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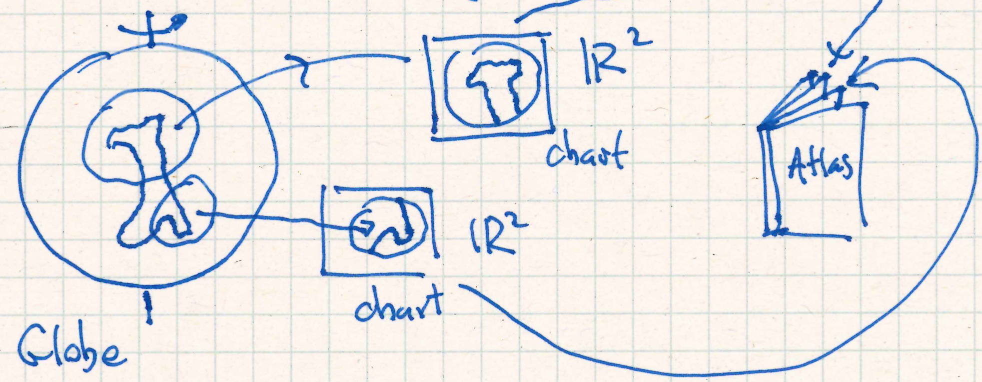
Lecture 1 13.04.2018

Pre Einstein spacetimes.

1) All classical theories assume that a spacetime is a differentiable manifold

- Euclidean space $\mathbb{R}^3$ is a manifold
- $\mathbb{R}^n$ is a manifold

but a typical example of a manifold is something less trivial, as e.g. 2-dimensional sphere. S^2.



To depict the Globe it is convenient to use charts, and collect them into an atlas.

Roughly speaking, a differentiable manifold is a topological space M, equipped with differentiably compatible charts which form an atlas of M.

We pass to precise definitions.

Topological space M is a set with a collection τ of its subsets, U_I, called open sets, satisfying the following axioms:

1) empty set ϕ and M belong to τ
2) any union (finite or infinite) of sets from τ belongs to τ
3) intersection of any finite number of sets from τ belongs to τ.

The collection τ is called topology in M

Topological Hausdorff space is a topological space (M, τ) such that for every two points $x \neq y$ in M there exist sets U_x, U_y in τ s.t.

$$x \in U_x, \quad y \in U_y, \quad U_x \cap U_y = \phi.$$

(Points can be distinguished by sets from the topology).

Example of no-Hausdorf: $M = \{x, y\}$, $\tau = \{\phi, \{x\}, \{x,y\}\}$

$x \neq y$ but none of pairs of the sets $\{x\}, \{x,y\}$ has property as in definition.

Continous map between two topological spaces

(M_1, τ_1) and (M_2, τ_2) two topological spaces.

Map $\varphi: M_1 \to M_2$ is continous iff for every $V \in \tau_2$

$$\varphi^{-1}(V) \subset \tau_1.$$

(preimage of an open set in τ_2 is open in τ_1).

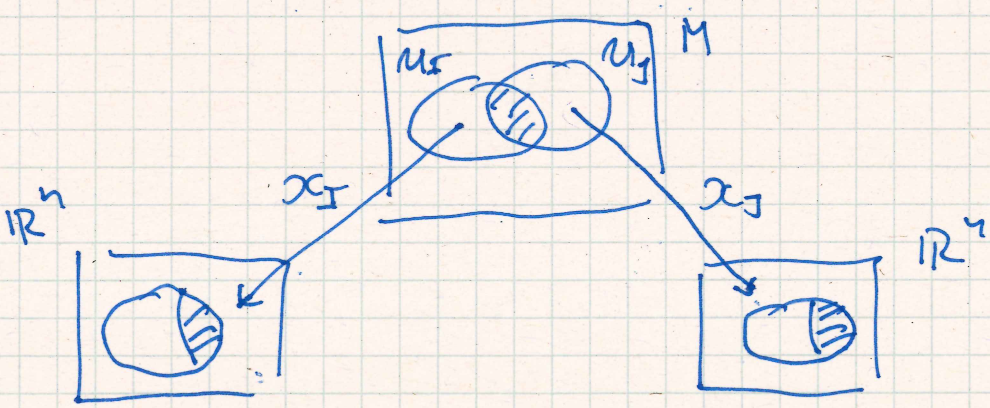
Homeomorphism is a continous map between topological spaces which a) has an inverse and b) its inverse is continous.

<u>Differentiable</u> <u>manifold</u> of <u>class C^k</u> is a topological Hausdorff space M equipped with a collection of pairs (U_I, x_I), called <u>charts</u>, such that

a) U_I are open and x_I is a homeomorphism between U_I ~~and~~ and an open set in $\mathbb{R}^n$,

$$x_I : U_I \xrightarrow[\text{homeo}]{} \mathbb{R}^n$$

b) any two charts are C^k compatible, i.e.



$$x_J \circ x_I^{-1}\Big|_{x_I(U_I \cap U_J)} \quad \text{and} \quad x_I \circ x_J^{-1}\Big|_{x_J(U_I \cap U_J)}$$

are differentiable maps ~~between~~ of class C^k (between $\mathbb{R}^n$'s).

c) $\bigcup_I U_I = M$

d) the collection of charts is maximal, meaning that addition of new charts will destroy C^k compatibility.

The collection of such pairs is called a maximal <u>atlas</u> of M.

$k = 1, 2, \ldots, \infty, \omega$ (∞ ← smooth; ω ← analytic.)

Number n is called <u>dimension</u> of M

If only a, b, c we have an <u>atlas</u>. Any atlas can be extended to a maximal atlas in a unique way.

We assume that a spacetime is a differentiable manifold of class C^∞ with a <u>countable</u> atlas.

<u>Examples</u>

① M = open subset of $\mathbb{R}^n$

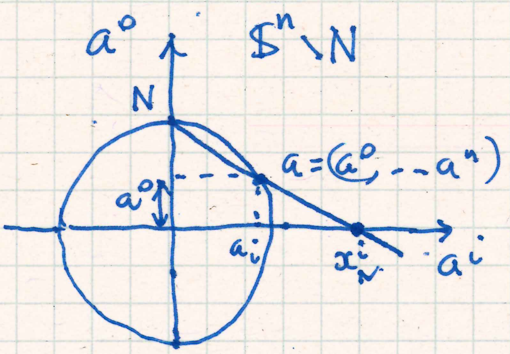
Define $x: M \to \mathbb{R}^n$ by $x(p) = p$

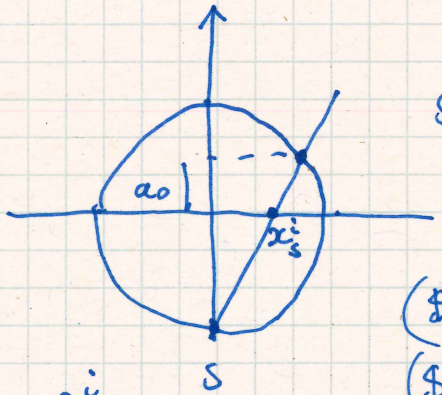
this is an atlas on M (consisting of <u>one</u> chart (M, x))

which gives M a structure of a manifold of class C^ω.

② n-dimensional sphere S^n.

$$S^n = \{ a \in \mathbb{R}^{n+1} \text{ s.t. } (a^0)^2 + (a^1)^2 + \dots + (a^n)^2 = 1 \}$$





$(S^n \setminus S, x_S)$

$(S^n \setminus N, x_N)$ atlas

$$\frac{1-a^0}{a^i} = \frac{1}{x^i_N} \Rightarrow x^i_N = \frac{a^i}{1-a^0}$$

$$\frac{1+a^0}{a^i} = \frac{1}{x^i_S} \Rightarrow x^i_S = \frac{a^i}{1+a^0}$$

$$x_N(a) = \frac{1}{1-a^0}(a^1, \dots, a^n) \qquad x_N : S^n_\bullet \to \mathbb{R}^n$$

$$x_S(a) = \frac{1}{1+a^0}(a^1, \dots, a^n) \qquad x_S : S^n_0 \to \mathbb{R}^n$$

$S^n_\bullet = S^n - N$

$S^n_0 = S^n - S$

<u>Check</u>

$$x_N \circ x_S^{-1}(c) = \frac{(c^1, \dots c^n)}{\sum_{i=1}^n (c^i)^2}$$ analytic on $S^n - \{N, S\}$.

2) Arystotelian and Galilean spacetime.

Spacetime is a 4 dimensional smooth manifold.
Points of this manifold = events in spacetime.
4-dimensions: because an event needs 4 coordinates to be identified (time, elevation, latitude, longitude) if on Earth.

No gravitational effects for today!

Galileo/Newton: First law of dynamics.

(1) There is a preferred class of motions, called free motions
(2) there exist reference frames relative to which the free motions have no acceleration. (inertial frames)

Remarks

Ad (1) one adds that a body is in free motion when no external influences act on it.

Ad (2) This is usually formulated that free motions are rectilinear (along straight lines) and uniform. relative to certain reference frames called inertial frames

This second formulation is equivalent to (2) if one defines what rectilinearity and uniformity means.

$\rightsquigarrow$ This requires additional mathematical (geometric) structure on a manifold.

$\Longrightarrow$ Notion of an AFFINE SPACE.

• <u>Affine space</u> (M, V) is a pair in which M is a set, and V is a vector space equipped with a map "+",

$$+ : M \times V \longrightarrow M$$

such that V acts in M as an Abelian group, i.e.

- $$(p+u)+v = p+(u+v)$$

and

- if θ is zero vector in V then

$$p + 0 = p \quad \text{for all } p \in M.$$

In addition

(freedom) • $p+u = p$ implies $u = 0$ for all $p \in M$

(transitivity). • for all $p, q \in M$ there exists $u \in V$ s.t.

$$q = p + u.$$

It follows that such u is unique ($q = p+u = p+u' \Rightarrow$
$\Rightarrow p + u + (-u') = p + (u') + (-u') \Rightarrow p = p + (u-u') \Rightarrow$
$\Rightarrow u = u'$).

u is thus called a "difference between q and p"

$$u := q - p$$

<u>The dimension</u> of affine space is the dimension of V.

All affine spaces of equal dimension are <u>isomorphic</u>.

<u>Indeed:</u> Distinguish a point θ in M. Then every other point p in M uniquely defines vector $u(p) = p - \theta$. This gives a map

$$M \ni p \longrightarrow u(p) \in V$$ showing an isomorphism between (M, V) and (V, V).

<u>Note</u> in V there is a distinguished point – θ vector in M no such point exists.

<u>Repère</u> (basis) in (M,V) is a pair (θ, e_μ) where θ is a point in M and e_μ is a basis in V.
Given a point $M \ni p$ and repere (θ, e_μ) we have

$$p = \theta + u(p) = \theta + \underbrace{x^\mu(p)}_{\text{components of vector } u(p) \text{ in basis } e_\mu.} e_\mu$$

(Summation convention: repeated indices on different altitude are summed over the range of indices).

coordinates of p in repere (θ, e_μ)

This formula gives a 1-1 map $x: M \to \mathbb{R}^n$

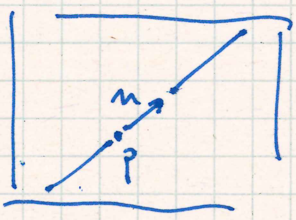
$$M \ni p \longrightarrow x^\mu(p) \in \mathbb{R}^n.$$

Then the pair (M, x) is a chart with domain being entire M. It gives atlas on M, showing that <u>affine space</u> (M,V) <u>is a differential manifold.</u>

<u>Straight lines</u> Given $p \in M$ and $u \in V$ in an affine space (M,V) we define a straight line to be a set

$$\ell = \{ p + \lambda u : \lambda \in \mathbb{R}^1 \}$$

u - is called directional vector of a line; it is given up to a scale.



<u>Affine transformation</u> is a pair (f, α_f) of bijections of M and V such that α_f is a linear transformation and

$$f(p+u) = f(p) + \alpha_f \cdot u$$

Fact Affine transformations transform lines into lines.

Proof

$$\ell = \{ p(\lambda) = p + u\lambda \ : \ \lambda \in \mathbb{R} \}$$

$$f(p(\lambda)) = f(p) + \alpha_f(u\lambda) = f(p) + \lambda\,\alpha_f(u) =$$
$$= p' + \lambda u'$$

$\ell' = \{ f(p(\lambda)) \ : \ \lambda \in \mathbb{R} \}$ is a line $\square$.

Affine transformation in coordinates: In repere (o, e_μ)

$$p = o + x^\mu e_\mu \quad , \quad f(p) = o + x'^\mu e_\mu$$

$$\parallel$$

$$f(o) + \alpha_f(x^\mu e_\mu) = f(o) + x^\mu \alpha_f(e_\mu) =$$

$$= f(o) + (o - o) + x^\mu \alpha_f(e_\mu) = o + \underbrace{(f(o) - o)}_{\in V} + x^\mu \alpha_f e_\mu =$$

$$= o + b + x^\mu \alpha_f e_\mu =$$

$$= o + b^\mu e_\mu + x^\mu e_\nu \alpha^\nu{}_\mu =$$

$$= o + (\alpha^\nu{}_\mu x^\mu + b^\nu) e_\nu$$

$$\boxed{x'^\mu = \alpha^\nu{}_\mu x^\mu + b^\nu}$$

Pre Einstein models of spacetime assume that spacetime is a 4-dimensional affine space (M, V)

~~In particular preferred motion~~

Motion in spacetime = world line of a particle = curves in (M, V)

Preferred motion = free motion = motion which is rectilinear
= motion along straight lines in (M, V)
= world line of a particle moving is a straight line.

Uniformity ?

⟹ Absolute time

(M,V) has to be equipped with a map $\overset{\circ}{\overset{\#}{\tau}} : V \to \mathbb{R}^1$ which is linear. (τ is a 1-form on V).

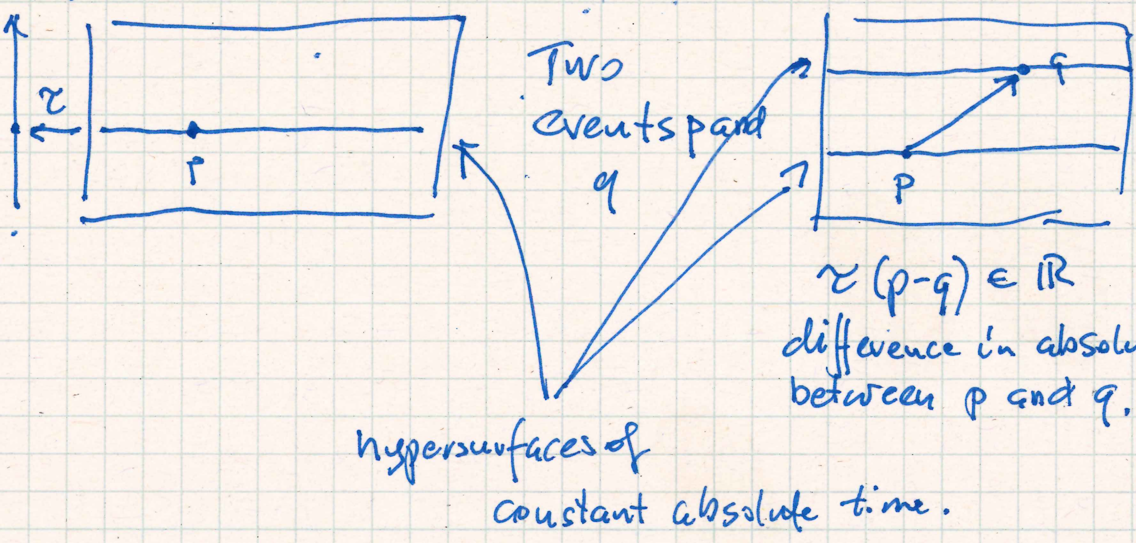
This defines $S = \{ v \in V \text{ s.t. } \tau(v) = 0 \}$. - space of spacelike vectors in V.

Enables a notion of simultaneity of events

Two events, points p and q in M, are simultaneous if $\tau(p-q) = 0$

Take a $p \in M$ and define $p+S = \{ p+u \text{ s.t. } u \in S \}$

$p+S$ is a space of events simultaneous with p.

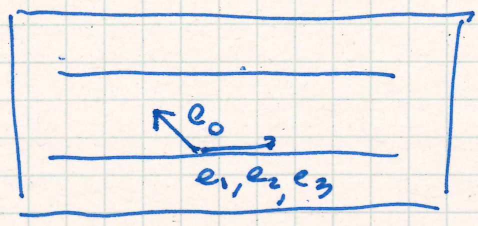


$\tau(p-q) \in \mathbb{R}$ difference in absolute time between p and q. $\tau(p-q) = t_{p\to q}$

~~In repere (o, e_μ) we have coordinates x^μ of a point. One of theese coordinates is an absolute time~~

In repere one of coordinates x^μ is a time. Say x^0.

Existence of absolute time distinguishes a class of reperes, namely those in which x^0 is the absolute time t.



e_0 unit vector in absolute time direction
e_1, e_2, e_3 ~~unit~~ vectors in space directions

then $x^0 = t$ – absolute time.

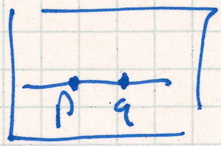
<u>Aristotle</u> $V = \mathbb{R} \oplus S$
↖ absolute time direction.

$\tau = pr_{\mathbb{R}}$

<u>Galileo</u> V is only equipped with $\tau : V \to \mathbb{R}$
S is defined as $S = \ker \tau$.

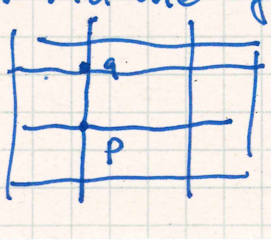
Both Galileo and Aristotle have S – space of spacelike vectors which enables to define simoultaneity. Moreover, both of them equipp S with a metric structure: $h : S \times S \to \mathbb{R}$, where h is (symmetric) bilinear form, positive definite. This enables to define <u>distances</u> between simultaneous events p, q:

$$|p-q| = \sqrt{h(p-q, p-q)}$$

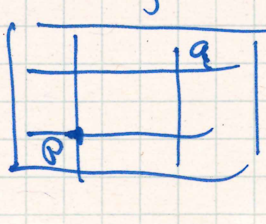


<u>In Galilean physics</u> one can not ~~distinguish between~~ atribute distances to <u>not</u> simultaneous events.

In Aristotelian physics one has also $\sigma = pr_S$.
This defines timelike vectors $T = \{v \in V : \sigma(v) = 0\} = \mathbb{R}$
This defines an <u>absolute rest</u>: p and q are on a worldline of absolute rest if $\sigma(p-q) = 0$



Then



$\sigma(p-q) \in S$
one can calculate distance!

Galilean spacetime $(M, V, \tau, h \text{ in } \ker\tau = S)$

Galilean transformation affine transformation of (M, V) preserving τ and h, i.e., (f, γ_f) s.t.

- $f(p+u) = f(p) + \gamma_f u$
- $\tau(\gamma_f(u)) = \tau(u)$
- $h(\gamma_f(v), \gamma_f(w)) = h(v, w)$ for $w, v \in S$

~~Def Galilean transformation~~ We have a distinguished class of repers:

Galilean repers $(\theta, (e_0, e_i))$ s.t.

1) $\tau(e_0) = \underline{1}$
2) $\tau(e_i) = 0$, $i = 1, 2, 3$
3) $h(e_i, e_j) = \delta_{ij}$.

Galilean transformation in coordinates in Galilean repers. (?).