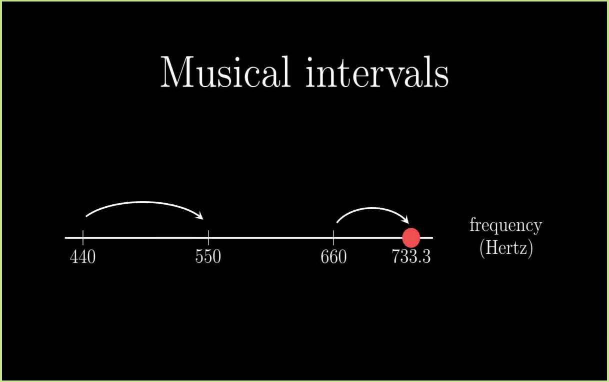


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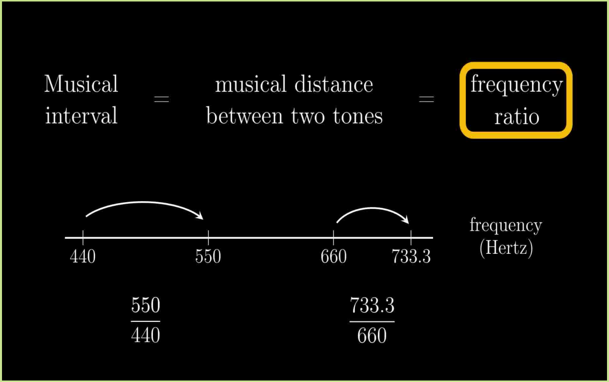


- We now introduce two mportant intervals.

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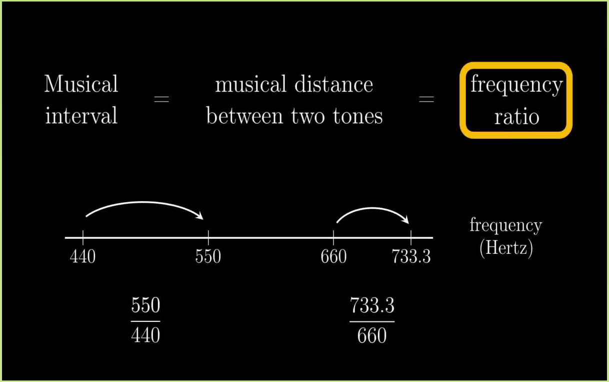


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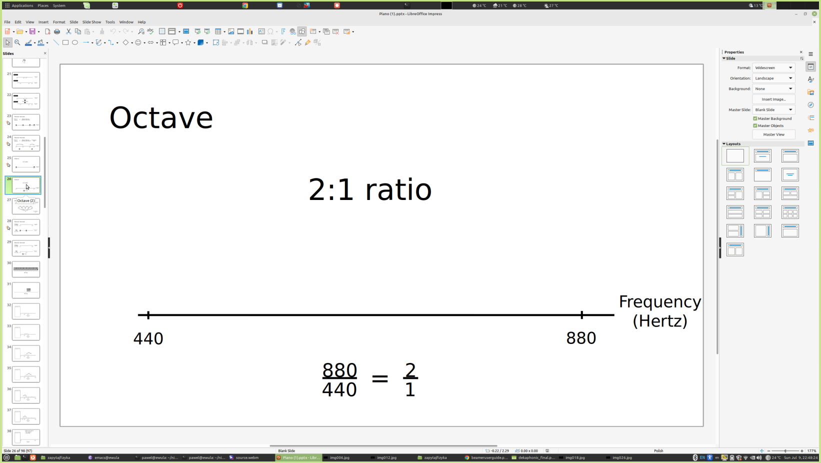
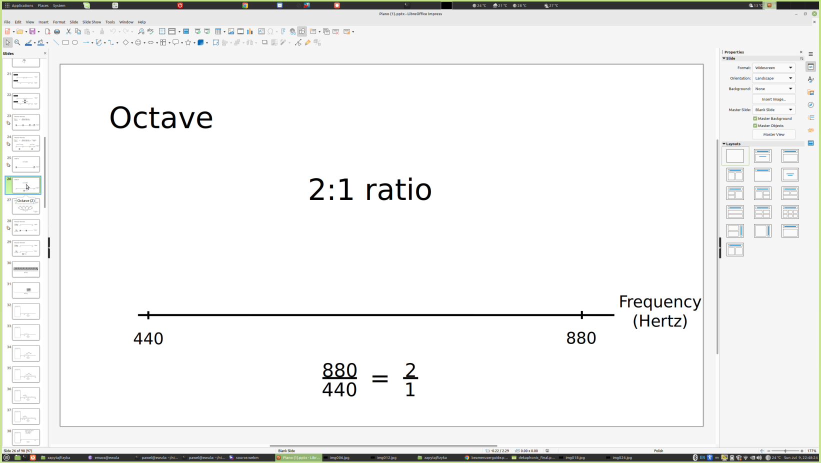



- **The octave** = musical interval with **ratio 2:1**

# The octave

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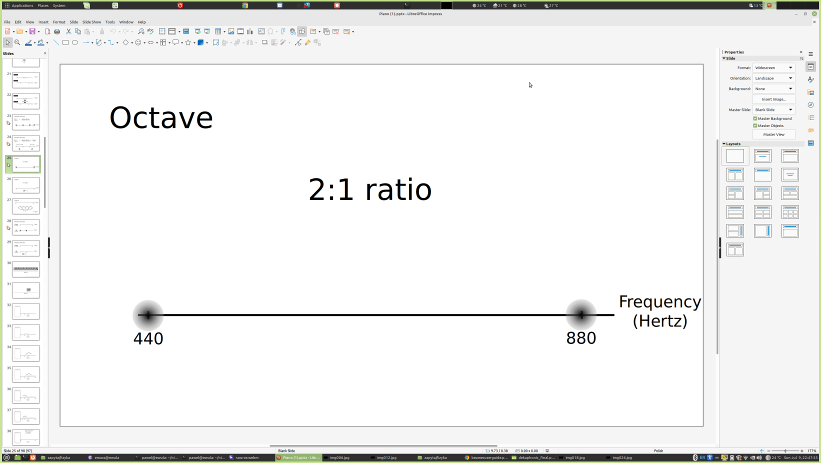


- The **octave is very pleasent** to the ear.
- Two tones, an **octave apart**, played together sound **very harmonious**

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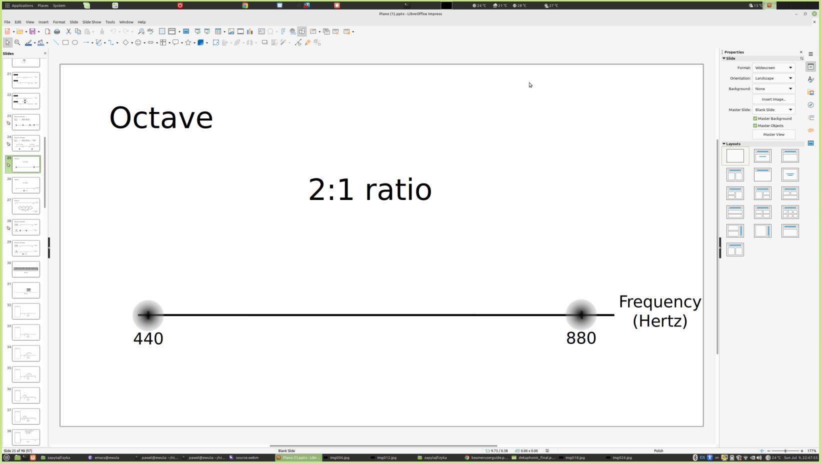


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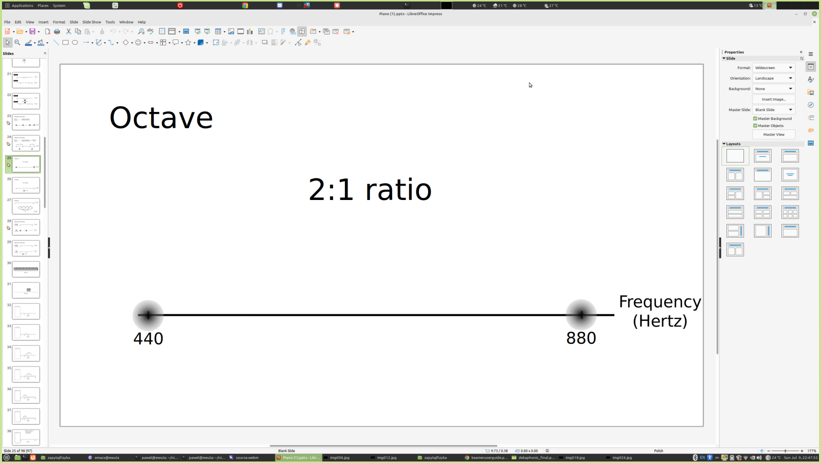


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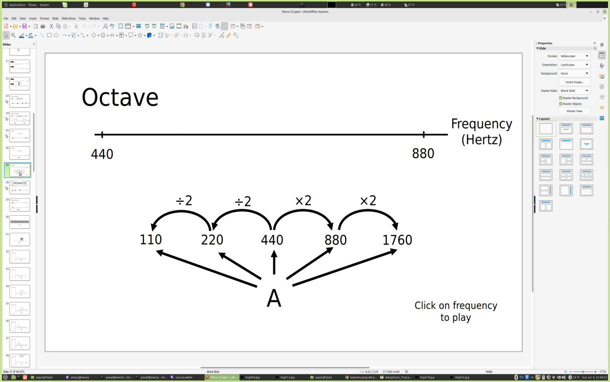


- They are **highly similar**; they are so similar that musicians denote them by **the same letter**, and musicologists say that they belong to the same **pitch class**.
- They are considered to be musically **equivalent**.
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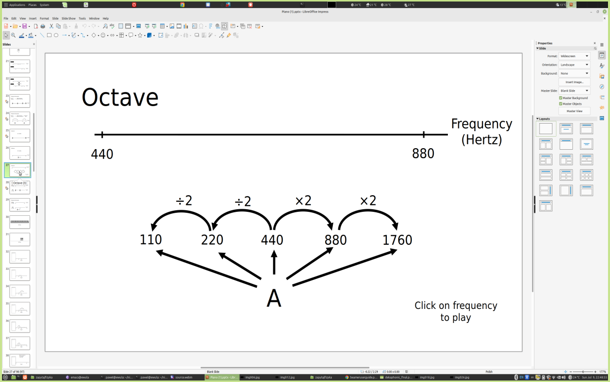


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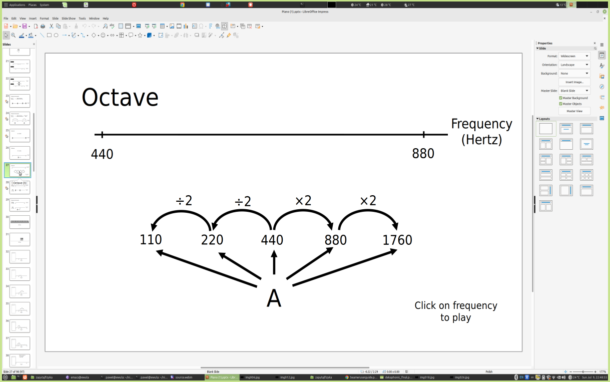


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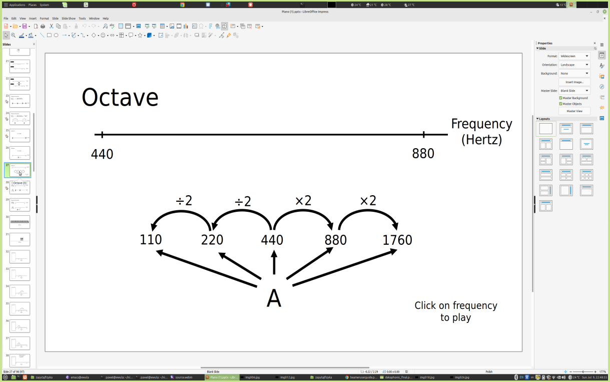


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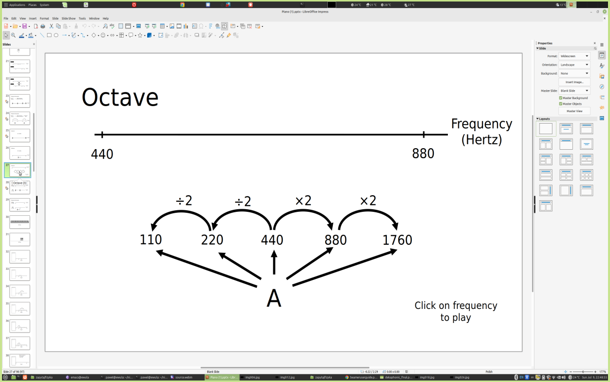


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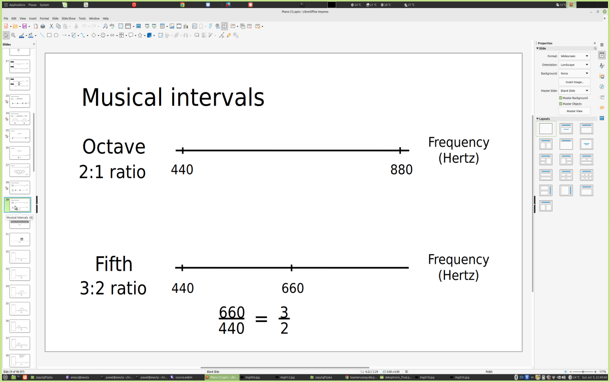


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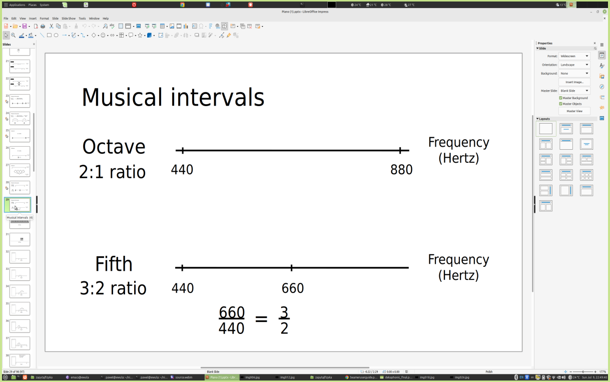
- It is called **the fifth**.

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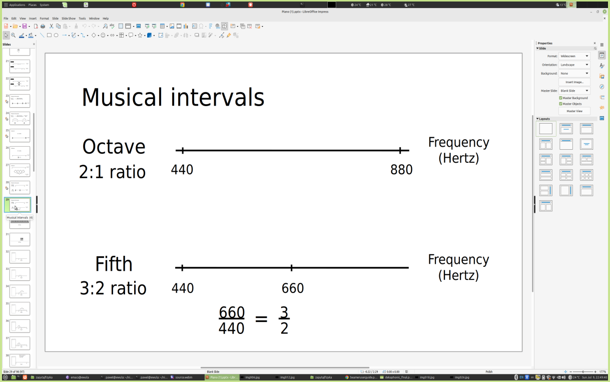


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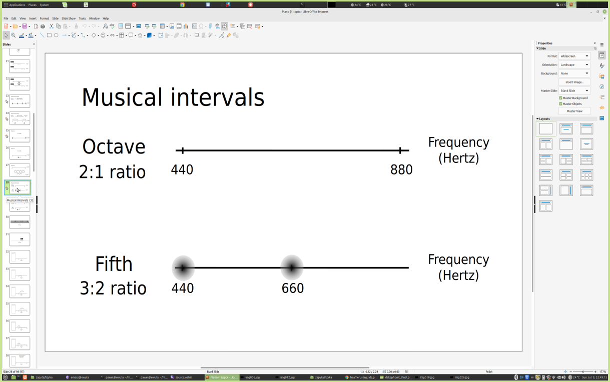


- Another important **musical interval** is given by the **ratio 3:2**.
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$$2f, 3f, 4f, 5f, 6f, 7f, 8f, 9f, 10f, 11f, 12f, 13f, \ldots,$$

  can be **placed between** $f$ **and** $2f$.
- Since for melodies only the **ratios between the tones** are important, so $f$, $2f$, $4f$, ..., $2^k f$ are **octave equivalent**, as well as $\frac{1}{2}f$, $\frac{1}{4}f$, ... $\frac{1}{2^k}f$, for every integer $k$.
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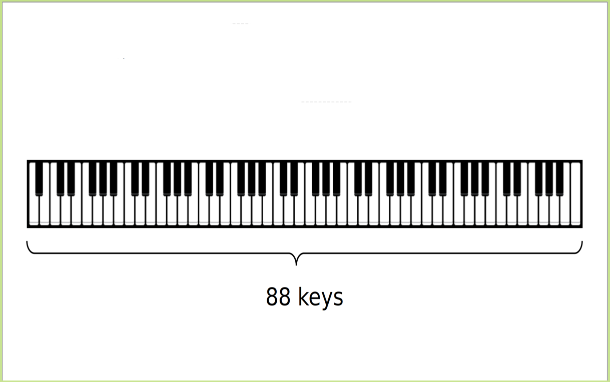
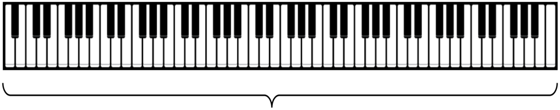
# The piano keyboard




- We are in a position to chose pleasent sounds for the piano keys.
- The piano keyboard has $2 + \mathbf{12} \times \mathbf{7} + 2 = 88$ keys.

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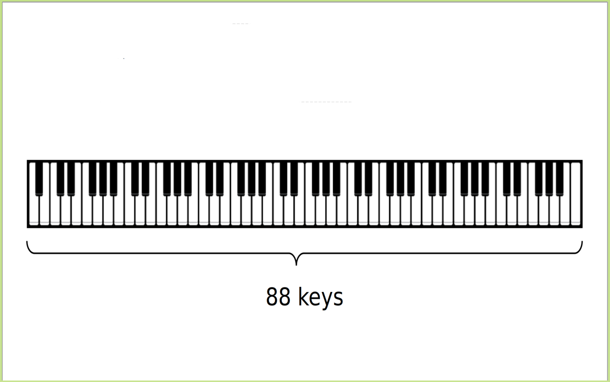


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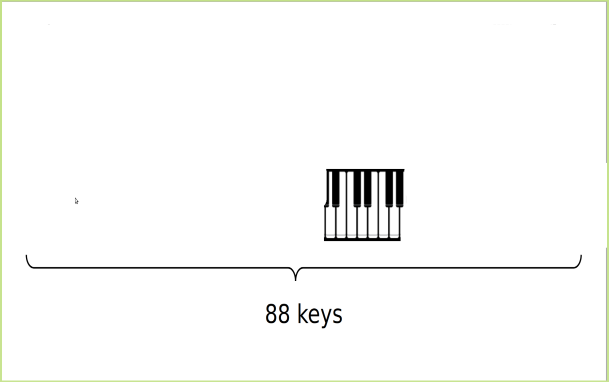


- Look at the above above patern of **twelve** keys. In the keyboard, the pattern is repeated **seven** times, an octave apart from each other. This gives $\mathbf{12} \times \mathbf{7} = \mathbf{84}$ keys. To these 84 keys, two additional keys are added on each of the lateral ends of the keyboard.

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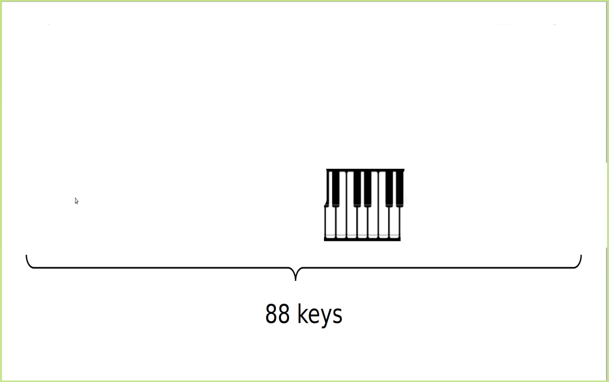


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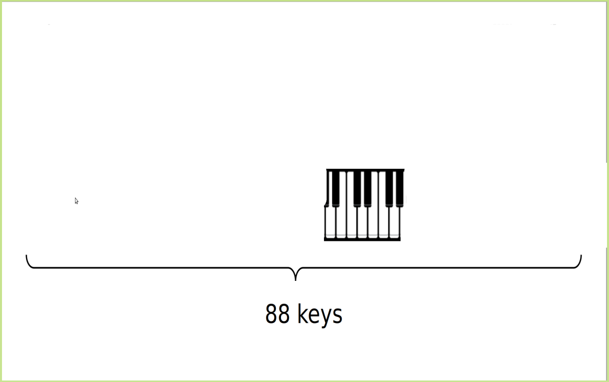


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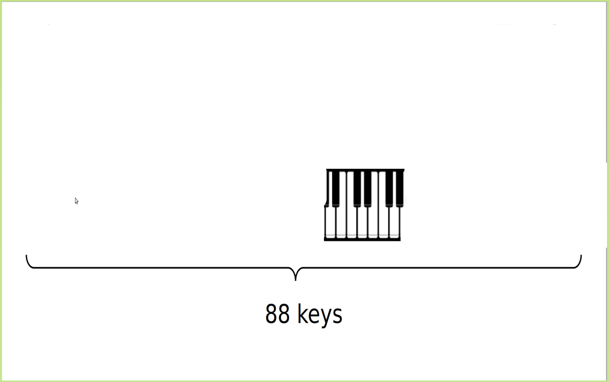


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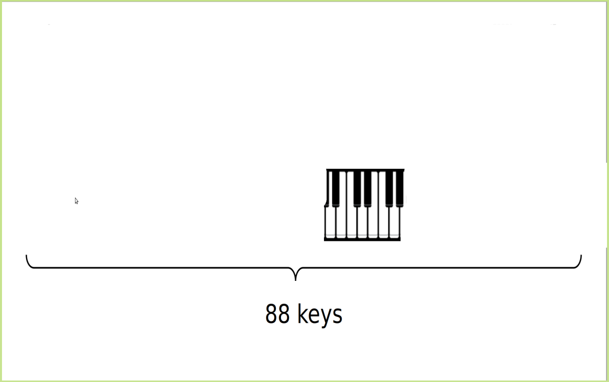


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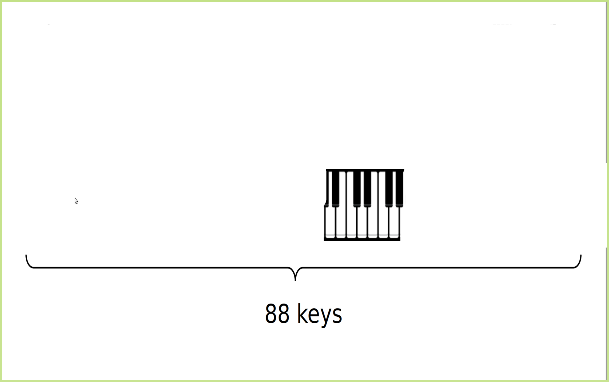


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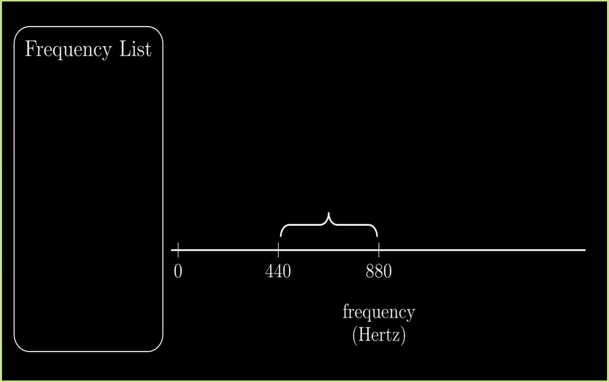
- Which tones are chosen for the **twelve** keys in an octave?




- We start with an octave, say 440Hz – 880Hz.
- We can restrict ourselves to chose frequencies in one octave only, because frequencies for the keys in other octaves will be obtained either

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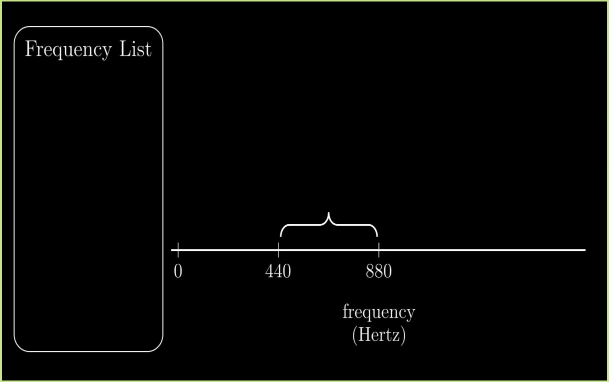


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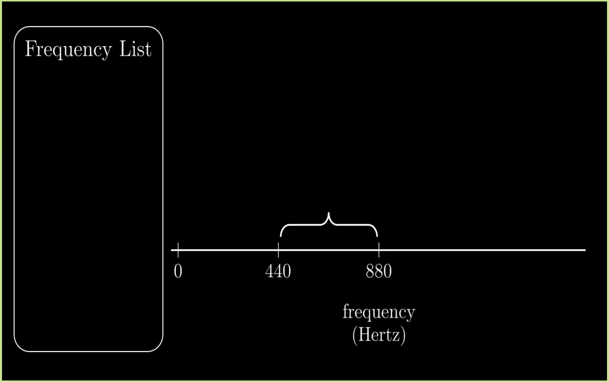


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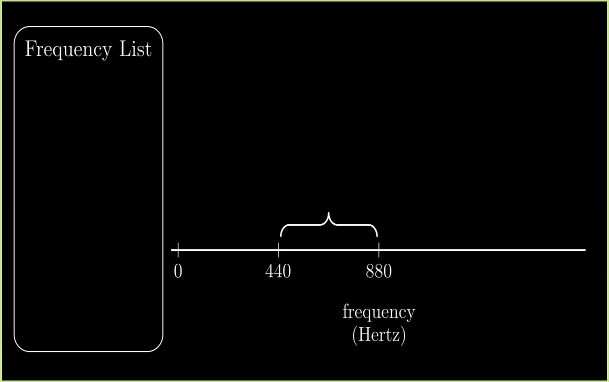


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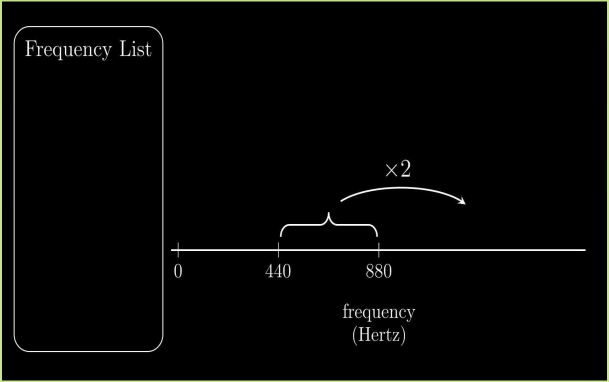
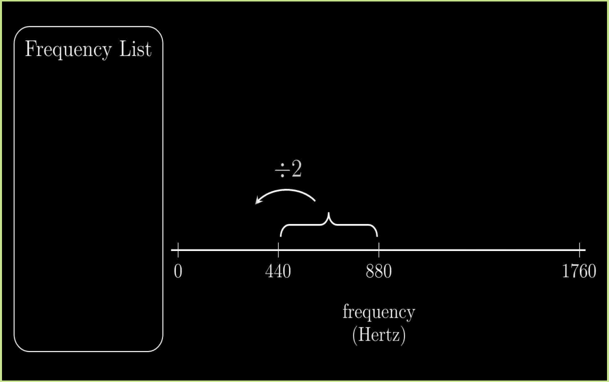
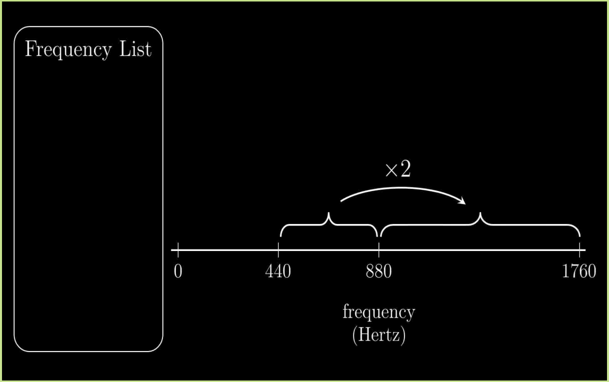


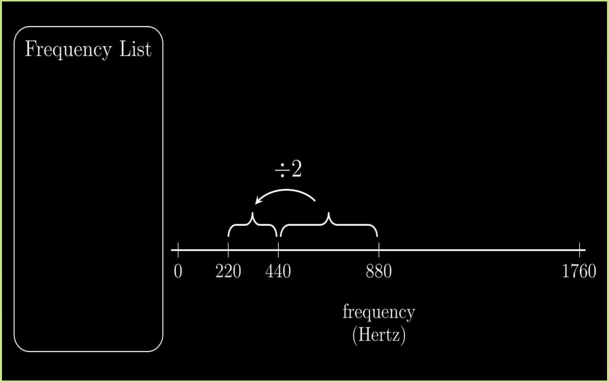

- by **multiplying** all the frequencies by 2, or



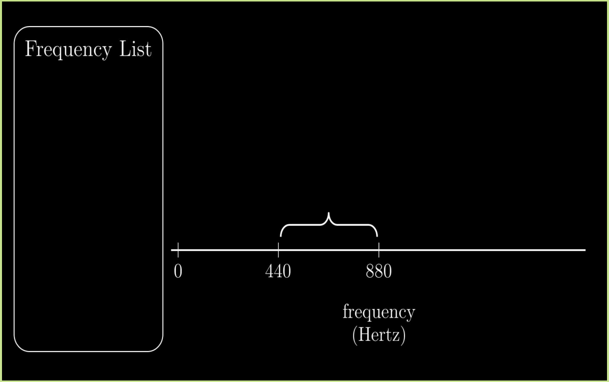

- by **dividing** frequencies by 2.




- For example, the **next octave** will have the frequency range: 880Hz–1760Hz,

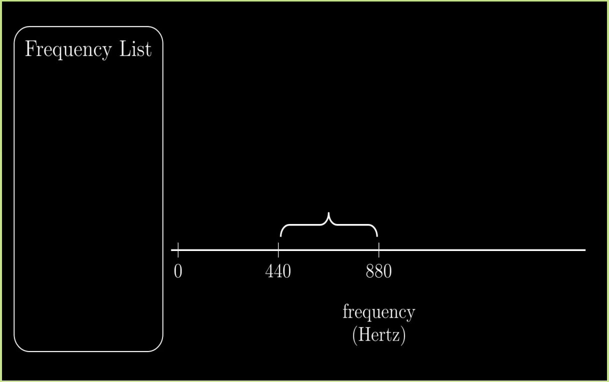



- and the **previous octave** will have the frequency range: 220Hz–440Hz.

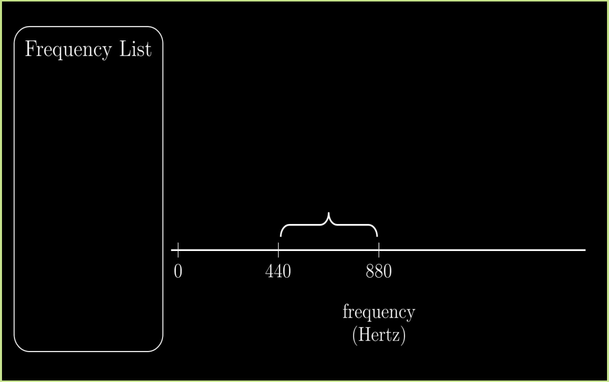



- We will now make frequency choices for the keys in our chosen octave 440Hz-880Hz.

# Choice of frequencies




- Of course we **want to play octaves**, so we have to have two frequencies octave apart in our frequency list.
- We also **want to play fifths**, so the corresponding frequency should also be in our list.

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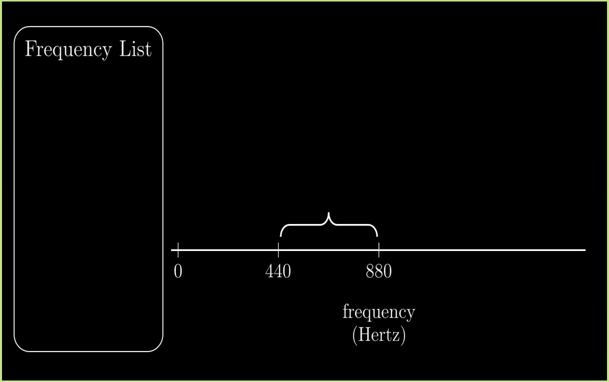


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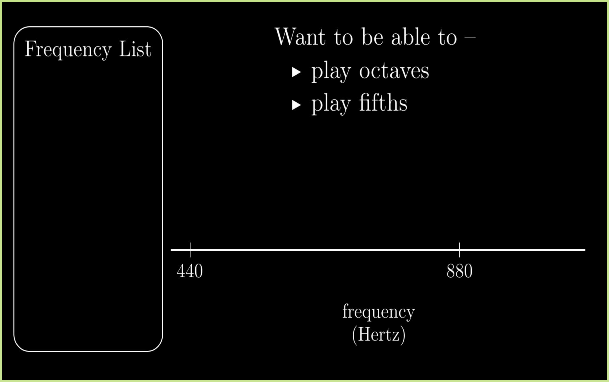

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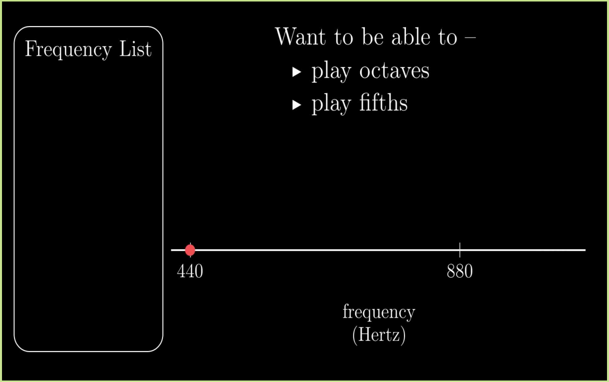

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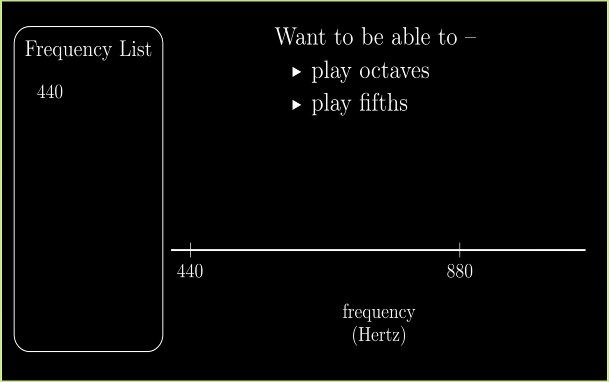

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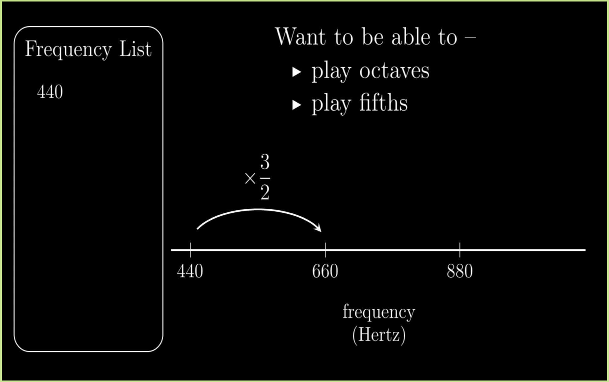

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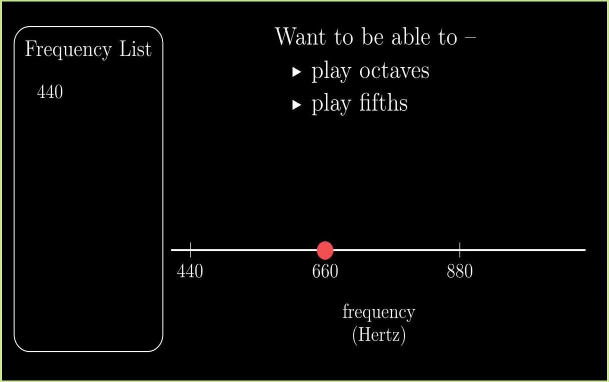

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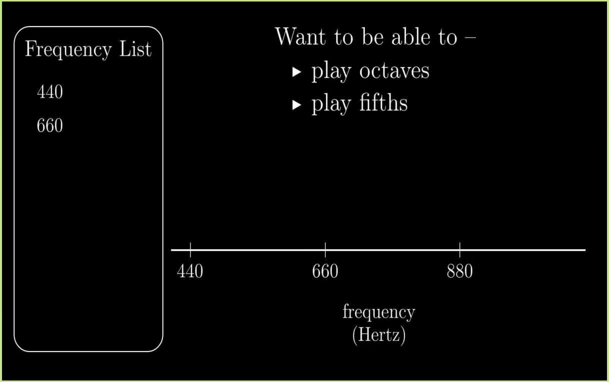

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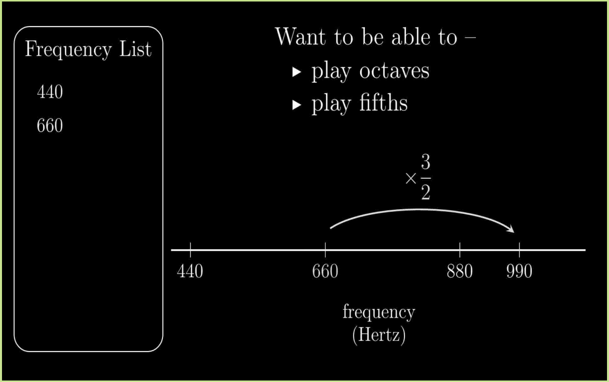

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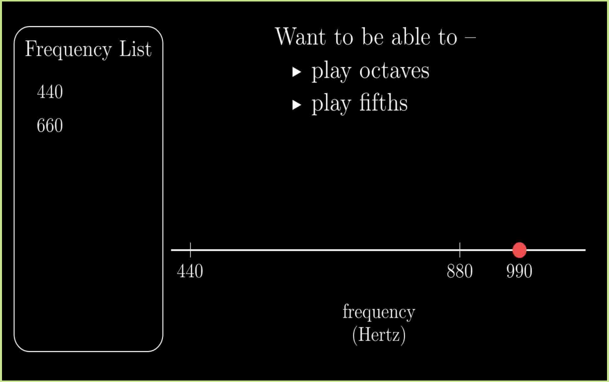

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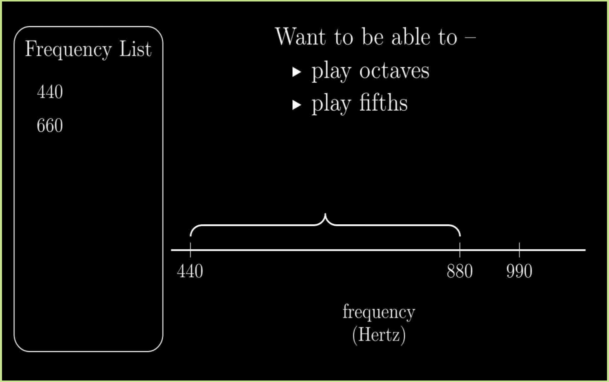

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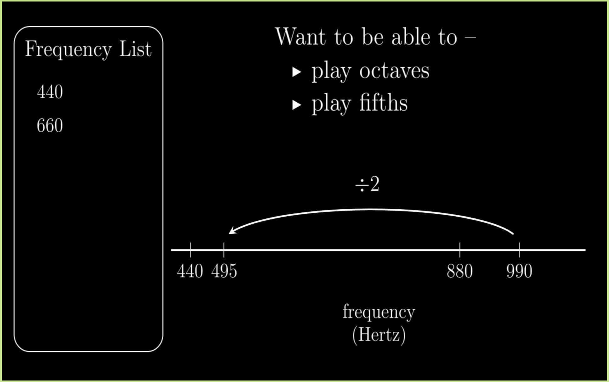

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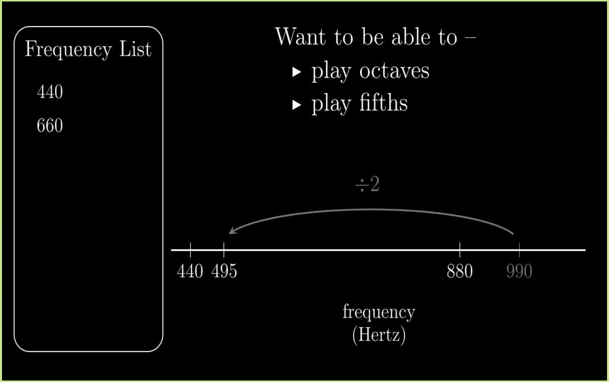

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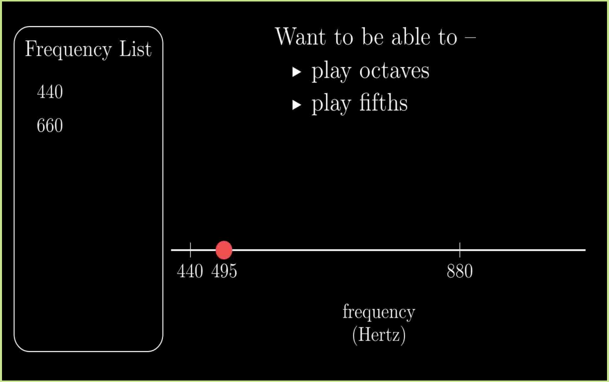

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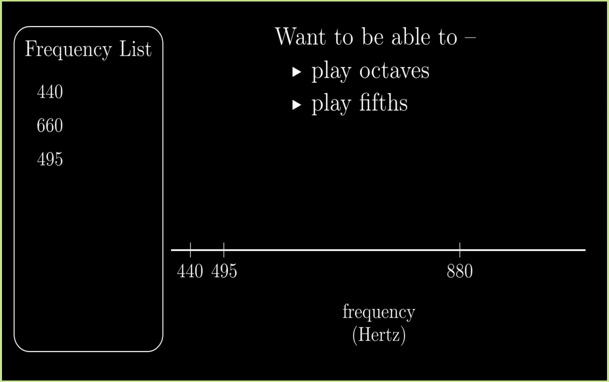


# Here is as it goes:

**Frequency List**

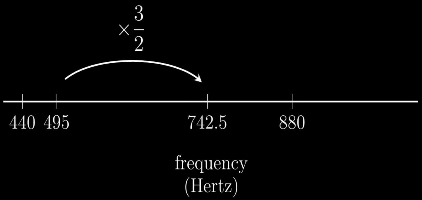
440

660

495

Want to be able to –

- play octaves
- play fifths



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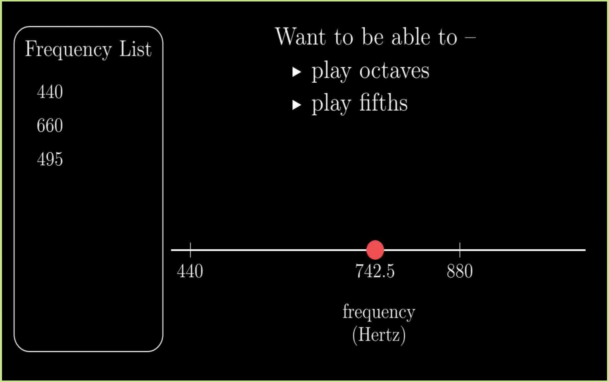

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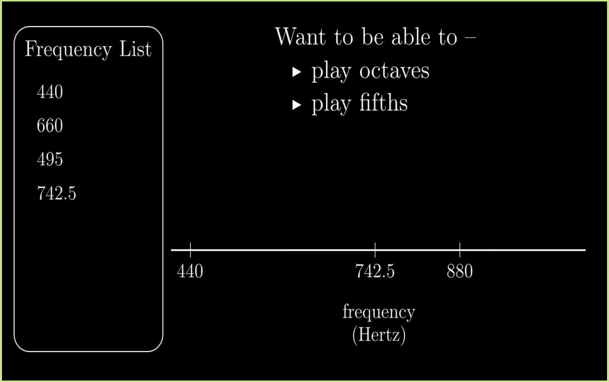

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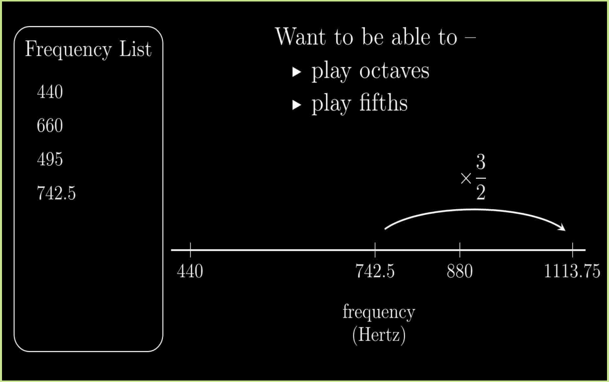

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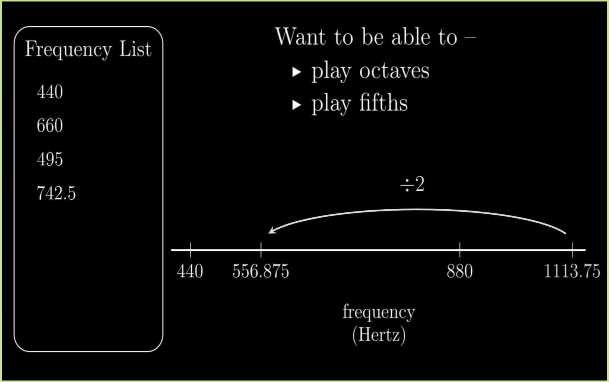

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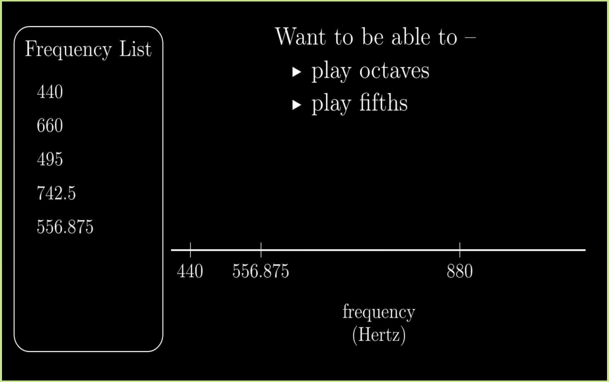


- When does this stop?

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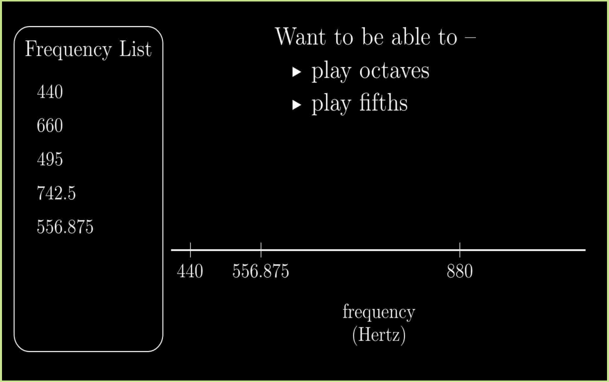


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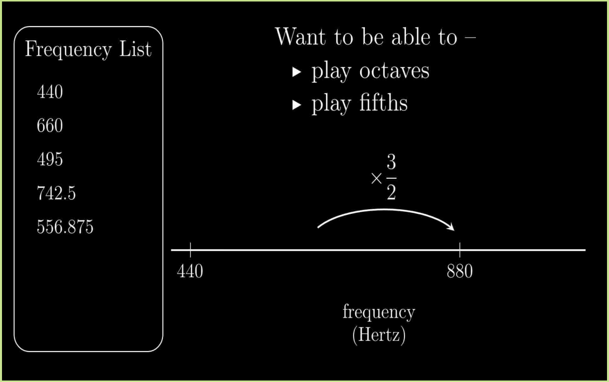
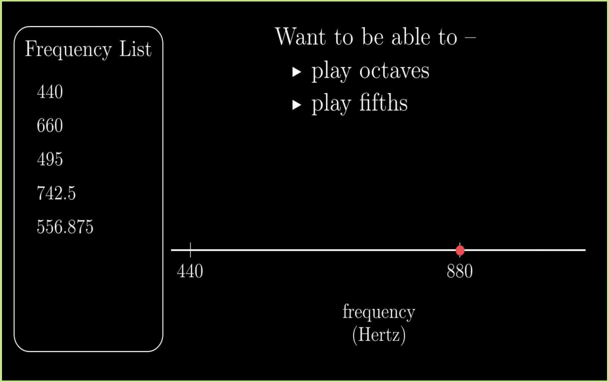



- Well... the best would be if we eventually arrived at 880Hz.

# When does it stop?



# When does it stop?

**Frequency List**

440

660

495

742.5

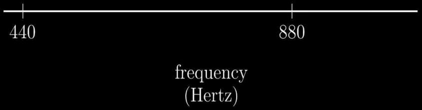
556.875

⋮

880

Want to be able to –

- play octaves
- play fifths



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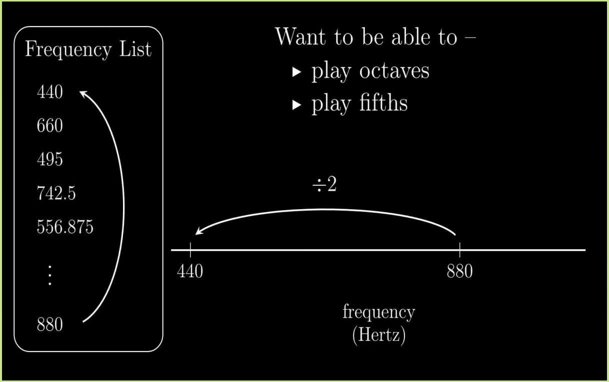
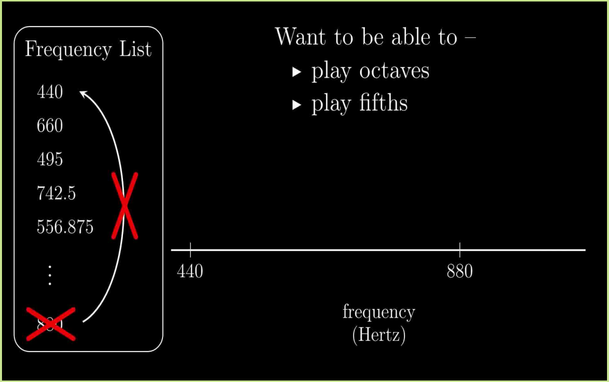


# It never stops!



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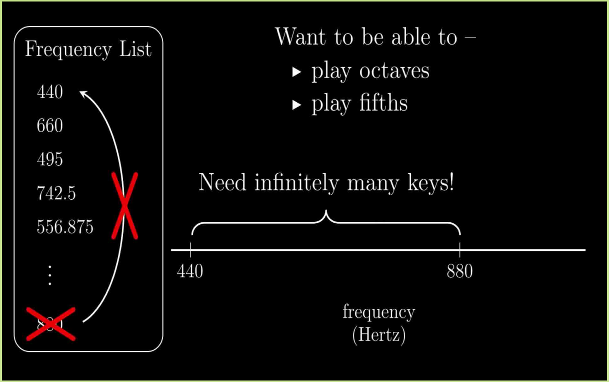


# It never stops!

If get back to 440 –

$n$ = no. of times went up a fifth

$k$ = no. of times went down an octave

$$440 \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = 440$$

# It never stops!

If get back to 440 –

$n$ = no. of times went up a fifth

$k$ = no. of times went down an octave

$$\cancel{440} \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = \cancel{440}$$

$$3^n = 2^{n+k}$$

# It never stops!

If get back to 440 –

$n =$ no. of times went up a fifth

$k =$ no. of times went down an octave

$$\cancel{440} \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{1}{2}\right)^k = \cancel{440}$$

$$\text{odd number} = 3^n = 2^{n+k} = \text{even number}$$

contradiction

## It never stops!

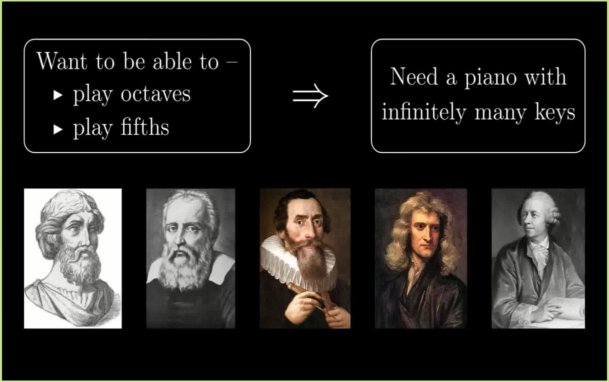
Want to be able to –

- play octaves
- play fifths

$\Rightarrow$

Need a piano with infinitely many keys

# It never stops!



# A choice of finite number of notes in an octave

A choice of note frequencies for an octave –

“tuning”,

“intonation”,

“temperament”

- There are numerous choices of tuning systems!

A choice of note frequencies for an octave –

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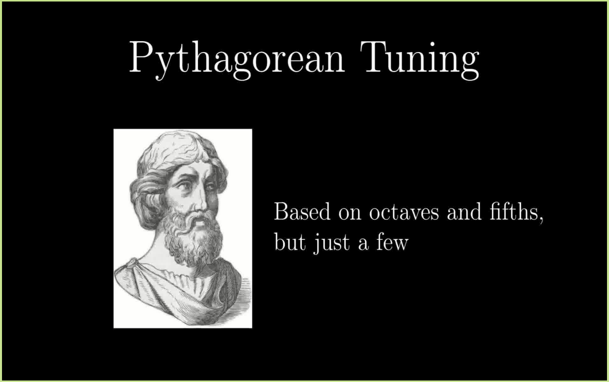
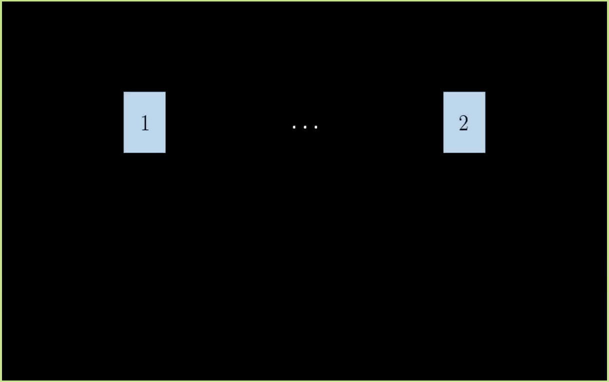

Pythagorean Tuning
Based on octaves and fifths,
but just a few

# Pythagorian tuning



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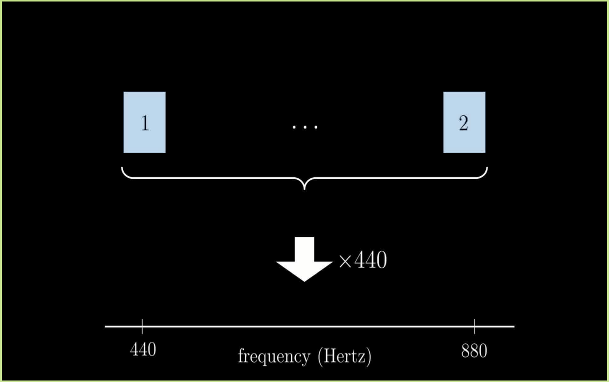
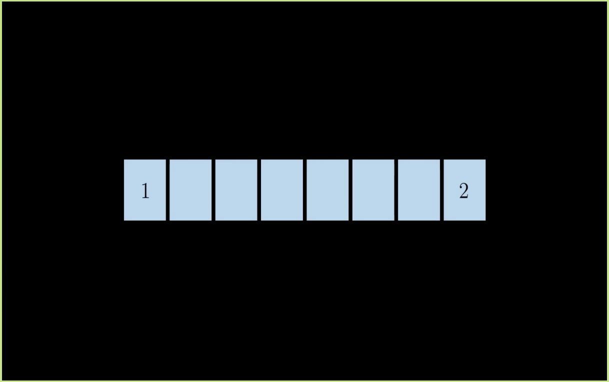
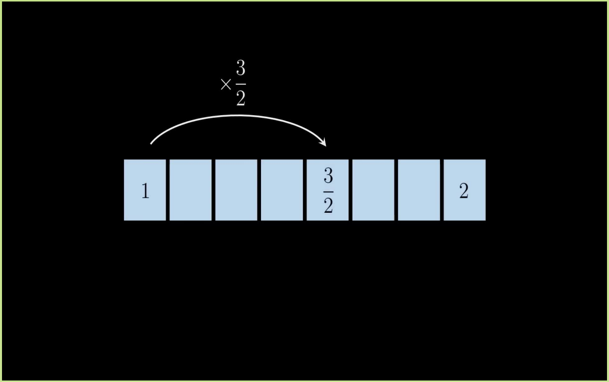



1
2

## Pythagorian tuning



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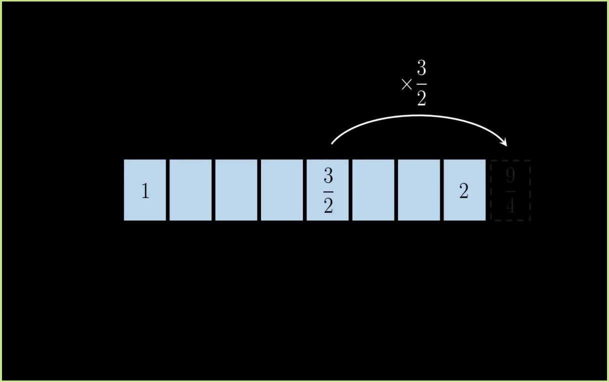
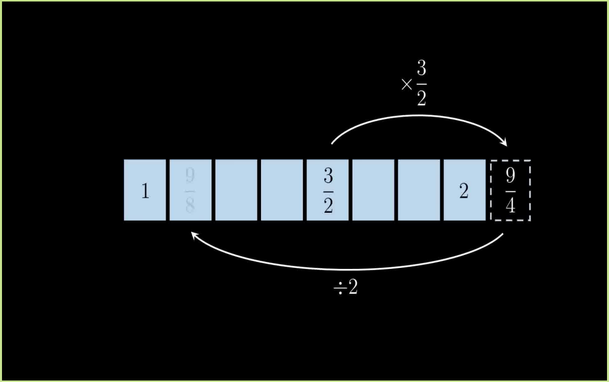
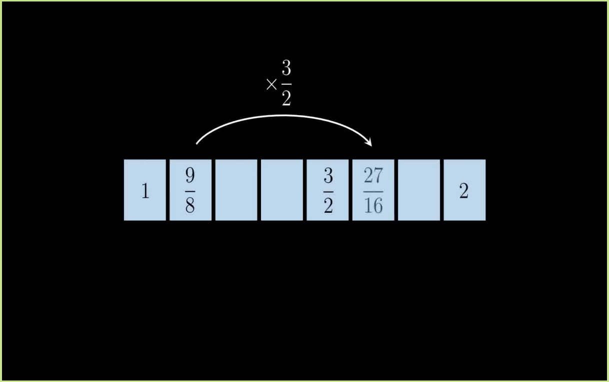
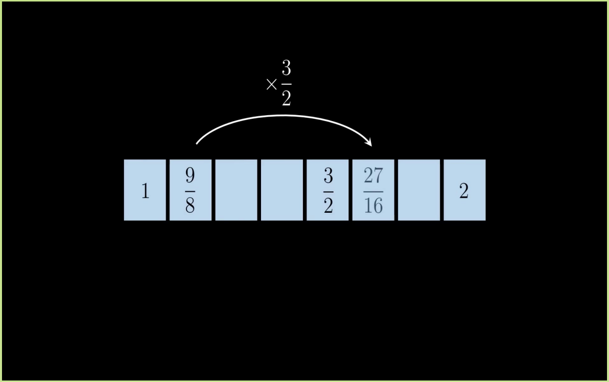
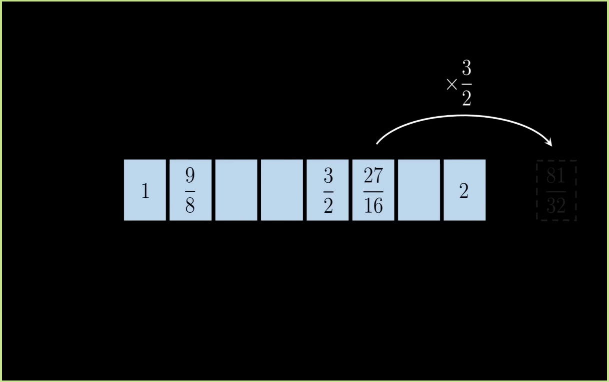


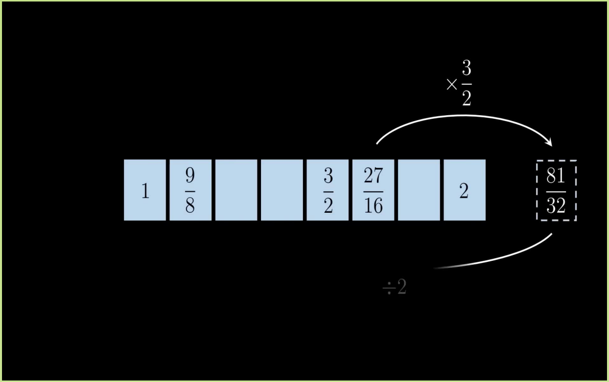

$\times \frac{3}{2}$
1
$\frac{9}{8}$
$\frac{3}{2}$
2
$\frac{9}{4}$
$\div 2$

# Pythagorian tuning




×3/2
1
9/8
3/2
27/16
2


$\times \frac{3}{2}$
1
$\frac{9}{8}$
$\frac{3}{2}$
$\frac{27}{16}$
2
$\frac{81}{32}$



$\times \frac{3}{2}$
1
$\frac{9}{8}$
$\frac{3}{2}$
$\frac{27}{16}$
2
$\frac{81}{32}$
$\div 2$

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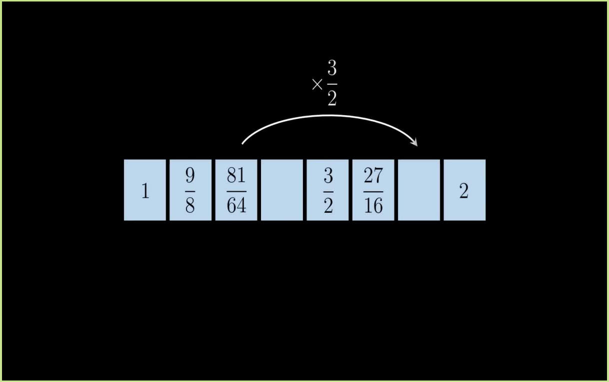

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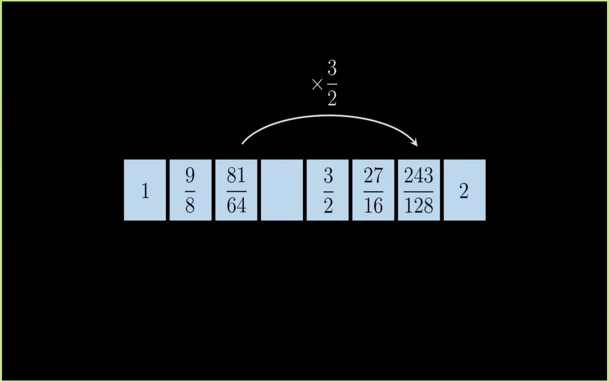
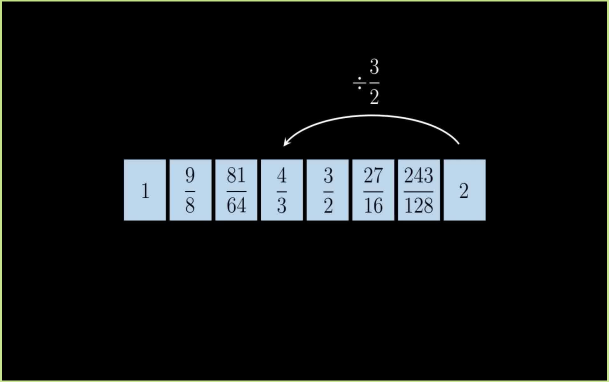


# Pythagorian tuning

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $1$ | $\frac{9}{8}$ | $\frac{81}{64}$ | | $\frac{3}{2}$ | $\frac{27}{16}$ | $\frac{243}{128}$ | $2$ |

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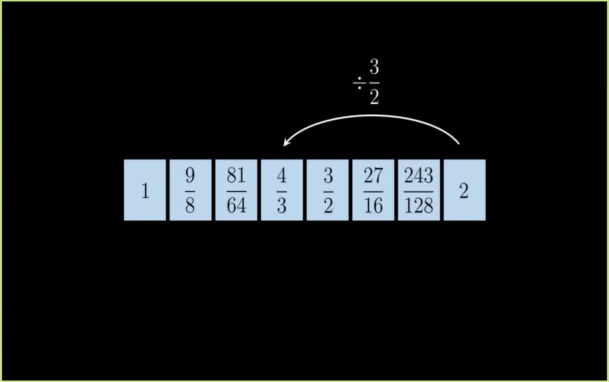


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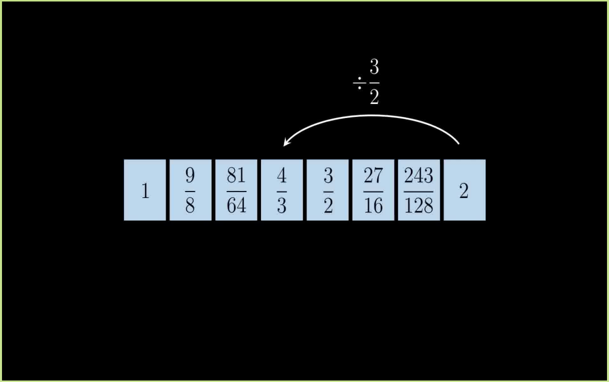


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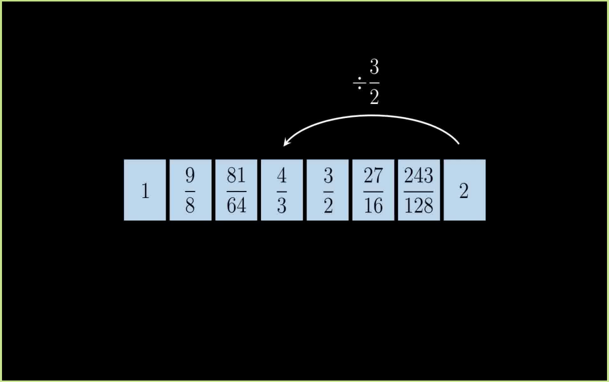


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## Pythagorian tuning: sounds of the seven white keys

Link

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- It is enough to play a simple melody.

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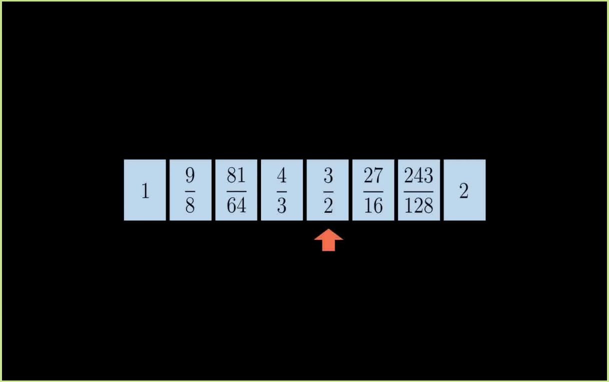
- There is however, a problem with a transposition: We started our melody from the fifth, i.e. $\frac{3}{2}$ key. If we started from the next key, $\frac{27}{16}$, we would be missing one note, to play the same melody :((.



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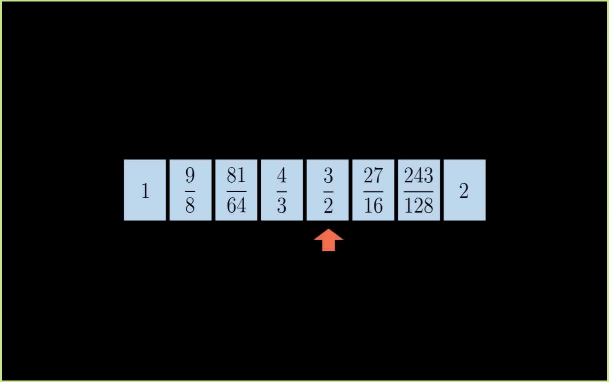

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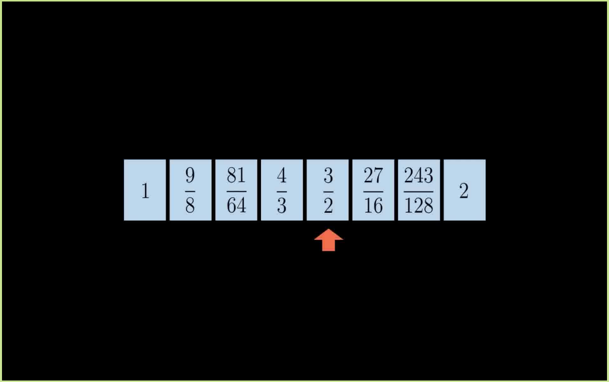

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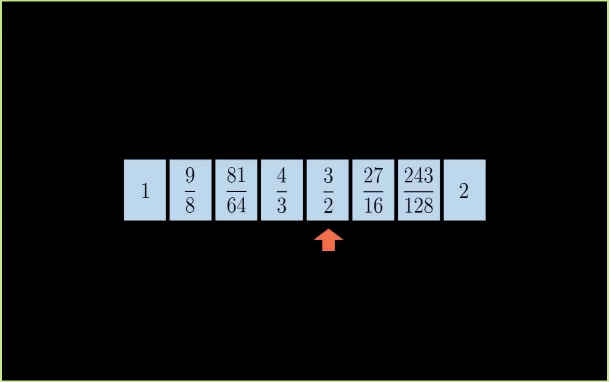

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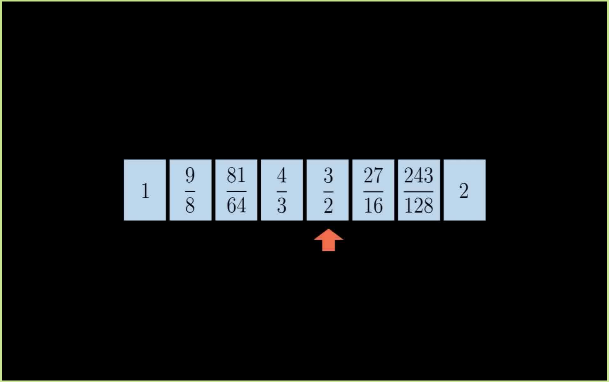

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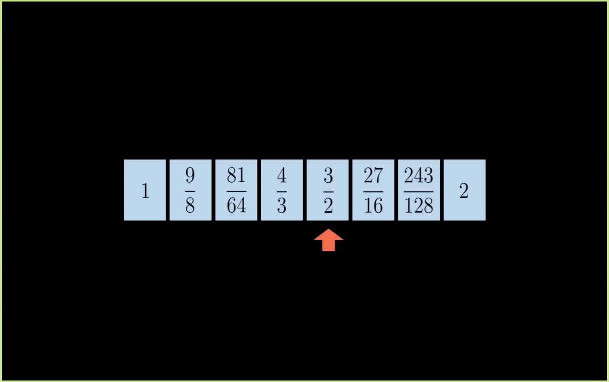

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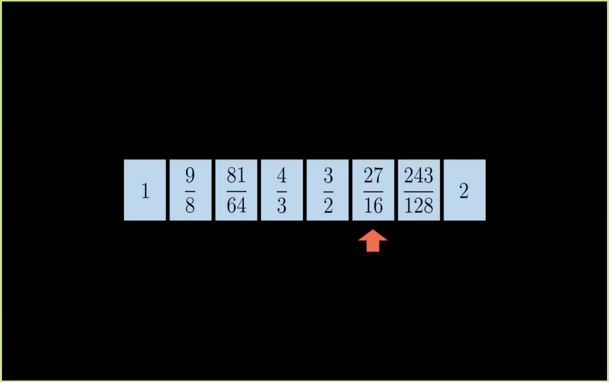

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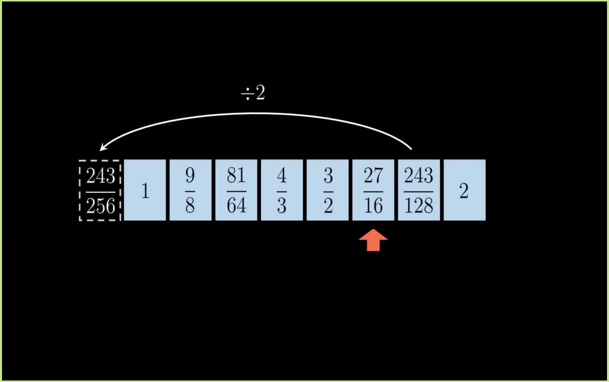

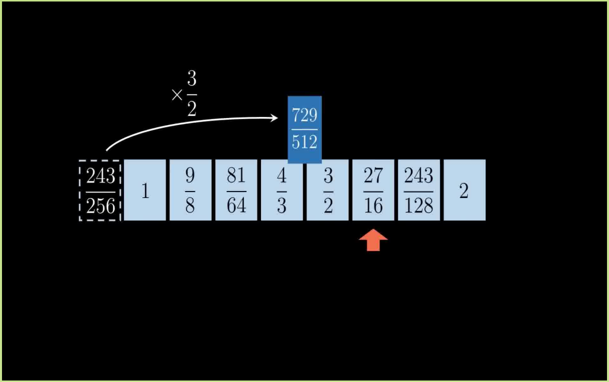


- We need to add only one key more, we again can play the same melody, now starting at $\frac{27}{16}$.
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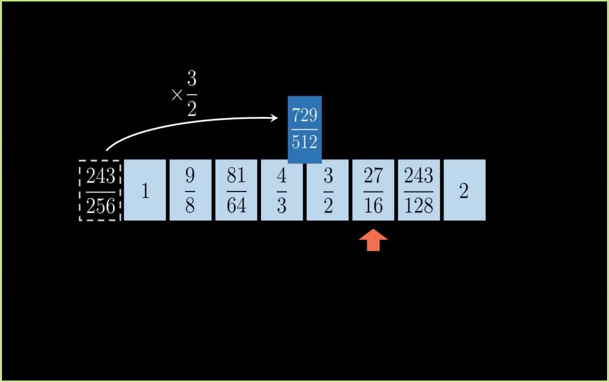


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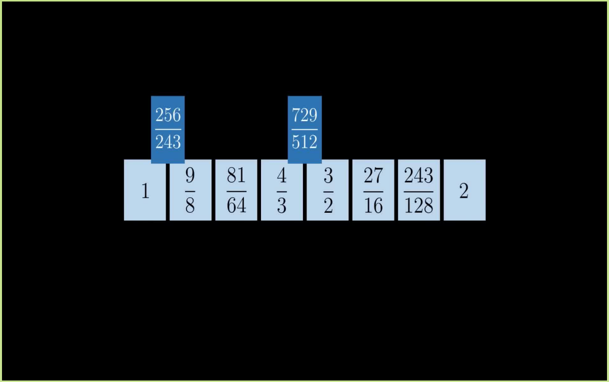

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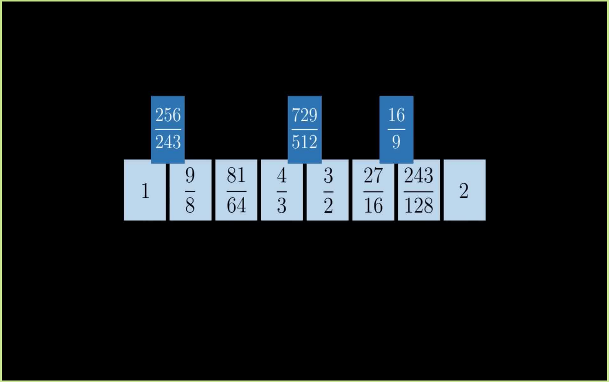
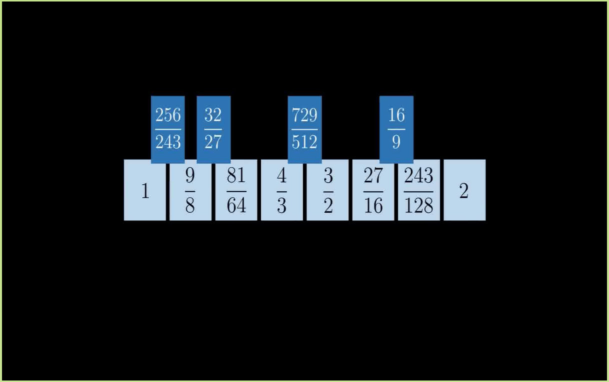
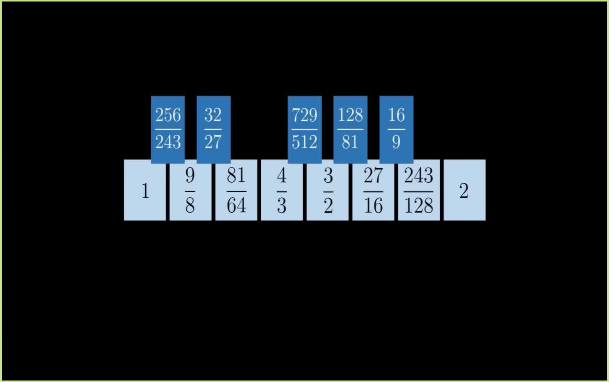



256/243
32/27
729/512
16/9
1
9/8
81/64
4/3
3/2
27/16
243/128
2




- Uff...We have 12 keys! 7 whites, and 5 blacks!

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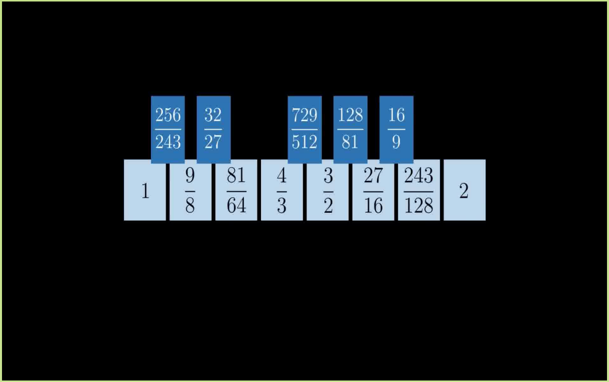


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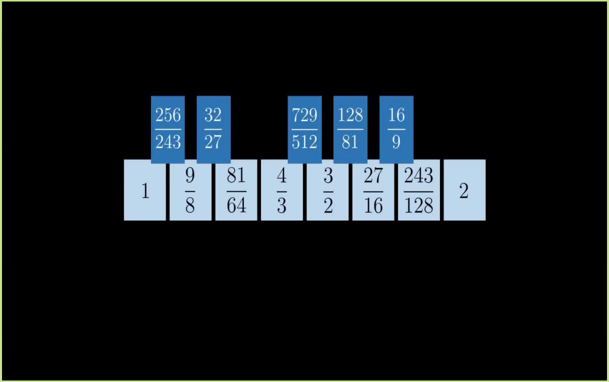


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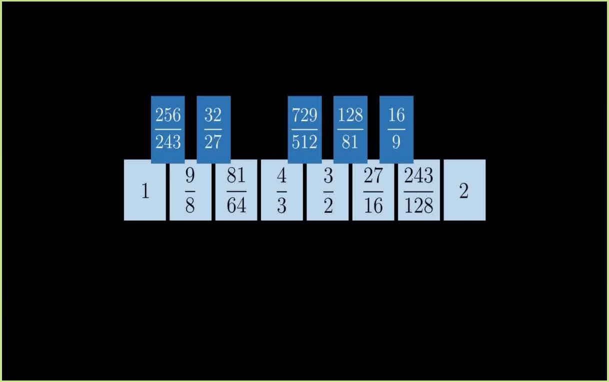


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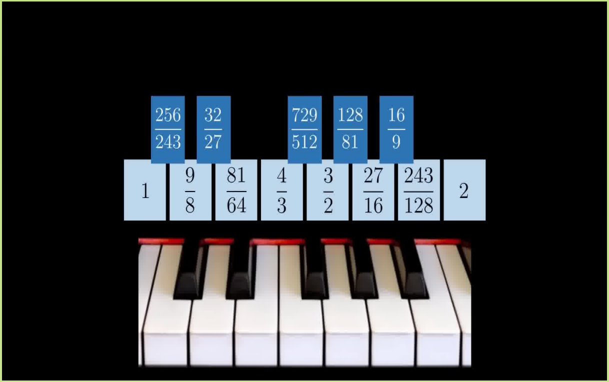
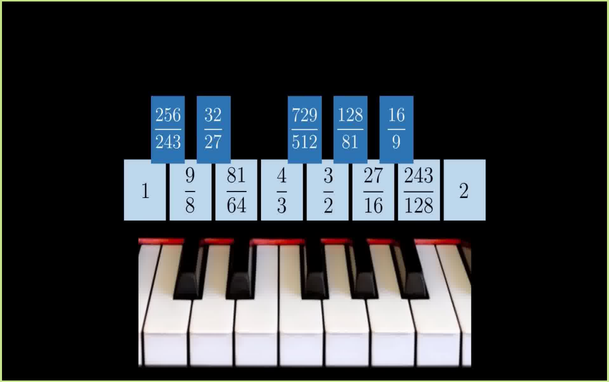



- What we have now is pretty much as is the fundamental pattern of the twelve keys of the piano keyboard.
- The Pythagorian system, has more trouble issues than just having troubles with transposition. The **just intonation** -yet another tuning system - is introduced to cure one of them.

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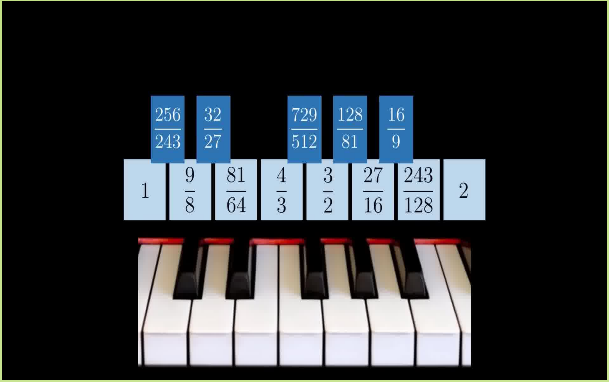


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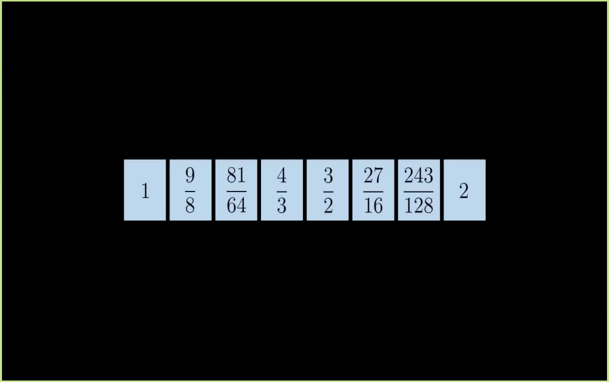


- One problem of the Pythagorian tuning is caused by the **beats.**
- Some important **chords** - the multiple harmoneous tones played at once - sound better in **just intonation.**

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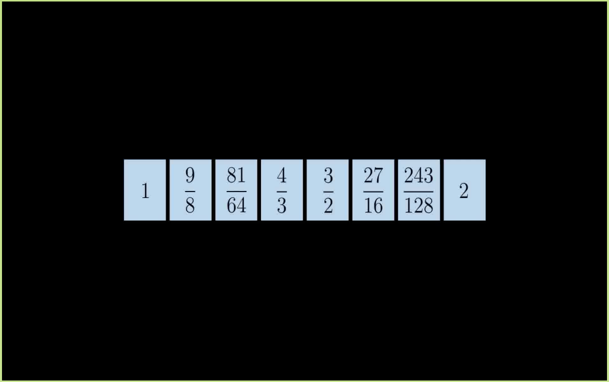


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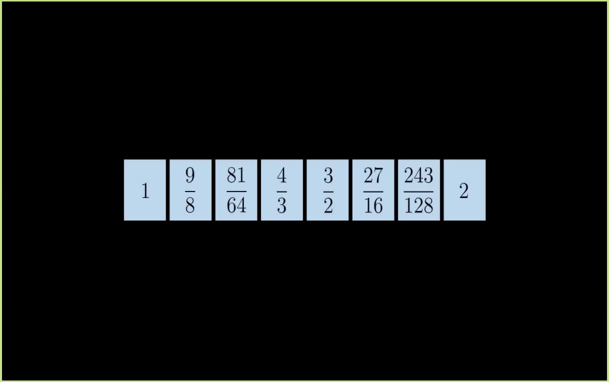


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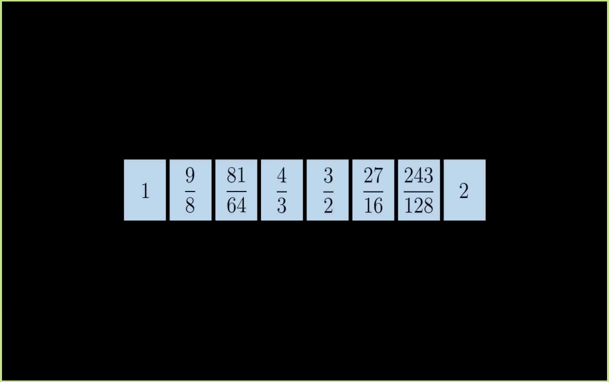


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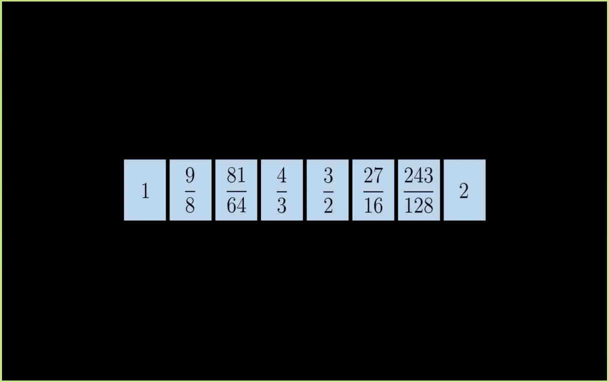


- In particular, instantenous play of the **1st, 3rd and 5th note**, in the white keys, is an important chord, called **the major chord**.
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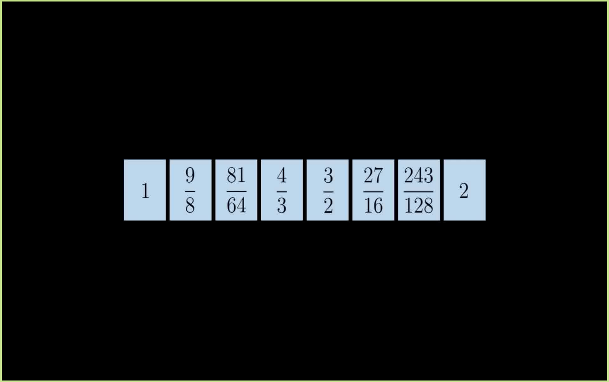
- And the problem is with the **5th overtone** of the first note, and the **4th overtone** of the 3rd note.

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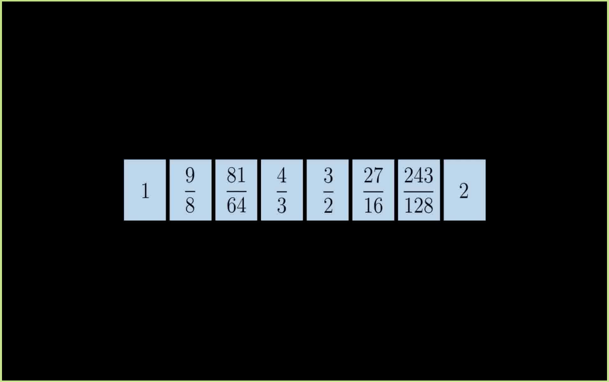


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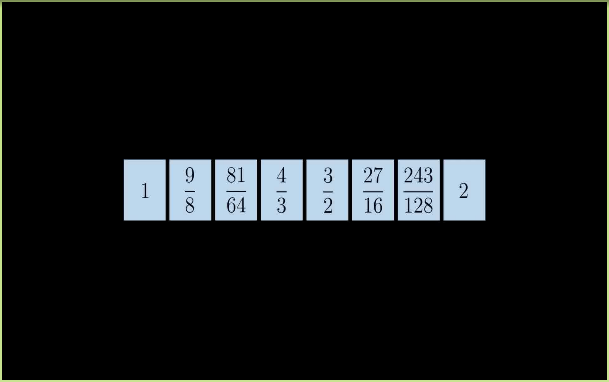


- It is as simple as:

$$4 \times \frac{81}{64} \simeq 5.0625 \simeq 5 \times 1.$$

- Because a superposition of two sinusoidal waves with so close frequencies produces this:
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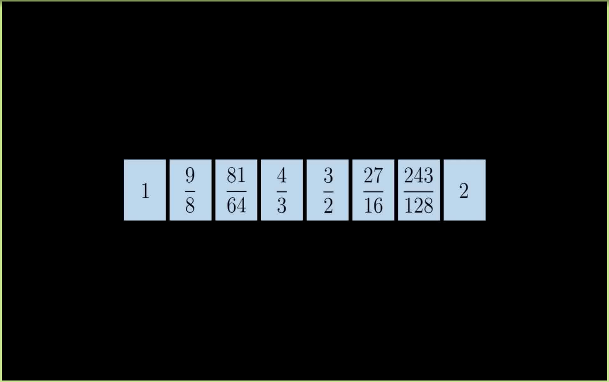


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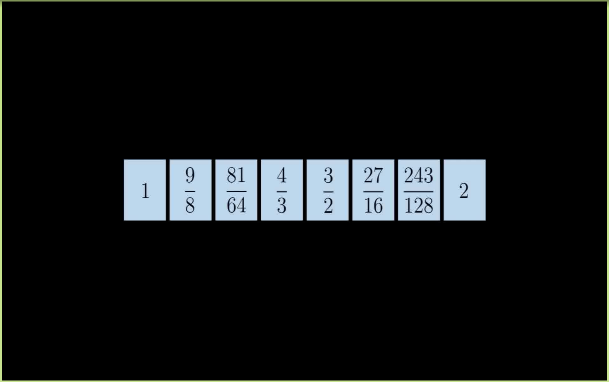


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|---|---|---|---|---|---|---|---|---|
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$$\frac{81/64}{5/4} = 1.0125$$

# Just intonation

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Just | 1 | $\frac{9}{8}$ | $\frac{5}{4}$ | $\frac{4}{3}$ | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{15}{8}$ | 2 |
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| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
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$$\left(\frac{3}{2}\right)^{12} \simeq 2^7.$$

- This is the same as saying that

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- Well...

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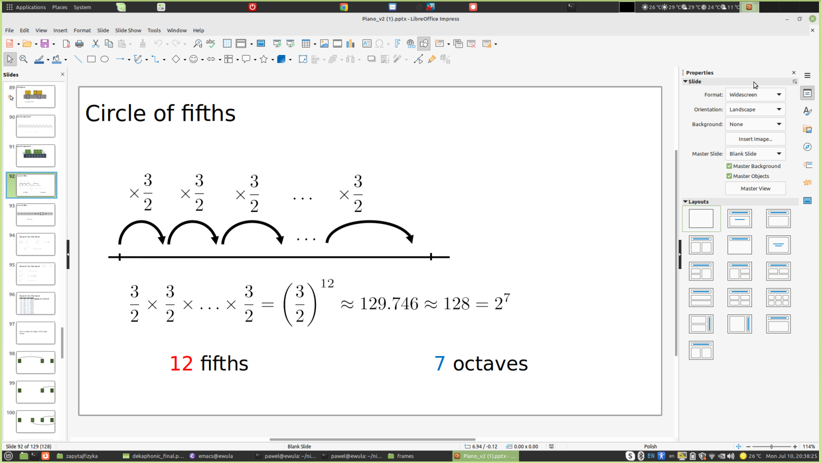


- This means that starting in the pitch class *C* and heating the successive key notes by the musical intervals of the fifth, the thirteenth heated fifth will sound as *C*.
- Well..., with a good aproximation.

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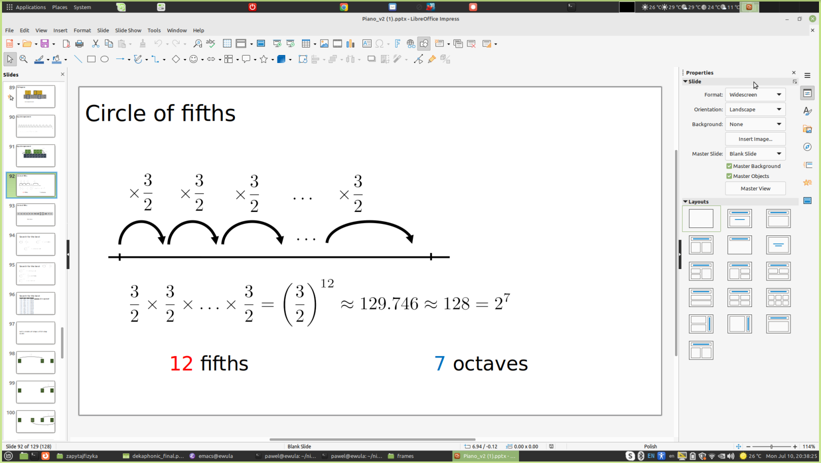


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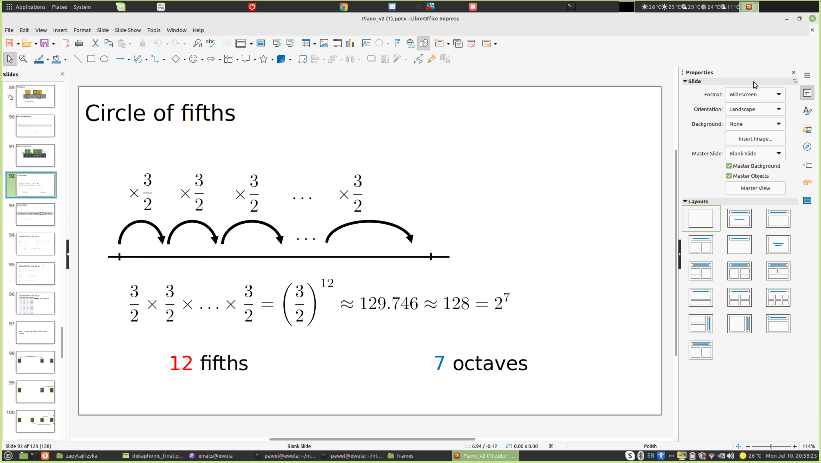


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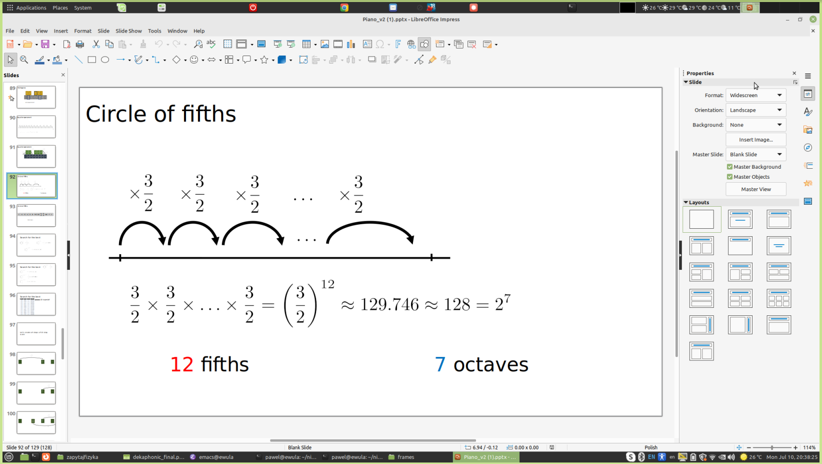


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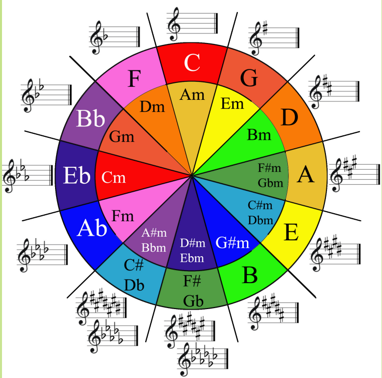
C
G
D
A
E
B
F#
Gb
C#
Db
Ab
Eb
Bb
F
Am
Em
Bm
F#m
Gbm
C#m
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G#m
D#m
Ebm
A#m
Bbm
Fm
Cm
Gm
Dm

Link

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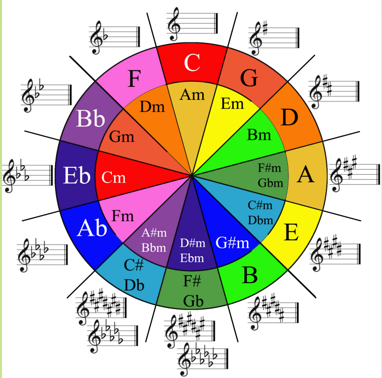


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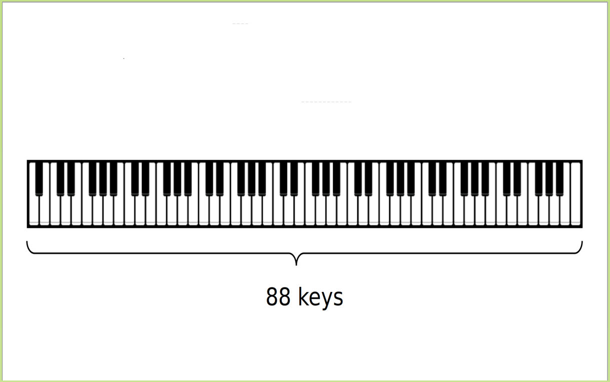
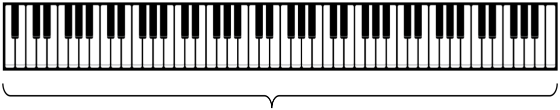



- This, in particular explains $88 = 2 + \mathbf{12} \times \mathbf{7} + 2$ keys of the piano!
- **twelve** keys in each octaves, and **seven** octaves, to go around in terms of the fifths to the same pitch class $C$.

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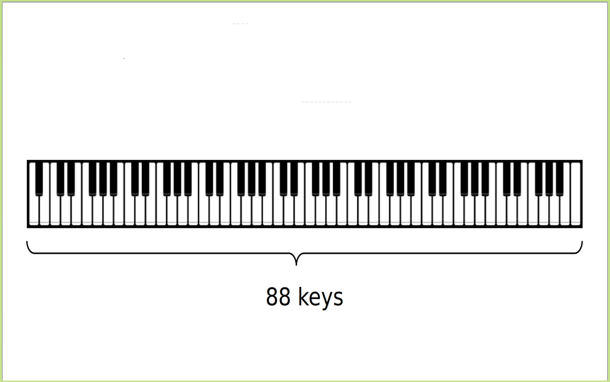


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- For the pourpose of this lecture, I will call the **modulus of the difference** $\left|\left(\frac{3}{2}\right)^{\frac{12}{7}} - 2\right| = \mu_{(3,12,7)}$, a **comma**, of the tuning system.
- In the tuning systems we considered here, **the comma** $\mu_{(3,12,7)} = 0.00387547$ is related to the **three natural numbers** $f$, $t$, $k$ which are:
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- **Question**: Can we have a Pythagorian tuning system with $t$ keys in every octave, which has $k$ octaves and which is generated from the first octave $1 \longleftrightarrow 2$ by a harmonic[2] $f$, such that
  - its comma $\mu_{(f,t,k)} < 0.01$,
  - the number of tones $t \times k < 100$,
  - its generating harmonic $f < 64$?
- I call this system Pythagorian, although it is based on $f$ rather than $f = 3$. But this is a straightforward generalization.

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**Theorem**
The only values of $(f, t, k)$ that answer the question in positive are in the following table:

| $f$ | $t$ | $k$ | $\mu$ |
|---|---|---|---|
| $\frac{3}{2}$ | 12 | 7 | 0.00387547 |
| $\frac{11}{8}$ | 13 | 6 | 0.00631819 |
| $\frac{13}{8}$ | 10 | 7 | 0.000871016 |
| $\frac{17}{16}$ | 23 | 2 | 0.00808825 |
| $\frac{27}{16}$ | 4 | 3 | 0.00905446 |
| $\frac{29}{16}$ | 7 | 6 | 0.00135602 |

| $f$ | $t$ | $k$ | $\mu$ |
|---|---|---|---|
| $\frac{33}{32}$ | 45 | 2 | 0.00156911 |
| $\frac{37}{32}$ | 19 | 4 | 0.0070528 |
| $\frac{39}{32}$ | 7 | 2 | 0.00151358 |
| $\frac{41}{32}$ | 14 | 5 | 0.00158879 |
| $\frac{43}{32}$ | 7 | 3 | 0.00744747 |
| $\frac{47}{32}$ | 9 | 5 | 0.00241079 |
| $\frac{53}{32}$ | 11 | 8 | 0.00123505 |
| $\frac{55}{32}$ | 9 | 7 | 0.00639457 |
| $\frac{57}{32}$ | 6 | 5 | 0.000737349 |
| $\frac{59}{32}$ | 9 | 8 | 0.00971721 |

## Pythagorian system with 10 keys and 7 octaves

- We see that among all Pythagorian systems, based on $f < 56$, the system with $f = 13$, $t = 10$, $k = 7$ is the system with the **smallest comma**.
- It is precisely the system which **Leszek Możdżer** wanted us to design for him. His motivation for using $t = 10$, $k = 7$ is kind of 'mystic'; totally unclear for us.
- The Pythagorian **decaphonic piano** has **six white** keys and **four black** keys.
- Its octave with **six white** keys is generated by $f = \frac{13}{8}$, which plays the role of the 'fifth', when compared to the $t = 12$, $k = 7$ Pythagorian system.
- Since we have **ten** keys, and $k = 7$, to cover a passage from $C$ to $C$ via the 'fifths' of $\frac{13}{8}$, the piano needs 70 keys, only.
- It should be stressed, that in our analysis we did not insist on having $t = 10$ keys! The **decimal/decaphonic** system, with **seven** octaves, was distinguished by pure mathematics.

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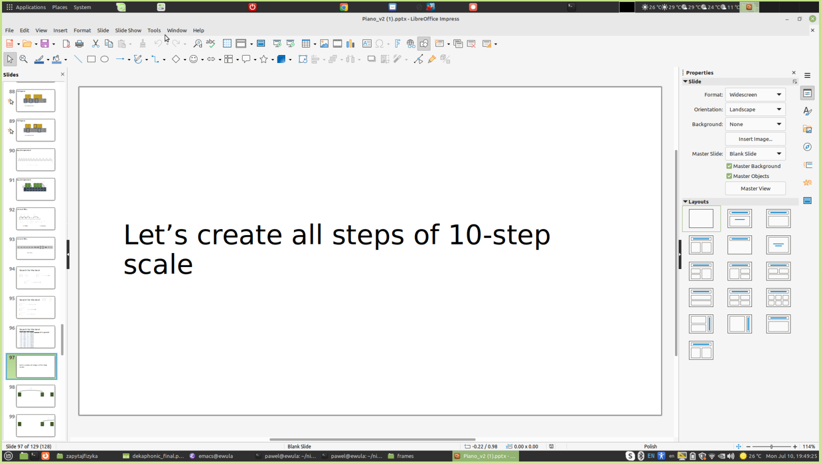
# Ten scale Pythagorian tuning



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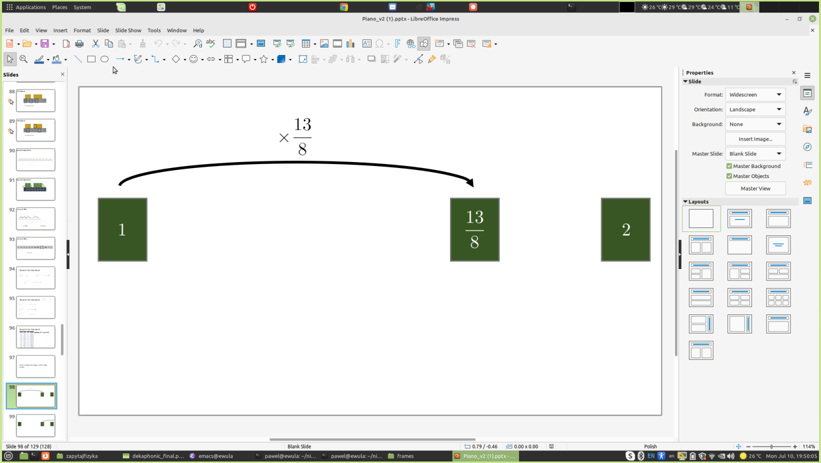

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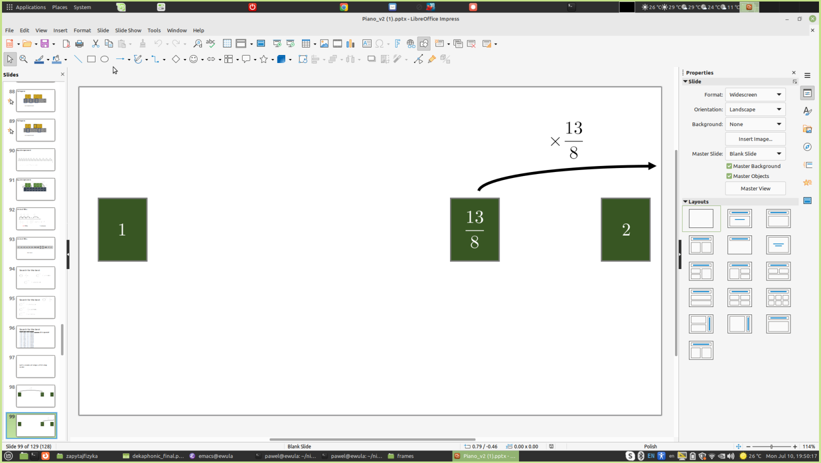

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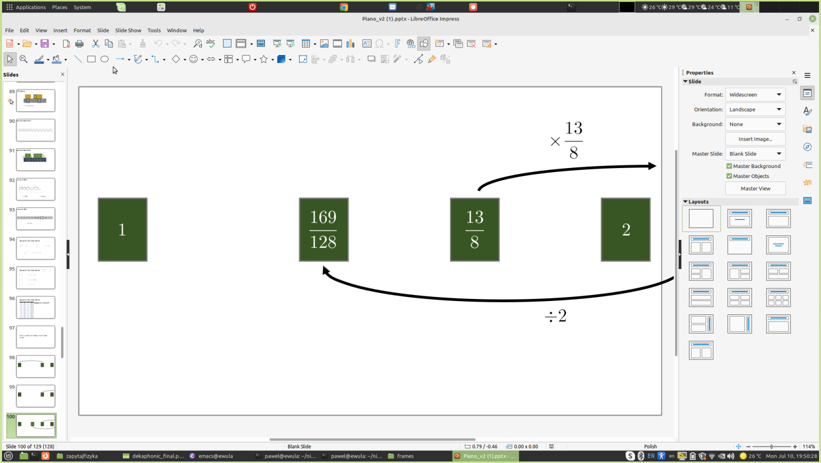

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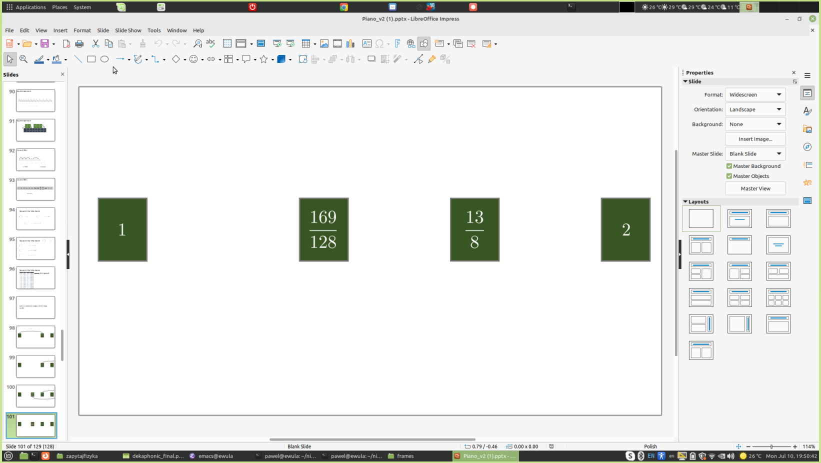

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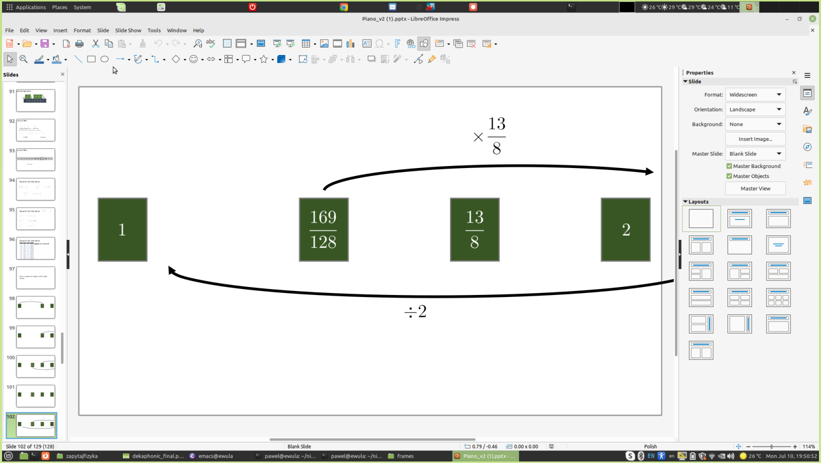

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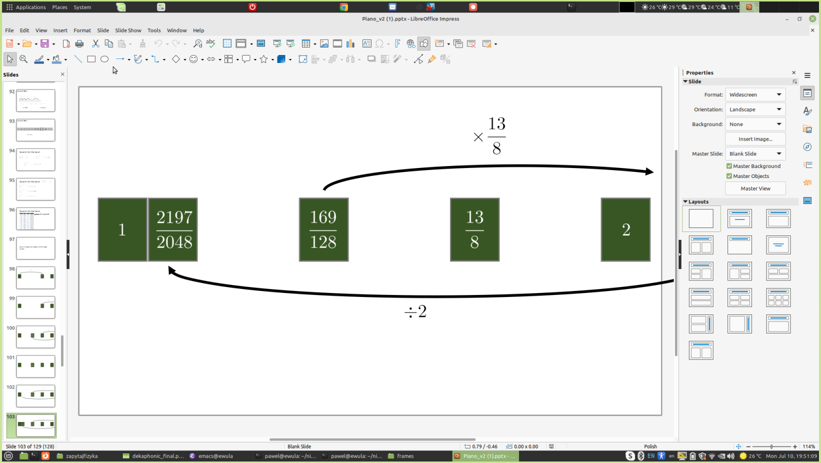

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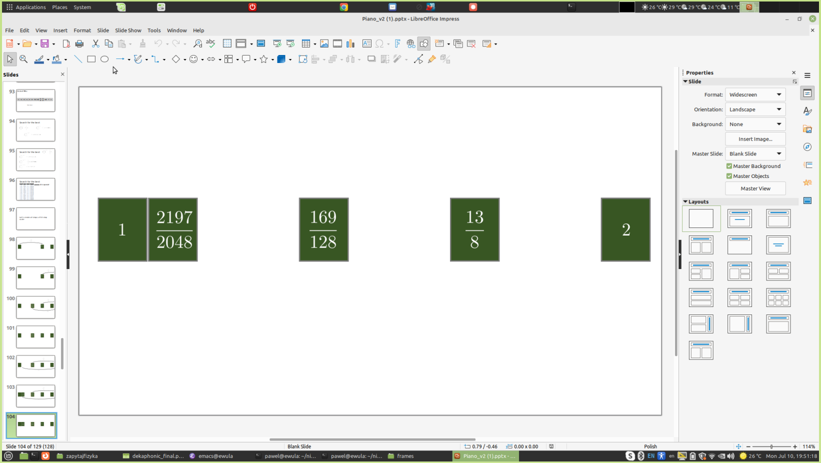

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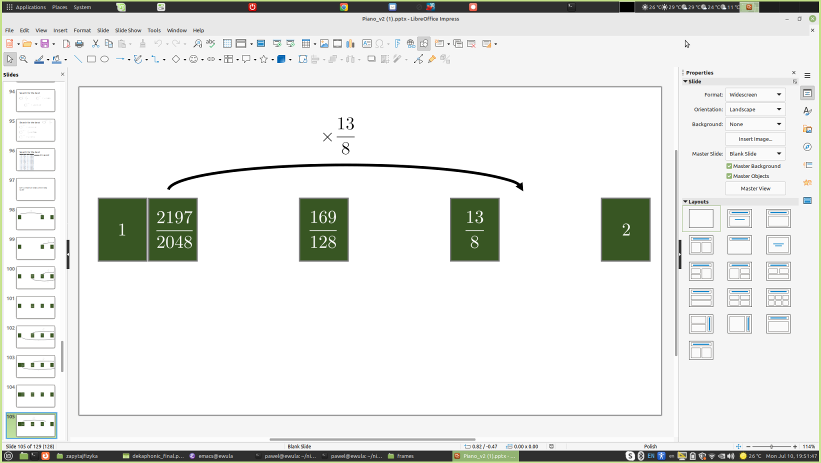

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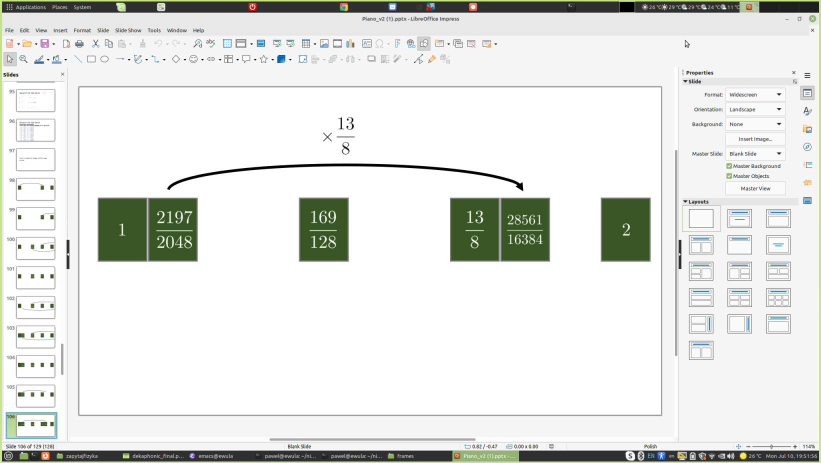

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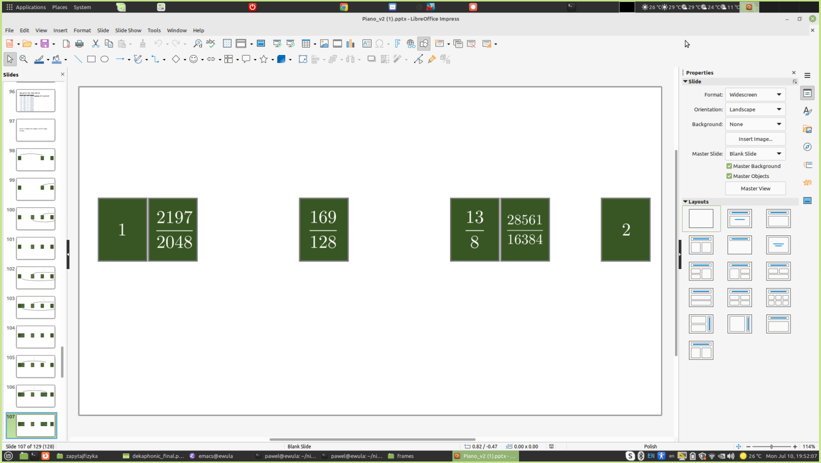

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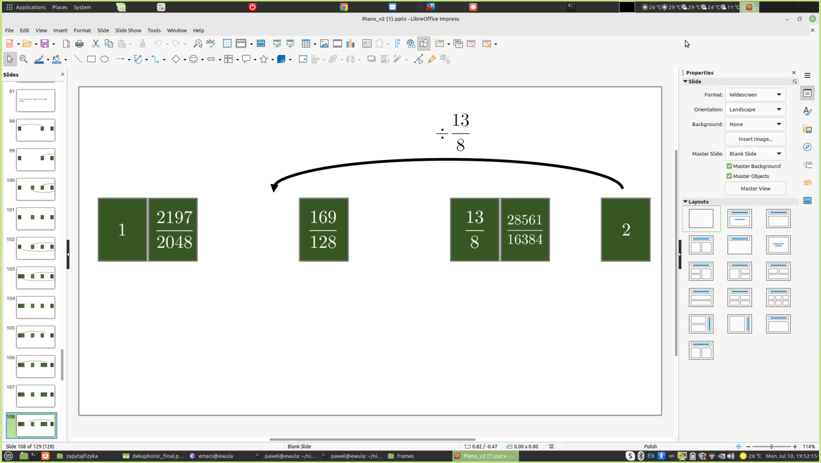

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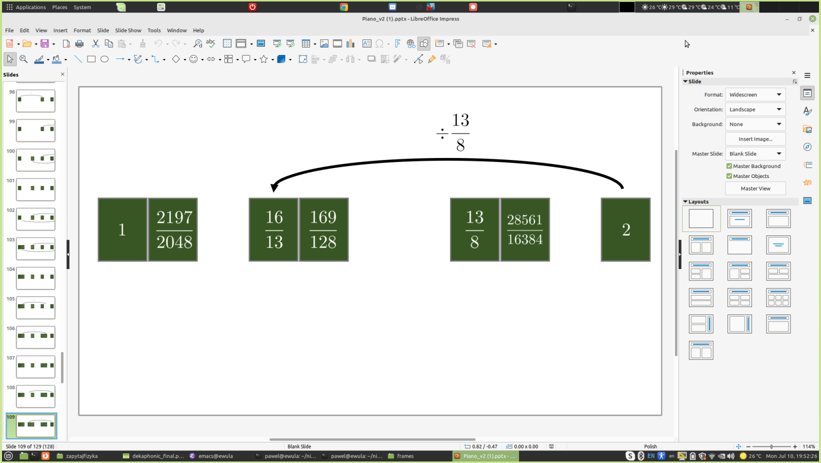

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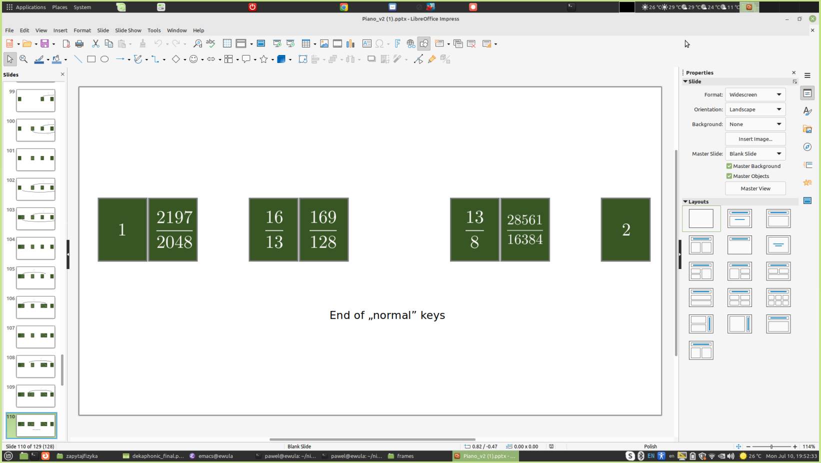

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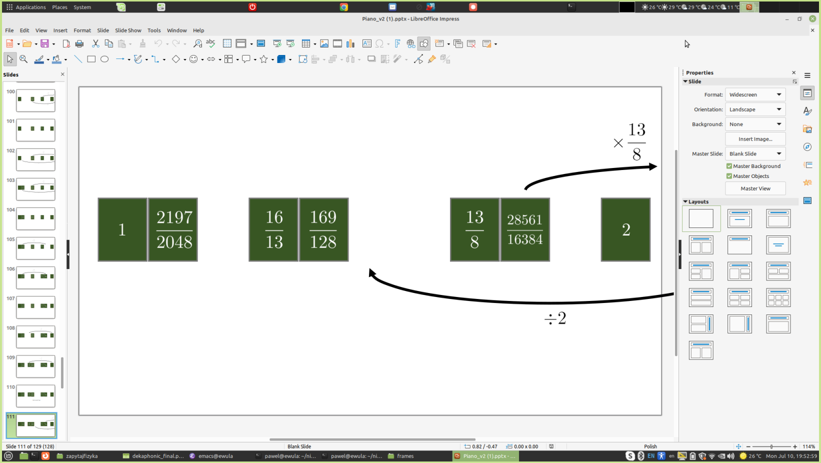

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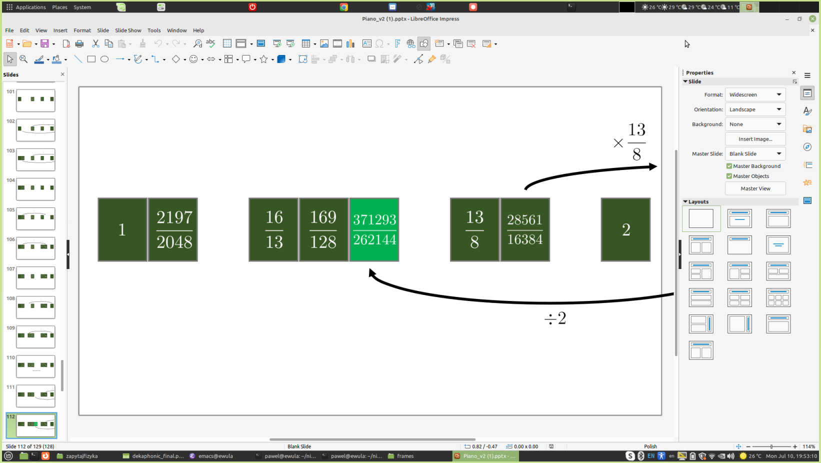

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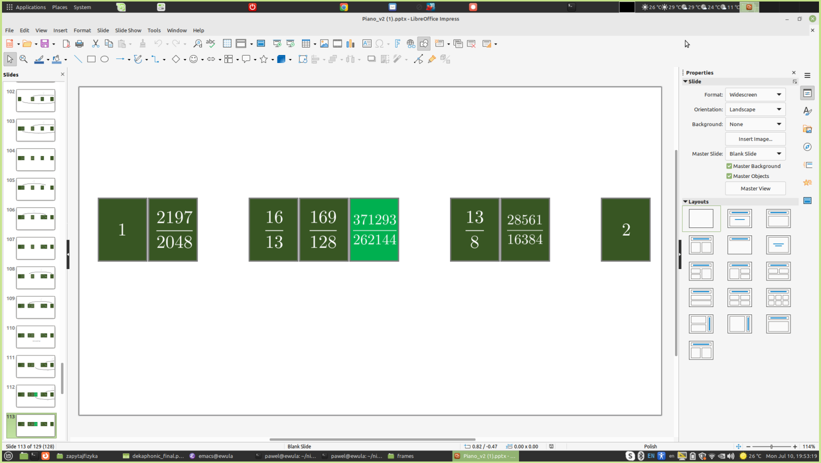

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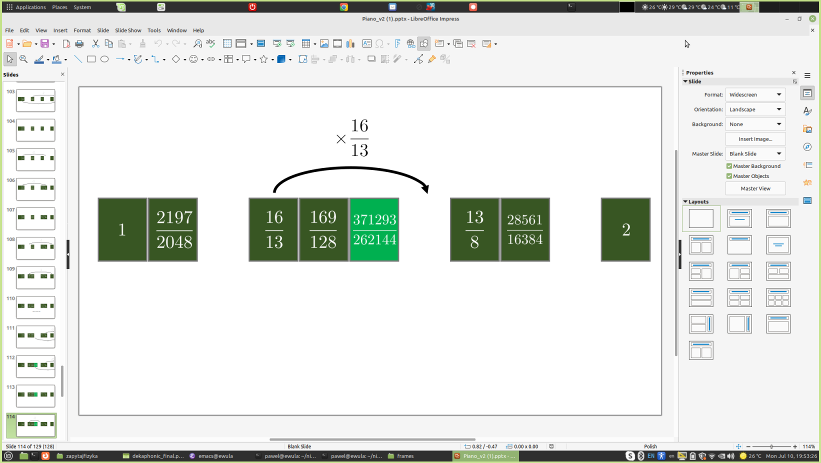

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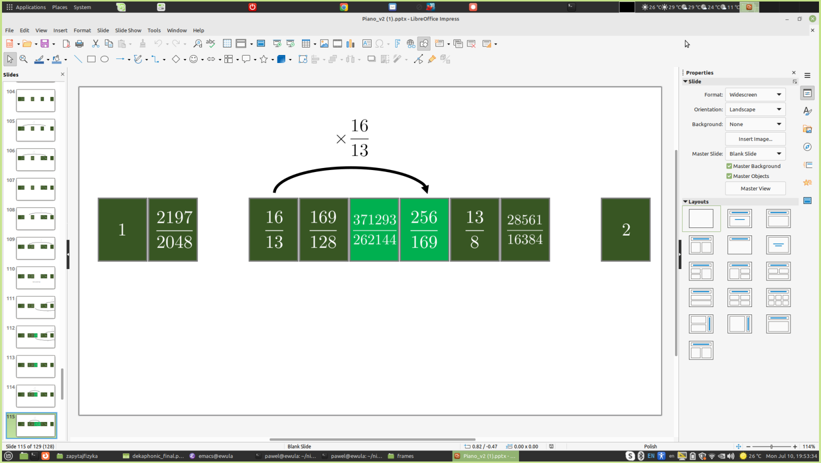

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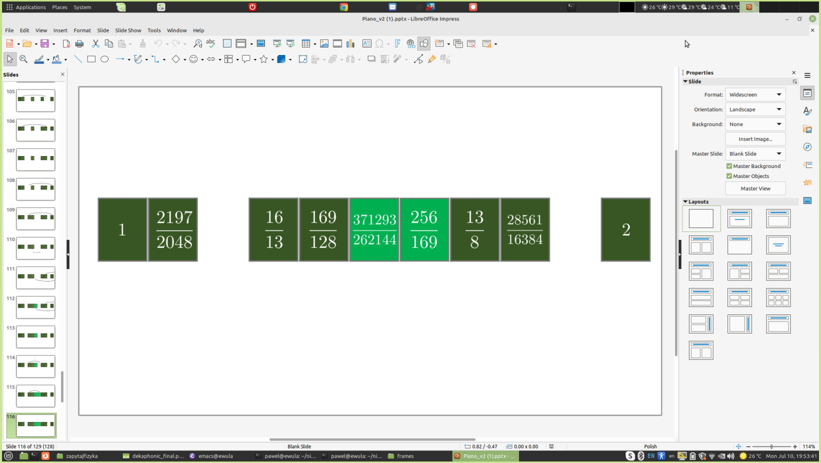

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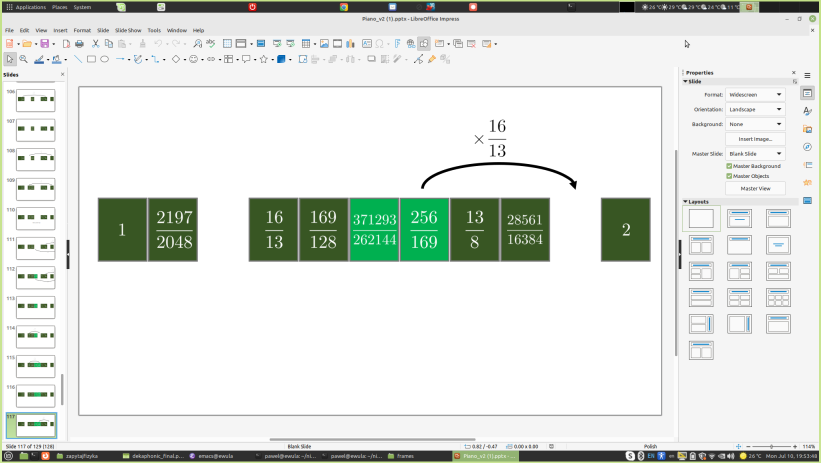

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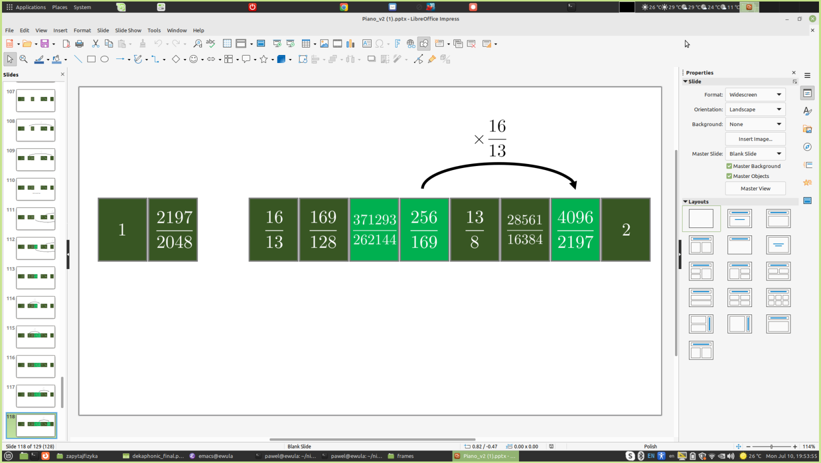

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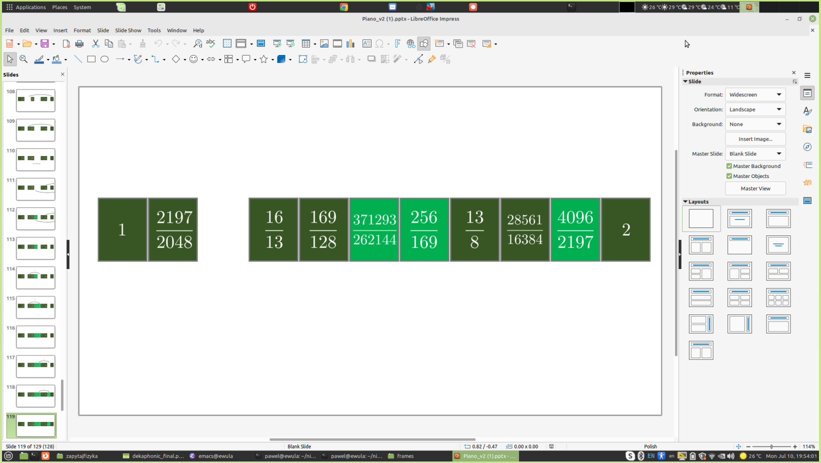

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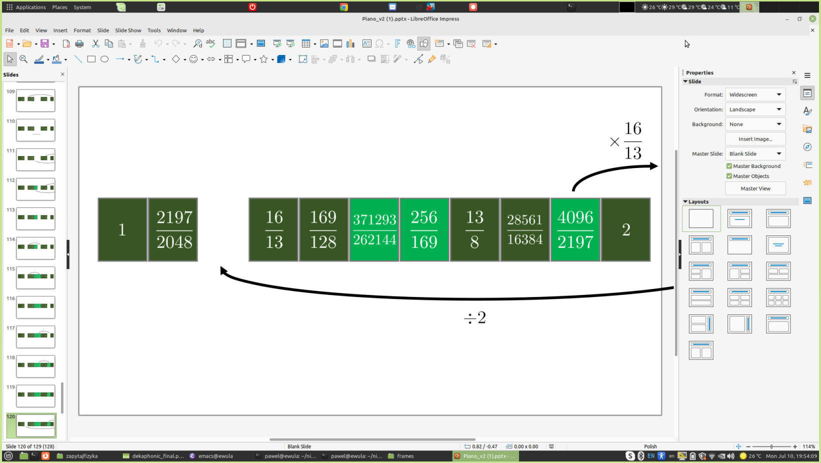

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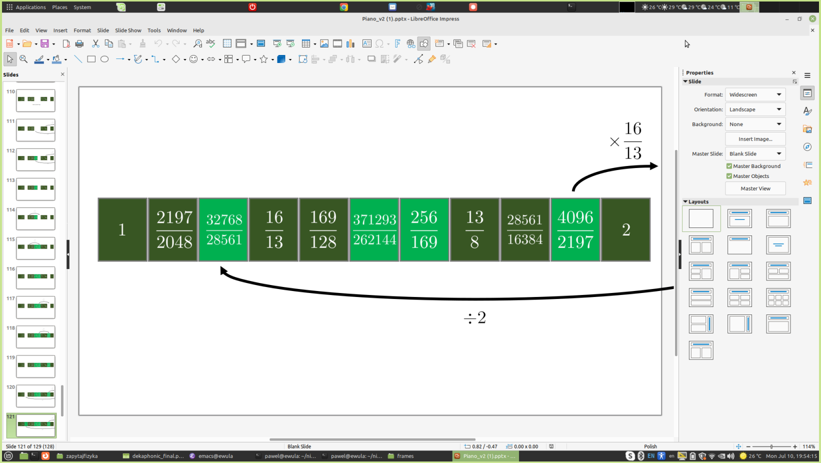

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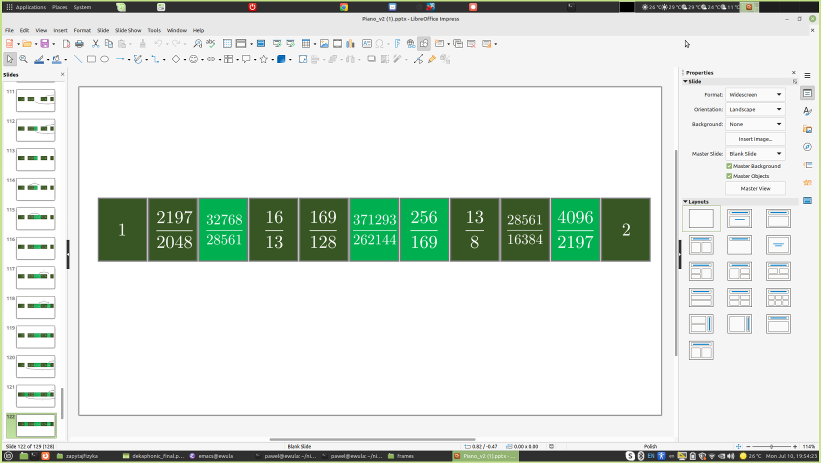

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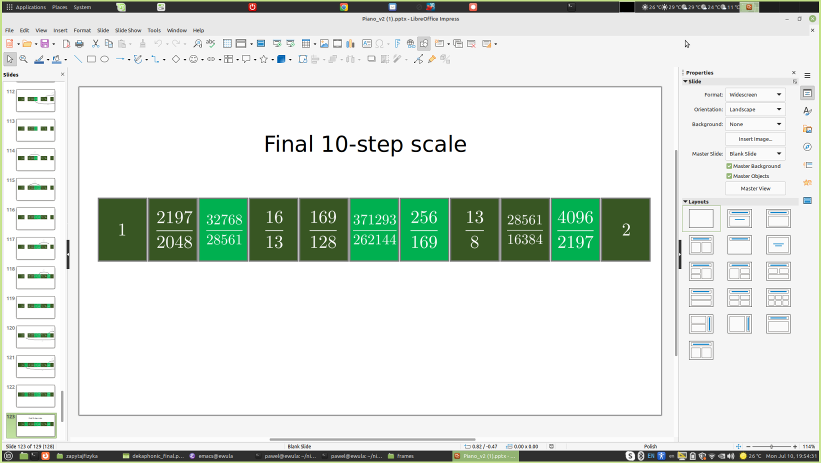

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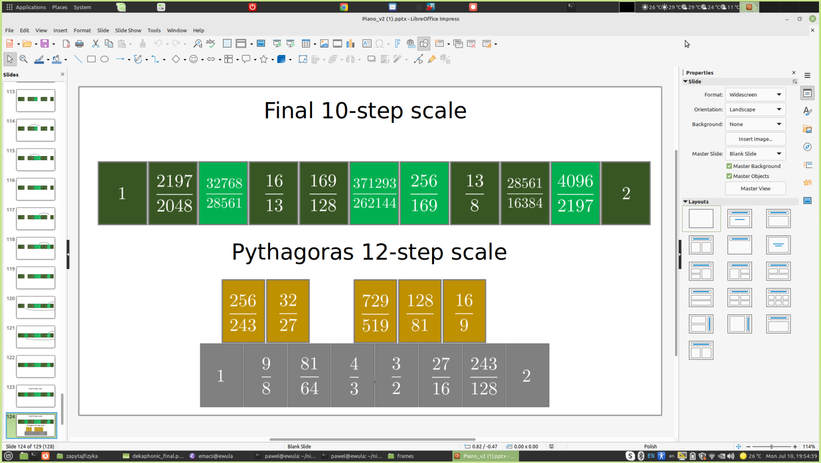

- Consider all possible musical systems with $t$ keys in an octave.
- As far as the **problem of transposing melodies** from one key to another is concerned, the **best** among these systems is the $t$-system in which all the **adjascent keys are apart the same interval**.
- Let this interval be given by a number $r$.
- We start with our first key, in the first octave, which has frequency value $1$.
- The second key has frequency value $1 \times r = r$, , the third key, $r \times r = r^2$, the fourth $r^2 \times r = r^3$, and so on, until the $t^{\text{th}}$ key, which will have frequency value $r^{t-2} \times r = r^{t-1}$.
- Since the scale has $t$ keys, the $(t+1)^{\text{th}}$ key **starts a new octave**, so that the value of this key is on one hand $r^{t-1} \times r = r^t$, but on the other hand is $2$.

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- We thus have $r^t = 2$, or $r$ is the $t^{\text{th}}$ **root of** $r$.
- Thus, the musical intervals in such equally distanced scale are equal to

$$r = (2)^{\frac{1}{t}}.$$

- A musical scale with $t$ equal intervals $r = (2)^{\frac{1}{t}}$ is called **equally tempered**, or **equal temperament**.

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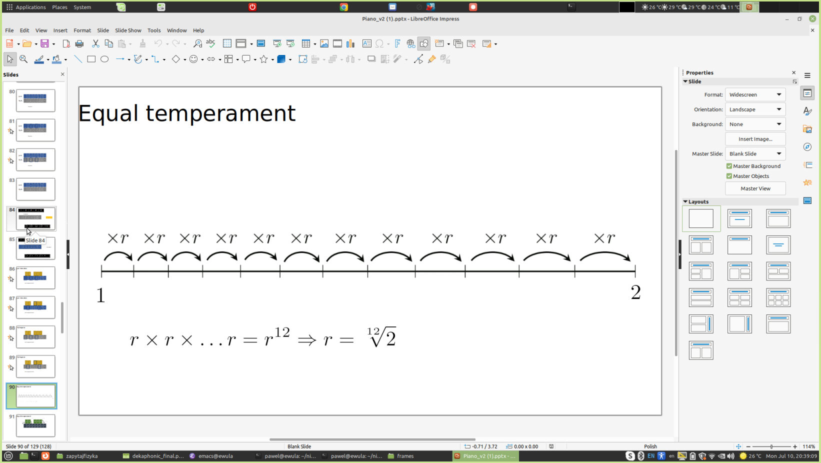
# Equal temperament for $t = 12$



## Equal temperament

- The **decaphonic piano of Leszek Mozdzer**, which physical implementation will be presented during the **concert at Nowa Miodowa Hall on Thursday, 13th July, at 19:00**, uses **equally tempered 10 scale musical system.**
- Mathematically the system has a remarkable property that the **differences in musical intervals** between **each of its ten keys** and the corresponding keys of **the Pythagorian** $f = 13$, $t = 10$, $k = 7$ **system**, are about **an order of magnitude smaller**, than the corresponding distances differences between keys of the **equally tempered 12 piano scale** and the keys of **its** $f = 3$, $t = 12$, $k = 7$ **Pytagorian counterpart.**

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