Integrability properties of the geodesic equation in sub-Riemannian spaces

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Aim

- To study integrability of the geodesic equation (adjoint equation) in sub-Riemannian problems.
- To show usefulness of the Morales-Ramis theory in proving non-integrability.
Plan

- Sub-Riemannian manifolds
- Geodesic equation
- Classification of integrable homogeneous sub-Riemannian problems in dimension 3
- Nilpotent approximations of 3-dim. sub-Riemannian manifolds
- Integrability and nonintegrability in the tangent case
- Morales-Ramis theorem and differential Galois group
- Optimal energy of the transfer pulses for the $n$-level quantum system and nonintegrability for $n \geq 4$
A sub-Riemannian manifold is a triple \((M, \mathcal{D}, B)\), where

- \(M\) is a smooth manifold,
- \(\mathcal{D}\) is a smooth distribution of rank \(m\) on \(M\)
- \(B\) is a smoothly varying positive definite bilinear form on \(\mathcal{D}\), that is, a smoothly varying scalar product on \(\mathcal{D}\).

### Controllability: Rashevsky and Chow

Put \(\mathcal{D}_0 = \mathcal{D}\) and \(\mathcal{D}_{s+1} = \mathcal{D}_s + [\mathcal{D}, \mathcal{D}_s]\). If for each point \(q \in M\), there exists an integer \(r(q)\) (called the nonholonomy degree at \(q\)) such that \(\mathcal{D}_{r(q)}(q) = T_q M\), then any two points in \(M\) can be joined by a curve that is almost everywhere tangent to \(\mathcal{D}\), called a horizontal curve.
Sub-Riemannian metric

Put $\|v\| = (B(v, v))^{1/2}$, for any $v \in \mathcal{D}(q) \subset T_qM$, and let $\gamma : I \rightarrow M$ be a horizontal curve. We define the length $l(\gamma)$ of $\gamma$ as

$$l(\gamma) = \int_I \|\dot{\gamma}(t)\| dt.$$ 

We can thus endow $M$ with a metric $d$: the sub-Riemannian distance $d(q_1, q_2)$ between two points $q_1$ and $q_2$ is the infimum of $l(\gamma)$ over all horizontal curves joining $q_1$ and $q_2$.

- Sub-Riemannian geometry problem: find horizontal curves minimizing the length $l(\gamma)$, i.e., find sub-Riemannian geodesics.
Minimizing: energy versus length

- The energy $E(\gamma)$ of a curve $\gamma$ is defined as
  \[ E(\gamma) = \frac{1}{2} \int_I \|\dot{\gamma}(t)\|^2 dt. \]

- Analytically it is more convenient to minimize the energy $E(\gamma)$ rather than the length $l(\gamma)$.

- As in Riemannian geometry, due to Cauchy-Schwartz inequality, the minimizers of both problems coincide. Namely, a horizontal curve $\gamma$ minimizes the energy $E$ among all horizontal curves joining $q_1$ and $q_2$ in time $T$ if and only if it minimizes the length $l$ among all horizontal curves joining $q_1$ and $q_2$ and is parameterized to have constant speed $c = d(q_1, q_2)/T$. 
Choose a local orthonormal frame $\langle X_1, \ldots, X_m \rangle$ of $\mathcal{D}$, that is, $B(X_i, X_j) = \delta_{ij}$.

Consider each $X_i$ as a fiber-linear function on $T^*M$. Then each $X_i^2$ can be interpreted as a fiber-quadratic function on $T^*M$.

We have

$$h = \frac{1}{2}(X_1^2 + \cdots + X_m^2).$$

The hamiltonian equation associated with $h$ will be called *geodesic equation*.

The projections to $M$ of its solutions are sub-Riemannian geodesics, called *normal geodesics*. Notice that in the general case there may exist length minimizing horizontal curves that are not projections of solutions of the geodesic equation (Montgomery).
Formulating an optimal control problem

- For a given framing $\mathcal{D} = \langle X_1, \ldots, X_m \rangle$ by $m$ orthonormal vector fields, any integral curve $q(t)$ of $\mathcal{D}$ satisfies

$$\Sigma : \dot{q}(t) = \sum_{i=1}^{m} X_i(q(t))u_i(t),$$

where $u_i(t)$, for $1 \leq i \leq m$, are controls.

- A geodesic is a trajectory of $\Sigma$ that minimizes the energy

$$E = \frac{1}{2} \int_{I} \sum_{i=1}^{m} u_i^2(t) dt.$$ 

- The geometric problem of minimizing the subriemannian distance is the optimal control problem of minimizing the energy $E$ for the control-linear system $\Sigma$. 
To solve this optimal control problem, we will apply the Pontryagin Maximum Principle (PMP) to the problem of minimization of $E$.

Define the Hamiltonian of the optimal control problem

$$
\hat{h} : T^* M \times \mathbb{R}^m \rightarrow \mathbb{R}, \quad \hat{h}(q, p, u) = \sum_{j=1}^{m} \left( \langle p, u_j X_j(x) \rangle - \frac{1}{2} u_j^2 \right).
$$

Define the maximized Hamiltonian $h$ (solve $\frac{\partial \hat{h}}{\partial u} = 0$ which gives $u_j = \langle p, X_j \rangle$) by

$$
h(x, p) = \max_u \hat{h}(q, p, u) = \frac{1}{2} \sum_{j=1}^{m} (\langle p, X_j(q) \rangle)^2
$$

(a quadratic function on fibres).
Pontryagin Maximum Principle - statement

**Theorem 1** If a control $u(t)$ and the corresponding normal trajectory $q(t)$ minimize the cost $E$, then there exists a curve $p(t) \in T^*_q(t)M$ in the cotangent bundle such that $\lambda(t) = (q(t), p(t))$ satisfies the following Hamiltonian equation $\dot{\lambda}(t) = \vec{h}(\lambda(t))$ on $T^*M$:

\[
\dot{q} = \frac{\partial h}{\partial p}(q(t), p(t))
\]

\[
\dot{p} = -\frac{\partial h}{\partial q}(q(t), p(t)),
\]

where $h$ is the maximized Hamiltonian, and $u_j(t) = \langle p(t), X_j(q(t)) \rangle$ are optimal controls.
Integrability of the geodesic equation

- Our main problem: study integrability of the geodesic equation.
- Brockett and Dai started a systematic study of integrability of the geodesic equation (in terms of elliptic functions) in SR-geometry.
- 3-dimensional nilpotent cases are integrable: Heisenberg (in terms of trigonometric functions) and Martinet (in terms of elliptic functions, Bonnard, Chyba, Trelat); and the tangent case?
- Jurdjevic has shown integrability (in terms of elliptic functions) of several invariant SR-problems on Lie groups.
- There exist nonintegrable sub-Riemannian geodesic equations in nilpotent cases (a 6-dim. example of Montgomery-Shapiro).
Our goal

- Classify all cases of integrable adjoint geodesic equation for homogeneous spaces in dimension 3
- Study integrability of the nilpotent tangent case in dimension 3.
- Integrability of some quantum systems on $SO(n)$
Homogenous and symmetric SR-spaces

• A sub-Riemannian isometry between SR-manifolds $(M, \mathcal{D}, B)$ and $(\tilde{M}, \tilde{\mathcal{D}}, \tilde{B})$ is a diffeomorphism $\psi : M \rightarrow \tilde{M}$ such that $\psi_* (\mathcal{D}) = \tilde{\mathcal{D}}$ and $B = \psi^* (\tilde{B})$.

• A homogeneous sub-Riemannian space, shortly, a SR-homogeneous space, is a sub-Riemannian manifold for which the group of its sub-Riemannian isometries is a Lie group that acts smoothly and transitively on the manifold.

• A SR-homogeneous space is said to be symmetric, shortly, SR-symmetric, if for each point $q \in M$ there exists an isometry $\psi$ such that $\psi(q) = q$ and $\psi_* |_{\mathcal{D}(q)} = -\text{Id}$. 
3-dimensional homogeneous sub-Riemannian spaces

**Lemma 1** (Falbel-Gorodski) To any 3-dimensional SR-homogenous space \((M, \mathcal{D}, B)\) there corresponds a Lie group \(G\) that acts simply and transitively on \(M\) (need not be the group of SR-isometries).
Pontryagin Maximum Principle on a Lie group $G$

Using the PMP we conclude that if $Q(t)$ is a minimizing curve in $G$, then there exits a curve $P(t) \in T^*_{Q(t)}G$ such that $(Q(t), P(t))$ satisfies the hamiltonian system

$$\dot{Q} = \frac{\partial H}{\partial Q}(Q(t), P(t))$$

$$\dot{P} = -\frac{\partial H}{\partial X}(Q(t), P(t)),$$

where $H : T^*G \to \mathbb{R}$ is given by

$$H(Q, P) = \frac{1}{2} \sum_{j=1}^{m} (\langle P, X_j \rangle)^2.$$
Poisson structure on $g^*$

- Upon the identification of the space of left invariant vector fields on $G$ with the Lie algebra $g$ of $G$, the hamiltonian $H(Q, P) = \frac{1}{2} \sum_{j=1}^{m} (< P, X_j >)^2$ becomes identified with a quadratic function on $g^*$.

- The dual $g^*$ of the Lie algebra $g$ carries a Poisson bracket defined, for any smooth functions $\varphi_1$ and $\varphi_2$ on $g^*$, by

$$\{ \varphi_1, \varphi_2 \}(\eta) = \langle \eta, [d\varphi_1, d\varphi_2](\eta) \rangle, \quad \text{for each } \eta \in g^*. $$
To the hamiltonian $H$ on $\mathfrak{g}^*$ (considered as a Poisson manifold) we associate the Hamiltonian vector field $\overrightarrow{H}$ on $\mathfrak{g}^*$ defined by

$$\overrightarrow{H}(\varphi) = \{\varphi, H\}, \quad \text{for each } \varphi \in C^\infty(\mathfrak{g}^*).$$

We will call the differential equation

$$\dot{\eta}(t) = \overrightarrow{H}(\eta(t)), \quad \eta(t) \in \mathfrak{g}^*,$$

defined on $\mathfrak{g}^*$ by the Hamiltonian vector field $\overrightarrow{H}$ associated to $H$, the adjoint equation of the hamiltonian system

$$\dot{Q} = \frac{\partial H}{\partial P}(Q(t), P(t))$$

$$\dot{P} = -\frac{\partial H}{\partial Q}(Q(t), P(t)) \quad \left(\dot{\eta}(t) = \overrightarrow{H}(\eta(t))\right).$$
Form a basis $X_1, \ldots, X_m, X_{m+1}, \ldots, X_n$ and put

$$H_j = \langle P, X_j \rangle,$$

for $1 \leq i \leq n$, which allows to rewrite the Hamiltonian as

$$H = \frac{1}{2} \sum_{j=1}^{m} H_j^2,$$

the optimal controls as

$$u_j(t) = H_j(t) = \langle P(t), X_j(Q(t)) \rangle,$$

and the corresponding Hamiltonian system as

$$\dot{Q} = \sum_{j=1}^{m} H_j X_j$$

$$\dot{H}_i = \{H, H_i\}, \quad 1 \leq i \leq n, \quad \left( \dot{\eta}(t) = \overrightarrow{H}(\eta(t)) \right).$$
Integrability

- The adjoint equation is a Lie-Poisson equation defined by a Poisson structure on $\mathfrak{g}^*$ whose structure constants $C_{i,j}^k$ are those defining the Lie algebra $\mathfrak{g}$.

- This Poisson structure is degenerated and of rank, say, $2r$.

- Since $\dim \mathfrak{g}^* = n$, the Poisson structure admits $k = n - 2r$ Casimir functions $C_1, \ldots, C_{n-2r}$ whose common constant level sets $M_c = \{ \eta \in \mathfrak{g}^* : C_1(\eta) = c_1, \ldots, C_{n-2r}(\eta) = c_{n-2r} \}$ are $2r$-dimensional submanifolds of $\mathfrak{g}^*$ equipped with a symplectic structure defined by the restriction of the Poisson structure to $M_c$.

- The adjoint equation restricted to $M_c$ is a hamiltonian equation.
Integrability - definition

- If a Lie-Poisson equation possesses $k + r$ functionally independent first integrals belonging to a category $\mathcal{C}$ such that the first $k$ integrals are Casimir functions and the remaining $r$ ones commute, then we will say that this equation is integrable in the category $\mathcal{C}$. 
The Lie algebra $\mathfrak{g}$ of $G$ has a decomposition $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$, where for a chosen base point $q \in M$ we identify $\mathfrak{g}$ with $T_q M$, the subspace $\mathfrak{p}$ of $\mathfrak{g}$ with $\mathcal{D}(q)$, and the quadratic form $\mathfrak{b}$ defined on $\mathfrak{p}$ with $B$. The triple $(\mathfrak{g}, \mathfrak{p}, \mathfrak{b})$ will be called a sub-Riemannian Lie algebra (does not depend on the chosen base point $q$).

The SR-Lie algebra in the SR-symmetric cases is given by the normal form (sub-symmetric Lie algebras):

\[
\begin{align*}
[X_1, X_2] &= X_3, \\
[X_1, X_3] &= aX_2, \\
[X_2, X_3] &= bX_1,
\end{align*}
\]

where $(a, b) \in \mathbb{R}^2$; above $\mathfrak{g} = \text{span} \{X_1, X_2, X_3\}$, $\mathfrak{p} = \text{span} \{X_1, X_2\}$, and $X_1, X_2$ are orthonormal.
Integrability of the SR-symmetric case

**Theorem 2** For any 3-dimensional sub-Riemannian homogeneous space, the following conditions are equivalent:

(i) *The sub-Riemannian space is symmetric.*

(ii) *The adjoint equation has two functionally independent quadratic first integrals;*

(iii) *The optimal controls are elliptic functions;*

(iv) *All solutions of the complexified adjoint equation are single-valued functions of the complex time;*
Nonintegrability of the SR-non symmetric spaces

The Lie algebra of an orthonormal frame can be brought in the SR-symmetric case to the following normal form

\[ [X_1, X_2] = X_3, \]
\[ [X_1, X_3] = aX_2 + bX_3, \]
\[ [X_2, X_3] = 0, \]

where \((a, b) \in \mathbb{R}^2\) and \(ab \neq 0\). When \(a = 0\) or \(b = 0\) the underlying space is isometric to a sub-symmetric space. By a proper rescaling we can assume \(b = 1\).

We distinguish two subsets of the classification parameter:
• $a \in \Lambda_p \subset \mathbb{R}$ if and only if there exist positive integers $m$ and $n$ such that $a = mn/(m - n)^2$

• $a \in \Lambda_r \subset \mathbb{R}$ if and only if there exist integers $m$ and $n$ such that $a = mn/(m - n)^2$ and $a \neq -1/4$.

**Theorem 3** For any non symmetric sub-homogeneous space defined by the parameter $a$ we have:

(i) The adjoint equation admits a polynomial fist integral independent with the hamiltonian $H$ if and only if $a \in \Lambda_p$;

(ii) The adjoint equation admits a rational fist integral independent with the hamiltonian $H$ if and only if $a \in \Lambda_r$;

(iii) If $a \in \mathbb{R} \setminus \Lambda_r$ then the adjoint equation does not admit any real-meromorphic first integral independent with the hamiltonian $H$. 
Consider the system

\[ \dot{\xi} = \sum_{i=1}^{m} X_i(\xi)u_i. \]

on a manifold \( M \). We have \( \mathcal{D} = \text{span} \{X_1, \ldots, X_m\} \).

- Let \( \mathcal{L}_1 = \text{span}_\mathbb{R} \{X_1, \ldots, X_m\} \).
- Define inductively

\[ \mathcal{L}_s = \mathcal{L}_{s-1} + [\mathcal{L}_{s-1}, \mathcal{L}_1] \quad \text{for } s \geq 2. \]

- Clearly \( \mathcal{L}_s(q) = \mathcal{D}_s(q) \) and the sum

\[ \mathcal{L}(X_1, \ldots, X_m) = \mathcal{L} = \sum_{s \geq 1} \mathcal{L}_s, \]

is the Lie algebra of the system.
Weights

- For \( q \in M \), put \( L_s(q) = \{ X(q) : X \in \mathcal{L}_s \} \)

- Denote \( n_s(q) = \dim L_s(q) \). For a completely nonholonomic system we have

\[
1 \leq n_1(q) \leq n_2(q) \leq \cdots \leq n_{r(q)}(q) = n
\]

and we will call \((n_1(q), n_2(q), \ldots, n_{r(q)}(q))\) the \textit{growth vector} of the system (we will omit indicating the point if it is not confusing).

- Define weights \( w_1 \leq \cdots \leq w_n \) by putting \( w_j = s \) if \( n_{s-1} < j \leq n_s \), with \( n_0 = 0 \).
Privileged coordinates

• We will call $X_1 \varphi, \ldots X_m \varphi$ the nonholonomic partial derivatives of order 1 of a function $\varphi$.

• $X_{i_1} X_{i_2} \varphi$ nonholonomic derivatives of order two of $\varphi$ etc.

• If all the nonholonomic derivatives of order $\leq s - 1$ of $\varphi$ vanish at $q$, we say that $\varphi$ is of order $\geq s$ at $q$. A function $\varphi$ is of order $s$ at $q$ if it is of order $\geq s$ but not of order $\geq s + 1$.

• Local coordinates $(\xi_1, \ldots, \xi_n)$ are privileged coordinates at $q$ if the order of $\xi_i$ is $w_i$ for $1 \leq i \leq n$.

• The integers $(w_1, \ldots, w_n)$ are the weights of the privileged coordinates $(\xi_1, \ldots, \xi_n)$. Homogeneity is considered with respect to them.
Nilpotent approximations

• Using privileged coordinates we can rewrite the system as

\[ \dot{\xi}_j = \sum_{i=1}^{m} X_{ij}(\xi_1, \ldots, \xi_{j-1})u_i + O(\|\xi\|^{w_j}) \]

for \(1 \leq j \leq n\), where the components \(X_{ij}\) are homogeneous polynomials of weighted degree \(w_j - 1\).

• By dropping the terms \(O(\|\xi\|^{w_j})\), we get

\[ \dot{\xi} = \sum_{i=1}^{m} \hat{X}_i(\xi)u_i, \text{ where } \hat{X}_i = \sum_{j=1}^{n} X_{ij}(\xi_1, \ldots, \xi_{j-1}) \frac{\partial}{\partial \xi_j}, \]

called the nilpotent approximation of the system. The Lie algebra \(\mathcal{L}(\hat{X}_1, \ldots, \hat{X}_m)\) is nilpotent.
Consider a 3-dimensional sub-Riemannian manifold \((M, \mathcal{D}, B)\), where

- \(M\) is a 3-dimensional manifold,
- \(\mathcal{D}\) is a rank 2 smooth distribution on \(M\)
- \(B\) is a smoothly varying positive definite quadratic form on \(\mathcal{D}\).
- Represent locally the sub-Riemannian structure \((M, \mathcal{D}, B)\) by the control system

\[
\dot{\xi} = X_1(\xi)u_1 + X_2(\xi)u_2,
\]

where the smooth vector fields \(X_1\) and \(X_2\) form an orthonormal frame of \(\mathcal{D}\).
An isometry between two sub-Riemannian manifolds \((M, \mathcal{D}, B)\) and \((\tilde{M}, \tilde{\mathcal{D}}, \tilde{B})\) is a diffeomorphism \(\phi : M \rightarrow \tilde{M}\) such that \(\phi_* (\mathcal{D}) = \tilde{\mathcal{D}}\) and \(B = \phi^* (\tilde{B})\). Agrachev et al have shown that there exists a sub-Riemannian isometry transforming the orthonormal frame \(\langle X_1, X_2 \rangle\) into an orthonormal frame, which in local coordinates \((x, y, z)\) takes the following normal form around \(0 \in \mathbb{R}^3\):

\[
X_1(x, y, z) = (1 + y^2 \beta(x, y, z)) \frac{\partial}{\partial x} - xy \beta(x, y, z) \frac{\partial}{\partial y} + \frac{y}{2} \gamma(x, y, z) \frac{\partial}{\partial z} \\
X_2(x, y, z) = -xy \beta(x, y, z) \frac{\partial}{\partial x} + (1 + x^2 \beta(x, y, z)) \frac{\partial}{\partial y} - \frac{x}{2} \gamma(x, y, z) \frac{\partial}{\partial z}.
\]
Contact case

• If $\gamma(0, 0, 0) \neq 0$, then we are in the contact case.

• The growth vector in the contact case is $(2, 3)$ and the variables $x, y, z$ have weights 1, 1, and 2, respectively.

• The normal form for the nilpotent approximation is
  \[
  \hat{X}_1(x, y, z) = \frac{\partial}{\partial x} + cy \frac{\partial}{2 \partial z},
  \]
  \[
  \hat{X}_2(x, y, z) = \frac{\partial}{\partial y} - cx \frac{\partial}{2 \partial z}.
  \]

• All cases are isometric to the Heisenberg case $c = 1$.

• The Heisenberg case is integrable in trigonometric functions.

• The general contact case (non nilpotent) has been completely analyzed by Agrachev, Gauthier, Kupka, and Chakir.
If $\gamma$ is of order 1 with respect to $(x, y)$, then we are in the Martinet case.

The growth vector at $0 \in \mathbb{R}^3$ in the Martinet case is $(2, 2, 3)$ and the weights of the variables $x, y, z$ are 1, 1, and 3, respectively.

The set of points, at which the growth vector is $(2, 2, 3)$, is a smooth surface (called Martinet surface) and the distribution $\mathcal{D}$ spanned by $X_1$ and $X_2$ is transversal to the Martinet surface.

The normal form for the nilpotent approximation is

$$\hat{X}_1(x, y, z) = \frac{\partial}{\partial x} + \frac{y}{2}(ax + by) \frac{\partial}{\partial z}$$
$$\hat{X}_2(x, y, z) = \frac{\partial}{\partial y} - \frac{x}{2}(ax + by) \frac{\partial}{\partial z}.$$
Martinet case - cont.

- All nilpotent Martinet cases are integrable in terms of elliptic functions.

- sub-Riemannian geometry in the general (non nilpotent) case has been intensively studied by Bonnard, Chyba, and Trélat.
Tangent case

- The next degeneration, *tangent case*, occurs at points at which the distribution $\mathcal{D}$ is tangent to the Martinet surface.

- Generically, the growth vector at such a tangency point is $(2, 2, 2, 3)$ and the variables $x, y, z$ are of weights 1, 1, and 4, respectively.

- $\gamma$ is of order 2 with respect to $(x, y)$.

- The normal form of the nilpotent approximation of the tangent case is

  \[
  \hat{X}_1(x, y, z) = \frac{\partial}{\partial x} + \frac{y}{2}(ax^2 + by^2) \frac{\partial}{\partial z}
  \]
  \[
  \hat{X}_2(x, y, z) = \frac{\partial}{\partial y} - \frac{x}{2}(ax^2 + by^2) \frac{\partial}{\partial z}.
  \]

  We can assume that $a = 1$ (by normalizing $z$).
Tangent case: geodesic equation

The geodesic equation in the nilpotent tangent case is:

\[
\begin{align*}
\dot{x} &= p + \frac{ry}{2}(x^2 + by^2), \\
\dot{y} &= q - \frac{rx}{2}(x^2 + by^2), \\
\dot{z} &= \frac{1}{2}(x^2 + by^2)(yp - xq) + \frac{r}{4}(x^2 + y^2)(x^2 + by^2)^2, \\
\dot{p} &= -rxyu_1 + \frac{r}{2}(3x^2 + by^2)u_2, \\
\dot{q} &= -\frac{r}{2}(x^2 + 3by^2)u_1 + brxyu_2, \\
\dot{r} &= 0
\end{align*}
\]

\textit{(GE)}

where \( u_1 = p + \frac{ry}{2}(x^2 + by^2) \) and \( u_2 = q - \frac{rx}{2}(x^2 + by^2) \).
Integrability problem

- The hamiltonian $H$ and $H_1 = r$ are first integrals.

- **Integrability problem:** find a third first integral $H_2$, commuting with $H$ and $H_1$, and functionally independent with $H$ and $H_1$ (Liouville integrability).

- We will distinguish the *elliptic nilpotent tangent case*, for which $a = 1$ and $b > 0$ and the *hyperbolic nilpotent tangent case*, for which $a = 1$ and $b < 0$. 
Tangent case: integrable cases

- M. Pelletier proved that if $b = 1$ (symmetric elliptic case), then the Hamiltonian (GE) is integrable in the Liouville sense with an additional first integral given by

\[ H_2 = xq - yp. \]

- Geometric reason: if $b = 1$, then the rotation in the $(x, y)$ space is a sub-Riemannian isometry.

- For $b = 0$, the geodesic equation (GE) is also integrable. In this case the third first integral has the form

\[ H_2 = 6q + rx^3. \]

- Both cases are integrable in terms of elliptic functions.
Main result

**Theorem 4** The complexified geodesic equation for the 3-dimensional nilpotent tangent case is not meromorphically integrable in the Liouville sense, except for $b = 1$ and $b = 0$, that is, for $b \in \mathbb{R} \setminus \{0, 1\}$ the complexified system (GE) does not possess a meromorphic first integral, commuting with $H$ and $H_1$ and functionally independent with $H$ and $H_1$.

- Our proof is based on the Morales-Ramis theory
Morales-Ramis theory

Consider a complex analytic hamiltonian differential equation

\[ \frac{dx}{dt} = v(x), \quad t \in \mathbb{C}, \]

on an analytic symplectic manifold \( M \) (say, \( \mathbb{C}^n \)). Let \( \varphi(t) \) be its non-stationary solution and \( \Gamma \) its maximal analytic prolongation (Riemann surface). Take the linearization (variational equation) along \( \Gamma \)

\[ \frac{d\xi}{dt} = \frac{\partial v}{\partial x}(\varphi(t))\xi \]

**Theorem 5** (Morales-Ramis) *If the hamiltonian system on \( M \) (\( \mathbb{C}^n \)) is Liouville integrable in the meromorphic category, then the identity component of the differential Galois group of the (normal) variational equation along \( \Gamma \) is abelian.*
Consider a homogeneous ordinary linear differential equation in \( \mathbb{C}^n \), over the field \( F = \mathbb{C}(z) \) of rational functions of \( z \in \mathbb{C} \)

\[
L(Y) = \frac{d}{dz} Y - A(z)Y = 0, \quad Y \in \mathbb{C}^n,
\]

where \( A^j_i \in \mathbb{C}(z) \)

- Where do the solutions live?

**Theorem 6** There exists a unique (up to isomorphism) \( PV_L \supset \mathbb{C}(z) \), the smallest differential field extension containing \( n \) linearly independent, over \( \mathbb{C} \), solutions of \( L(Y) = 0 \) (Picard-Vessiot extension).

We have \( (PV_L, D) \supset (\mathbb{C}(z), \frac{d}{dz}) \), where the derivation \( D \) restricted to \( \mathbb{C}(z) \) is \( \frac{d}{dz} \).
The space of solutions $V = \{ Y \in PV_L \mid L(Y) = 0 \}$ is a linear space over $\mathbb{C}$.

**Definition 1** Differential Galois group of $L$ is the group of differential automorphisms of $PV_L$ (i.e., commuting with the derivation $D$) preserving all elements of $\mathbb{C}(z)$.

The differential Galois group, denoted $Gal(PV_L \setminus \mathbb{C}(z))$

- preserves solutions
- preserves polynomial relations among them
- is an algebraic subgroup of $GL(n, \mathbb{C})$ (in the hamiltonian case of $Sp(n, \mathbb{C})$).
The \((x, y, p, q)\)-part of the geodesic equation can be transformed to

\[
\begin{align*}
\dot{z}_1 &= z_3, \\
\dot{z}_2 &= z_4, \\
\dot{z}_3 &= r\gamma z_1 z_2 [(z_4 - z_3) - b(z_3 + z_4)], \\
\dot{z}_4 &= r\gamma z_1 z_2 [(z_4 - z_3) + b(z_3 + z_4)].
\end{align*}
\]

It is obvious that \(z(t) = (0, ct, 0, c)\) with \(c \neq 0\) is a solution of the above equations.

The normal variational equation can be represented as

\[
\ddot{\xi}_1 = (1 - b)\gamma rc^2 t\xi_1.
\]

where \((1 - b)\gamma rc^2 \neq 0\), which gives the Airy equation. It is known that the differential Galois group of this equation is \(\text{Sl}(2, \mathbb{C})\) and thus non Abelian.
\textbf{\textit{n}-level quantum system}

- Consider a quantum system with a finite number of (distinct) levels in interaction with a time dependent external field.
- The energies of the system state appearing on the diagonal, we put $\mathcal{H}_0 = \text{diag} (E_1, \ldots, E_n)$.
- The time-functions $\Omega_j(\cdot) : \mathbb{R} \rightarrow \mathbb{C}$, for $1 \leq j \leq n - 1$ have their supports in $[t_0, t_1]$. They couple the states by pairs.
- The hamiltonian $\mathcal{H}$ is given by:
\[ H = \begin{pmatrix}
E_1 & \Omega_1(t) & 0 & \ldots & 0 \\
\Omega_1^*(t) & E_2 & \Omega_2(t) & \ddots & \vdots \\
0 & \Omega_2^*(t) & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & E_{n-1} & \Omega_{n-1}(t) \\
0 & \ldots & 0 & \Omega_{n-1}^*(t) & E_n
\end{pmatrix} \]

\[ = H_0 + \begin{pmatrix}
0 & \Omega_1(t) & 0 & \ldots & 0 \\
\Omega_1^*(t) & 0 & \Omega_2(t) & \ddots & \vdots \\
0 & \Omega_2^*(t) & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & \Omega_{n-1}(t) \\
0 & \ldots & 0 & \Omega_{n-1}^*(t) & 0
\end{pmatrix} \]
The state vector $\psi(\cdot) : \mathbb{R} \rightarrow \mathbb{C}^n$ satisfies the Schrödinger equation

$$i \frac{d\psi(t)}{dt} = \mathcal{H}\psi = (\mathcal{H}_0 + \sum_{j=1}^{n-1} \Omega_j(t)\mathcal{H}_j)\psi$$

(we have assumed coupling of neighboring levels only).

We represent

$$\psi(t) = \psi_1(t)e_1 + \psi_2(t)e_2 + \cdots \psi_n(t)e_n,$$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{C}^n$.

We have $|\psi_1(t)|^2 + |\psi_2(t)|^2 + \cdots + |\psi_n(t)|^2 = 1$.

For $t < t_0$ and $t > t_1$, $|\psi_j(t)|^2$ is the probability of measuring the energy $E_j$. Notice that $\frac{d}{dt} |\psi_j(t)|^2 = 0$, for $t < t_0$ and $t > t_1$. 

Schrödinger equation
Optimal problem

Problem:

Assuming that
\[ |\psi_1(t)|^2 = 1, \quad \text{for} \quad t < t_0 \]
find suitable interaction functions \( \Omega_j(t), 1 \leq j \leq n - 1 \), such that
\[ |\psi_i(t)|^2 = 1, \quad \text{for} \quad t > t_1 \]
for some chosen \( i \in \{2, \ldots, n\} \), say \( i = n \), and such that the cost
\[
E = \frac{1}{2} \int_{t_0}^{t_1} \sum_{j=1}^{n-1} |\Omega_j(t)|^2 \, dt \longrightarrow \text{min}.
\]
(minimize the energy of the transfer pulses).
**Resonant case**

Optimal interaction functions $\Omega_j$ correspond to lasers that are in resonance (real resonant case, Brockett, Khaneja, Glaser, and Boscain, Charlot, Gauthier):

$$\Omega_j(t) = u_j(t)e^{i\omega_j t}, \quad \omega_j = E_{j+1} - E_j,$$

for $1 \leq j \leq n - 1$, where $u_j(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are real controls. The cost function becomes

$$E = \frac{1}{2} \int_{t_0}^{t_1} \sum_{j=1}^{n-1} u_j^2(t) dt.$$
Simplifications of the problem

• We apply the unitary transformation

\[ \psi(t) = U(t)\tilde{\psi}(t). \]

to eliminate the drift \( \mathcal{H}_0 = \text{diag}(E_1, \ldots, E_n) \).

• We pass from \( \mathbb{C}^n \) to \( \mathbb{R}^n \) to get finally the system

\[ \dot{x} = \mathcal{H}_\mathbb{R}x, \quad x \in \mathbb{R}^n, \]
where

\[
\mathcal{H}_R = \begin{pmatrix}
0 & u_1(t) & 0 & \ldots & 0 \\
-u_1(t) & 0 & u_2(t) & \ddots & \\
0 & -u_2(t) & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 & u_{n-1}(t) \\
0 & \cdots & 0 & -u_{n-1}(t) & 0
\end{pmatrix}.
\]

Introduce the vector fields (infinitesimal generators of rotation in the \((x_i,x_j)\)-space)

\[
f_{i,j} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad 1 \leq i, j \leq n
\]
Optimal problem in $\mathbb{R}^n$

The problem is now: find real controls $u_1(t), \ldots, u_{n-1}(t)$ such that the corresponding trajectory of

$$\dot{q} = \mathcal{H}_{\mathbb{R}} q = \sum_{j=1}^{n-1} u_j f_{j,j+1}(q), \quad q \in \mathbb{R}^n,$$

joins given $q_0$ and $q_T$ and

$$E = \frac{1}{2} \int_{t_0}^{t_1} \left( \sum_{j=1}^{n-1} u_j^2(t) \right) dt \rightarrow \min.$$
Lifting the problem to $\text{SO}(n)$

- The Lie algebra

\[
\{f_1,2, \ldots, f_{n-1},n\}_{LA} = \text{vect}_\mathbb{R} \{f_{i,k}, \ 1 \leq i < k \leq n\} = \mathfrak{so}(n)
\]

- Let $F_{i,k}$ stand for the left invariant vector fields on $\text{SO}(n)$ that satisfy exactly the same commutation relations as $f_{i,k}$.

- We lift our optimal control problem to the following left invariant on $G=\text{SO}(n)$: find controls $u_j(t)$ that minimize the energy $E$ of the curve $Q(t) \in G = \text{SO}(n)$ (time evolution operator) satisfying

\[
\dot{Q} = \sum_{j=1}^{n-1} u_j F_{j,j+1}, \quad E = \frac{1}{2} \int_{t_0}^{t_1} \sum_{j=1}^{n-1} u_j^2(t) \, dt \rightarrow \text{min}.
\]

- It is a sub-Riemannian problem!!!
3-level system

Easy to integrate (Brockett, Boscain et al. for the quantum system).

The adjoint equation takes the form

\[
\dot{H}_{1,2} = H_{1,3}H_{2,3} \\
\dot{H}_{2,3} = -H_{1,3}H_{1,2} \\
\dot{H}_{1,3} = 0
\]

We get \( H_{1,3}(t) = \text{const.} = a \) and

\[
\begin{align*}
    u_1(t) &= H_{1,2}(t) = r \cos(at + \varphi) \\
    u_2(t) &= H_{1,2}(t) = -r \sin(at + \varphi).
\end{align*}
\]

\( H_{1,3} \) is a Casimir function; we integrate the system on its constant level sets.
Now it suffices to integrate the linear time-varying system

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = u_1 \begin{pmatrix}
-x_2 \\
x_1 \\
0
\end{pmatrix} + u_2 \begin{pmatrix}
0 \\
-x_3 \\
x_2
\end{pmatrix}
\]

which has the first integral:

\[
h = x_1^2 + x_2^2 + x_3^2.
\]
Main result

**Theorem 7** For the $n$-level system, $n \geq 4$, the complexification of the adjoint equation on $\mathfrak{so}(n)^*$ is not integrable in the meromorphic category. More precisely, restricted to the leaves $M_c$ of the symplectic foliation on $\mathfrak{so}(n)^*$, does not possess any meromorphic first integral independent of the hamiltonian, i.e. is not Liouville integrable on $M_c$. 
4-level system: Adjoint equation on $\mathfrak{so}(4)^*$

- By restricting the $AE$ to $\{H_{i,k} = 0\}$, where $i \geq 5$ or $k \geq 5$, the nonintegrability problem of the general $n$-level system reduces to that of the 4-level system.

- We will consider the complexification $AE_\mathbb{C}$ of $AE$ on $\mathfrak{so}(4)^*$ by taking $x_i \in \mathbb{C}$ and $t \in \mathbb{C}$, where $x_1 = H_{1,2}$, $x_2 = H_{2,3}$, $x_3 = H_{1,3}$, $x_4 = H_{3,4}$, $x_5 = H_{1,4}$, and $x_6 = H_{4,2}$.

- The complexified $AE_\mathbb{C}$ reads as
\[
\frac{d}{dt} x = J(x) \nabla H(x), \quad x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{C}^6, \quad t \in \mathbb{C}
\]
where
\[
H = H(x) = \frac{1}{2} (x_1^2 + x_2^2 + x_4^2),
\]
and
\[
J(x) = \begin{bmatrix}
0 & x_3 & -x_2 & 0 & x_6 & -x_5 \\
-x_3 & 0 & x_1 & -x_6 & 0 & x_4 \\
x_2 & -x_1 & 0 & x_5 & -x_4 & 0 \\
0 & x_6 & -x_5 & 0 & x_3 & -x_2 \\
-x_6 & 0 & x_4 & -x_3 & 0 & x_1 \\
x_5 & -x_4 & 0 & x_2 & -x_1 & 0 \\
\end{bmatrix}
\]
It is a Lie-Poisson system: \( \text{rank } J(x) = 4 \) so \( J(x) \) defines a Poisson structure (a ”degenerated symplectic structure”).
• Besides the Hamiltonian $H$, $AE_{\mathbb{C}}$ admits two additional first integrals

$$C_1 = \sum_{i=1}^{6} x_i^2, \quad C_2 = x_1 x_4 + x_2 x_5 + x_3 x_6,$$

which are actually the Casimir function of the Poisson structure defined by $J(x)$; the first integrability requirement is satisfied.

• Each level set

$$\mathcal{M}_{a,b} := \{ x \in \mathbb{C}^6 \mid C_1(x) = a, \quad C_2(x) = b \},$$

is a 4-dimensional symplectic manifold on which $AE_{\mathbb{C}}$ is hamiltonian with Hamiltonian function $H|_{\mathcal{M}_{a,b}}$. We need one more first integral!
$AE_{\mathbb{C}}$ admits the invariant space

$$\mathcal{M}^3 = \{ x \in \mathbb{C}^6 \mid x_4 = x_5 = x_6 = 0 \},$$

foliated by the phase curves $\Gamma_{h,f} = S^1_{\mathbb{C}}$, complex circles, given by

$$x_1^2 + x_2^2 = h, \quad x_3 = f$$

The normal variational equations along $\Gamma_{h,f}$ reduces to the form

$$w'' = r(z)w, \quad r(z) = \frac{\alpha_0}{z^2} + \frac{\alpha_h}{(z-h)^2} + \frac{\beta_0}{z} + \frac{\beta_h}{z-h}$$

Singular points at $z = 0$ and $z = h$ are regular but at $\infty$ is irregular. Indeed, we have (using Kovacic algorithm)

**Lemma 2** The differential Galois group of $w'' = r(z)w$ is $SL(2, \mathbb{C})$.

$SL^0(2, \mathbb{C})$ is non-abelian, hence the adjoint equation is not integrable.
\( n \)-level quantum system

\[ \Downarrow \]

optimal control problem: Pontryagin Maximum Principle

\[ \Downarrow \]

sub-Riemannian problem on \( \text{SO}(n) \)

\[ \Downarrow \]

nonintegrability of a hamiltonian system

\[ \Downarrow \]

Differential Galois group and complex analysis
Conclusions

• We discussed (non)integrability of the geodesic equation (adjoint equation) for various Sub-Riemannian problems

• We show usefulness of the Morales-Ramis theory in proving non-integrability

• open problems: homogenous 4-dimensional SR-problems, general contact and quasi-contact SR-problems,...