Generalized Stäckel Transform: Integrability, Reciprocal Transformations and More

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Background

Classification of integrable systems involves

- search for new (super)integrable systems
- search for the (super)integrability-preserving *equivalence transformations*

Examples of such transformations include

- canonical transformations
- classical Stäckel transform

Our goal: to present a (super)integrability-preserving generalization of the ST:

multiparameter generalized Stäckel transform

The classical Stäckel transform

Hietarinta et al. 1984, Boyer et al. 1986:
let $x = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}^{2n}$, $\{\lambda^i, \mu_j\} = \delta_i^j$, and

$$H = H(x, \alpha) = T(x) + \alpha V(x).$$

Then the stationary Hamilton–Jacobi equation for H reads

$$H(\boldsymbol{\lambda}, \partial S/\partial\boldsymbol{\lambda}, \alpha) = \tilde{\alpha}, \qquad (1)$$

Suppose we know a solution S for (1).

Let $\tilde{H}(x, \tilde{\alpha}) = (\tilde{\alpha} - T(x))/V(x)$.

Then the above S solves the stationary Hamilton–Jacobi equation for $\tilde{H}$:

$$\tilde{H}(\boldsymbol{\lambda}, \partial S/\partial\boldsymbol{\lambda}, \tilde{\alpha}) = \alpha,$$

as this equation is equivalent to (1).

The classical Stäckel transform: an easy example

$$H = p^2 + \alpha q^2 \to \tilde{H} = -p^2/q^2 + \tilde{\alpha}/q^2$$

$$(\partial S/\partial q)^2 + \alpha q^2 = \tilde{\alpha}$$
$$\Updownarrow$$
$$-(1/q^2)(\partial S/\partial q)^2 + \tilde{\alpha}/q^2 = \alpha$$

The Hamilton–Jacobi equations for H and $\tilde{H}$ are *identical* modulo the interchange of roles of α and $\tilde{\alpha}$.

Note that the point transformation $z = q^2/2$ turns the above H–J equation into the one for the Coulomb potential

$$(\partial S/\partial z)^2 + \frac{\tilde{\alpha}}{2z} = \alpha$$

MGST: Preliminaries

Let (M, P) be a Poisson manifold with the Poisson bracket

$$\{f, g\} = \langle df, Pdg \rangle.$$

Consider r functionally independent Hamiltonians on M:

$$H_i = H_i(x, \alpha_1, \ldots, \alpha_k), \quad i = 1, \ldots, r, \tag{2}$$

where $x \in M$ and $k \le r$.

Suppose that there exists a k-tuple of pairwise distinct numbers $s_i \in \{1, \ldots, r\}$ such that

$$\det \left(\| \partial H_{s_i} / \partial \alpha_j \|_{i,j=1,\ldots,k} \right) \neq 0. \tag{3}$$

In what follows we will fix this k-tuple and call the Hamiltonians H_{s_i}, $i = 1, \ldots, k$, *distinguished*.

MGST: definition

$$H_{s_i}(x,\alpha_1,\dots,\alpha_k)=\tilde{\alpha}_i \quad \stackrel{(3)}{\Rightarrow} \alpha_i = A_i(x,\tilde{\alpha}_1,\dots,\tilde{\alpha}_k),$$

$$i=1,\dots,k \qquad\qquad\qquad i=1,\dots,k.$$

Define *new Hamiltonians* $\tilde{H}_{s_i}=A_i$, $i=1,\dots,k$. Then

$$H_{s_i}|_{[\Phi]}\equiv H_{s_i}(x,\tilde{H}_{s_1},\dots,\tilde{H}_{s_k})=\tilde{\alpha}_i, \quad i=1,\dots,k, \quad (4)$$

The subscript $[\Phi]$ means that we have substituted $\tilde{H}_{s_i}$ for α_i for all $i=1,\dots,k$. Let also

$$\tilde{H}_i := H_i|_{[\Phi]}\equiv H_i(x,,\tilde{H}_{s_1},\dots,\tilde{H}_{s_k}),$$
$$i=1,\dots,r, \quad i\neq s_j \quad \text{for} \quad j=1,\dots,k. \quad (5)$$

Note that $\tilde{H}_i$ involve k parameters $\tilde{\alpha}_i$, $i=1,\dots,k$:

$$\tilde{H}_i=\tilde{H}_i(x,\tilde{\alpha}_1,\dots,\tilde{\alpha}_k), \quad i=1,\dots,r.$$

We shall refer to the above transformation from H_i, $i=1,\dots,r$, to $\tilde{H}_i$, $i=1,\dots,r$, as to the k-parameter **generalized Stäckel transform** generated by $H_{s_1},\dots,H_{s_k}$. We shall say that the r-tuples H_i, $i=1,\dots,r$, and $\tilde{H}_i$, $i=1,\dots,r$, are *Stäckel-equivalent*.

Duality

Consider the *duals* of $H_{s_i}|_{[\Phi]} = \tilde{\alpha}_i, \quad i = 1, \ldots, k$:

$$\tilde{H}_{s_i}|_{[\tilde{\Phi}]} \equiv \tilde{H}_{s_i}(x, H_{s_1}, \ldots, H_{s_k}) = \alpha_i, \quad i = 1, \ldots, k, \tag{6}$$

where the subscript $[\tilde{\Phi}]$ means that we have substituted H_{s_i} for $\tilde{\alpha}_i$ for all $i = 1, \ldots, k$.
Solve (6) with respect to H_{s_i}, $i = 1, \ldots, k$ and define the remaining Hamiltonians H_i by the formulas

$$\begin{gathered} H_i = \tilde{H}_i|_{[\tilde{\Phi}]} \equiv \tilde{H}_{s_i}(x, H_{s_1}, \ldots, H_{s_k}), \\ i = 1, \ldots, r, \quad i \neq s_j \quad \text{for} \quad j = 1, \ldots, k. \end{gathered} \tag{7}$$

Then the conditions

$$\begin{aligned} & H_{s_i}|_{[\Phi]} = \tilde{\alpha}_i, \ i = 1, \ldots, k, \\ & \tilde{H}_i = H_i|_{[\Phi]}, \ i = 1, \ldots, r, \ i \neq s_j \ \text{ for } j = 1, \ldots, k \end{aligned} \tag{*}$$

hold identically, so (6) and (7) define the inverse of $(*)$.

The two transformations are *dual*, with the duality transformation swapping H_i and $\tilde{H}_i$ for all $i = 1, \ldots, r$ and swapping α_j and $\tilde{\alpha}_j$ for all $j = 1, \ldots, k$.

MGST: Example

The (extended) Hénon–Heiles system has a Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \alpha_1\left(q_1^3 + \frac{q_1 q_2^2}{2}\right) - \alpha_2 q_1,$$

that Poisson commutes with

$$H_2 = \frac{1}{2}q_2 p_1 p_2 - \frac{1}{2}q_1 p_2^2 - \alpha_1\left(\frac{q_2^4}{16} + \frac{q_1^2 q_2^2}{4}\right) - \alpha_2\frac{q_2^2}{4}.$$

Let $s_1 = 1$, $s_2 = 2$, $k = r = 2$. Then

$$\tilde{H}_1 = \frac{2}{q_1 q_2^2}p_1^2 - \frac{8}{q_2^3}p_1 p_2 - \frac{2(q_2^2 + 4q_1^2)}{q_1 q_2^4}p_2^2 - \frac{4}{q_1 q_2^2}\tilde{\alpha}_1 + \frac{16}{q_2^4}\tilde{\alpha}_2,$$

$$\tilde{H}_2 = -\frac{4q_1^2 + q_2^2}{2q_1 q_2^2}p_1^2 - \frac{4(q_2^2 + 2q_1^2)}{q_2^3}p_1 p_2$$

$$+\frac{16q_1^4 + 12q_1^2 q_2^2 + q_2^4}{q_1 q_2^4}p_2^2 + \frac{(q_2^2 + 4q_1^2)}{q_1 q_2^2}\tilde{\alpha}_1 - \frac{8(q_2^2 + 2q_1^2)}{q_2^4}\tilde{\alpha}_2,$$

and $\{\tilde{H}_1, \tilde{H}_2\} = 0$.

Special case: transformations linear in parameters

Let $$H_i = H_i^{(0)} + \sum_{j=1}^{k} \alpha_j H_i^{(j)}, \quad i = 1, \ldots, r. \qquad (8)$$

Then $H_{s_i}|_{[\Phi]} = \tilde{\alpha}_i, \quad i = 1, \ldots, k$, yields

$$H_{s_i}^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} H_{s_i}^{(j)} = \tilde{\alpha}_i, \quad i = 1, \ldots, k, \qquad (9)$$

whence $$\tilde{H}_{s_i} = \det W_i / \det W, \qquad (10)$$

where $$W = \begin{Vmatrix} H_{s_1}^{(1)} & \cdots & H_{s_1}^{(k)} \\ \vdots & \ddots & \vdots \\ H_{s_k}^{(1)} & \cdots & H_{s_k}^{(k)} \end{Vmatrix}$$ is a $k \times k$ matrix;

W_i are obtained from W by replacing $H_{s_j}^{(i)}$ by $H_{s_j}^{(0)} - \tilde{\alpha}_j$ for all $j = 1, \ldots, k$;

$$\tilde{H}_i = H_i^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} H_i^{(j)}, \; i = 1, \ldots, r, \; i \neq s_j \text{ for } j = 1, \ldots, k. \qquad (11)$$

MGST preserves (super)integrability – main result

Proposition 1 *Let* H_i, $i = 1, \ldots, r$, *be functionally independent and let* $\tilde{H}_i$, $i = 1, \ldots, r$, *be related to* H_i, $i = 1, \ldots, r$, *by a* k*-parameter generalized Stäckel transform (4), (5) generated by* $H_{s_1}, \ldots, H_{s_k}$, *where* $k \le \operatorname{corank} P + (1/2) \operatorname{rank} P$. *Then the following assertions hold:*

i) if $\{H_{s_i}, H_{s_j}\} = 0$ *for all* $i, j = 1, \ldots, k$ *then* $\{\tilde{H}_{s_i}, \tilde{H}_{s_j}\} = 0$ *for all* $i, j = 1, \ldots, k$;

ii) under the assumptions of i) suppose that $k + 1 \le \operatorname{corank} P + (1/2) \operatorname{rank} P$ *and for a* $j_0 \in \{1, \ldots, r\}$, $j_0 \neq s_1, \ldots, s_k$ *we have* $\{H_{s_i}, H_{j_0}\} = 0$ *for all* $i = 1, \ldots, k$; *then* $\{\tilde{H}_{s_i}, \tilde{H}_{j_0}\} = 0$ *for all* $i = 1, \ldots, k$;

iii) under the assumptions of i) let $k + 2 \le \operatorname{corank} P + (1/2) \operatorname{rank} P$ *and for* $j_q \in \{1, \ldots, r\}$, $j_q \neq s_1, \ldots, s_k$, $q = 1, 2$, $j_1 \neq j_2$, *we have* $\{H_{s_i}, H_{j_q}\} = 0$, $i = 1, \ldots, k$, $q = 1, 2$, *and* $\{H_{j_1}, H_{j_2}\} = 0$; *then* $\{\tilde{H}_{s_i}, \tilde{H}_{j_q}\} = 0$, $i = 1, \ldots, k$, $q = 1, 2$, *and* $\{\tilde{H}_{j_1}, \tilde{H}_{j_2}\} = 0$.

MGST preserves integrability

Under the assumptions of Proposition 1 (i) let $\dim M = 2n$, $\operatorname{rank} P = 2n$, and consider r functionally independent Hamiltonians H_i, $i = 1, \ldots, r$, on M. Suppose that $\{H_{l_p}, H_{l_q}\} = 0$ for $p, q = 1, \ldots, m$, where $m \geq k$. Here $l_p \in \{1, \ldots, r\}$ are distinct integers such that $s_i \in \{l_1, \ldots, l_m\}$ for all $i = 1, \ldots, k$.

If $m = n$ then the dynamical system associated with any of H_{l_i} is Liouville integrable, as it has n commuting functionally independent integrals, H_{l_j}, $j = 1, \ldots, n$. By Proposition 1 the dynamical system associated with any of $\tilde{H}_{l_i}$ enjoys the same property, the required integrals of motion in involution now being $\tilde{H}_{l_i}$, $i = 1, \ldots, n$.

MGST preserves superintegrability

Let again $m = n$. Further assume that $k < n$, $n < r \leq 2n - k$, and $\{H_{s_i}, H_j\} = 0$ for all $i = 1, \ldots, k$ and for all $j = 1, \ldots, r$. Note that the condition $r \leq 2n - k$ enables the relations $\{H_{s_i}, H_j\} = 0$, $i = 1, \ldots, k$, $j = 1, \ldots, r$, to hold without losing the functional independence of H_i, $i = 1, \ldots, r$, as the latter must hold by assumption.

Then the Hamiltonian H_{s_i} is superintegrable for any $i \in \{1, \ldots, k\}$ as it has $r > n$ integrals of motion H_j, $j = 1, \ldots, r$, and, moreover, there are n commuting integrals of motion H_{l_p}, $p = 1, \ldots, n$.

By Proposition 1, i) – iii), the Hamiltonian $\tilde{H}_{s_j}$ is superintegrable for any $j \in \{1, \ldots, k\}$ as well, the integrals of motion now being $\tilde{H}_i$, $i = 1, \ldots, r$, and we have n *commuting* integrals of motion $\tilde{H}_{l_i}$, $i = 1, \ldots, n$. Thus, under certain technical assumptions the generalized Stäckel transform preserves superintegrability.

Basics of noncommutative integrability

References: Mishchenko Fomenko Bolsinov etc.

Consider an algebra $\mathcal{A}$ of functions on a *symplectic* manifold M and assume that $\mathcal{A}$ is closed under the Poisson bracket.

The *differential dimension* $\operatorname{ddim}\mathcal{A}$ of $\mathcal{A}$ is, informally, the number of functionally independent generators of $\mathcal{A}$.

The *differential index* $\operatorname{dind}\mathcal{A}$ can be (informally) defined as $\operatorname{dind}\mathcal{A} = \operatorname{ddim}\ker\{,\}|_{\mathcal{A}}$, and $\mathcal{A}$ is *complete* if $\operatorname{ddim}\mathcal{A} + \operatorname{dind}\mathcal{A} = \dim M$ on an open dense subset $U \subset M$.

As H_i, $i = 1, \dots, r$, are functionally independent generators of $\mathcal{F}$, we have $\operatorname{dind}\mathcal{F} = \operatorname{corank}||\{H_i, H_j\}||_{i,j=\overline{1,r}}$.

A Hamiltonian dynamical system is said to be *integrable in the noncommutative sense* if this system possesses an algebra of integrals of motion which is closed under the Poisson bracket and complete.

MGST preserves noncommutative integrability

Proposition 2 *Under the assumptions of Proposition 1, i) suppose that* $\dim M = 2n$, P *is nondegenerate* ($\operatorname{rank} P = 2n$)*, and the algebra* $\mathcal{F}$ *of functions on* M *generated by* $H_1, \ldots, H_r$ *is closed under the Poisson bracket and complete. Further suppose that* $\ker\{,\}|_{\mathcal{F}} = \mathcal{F}_0$*, where* $\mathcal{F}_0$ *is the algebra of functions on* M *generated by* $H_{l_1}, \ldots, H_{l_m}$*,* $m \leq n$*, where* $l_j \in \{1, \ldots, r\}$*,* $j = 1, \ldots, m$*, are distinct integers,* H_{l_j}*,* $j = 1, \ldots, m$*, are functionally independent, and* $s_p \in \{l_1, \ldots, l_m\}$ *for all* $p = 1, \ldots, k$*.*

Then the algebra $\tilde{\mathcal{F}}$ *of functions on* M *generated by* $\tilde{H}_1, \ldots, \tilde{H}_r$ *is also closed under the Poisson bracket and complete.*

Equations of motion: brief recap

Let $dx^b/dt_H = (X_H)^b, \quad b = 1, \ldots, \dim M, \qquad (12)$

where x^b are local coordinates on M, $X_H = PdH$ is the Hamiltonian vector field associated with H, and t_H is the corresponding evolution parameter (time), and for $H = H(x, \alpha_1, \ldots, \alpha_k)$ we set

$$dH \stackrel{\text{def}}{=} \sum_{b=1}^{\dim M} \frac{\partial H}{\partial x^b} dx^b,$$

i.e., the parameters α_i are assumed to be constant while computing dH.

Historical remark: time-changing transformations for the equations of motion date back to classical authors (Maupertuis, Lagrange, Jacobi, etc.) and were also studied more recently (Hietarinta et al., Veselov, Tsiganov, and others).

Reciprocal transformations for the equations of motion

Proposition 3 *Let* $\{H_{s_i}, H_{s_j}\} = 0$ *for all* $i, j = 1, \ldots, k$. *Consider the equations of motion*

$$dx^b/dt_{s_i} = (X_{H_{s_i}})^b, \; b = 1, \ldots, \dim M, \; i = 1, \ldots, k,$$

for H_{s_i}, $i = 1, \ldots, k$, *restricted onto the common level surface of* H_{s_i}

$$N_{\tilde{\alpha}} = \{x \in M | H_{s_i}(x, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \quad i = 1, \ldots, k\}.$$

Then the reciprocal transformation

$$d\tilde{t}_{s_i} = -\sum_{j=1}^{k} \left(\frac{\partial H_{s_j}}{\partial \alpha_i}\right)\Bigg|_{[\Phi]} dt_{s_j}, \quad i = 1, \ldots, k. \tag{13}$$

is well defined on these restricted equations of motion and sends them into the equations of motion

$$dx^b/d\tilde{t}_{s_i} = (X_{\tilde{H}_{s_i}})^b, \; b = 1, \ldots, \dim M, \; i = 1, \ldots, k.$$

for $\tilde{H}_{s_i}$, $i = 1, \ldots, k$, *restricted onto the common level surface of* $\tilde{H}_{s_i}$

$$\tilde{N}_{\alpha} = \{x \in M | \tilde{H}_{s_i}(x, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \alpha_i, \quad i = 1, \ldots, k\}.$$

Extended reciprocal transformation

Proposition 4 *Let* $\{H_i, H_j\} = 0$, $i, j = 1, \ldots, r$. *Consider the equations of motion*

$$dx^b/dt_i = (X_{H_i})^b, \quad b = 1, \ldots, \dim M, \quad i = 1, \ldots, r,$$

for H_i, $i = 1, \ldots, r$, *restricted onto* $N_{\tilde{\alpha}}$.

Then the reciprocal transformation

$$d\tilde{t}_{s_i} = -\sum_{j=1}^{r} \left(\frac{\partial H_j}{\partial \alpha_i}\right)\Bigg|_{[\Phi]} dt_j, \; i = 1, \ldots, k, \tag{14}$$

$$\tilde{t}_q = t_q, \; q = 1, 2, \ldots, r, \; q \neq s_p \text{ for any } p = 1, \ldots, k,$$

is well defined on these restricted equations of motion and sends them into the equations of motion

$$dx^b/d\tilde{t}_i = (X_{\tilde{H}_i})^b, \quad b = 1, \ldots, \dim M, \quad i = 1, \ldots, r,$$

for $\tilde{H}_i$, $i = 1, \ldots, r$, *restricted onto* $\tilde{N}_\alpha$.

Canonical Poisson structure

Corollary 1 *Let* rank P = dim $M = 2n$, $\{\lambda^i, \mu_j\} = \delta^i_j$, $r = n$, $\{H_i, H_j\} = 0$ *for all* $i, j = 1, \ldots, n$, $\partial^2 H_i / \partial\alpha_j \partial\boldsymbol{\mu} = 0$ *for all* $i = 1, \ldots, n$ *and all* $j = 1, \ldots, k$, *and that* λ_j, $j = 1, \ldots, n$, *can be chosen as local coordinates on the Lagrangian submanifold*

$$N_E = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in M | H_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \alpha_1, \ldots, \alpha_k) = E_i, \quad i = 1, \ldots, n\}$$

(i.e., the system $H_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \alpha_1, \ldots, \alpha_k) = E_i$, $i = 1, \ldots, n$, *can be solved for* $\boldsymbol{\mu}$*), and that we have*

$$\alpha_j = \tilde{E}_{s_j}, \quad E_{s_j} = \tilde{\alpha}_j, \quad j = 1, \ldots, k,$$

$$E_i = \tilde{E}_i, \quad i = 1, \ldots, n, \quad i \neq s_p \quad \text{for all} \quad p = 1, \ldots, k.$$

Then the reciprocal transformation (14) turns the system

$$d\boldsymbol{\lambda}/dt_i = (\partial H_i / \partial\boldsymbol{\mu})|_{N_E}, \quad i = 1, \ldots, n,$$

$$\text{into } d\boldsymbol{\lambda}/d\tilde{t}_i = (\partial \tilde{H}_i / \partial\boldsymbol{\mu})|_{\tilde{N}_{\tilde{E}}}, \quad i = 1, \ldots, n,$$

$$\tilde{N}_{\tilde{E}} = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in M | \tilde{H}_i(\boldsymbol{\lambda}, \boldsymbol{\mu}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \tilde{E}_i, \ i = 1, \ldots, n\}.$$

Natural Hamiltonians

In particular, if H_i are quadratic in the momenta μ_j, and

$$H_i = \sum_{j,k=1}^{n} K_i^{jk}(\boldsymbol{\lambda})\mu_j\mu_k + \sum_{q=1}^{k} \alpha_q W_i^{(q)}(\boldsymbol{\lambda}), \quad i = 1, \ldots, n,$$

Then the systems

$$d\boldsymbol{\lambda}/dt_i = (\partial H_i/\partial\boldsymbol{\mu})|_{N_E}, \quad i = 1, \ldots, n,$$

$$d\boldsymbol{\lambda}/d\tilde{t}_i = (\partial \tilde{H}_i/\partial\boldsymbol{\mu})|_{\tilde{N}_{\tilde{E}}}, \quad i = 1, \ldots, n,$$

are nothing but the sets of dispersionless (hydrodynamic-type) systems, and the transformation (14), i.e.,

$$d\tilde{t}_{s_i} = -\sum_{j=1}^{r} \left(\frac{\partial H_j}{\partial \alpha_i}\right)\bigg|_{[\Phi]} dt_j, \ i = 1, \ldots, k,$$
$$\tilde{t}_q = t_q, \ q = 1, 2, \ldots, r, \ q \neq s_p \text{ for any } p = 1, \ldots, k,$$

is a reciprocal transformation relating these two sets.

Reduced equations of motion: example

Let $k=1$, $\alpha_1 \equiv \alpha$, $s_1 = s$, $r = n$,

$$H_i = \frac{1}{2}(\boldsymbol{\mu}, G_i(\boldsymbol{\lambda})\boldsymbol{\mu}) + V_i(\boldsymbol{\lambda}) + \alpha W_i(\boldsymbol{\lambda}), \quad i = 1, \ldots, n, \quad (15)$$

where $(\cdot,\cdot)$ stands for the standard scalar product in $\mathbb{R}^n$ and $G_i(\boldsymbol{\lambda})$ are $n \times n$ matrices, then the reduced equations of motion $d\boldsymbol{\lambda}/dt_i = (\partial H_i/\partial \boldsymbol{\mu})|_{N_E}$, $i = 1, \ldots, n$, read

$$d\boldsymbol{\lambda}/dt_i = G_i(\boldsymbol{\lambda})\boldsymbol{M}, \quad (16)$$

where $\boldsymbol{\mu} = \boldsymbol{M}(\boldsymbol{\lambda}, \alpha, E_1, \ldots, E_n)$ is a general solution of the system $H_i(\alpha, \boldsymbol{\lambda}, \boldsymbol{\mu}) = E_i$, $i = 1, \ldots, n$.

If we eliminate $\boldsymbol{M}$ from (16) then we obtain the *dispersionless Killing systems*

$$\boldsymbol{\lambda}_{t_i} = G_i(G_s)^{-1}\boldsymbol{\lambda}_{t_s}, \quad i = 1, 2, \ldots, s-1, s+1, \ldots, n, \quad (17)$$

and the reciprocal transformation $d\tilde{t}_s = -\sum_{i=1}^{n} W_i(\boldsymbol{\lambda}) dt_i$, $\tilde{t}_i = t_i, \quad i \neq s$, turns (17) into

$$\boldsymbol{\lambda}_{\tilde{t}_i} = \tilde{G}_i(\tilde{G}_s)^{-1}\boldsymbol{\lambda}_{\tilde{t}_s}, \quad i = 1, 2, \ldots, s-1, s+1, \ldots, n. \quad (18)$$

Reduced equations of motion: example continued

Here the contravariant metrics

$$\tilde{G}_s = -G_s/W_s,$$

$$\tilde{G}_i = G_i - W_i G_s/W_s,\ \ i = 1, 2, \ldots, s-1, s+1, \ldots, n,$$

are related to the Hamiltonians

$$\tilde{H}_i = \frac{1}{2}(\boldsymbol{\mu}, \tilde{G}_i(\boldsymbol{\lambda})\boldsymbol{\mu}) + \tilde{V}_i(\boldsymbol{\lambda}) + \tilde{\alpha}\tilde{W}_i(\boldsymbol{\lambda}), \quad i = 1, \ldots, n, \ (19)$$

which are Stäckel-equivalent to H_i, $i = 1, \ldots, n$.

MGST and deformations of separation curves

Under the assumptions of Corollary 1, suppose that λ_i, μ_i, $i = 1, \ldots, n$, are *separation coordinates* for the n-tuple of commuting Hamiltonians H_i, $i = 1, \ldots, n$.
If the relations

$$\varphi(\lambda_i, \mu_i, \alpha_1, \ldots, \alpha_k, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n,$$

uniquely determine the Hamiltonians H_i for $i = 1, \ldots, n$, then for the sake of brevity we shall say that H_i for $i = 1, \ldots, n$ have the *separation curve*

$$\varphi(\lambda, \mu, \alpha_1, \ldots, \alpha_k, H_1, \ldots, H_n) = 0. \tag{20}$$

Fixing values of all Hamiltonians $H_i = E_i$, $i = 1, \ldots, n$, picks a particular Lagrangian submanifold from the Lagrangian foliation. It is also clear that the Stäckel-equivalent n-tuples of the Hamiltonians H_i, $i = 1, \ldots, n$, and $\tilde{H}_i$, $i = 1, \ldots, n$, share the separation curve (20) provided (4) and (6) hold.

MGST and deformations of separation curves II

Define first an operator R_k^f that acts as follows:

$$R_k^f(F) = F + f\lambda^k - (\lambda^k/k!)(\partial^k F/\partial\lambda^k)|_{\lambda=0}.$$

For instance, we have

$$R_k^f\left(\sum_{j=0}^{s} a_j\lambda^j\right) = f\lambda^k + \sum_{j=0, j\neq k}^{s} a_j\lambda^j.$$

Let $F_0 = \sum_{j=1}^{n} H_j\lambda^{n-j}$ and $\tilde{F}_0 = \sum_{j=1}^{n} \tilde{H}_j\lambda^{n-j}$.

For any integer m define the so-called *basic separable potentials* $V_j^{(m)}$ by means of the relations

$$\lambda^m + \sum_{j=1}^{n} V_j^{(m)}\lambda^{n-j} = 0 \qquad (21)$$

that must hold for $\lambda = \lambda_i$, $i = 1, \ldots, n$.

MGST and deformations of separation curves III

Under the assumptions of Corollary 1, consider an n-tuple of Poisson commuting Hamiltonians of the form

$$H_i = H_i^{(0)} + \sum_{j=1}^{k} \alpha_j V_i^{(\gamma_j)}, \tag{22}$$

where γ_j, $j = 1, \ldots, k$, are pairwise distinct integers. Suppose that H_i have the separation curve of the form

$$\sum_{j=1}^{k} \alpha_j \lambda^{\gamma_j} + F_0 = \psi(\lambda, \mu), \quad F_0 = \sum_{j=1}^{n} H_j \lambda^{n-j} \tag{23}$$

where $\gamma_j > n - 1$ for all $j = 1, \ldots, k$ and $\gamma_i \neq \gamma_j$ if $i \neq j$ for all $i, j = 1, \ldots, k$.

Now pick $k \leq n$ distinct $s_i \in \{1, \ldots, n\}$ and define the Hamiltonians $\tilde{H}_i$ via the following separation curve

$$\begin{gathered} \sum_{j=1}^{k} \tilde{H}_{s_j} \lambda^{\gamma_j} + R_{n-s_1}^{\tilde{\alpha}_1} \cdots R_{n-s_k}^{\tilde{\alpha}_k}(\tilde{F}_0) = \psi(\lambda, \mu), \\ \tilde{F}_0 = \sum_{j=1}^{n} \tilde{H}_j \lambda^{n-j}. \end{gathered} \tag{24}$$

MGST and deformations of separation curves IV

Proposition 5 *Under the above assumptions the n-tuple of Hamiltonians $\tilde{H}_i$, $i = 1, \ldots, n$, is Stäckel-equivalent to H_i, $i = 1, \ldots, n$. The n-parameter generalized Stäckel transform in question reads as follows:*

$$\tilde{H}_{s_i} = \det B_i / \det B, \tag{25}$$

where $B = \left\| \begin{matrix} V_{s_1}^{(\gamma_1)} & \ldots & V_{s_1}^{(\gamma_k)} \\ \vdots & \ddots & \vdots \\ V_{s_k}^{(\gamma_1)} & \ldots & V_{s_k}^{(\gamma_k)} \end{matrix} \right\|$ *is a $k \times k$ matrix,*

and B_i are obtained from B by replacing $V_{s_j}^{(\gamma_i)}$ by $H_{s_j}^{(0)} - \tilde{\alpha}_j$ for all $j = 1, \ldots, k$;

$$\tilde{H}_i = H_i^{(0)} + \sum_{j=1}^{k} \tilde{H}_{s_j} V_i^{(\gamma_j)}, \; i = 1, \ldots, r, \; i \neq s_j \text{ for } j = 1, \ldots, k. \tag{26}$$

Example

Let $M = \mathbb{R}^4$ with the coordinates (p_1, p_2, q_1, q_2) and canonical Poisson structure. Let $k = 1$, $r = 2$, $s_1 = 2$, $\alpha_1 \equiv \alpha$ and $\tilde{\alpha}_1 \equiv \tilde{\alpha}$. Consider the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{\alpha(q_1^2 - q_2^2)}{q_2}p_2 - 2\alpha^2 q_1^2,$$

which Poisson commutes with $H_2 = \dfrac{q_1 p_2 - q_2 p_1 - 2\alpha q_1 q_2}{p_2}$.

The relation $H_2|_{[\Phi]} = \tilde{\alpha}$ in our case takes the form

$$\frac{q_1 p_2 - q_2 p_1 - 2\tilde{H}_2 q_1 q_2}{p_2} = \tilde{\alpha},$$

whence $$\tilde{H}_1 = \frac{q_1^2 + q_2^2 - 2\tilde{\alpha} q_1}{2 q_1 q_2} p_1 p_2 + \frac{\tilde{\alpha}(q_1^2 - \tilde{\alpha} q_1 + q_2^2)}{2 q_1 q_2^2} p_2^2,$$

$$\tilde{H}_2 = \frac{q_1 p_2 - q_2 p_1 - \tilde{\alpha} p_2}{2 q_1 q_2}.$$

It is easily verified that $\{H_1, H_2\} = 0$.

Example: continued

By Proposition 3 the reciprocal transformation

$$\tilde{t}_1 = t_1,\ d\tilde{t}_2 = \left(-2q_1p_1 + \frac{(q_1^2 - 2\tilde{\alpha}q_1 + q_2^2)p_2}{q_2}\right)dt_1 + \frac{2q_1q_2}{p_2}dt_2$$

takes the equations of motion for H_1 and H_2, with the respective evolution parameters t_1 and t_2, restricted onto the common level surface $N_{\tilde{\alpha}} = \{x \in \mathbb{R}^4 | H_2(x,\alpha) = \tilde{\alpha}\}$ into the equations of motion for $\tilde{H}_1$ and $\tilde{H}_2$, with the respective evolution parameters $\tilde{t}_1$ and $\tilde{t}_2$, restricted onto the common level surface $\tilde{N}_\alpha = \{x \in \mathbb{R}^4 | \tilde{H}_2(x,\tilde{\alpha}) = \alpha\}$. It is easily seen that $\tilde{N}_\alpha$ and $N_{\tilde{\alpha}}$ represent identical submanifolds of $\mathbb{R}^4$.

Reciprocal transformations and integrable hydrodynamic-type systems

Let L be a (1,1)-tensor with the maximal possible number, n, of distinct, *functionally independent* eigenvalues and vanishing Nijenhuis torsion on an n-dimensional manifold Q.

Consider the following set of tensors of type (1,1) on M:

$$K_1 = \mathbb{I}, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n, \qquad (27)$$

where $\mathbb{I}$ is the $n \times n$ unit matrix, and ρ_i are coefficients of the characteristic polynomial of the tensor L, i.e.,

$$\det(\xi \mathbb{I} - L) = \sum_{i=0}^{n} \rho_i \xi^{n-i}. \qquad (28)$$

Basic separable potentials: an alternative definition

$$\text{Let} \quad V_r^{(k)} = V_{r+1}^{(k-1)} - \rho_r V_1^{(k-1)}, k \in \mathbb{Z}, \tag{29}$$

with the initial condition

$$V_r^{(0)} = -\delta_r^n, \qquad r = 1, \ldots, n. \tag{30}$$

Here and below we tacitly assume that $V_r^{(k)} \equiv 0$ for $r < 1$ or $r > n$.

The recursion (29) can be reversed. The inverse recursion is given by

$$V_r^{(k)} = V_{r-1}^{(k+1)} - \frac{\rho_{r-1}}{\rho_n} V_n^{(k+1)}, \quad k \in \mathbb{Z}, \quad r = 1, \ldots, n. \tag{31}$$

Hence, the first nonconstant potentials are $V_r^{(n)} = \rho_r$ for $k > 0$ and $V_r^{(-1)} = \frac{\rho_{r-1}}{\rho_n}$ for $k < 0$, respectively.

The seed systems

Consider a set of hydrodynamic-type systems of the form

$$K_1^{-1}\boldsymbol{u}_{t_1} = K_2^{-1}\boldsymbol{u}_{t_2} = \cdots = K_n^{-1}\boldsymbol{u}_{t_n} \qquad (32)$$

where $\boldsymbol{u} = (u^1, \ldots, u^n)^T$ are local coordinates on Q, and the superscript T refers to the matrix transposition, t_i are independent variables, K_i^{-1} are tensors of type (1,1) such that $K_i K_i^{-1} = \mathbb{I}$, $i = 1, \ldots, n$, and K_i are given by (27), i.e.,

$$K_1 = \mathbb{I}, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n.$$

Eq.(32) has infinitely many conservation laws of the form

$$D_{t_i}(V_j^{(k)}) = D_{t_j}(V_i^{(k)}),\ i,j = 1,\ldots,n,\ i \neq j,\ k \in \mathbb{Z}, \quad (33)$$

where D_{t_i} are total derivatives computed by virtue of (32). These conservation laws are obviously nontrivial for all integer $k \neq 0, \ldots, n-1$. Most importantly, these conservation laws can be written down *explicitly* in *arbitrary* coordinates, not just in the Riemann invariants.

Reciprocal transformation

Consider the reciprocal transformations we found earlier:

$$
\begin{aligned}
d\tilde{t}_{s_i} &= -\sum_{j=1}^{n} V_j^{(\gamma_i)} dt_j, \quad i = 1, \ldots, k, \\
\tilde{t}_m &= t_m, \; m = 1, 2, \ldots, n, \; m \neq s_a, \; a = 1, \ldots, k.
\end{aligned}
\tag{34}
$$

Here $1 \leq k \leq n$; the numbers s_a, $a = 1, \ldots, k$, are a k-tuple of distinct integers from the set $\{1, \ldots, n\}$, and γ_j are arbitrary positive integers that satisfy the following conditions:

$$\gamma_1 > \gamma_2 > \cdots > \gamma_k > n - 1. \tag{35}$$

The choice of numbers $k \in \{1, \ldots, n\}$, s_a, and γ_a that satisfy the above conditions uniquely determines the transformation (34).

Inverse reciprocal transformation

The inverse of (34) reads

$$dt_{s_i} = -\sum_{j=1}^{n} \tilde{V}_j^{(n-s_i)} d\tilde{t}_j, \quad i = 1, \dots, k,$$
$$t_l = \tilde{t}_l, \quad q = 1, 2, \dots, n, \quad l \neq s_a, \quad a = 1, \dots, k, \tag{36}$$

$\tilde{V}_j^{(m)}$ are *deformed separable potentials* def'd as follows:

1) for $j = s_1, \dots, s_k$ we define $\tilde{V}_{s_i}^{(m)}$ by the relations

$$V_{s_i}^{(m)} + \sum_{p=1}^{k} \tilde{V}_{s_p}^{(m)} V_{s_i}^{(\gamma_p)} = 0, \tag{37}$$

$$\text{whence} \quad \tilde{V}_{s_i}^{(m)} = -\det W_i^{(m)} / \det W, \tag{38}$$

where W is a $k \times k$ matrix of the form

$$W = \left\| \begin{matrix} V_{s_1}^{(\gamma_1)} & \dots & V_{s_1}^{(\gamma_k)} \\ \vdots & \ddots & \vdots \\ V_{s_k}^{(\gamma_1)} & \dots & V_{s_k}^{(\gamma_k)} \end{matrix} \right\|, \tag{39}$$

and $W_i^{(m)}$ are obtained from W by replacing $V_{s_j}^{(\gamma_i)}$ by $V_{s_j}^{(m)}$ for all $j = 1, \dots, k$;

Inverse reciprocal transformation – continued

2) for $j \neq s_1, \dots, s_k$ we set

$$\tilde{V}_j^{(m)} = V_j^{(m)} + \sum_{p=1}^{k} \tilde{V}_{s_p}^{(m)} V_j^{(\gamma_p)}, \qquad (40)$$

or equivalently

$$\tilde{V}_j^{(m)} = \det \widehat{W}_j^{(m)} / \det W, \qquad (41)$$

where $\widehat{W}_j^{(m)}$ is a $(k+1) \times (k+1)$ matrix of the form

$$\widehat{W}_j^{(m)} = \left\| \begin{matrix} V_j^{(m)} & V_j^{(\gamma_1)} & \dots & V_j^{(\gamma_k)} \\ V_{s_1}^{(m)} & V_{s_1}^{(\gamma_1)} & \dots & V_{s_1}^{(\gamma_k)} \\ \vdots & \vdots & \ddots & \vdots \\ V_{s_1}^{(m)} & V_{s_k}^{(\gamma_1)} & \dots & V_{s_k}^{(\gamma_k)} \end{matrix} \right\|. \qquad (42)$$

It can be shown that the above definition of $\tilde{V}_i^{(j)}$ is equivalent to the one we used earlier, with λ_i being the eigenvalues of L.

Transformed seed systems

The reciprocal transformation (34), i.e.,

$$d\tilde{t}_{s_i} = -\sum_{j=1}^{n} V_j^{(\gamma_i)} dt_j, \quad i = 1, \ldots, k,$$

$$\tilde{t}_m = t_m, \; m = 1, 2, \ldots, n, \; m \neq s_a, \; a = 1, \ldots, k,$$

sends the set (32) of seed systems into the following set:

$$\tilde{K}_1^{-1} \boldsymbol{u}_{\tilde{t}_1} = \tilde{K}_2^{-1} \boldsymbol{u}_{\tilde{t}_2} = \cdots = \tilde{K}_n^{-1} \boldsymbol{u}_{\tilde{t}_n}, \tag{43}$$

where
$$\tilde{K}_{s_i} = -\sum_{j=1}^{k} \tilde{V}_{s_i}^{(n-s_j)} K_{s_j} B^{-1}, \quad i = 1, \ldots, k,$$

$$\tilde{K}_m = K_m B^{-1} - \sum_{l=1}^{k} \tilde{V}_m^{(n-s_l)} K_{s_l} B^{-1}, \tag{44}$$

$$m = 1, 2, \ldots, n, \; m \neq s_a \quad \text{for any} \quad a = 1, \ldots, k,$$

$$B = -\det W_{s_1} / \det W, \tag{45}$$

W is given by (39), and W_{s_1} is obtained from W by replacing $V_{s_j}^{(\gamma_1)}$ by K_{s_j} for all $j = 1, \ldots, k$. Here $\det W_{s_1}$ is a formal determinant with matrix-valued entries.

Conservation laws for the transformed seed systems

Eq.(43), i.e.,

$$\tilde{K}_1^{-1}\boldsymbol{u}_{\tilde{t}_1} = \tilde{K}_2^{-1}\boldsymbol{u}_{\tilde{t}_2} = \cdots = \tilde{K}_n^{-1}\boldsymbol{u}_{\tilde{t}_n},$$

possesses the following infinite set of nontrivial conservation laws similar to (33):

$$D_{\tilde{t}_i}(\tilde{V}_j^{(m)}) = D_{\tilde{t}_j}(\tilde{V}_i^{(m)}), \quad i,j = 1,\ldots,n, \quad i \neq j,$$
$$m \in \mathbb{Z}, \quad m \neq \gamma_l, \quad l = 1,\ldots,k, \qquad (46)$$

$$m \notin (\{1,\ldots,n\}/\{s_1,\ldots,s_k\}),$$

where the derivatives $D_{\tilde{t}_i}$ are computed by virtue of (43).

The hidden metric

Any tensor L of type (1,1) with zero Nijenhuis torsion and n distinct, functionally independent eigenvalues always is an L-tensor (i.e., a special conformal Killing tensor) for some family of metrics on Q. In fact, in the coordinate frame associated with the eigenvalues λ^i, $i = 1, \ldots, n$, of L, the most general family of such contravariant metrics is given by

$$G = \text{diag}\left(\frac{f_1(\lambda_1)}{\Delta_1}, \ldots, \frac{f_n(\lambda_n)}{\Delta_n}\right), \qquad (47)$$

where $\Delta_i = \prod\limits_{j \neq i} (\lambda^i - \lambda^j)$. The quantities K_i (27) are then Killing tensors of type (1,1) for any metric tensor from the family (47).

The Hidden Metric II

Likewise, the quantities K_i (27) are the Killing tensors of type (1,1) for any metric tensor from the family $\tilde{G} = BG$, where G is given by (47) and, as before,

$$B = -\det W_{s_1} / \det W,$$

where

$$W = \left\| \begin{array}{ccc} V_{s_1}^{(\gamma_1)} & \dots & V_{s_1}^{(\gamma_k)} \\ \vdots & \ddots & \vdots \\ V_{s_k}^{(\gamma_1)} & \dots & V_{s_k}^{(\gamma_k)} \end{array} \right\|,$$

and W_{s_1} is obtained from W by replacing $V_{s_j}^{(\gamma_1)}$ by K_{s_j} for all $j = 1, \dots, k$.

Back to the Hamiltonians

Thus, the reciprocal transformation for the seed hydrodynamic-type systems is the same one as discussed in the context of the separation curves.

Namely, the reciprocal transformation in question is induced by the multiparameter generalized Stäckel transform relating the family of Hamiltonians on $M = T^*Q$

$$H_i = \frac{1}{2}(\boldsymbol{\mu}, K_i G \boldsymbol{\mu}) + \sum_{j=1}^{k} \alpha_j V_i^{(\gamma_j)},$$

to the family

$$\tilde{H}_i = \frac{1}{2}(\boldsymbol{\mu}, \tilde{K}_i \tilde{G} \boldsymbol{\mu}) + \sum_{j=1}^{k} \tilde{\alpha}_j \tilde{V}_i^{(\gamma_j)}.$$

The Riemann invariants

The Riemann invariants for the systems we dealt with,

$$K_1^{-1}\boldsymbol{u}_{t_1} = K_2^{-1}\boldsymbol{u}_{t_2} = \cdots = K_n^{-1}\boldsymbol{u}_{t_n},$$

$$\tilde{K}_1^{-1}\boldsymbol{u}_{\tilde{t}_1} = \tilde{K}_2^{-1}\boldsymbol{u}_{\tilde{t}_2} = \cdots = \tilde{K}_n^{-1}\boldsymbol{u}_{\tilde{t}_n},$$

where $K_1 = \mathbb{I}, \quad K_r = \sum_{k=0}^{r-1} \rho_k L^{r-1-k}, \quad r = 2, \ldots, n,$

are simply the eigenvalues λ^i of L.

Upon introducing new dependent variables λ^i (i.e., new coordinates on Q) instead of u^i we readily find that the above systems are weakly nonlinear (=linearly degenerate) and semi-Hamiltonian, and thus can be solved using the generalized hodograph method of Tsarev.

Conclusions

- We found a *multiparameter generalization* of the classical Stäckel transform which, under certain technical assumptions, preserves (super)integrability and noncommutative integrability

- This *multiparameter generalized Stäckel transform* includes the separation of variables in the Hamilton–Jacobi equation as a particular case

- The corresponding transformation for the equations of motion is the *reciprocal transformation* of a special form with interesting applications in the theory of hydrodynamic-type systems

For further details see *J. Phys. A: Math. Theor.* **41**(2008) paper 105205 (arXiv: **0706.1473**) and arXiv: **0803.0308**