Progress in integrability studies of Hamiltonian systems with three degrees of freedom

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Outline

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Integrability of homogeneous Hamiltonian equations

Integrability of Hamiltonian systems given by

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\boldsymbol{q}), \qquad (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{C}^{2n},$$

$$V \in \mathbb{C}[\boldsymbol{q}], \qquad \deg V = k > 2,$$

V — homogeneous.

Equivalent potentials

Problem

$$V = \sum_{i_1,\ldots,i_n} v_{i_1\cdots i_n} q_1^{i_1}\cdots q_n^{i_n},$$

where $i_1,\ldots,i_n \in \{0,1,\ldots,k\}$ and the sum is taken over such elements that $i_1+\cdots+i_n = k$.
How to find coefficients $v_{i_1\cdots i_n}$ for which potential is integrable?

- Equivalent potentials:

$$V(\boldsymbol{q}) \sim V_A(\boldsymbol{q}) = V(A\boldsymbol{q}), \qquad A \in \mathrm{PO}(n,\mathbb{C}),$$

$$\mathrm{PO}(n,\mathbb{C}) := \{A \in \mathrm{GL}(n,\mathbb{C}) \,|\, AA^T = \alpha E, \quad \alpha \in \mathbb{C}^\star\}.$$

Morales-Ramis theorem

Theorem

Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of the analytic phase curve Γ. Then the identity component of the differential Galois group of the variational equations along Γ is Abelian.

Outline

Particular solutions

Definition (standard)

Darboux point $\boldsymbol{d} \in \mathbb{C}^n$ is a non-zero solution of

$$V'(\boldsymbol{d}) = \boldsymbol{d}$$

Particular solution

$$\boldsymbol{q}(t) = \varphi(t)\boldsymbol{d}, \ \boldsymbol{p}(t) = \dot{\varphi}(t)\boldsymbol{d} \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

On the energy level:

$$H(\varphi(t)\boldsymbol{d}, \dot{\varphi}(t)\boldsymbol{d}) = e \in \mathbb{C}^\star,$$

hyperelliptic curve

$$\dot{\varphi}^2 = \frac{2}{k}\left(\varepsilon - \varphi^k\right), \quad \varepsilon = ke \in \mathbb{C}^*.$$

Variational equations

$$\ddot{\boldsymbol{x}} = -\varphi(t)^{k-2} V''(\boldsymbol{d})\boldsymbol{x},$$

where $V''(\boldsymbol{d})$ is the Hessian of V calculated at $\boldsymbol{d}$. If $V''(\boldsymbol{d})$ is diagonalisable

$$\ddot{\eta}_i = \lambda_i \varphi(t)^{k-2}\eta_i, \qquad i = 1, \ldots, n,$$

where λ_i for $i = 1, \ldots, n$ are eigenvalues of V''.

By homogeneity of V, $\lambda_n = k - 1$.

Other Morales-Ramis Theorem

If the Hamiltonian system with homogeneous potential is meromorphically integrable then each (k, λ_i) belong to the following list:

$$\left(k, p + \frac{k}{2}p(p-1)\right), \qquad \left(k, \frac{1}{2}\left[\frac{k-1}{k} + p(p+1)k\right]\right),$$

$$\left(3, -\frac{1}{24} + \frac{1}{6}(1+3p)^2\right), \qquad \left(3, -\frac{1}{24} + \frac{3}{32}(1+4p)^2\right),$$

$$\left(3, -\frac{1}{24} + \frac{3}{50}(1+5p)^2\right), \qquad \left(3, -\frac{1}{24} + \frac{3}{50}(2+5p)^2\right),$$

$$\left(4, -\frac{1}{8} + \frac{2}{9}(1+3p)^2\right), \qquad \left(5, -\frac{9}{40} + \frac{5}{18}(1+3p)^2\right),$$

$$\left(5, -\frac{9}{40} + \frac{1}{10}(2+5p)^2\right), \quad p \in \mathbb{Z}.$$

Weakness of this theorem in applications

$$V = \frac{1}{3}aq_1^3 + \frac{1}{2}q_1^2 q_2 + \frac{1}{3}cq_2^3.$$

$$\lambda_1 = \frac{1}{c}, \qquad \lambda_{2,3} = \frac{2c-1}{1+a^2 \mp \Delta}, \qquad \Delta = \sqrt{a^2(2+a^2-2c)}$$

$$\begin{aligned}\lambda_1, \lambda_2, \lambda_3 \in &\left\{p + \frac{3}{2}p(p-1)\right\} \cup \left\{\frac{1}{2}\left(\frac{2}{3} + p(p+1)k\right)\right\} \\ &\cup \left\{-\frac{1}{24} + \frac{1}{6}(1+3p)^2\right\} \cup \left\{-\frac{1}{24} + \frac{3}{32}(1+4p)^2\right\} \\ &\cup \left\{-\frac{1}{24} + \frac{3}{50}(1+5p)^2\right\} \cup \left\{-\frac{1}{24} + \frac{3}{50}(2+5p)^2\right\}.\end{aligned}$$

$$c = \frac{1}{\lambda_1}, \quad a = \pm\sqrt{\frac{(\lambda_1 + \lambda_1\lambda_i - 2)^2}{2\lambda_1\lambda_i(2-\lambda_1-\lambda_i)}}, \qquad i = 2, 3.$$

Our observations

For $\boldsymbol{q} = (q_1, \ldots, q_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, we denote $[\boldsymbol{q}] := [q_1 : \cdots : q_n] \in \mathbb{CP}^{n-1}$.

Definition (version 1)

$[\boldsymbol{d}] \in \mathbb{CP}^{n-1}$ is a Darboux point of potential V, iff $V'(\boldsymbol{d}) = \gamma \boldsymbol{d}$ for a certain $\gamma \in \mathbb{C}^\star$.
$\mathcal{D}^\star(V)$ = all Darboux points of V.

Our observations for $n = 2$

For $n = 2$, each $[\boldsymbol{d}] \in \mathcal{D}^\star(V)$ gives one non-trivial eigenvalue $\lambda(\boldsymbol{d})$. Set $\Lambda(\boldsymbol{d}) = \lambda(\boldsymbol{d}) - 1$.

Theorem

For a generic homogeneous $V \in \mathbb{C}[\boldsymbol{q}]$ of degree k, card $\mathcal{D}^\star(V) = k$, *and*

$$\sum_{[\boldsymbol{d}] \in \mathcal{D}^\star(V)} \frac{1}{\Lambda(\boldsymbol{d})} = -1.$$

Our observations for $n = 2$

Theorem

For a generic homogeneous $V \in \mathbb{C}[\boldsymbol{q}]$ *of degree* k*, the set of admissible* $\{\Lambda(\boldsymbol{d}) \mid [\boldsymbol{d}] \in \mathcal{D}^\star(V)\} =: \mathcal{I}_{2,k}$ *is finite.*

$\Downarrow$ +analysis of non-generic cases $+ \cdots$

Theorem

For $k = 3$ *all integrable potentials are already known!*

Our results for an arbitrary n

Fact

A generic homogeneous $V \in \mathbb{C}[\boldsymbol{q}]$ has exactly $D(n,k) = [(k-1)^n - 1]/(k-2)$ Darboux points.

$$\mathcal{D}^{\star}(V) \ni [\boldsymbol{d}] \longmapsto \boldsymbol{\Lambda}(\boldsymbol{d}) = (\Lambda_1(\boldsymbol{d}), \ldots, \Lambda_{n-1}(\boldsymbol{d}))$$

where $\lambda_i(\boldsymbol{d}) := \Lambda_i(\boldsymbol{d}) + 1$, are the non-trivial eigenvalues of $V''(\boldsymbol{d})$.
τ_i is the elementary symmetric polynomial of degree i in $(n-1)$ variables.

Our results for an arbitrary n

Theorem

For a generic homogeneous $V \in \mathbb{C}[\boldsymbol{q}]$ of degree $k > 2$ we have

$$\sum_{[\boldsymbol{d}]\in\mathcal{D}^\star(V)} \frac{\tau_1(\boldsymbol{\Lambda}(\boldsymbol{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^n(n+k-2)^r, \quad 0 \leq r \leq n-1,$$

or, alternatively

$$\sum_{[\boldsymbol{d}]\in\mathcal{D}^\star(V)} \frac{\tau_r(\boldsymbol{\Lambda}(\boldsymbol{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\boldsymbol{d}))} = (-1)^{n-r-1}\sum_{i=0}^{r}\binom{n-r-1}{r-i}(k-1)^i.$$

Our results for an arbitrary n

Theorem

For a generic homogeneous $V \in \mathbb{C}[\boldsymbol{q}]$ of degree k set of admissible $\{\boldsymbol{\Lambda}(\boldsymbol{d}) \mid [\boldsymbol{d}] \in \mathcal{D}^\star(V)\} =: \mathcal{I}_{n,k}$ is finite.

$\Downarrow$ +many other things

New integrable potentials for $k = n = 3$

Outline

Darboux points once more

Definition

$[\boldsymbol{d}] \in \mathbb{CP}^{n-1}$ is a Darboux point of V iff $\boldsymbol{d} \wedge V'(\boldsymbol{d}) = \boldsymbol{0}$.
$\mathcal{D}(V)$ denotes the set of all Darboux points of V.

$$
\begin{aligned}
[\boldsymbol{d}] \in \mathcal{D}(V) = & \mathcal{V}(R_{1,2}, \ldots, R_{n-1,n}) = \\
& \{[\boldsymbol{q}] \in \mathbb{CP}^{n-1} \mid R_{i,j}(\boldsymbol{q}) = 0, \quad 1 \leq i < j \leq n\},
\end{aligned}
$$

where

$$
R_{i,j} := q_i \frac{\partial V}{\partial q_j} - q_j \frac{\partial V}{\partial q_i},
$$

for $1 \leq i < j \leq n$.

Classification of Darboux points

- Proper Darboux points:
  $\mathcal{D}^\star(V) := \{[\boldsymbol{d}] \in \mathcal{D}(V) \mid V'(\boldsymbol{d}) \neq \boldsymbol{0}\}$,

$$[\boldsymbol{d}] \in \mathcal{D}^\star(V) \iff \boldsymbol{f}(\boldsymbol{d}) = V'(\boldsymbol{d}) - \boldsymbol{d} = 0.$$

- Improper Darboux points: $\mathcal{S}(V) := \mathcal{D}(V) \setminus \mathcal{D}^\star(V)$.

$$[\boldsymbol{d}] \in \mathcal{S}(V) \iff V'(\boldsymbol{d}) = 0.$$

Obstruction to the integrability due to improper Darboux point *d*

Theorem

Assume that a homogeneous potential $V \in \mathbb{C}_k[\boldsymbol{q}]$ of degree $k > 2$ admits an improper Darboux point $[\boldsymbol{d}] \in \mathbb{CP}^{n-1}$. If V is integrable, then matrix $V''(\boldsymbol{d})$ is nilpotent, i.e., all its eigenvalues vanish.

particular solution

$$\boldsymbol{q}(t) = t\boldsymbol{d}, \qquad \boldsymbol{p}(t) = \boldsymbol{d},$$

variational equations after transformation of $V''(\boldsymbol{d})$ into the Jordan form contain a subsystem

$$\ddot{\eta}_i = -\lambda_i t^{k-2}\eta_i, \qquad i = 1, \dots, s$$

Passage to $\mathbb{CP}^n$

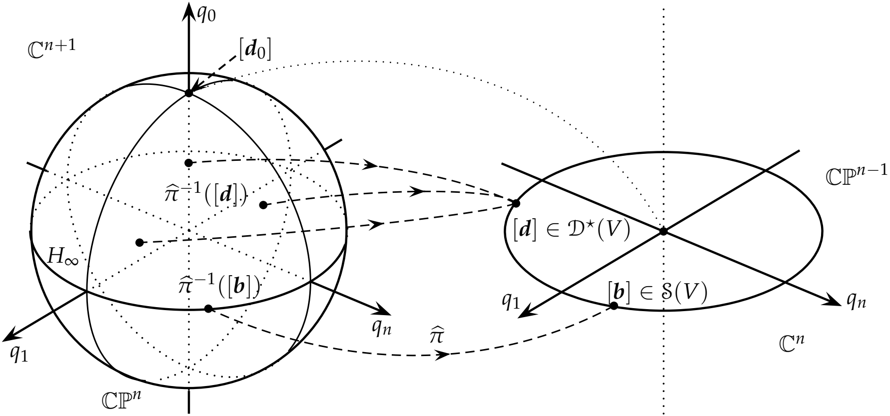
Define

$$F_i := \frac{\partial V}{\partial q_i} - q_0^{k-2} q_i, \qquad i = 1, \ldots, n,$$

and the algebraic set $\widehat{\mathcal{D}}(V) = \mathcal{V}(F_1, \ldots, F_n) \subset \mathbb{CP}^n$.

- $[\boldsymbol{d}_0] = [1 : 0 : \cdots : 0] \in \widehat{\mathcal{D}}(V)$.
- Let $[q_0 : q_1 : \cdots : q_n] \in \widehat{\mathcal{D}}(V)$ and $\boldsymbol{q} = (q_1, \ldots, q_n) \neq \boldsymbol{0}$, then $[\boldsymbol{q}] \in \mathcal{D}(V)$.
- If $[\boldsymbol{d}] \in \mathcal{D}^\star(V)$, then $\exists\, \gamma \in \mathbb{C}^\star$, such that $V'(\boldsymbol{d}) = \gamma \boldsymbol{d}$, so $k-2$ points $[\sqrt[k-2]{\gamma} : d_1 : \cdots : d_n] \in \mathbb{CP}^n$ belong to $\widehat{\mathcal{D}}(V)$.
- If $[\boldsymbol{d}] \in \mathcal{S}(V)$, then it defines just one point $[0 : d_1 : \cdots : d_n] \in \mathbb{CP}^n$ which is a point of $\widehat{\mathcal{D}}(V)$.

Passage to $\mathbb{CP}^n$



$$\widehat{\pi} : \mathbb{CP}^n \setminus \{[\boldsymbol{d}_0]\} \to \mathbb{CP}^{n-1}, \quad \widehat{\pi}([q_0 : q_1 : \cdots : q_n]) = [q_1 : \cdots : q_n]$$

Aim – to find universal relations between Λ_i

We look for relations of the form

$$\sum_{[\boldsymbol{d}]\in\mathcal{D}^\star(V)} \frac{S(\Lambda_1(\boldsymbol{d}),\ldots,\Lambda_{n-1}(\boldsymbol{d}))}{\Lambda_1(\boldsymbol{d})\cdots\Lambda_{n-1}(\boldsymbol{d})} = R(n,k,S),$$

where S – a symmetric polynomial.
Hint 1: Residue calculus.

Let $f_1, \ldots, f_n \in \mathbb{C}[\boldsymbol{q}]$ have an isolated common zero at $\boldsymbol{q} = \boldsymbol{0}$, and

$$\omega = \frac{f_0}{f_1 \cdots f_n} \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_n$$

In an affine part of $\mathbb{CP}^n$ (where $q_0 \neq 0$) we have form ω. We extend it to a form Ω on $\mathbb{CP}^n$ using standard transformations rules.

Theorem (Global residue theorem)

Assume that $\mathcal{V}(F_1, \ldots, F_n)$ is finite and $\deg F_0 \leq \prod_{i=1}^n \deg F_i - n - 1$, then

$$\sum_{\boldsymbol{d} \in \mathcal{V}(F_1, \ldots, F_n)} \operatorname{res}(\Omega, \boldsymbol{d}) = 0.$$

If $\boldsymbol{q} = \boldsymbol{0}$ is a simple intersection then

$$\operatorname{res}(\omega, \boldsymbol{0}) = \frac{f_0(\boldsymbol{0})}{\det \boldsymbol{f}'(\boldsymbol{0})}.$$

Hint 2: $\det \boldsymbol{f}'(\boldsymbol{d}) = (k-2)\Lambda_1(\boldsymbol{d})\cdots\Lambda_{n-1}(\boldsymbol{d})$.
Hint 3: Homogenisation of polynomials $\boldsymbol{f} = (f_1,\ldots,f_n)$

$$f_i := \frac{\partial V}{\partial q_i} - q_i, \qquad i = 1,\ldots,n,$$

are $(F_1,\ldots,F_n)$ where

$$F_i := \frac{\partial V}{\partial q_i} - q_0^{k-2}q_i, \qquad i = 1,\ldots,n.$$

So, we have to 'investigate' all points of $\widehat{\mathcal{D}}(V) := \mathcal{V}(F_1,\ldots,F_n)$
Hint 4: $f_0 = (\operatorname{Tr}\boldsymbol{f}' - (k-2))^r$, for $r = 0,\ldots,n-1$.

Outline

Integrability analysis for $n = 3$ and $k = 3$ – just beginning

$$V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_3 q_1^2 q_3 + a_4 q_1 q_2^2 + a_5 q_2^3 + a_6 q_2^2 q_3 + a_7 q_1 q_3^2 + a_8 q_1 q_3^2 + a_9 q_2 q_3^2 + a_{10} q_3^3$$

using the generalised rotation group one can reduced this potential to the form

$$V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_3 q_1^2 q_3 + a_4 q_1 q_2^2 + a_5 q_2^3 + a_6 q_2^2 q_3 + \frac{1}{3} q_3^3.$$

We moved one proper Darboux point $[0 : 0 : 1]$.

Integrability obstructions in action

$$\Lambda_1(\boldsymbol{d}_j), \Lambda_2(\boldsymbol{d}_j) \in B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6, \quad j = 1, \ldots, 7,$$

$$B_1 = -1 + \frac{1}{2}p(3p-1), \qquad B_2 = -\frac{2}{3} + \frac{3}{2}p(p+1),$$

$$B_3 = -\frac{25}{24} + \frac{1}{6}(1+3p)^2, \quad B_4 = -\frac{25}{24} + \frac{3}{32}(1+4p)^2,$$

$$B_5 = -\frac{25}{24} + \frac{3}{50}(1+5p)^2, \; B_6 = -\frac{25}{24} + \frac{3}{50}(2+5p)^2, \quad p \in \mathbb{Z}.$$

$$\sum_{j=1}^{7} \frac{1}{\Lambda_1(\boldsymbol{d}_j)\Lambda_2(\boldsymbol{d}_j)} = 1,$$

$$\sum_{j=1}^{7} \frac{\Lambda_1(\boldsymbol{d}_j) + \Lambda_2(\boldsymbol{d}_j)}{\Lambda_1(\boldsymbol{d}_j)\Lambda_2(\boldsymbol{d}_j)} = \sum_{j=1}^{7} \frac{1}{\Lambda_1(\boldsymbol{d}_j)} + \frac{1}{\Lambda_2(\boldsymbol{d}_j)} = -4,$$

$$\sum_{j=1}^{7} \frac{(\Lambda_1(\boldsymbol{d}_j) + \Lambda_2(\boldsymbol{d}_j))^2}{\Lambda_1(\boldsymbol{d}_j)\Lambda_2(\boldsymbol{d}_j)} = 16.$$

Admissible pairs $\{\Lambda_1, \Lambda_2\}$ for $n = 3$ and $k = 3$

1. $\{3 \times \{-1,-1\}, 3 \times \{-1,1\}, \{1,1\}\},$
2. $\left\{\{-1,-1\}, \left\{-1,-\frac{2}{3}\right\}, \left\{1,-\frac{2}{3}\right\}, 2 \times \{-1,4\}, 2 \times \{1,4\}\right\},$
3. $\left\{\{-1,-1\}, \left\{-1,-\frac{7}{8}\right\}, \left\{1,-\frac{7}{8}\right\}, 2 \times \{-1,14\}, 2 \times \{1,14\}\right\},$
4. $\left\{\{-1,-1\}, \left\{-1,-\frac{2}{3}\right\}, \left\{1,-\frac{2}{3}\right\}, \left\{-1,\frac{7}{3}\right\}, \left\{1,\frac{7}{3}\right\}, \{\pm 1,14\}\right\}$
5. $\left\{\left\{-\frac{2}{3},-\frac{2}{3}\right\}, 2 \times \left\{-\frac{2}{3},\frac{7}{3}\right\}, \{-1,4\}, \{1,4\}, 2 \times \{4,14\}\right\},$
6. $\left\{\left\{-\frac{2}{3},-\frac{7}{8}\right\}, \left\{-\frac{7}{8},4\right\}, \left\{-\frac{3}{8},4\right\}, 2 \times \left\{\frac{7}{3},4\right\} 2 \times \{4,21\}\right\},$

7. $\left\{2\times\{-1,6\},\left\{-1,\frac{7}{3}\right\},\left\{-\frac{7}{8},-\frac{2}{3}\right\},\left\{\frac{13}{8},14\right\},2\times\{14,39\}\right\},$

8. $\left\{\left\{-\frac{2}{3},-\frac{2}{3}\right\},3\times\left\{-1,\frac{7}{3}\right\},3\times\{6,14\}\right\},$

9. $\left\{\left\{-\frac{2}{3},-\frac{7}{8}\right\},\left(-\frac{2}{3},\frac{7}{3}\right\},\left\{-\frac{3}{8},14\right\},2\times\left\{1,\frac{52}{3}\right\},2\times(14,39)\right\}$

10. $\left\{\{-1,1\},\left\{-1,-\frac{3}{8}\right\},2\times\left\{-\frac{2}{3},\frac{7}{3}\right\},\left\{1,\frac{13}{8}\right\}2\times\{14,39\}\right\}.$

$$V_5 = \frac{3\mathrm{i}}{4}q_1^2q_2 + \frac{7\mathrm{i}}{3}q_2^3 + \frac{5}{2}q_2^2q_3 + \frac{1}{3}q_3^3,$$

$$\begin{aligned}
I_1 &= 48p_1(\mathrm{i}p_2 + p_3)q_1 - (3q_1^2 + 2(q_2 - 2\mathrm{i}q_3)^2)(3q_1^2 + 2q_2(5q_2\\
&- 4\mathrm{i}q_3)) - 48p_1^2(\mathrm{i}q_2 + q_3) + 32(p_2 - 2\mathrm{i}p_3)(p_2q_3 - p_3q_2),\\
I_2 &= 24p_1^4 - 8p_1^2(12(p_2 - \mathrm{i}p_3)(p_2 - 2\mathrm{i}p_3) - 9\mathrm{i}q_1^2q_2 + 2(2q_2 - \mathrm{i}q_3)\\
&\times (q_2 - 2\mathrm{i}q_3)(5\mathrm{i}q_2 + 4q_3)) + 3q_1^2(3q_1^4 + 16(p_2 + \mathrm{i}p_3)(p_3q_2 - p_2q_3)\\
&+ 6q_1^2(3q_2^2 - 4\mathrm{i}q_2q_3 + q_3^2) + 4q_2(q_2 - 2\mathrm{i}q_3)(7q_2^2 - 4\mathrm{i}q_2q_3 + 2q_3^2))\\
&+ 24p_1q_1(p_3(-3q_1^2 - 22q_2^2 + 20\mathrm{i}q_2q_3) - \mathrm{i}p_2(3q_1^2 + 6q_2^2 + 4\mathrm{i}q_2q_3\\
&+ 8q_3^2)).
\end{aligned}$$

$$
\begin{aligned}
V_6 = {} & 364\sqrt{17}q_1^3 + 2835\mathrm{i}\sqrt{17}q_1^2q_2 + 1560\sqrt{17}q_1q_2^2 + 6552\mathrm{i}\sqrt{17}q_2^3 \\
& + 4335q_1^2q_3 + 19074q_2^2q_3 + 578q_3^3, \\
I_1 = {} & 34\sqrt{17}p_3^2q_1 + 8p_2p_3(18\mathrm{i}q_1 + q_2) - 2p_2^2(7\sqrt{17}q_1 + 4q_3) \\
& - p_1^2(54\mathrm{i}\sqrt{17}q_2 + 26q_3) + 2p_1(p_2(27\mathrm{i}\sqrt{17}q_1 + 7\sqrt{17}q_2 + 9\mathrm{i}q_3) \\
& + p_3(13q_1 - 81\mathrm{i}q_2 - 17\sqrt{17}q_3)) - 51(-2568q_2^4 + 36\mathrm{i}q_2^3(695q_1 \\
& + 52\sqrt{17}q_3) - 36\mathrm{i}q_1q_2(151q_1^2 + 241\sqrt{17}q_1q_3 + 221q_3^2) \\
& + 2q_2^2(31351q_1^2 + 5386\sqrt{17}q_1q_3 + 2924q_3^2) + q_1(12205q_1^3 \\
& + 1668\sqrt{17}q_1^2q_3 + 3978q_1q_3^2 + 1156\sqrt{17}q_3^3)),
\end{aligned}
$$

- I_2 of degree four in the momenta.

$$V_7 = 44\sqrt{7}q_1^3 + 240\mathrm{i}\sqrt{14}q_1^2q_2 + 330\sqrt{7}q_1q_2^2 + 935\mathrm{i}\sqrt{14}q_2^3 + 3087q_2^2q_3 + 294q_3^3,$$

1. I_1 quartic in the momenta,
2. I_2 of degree 6 in the momenta.

$$V_8 = \frac{7}{2}q_1^2q_3 - \frac{5\mathrm{i}\sqrt{3}}{2}q_1^2q_2 - \frac{9\mathrm{i}\sqrt{3}}{2}q_2^3 + \frac{15}{2}q_2^2q_3 + \frac{1}{3}q_3^3,$$

$$\begin{aligned}I_1 &= 26\mathrm{i}\sqrt{3}p_1^3 + 3p_1(6\mathrm{i}\sqrt{3}p_2^2 - 48p_2p_3 - 32\mathrm{i}\sqrt{3}p_3^2 + 27q_2^3\\ &+ 69\mathrm{i}\sqrt{3}q_2^2q_3 + 15q_2(13q_1^2 - 8q_3^2) + \mathrm{i}\sqrt{3}q_3(91q_1^2 + 16q_3^2))\\ &+ 3q_1(p_2(-91q_1^2 + 45q_2^2 + 42\mathrm{i}\sqrt{3}q_2q_3 + 72q_3^2)\\ &- \mathrm{i}p_3(65\sqrt{3}q_1^2 - 99\sqrt{3}q_2^2 - 480\mathrm{i}q_2q_3 + 112\sqrt{3}q_3^2)),\end{aligned}$$

- I_2 of degree four in the momenta.

$$V_9 = 27\mathrm{i}\sqrt{3990}q_1^3 + 3726\sqrt{15}q_1^2q_2 - 456\mathrm{i}\sqrt{3990}q_1q_2^2 - 4092\sqrt{15}q_2^3 - 1125q_1^2q_3 - 3000q_2^2q_3 - 50q_3^3,$$

with very complicated first integrals of degree 4 and 6 in the momenta.

$$V_{10} = \frac{4\sqrt{2}q_1^3}{3} + \frac{5q_1q_2^2}{2\sqrt{2}} + q_2^2q_3 + \frac{1}{3}q_3^3,$$

with first integrals of degree 4 and 6 in the momenta.

Outline

Nongeneric cases

when the number of proper Darboux points is smaller than

$$D(n,k) = \frac{(k-1)^n - 1}{k-2}.$$

- the presence of improper Darboux points $\Longrightarrow$ all eigenvalues of Hessian at this point are zero,
- presence of a multiple proper Darboux point.

$$V = a_1 q_1^3 + a_2 q_1^2 q_2 + a_4 q_1 q_2^2 + a_5 q_2^3 - \frac{\mathrm{i}}{2}(q_1 + \mathrm{i} q_2)^2 q_3,$$

$$\left.\begin{aligned} &\sum_{i=1}^{5} \frac{1}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = 1,\\ &\sum_{i=1}^{5} \frac{\Lambda_1^{(i)}+\Lambda_2^{(i)}}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = -4,\\ &\sum_{i=1}^{5} \frac{(\Lambda_1^{(i)}+\Lambda_2^{(i)})^2}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = 16, \end{aligned}\right\}, \qquad \left.\begin{aligned} &\sum_{i=1}^{4} \frac{1}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = 1,\\ &\sum_{i=1}^{4} \frac{\Lambda_1^{(i)}+\Lambda_2^{(i)}}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = -4,\\ &\sum_{i=1}^{4} \frac{(\Lambda_1^{(i)}+\Lambda_2^{(i)})^2}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = 8, \end{aligned}\right\}$$

Cases $a_5 \neq a_2 + \mathrm{i}(a_4 - a_1)$ and $a_5 = a_2 + \mathrm{i}(a_4 - a_1)$.

Biernat reduction

$$\omega = \frac{f_0}{f_1\cdots f_n}\mathrm{d}q_1\wedge\cdots\wedge\mathrm{d}q_n.$$

Assume that: $\mathbf{0}\in\mathcal{V}(f_1,\ldots,f_n)$, and $\det\boldsymbol{f}'(\mathbf{0})=0$.

How to calculate $\operatorname{res}(\omega,\mathbf{0})$?

Assume that the analytic set $A:=\{\boldsymbol{q}\in U \mid f_2(\boldsymbol{q})=0,\ldots,f_n(\boldsymbol{q})=0\}$, $U\subset\mathbb{C}^n$ is a neighbourhood of $\mathbf{0}$, is irreducible. Then there is an injective parametrisation $s\mapsto\boldsymbol{q}(s)\in A$, $\boldsymbol{q}(0)=\mathbf{0}$.

Biernat formula

$$\operatorname{res}(\omega,\mathbf{0})=\operatorname{res}\left(\frac{f_0(\boldsymbol{q}(s))}{f_1(\boldsymbol{q}(s))\det\boldsymbol{f}'(\boldsymbol{q}(s))}\frac{\mathrm{d}f_1(\boldsymbol{q}(s))}{\mathrm{d}s}\mathrm{d}s,0\right).$$

$$
\begin{aligned}
V &= \frac{1-\mathrm{i}}{3(1+\Lambda)}q_1^3 + a_2 q_1^2(q_2 - \mathrm{i}q_3) \\
&- \frac{1+\mathrm{i}}{4\Lambda^2}(q_2 - \mathrm{i}q_3)q_1[a_2^2(\Lambda^2-1)(q_2-\mathrm{i}q_3) + 2\mathrm{i}\Lambda^2(q_2+\mathrm{i}q_3)] \\
&+ a_5(q_2-\mathrm{i}q_3)^3,
\end{aligned}
$$

$$
\left.
\begin{aligned}
&\sum_{i=1}^{3}\frac{1}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = -1 + \frac{2}{\Lambda^2}, \\
&\sum_{i=1}^{3}\frac{\Lambda_1^{(i)}+\Lambda_2^{(i)}}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = 0, \\
&\sum_{i=1}^{3}\frac{(\Lambda_1^{(i)}+\Lambda_2^{(i)})^2}{\Lambda_1^{(i)}\Lambda_2^{(i)}} = 8.
\end{aligned}
\right\}
$$

Potential with a multiple nonisotropic Darboux point

$$V = a_1 q_1^3 + a_2 q_1^2 q_2 + \frac{1}{2} q_1^2 q_3 + a_4 q_1 q_2^2 + a_5 q_2^3 + a_6 q_2^2 q_3 + \frac{1}{3} q_3^3.$$

$[\boldsymbol{d}_0] := [0 : 0 : 1]$

Theorem

Let us denote

$$D_r = \sum_{\boldsymbol{d} \in \mathcal{D}^\star(V) \setminus \{[\boldsymbol{d}_0]\}} \frac{(\Lambda_1 + \Lambda_2)^r}{\Lambda_1 \Lambda_2}(\boldsymbol{d}), \qquad r = 0, 1, 2.$$

Then

$$D_2 = 16 + 4\Lambda(D_2 + 4) - \Lambda^2(D_0 - 1),$$

where $\Lambda = 2a_6 - 1.$