BI-HAMILTONIAN STRUCTURES FOR INTEGRABLE SYSTEMS ON TIME SCALES

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"30 YEARS OF BI-HAMILTONIAN SYSTEMS"

Time Scale Calculus

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$*$ $\mathbb{Q}, \quad \mathbb{R} - \mathbb{Q}$,

$*$ The open intervals

are not time scales.

Jump Operators

Let $\mathbb{T}$ be a time scale. $\forall x \in \mathbb{T}$, the forward and the backward jump operators are defined respectively

$$\sigma(x) := \inf\{y \in \mathbb{T} : y > x\}$$
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x is called

$*$ right dense if $\sigma(x) = x$, right scattered if $\sigma(x) > x$,

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The graininess functions

$$\mu(x) := \sigma(x) - x, \quad \nu(x) := x - \rho(x), \quad \text{for all } x \in \mathbb{T}.$$

Δ-Derivative

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A function $f : \mathbb{T} \to \mathbb{R}$ is Δ-smooth if it is infinitely Δ-differentiable at all points of $\mathbb{T}^\kappa$.

Example

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If $\mathbb{T} = \mathbb{K}_q$ for $x \neq 0$ then $\quad \sigma(x) = qx, \quad \rho(x) = q^{-1}x$

$$\Delta f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

Δ-Integral

A function $F : \mathbb{T} \to \mathbb{R}$ is called **Δ-antiderivative** of $f : \mathbb{T} \to \mathbb{R}$ if $\Delta F(x) = f(x)$, $\forall\, x \in \mathbb{T}^{\kappa}$. Thus Δ-integral of f from a to b is

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Δ-integral over an whole time scale is given as

$$\int_{\mathbb{T}} f(x)\Delta x := \int_{x_*}^{x^*} f(x)\Delta x = \lim_{x \to x^*} F(x) - \lim_{x \to x_*} F(x)$$

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Thus the **inverse** E^{-1} is deduced

$$(E^{-1}f)(x) = f(\sigma^{-1}(x)) = f(\rho(x)).$$

Algebra of δ-differential operators

Let $\mathcal{G}$ be an algebra of δ-**differential operators** equipped with the commutator,

$$\mathcal{G} = \mathcal{G}_{\geqslant k} \oplus \mathcal{G}_{<k} = \{\sum_{i\geqslant k} u_i(x)\delta^i\} \oplus \{\sum_{i<k} u_i(x)\delta^i\}.$$

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The classical R-matrix

$$R := \frac{1}{2}(P_{\geqslant k} - P_{<k}) \qquad k = 0, 1.$$

The Lax Hierarchy

The Lax hierarchy of commuting evolution equations is

$$\frac{dL}{dt_n} = \left[\left(L^{\frac{n}{N}}\right)_{\geqslant k}, L\right] \qquad k = 0, 1.$$

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The admissible finite-field Lax operators are

$$L = u_N \delta^N + u_{N-1} \delta^{N-1} + \ldots + u_1 \delta + u_0 + \delta^{-1} u_{-1} + \sum_s \psi_s \delta^{-1} \varphi_s,$$

where for $k = 0$, u_N is time-independent and $u_{-1} = 0$.

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The natural **constraints** between dynamical fields

$$\sum_{i=-k}^{N+k-1} (-\mu)^i u_i - \mu \sum_s \psi_s \varphi_s = a,$$

The Continuous Limit of a Time Scale

$$\mathbb{T} = \hbar\mathbb{Z} \xrightarrow{\hbar \to 0} \mathbb{T} = \mathbb{R};$$

while

$$\mathbb{T} = \mathbb{K}_q \xrightarrow{q \to 1} \mathbb{T} = \mathbb{R}_+ \cup \{0\}.$$

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In the continuous time scale, the algebra of δ-differential operators turns out to be the algebra of pseudo-differential operators

$$\mathcal{G} = \mathcal{G}_{\geqslant k} \oplus \mathcal{G}_{<k} = \{\sum_{i\geqslant k} u_i(x)\partial^i\} \oplus \{\sum_{i<k} u_i(x)\partial^i\},$$

which is valid for $k = 0, 1, 2$ and $\partial u = u_x + u\partial$.

The Trace Functional

The **trace** form is introduced by

$$\mathrm{Tr} A := -\int_{\mathbb{T}} \frac{1}{\mu} \left(A_{<0}\right)\big|_{\delta=-\frac{1}{\mu}} \Delta x \equiv \int_{\mathbb{T}} \sum_{i<0} a_i (-\mu)^{-i-1} \Delta x,$$

where $A = \sum_i a_i \delta^i$.

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Note: When $\mu = 0$, $\mathrm{Tr} A = \int_{\mathbb{R}} a_{-1} dx$ for $A = \sum_i a_i \partial^i$.
Theorem: The **inner product** on $\mathcal{G}$ defined by bilinear map

$$(\cdot,\cdot)_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \to \mathbb{K} \qquad (A, B)_{\mathcal{G}} := \mathrm{Tr}(AB),$$

is **nondegenerate**, **symmetric** and **ad-invariant**.

The Poisson Tensors

Theorem: The **Linear Poisson tensor** has the form

$$\begin{aligned}\pi_0 dH &= [RdH, L] + R^\dagger [dH, L] \\ &= [L, dH_{<k}] + \left([dH, L](1+\mu\delta)^{-1}\right)_{<-k}(1+\mu\delta).\end{aligned}$$

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$$= [L, dH_{<k}] + \left([dH, L](1+\mu\delta)^{-1}\right)_{<-k}(1+\mu\delta).$$

Since the recursion operators Φ s.t

$$\Phi L_{t_n} = L_{t_{n+N}}.$$

are hereditary, the **quadratic Poisson tensor** π_1 can be reconstructed from Φ:

$$\pi_1 = \Phi\pi_0.$$

Bi-Hamiltonian Structure

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where the related **Hamiltonians** are given by

$$H_n(L) = \frac{N}{n+N} \mathrm{Tr}\left(L^{\frac{n}{N}+1}\right).$$

such that

$$dH_n = L^{\frac{n}{N}}$$

Example: Δ-differential AKNS

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The **first flow** is

$$\psi_{t_1} = \mu\psi^2\varphi + \Delta\psi,$$
$$\varphi_{t_1} = -\mu\varphi^2\psi + \Delta E^{-1}\varphi.$$

The **second flow** is

$$\psi_{t_2} = \psi(\Delta\mu\psi\varphi + (\mu\psi\varphi)^2 + \varphi E(\psi) + \psi E^{-1}(\varphi))$$
$$+(E+1)(\mu\psi\varphi\Delta\psi) + \Delta^2\psi,$$
$$\varphi_{t_2} = -\varphi(\Delta\mu\psi\varphi + (\mu\psi\varphi)^2 + \varphi E(\psi) + \psi E^{-1}(\varphi))$$
$$+\Delta E^{-1}(\varphi(E+1)\mu\psi\varphi) - (\Delta E^{-1})^2\varphi.$$

which is Δ-differential counterpart of soliton AKNS.

Continuous AKNS

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$$\frac{d\psi}{dt_1} = \psi_x, \qquad \frac{d\varphi}{dt_1} = \varphi_x.$$

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The **second flow** implies the AKNS equation

$$\frac{d\psi}{dt_2} = \psi_{xx} + 2\psi^2\varphi,$$
$$\frac{d\varphi}{dt_2} = -\varphi_{xx} - 2\varphi^2\psi$$

Differentials

The implicit form of the differentials

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such that the following holds

$$(dH, L_t)_{\mathfrak{g}} = \int_{\mathbb{T}} \left(\sum_{i-k}^{N+k-2} \frac{\delta F}{\delta u_i}(u_i)_t + \sum_{s} \left(\frac{\delta F}{\delta \psi_s}(\psi_s)_t + \frac{\delta F}{\delta \phi_s}(\psi_s)_t \right) \right) \Delta x.$$

Differential of AKNS

For AKNS: $N = 1$, $n = 2$, $k = 0$

The differential is of the form

$$dH = \gamma_1 + \delta\gamma_2$$

where

$$\begin{aligned}
\gamma_1 &= \frac{1}{\varphi}\frac{\delta H}{\delta\psi} - \frac{\Delta E^{-1}(\varphi)}{\psi\varphi E^{-1}(\varphi)}\Delta^{-1}(A) \\
\gamma_2 &= -\frac{1}{\psi E^{-1}(\varphi)}\Delta^{-1}(A) \\
A &= \frac{1}{\psi}\frac{\delta H}{\delta\psi} - \varphi\frac{\delta H}{\delta\varphi}
\end{aligned}$$

Bi-Hamiltonian structure for AKNS

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$$\pi_1 = \Phi\pi_0 = \begin{pmatrix} -\mu\psi^2 - 2\psi\Delta^{-1}\psi & \Delta + 2\mu\psi\varphi + 2\psi\Delta^{-1}\varphi \\ -\Delta^\dagger + 2\varphi\Delta^{-1}\psi & -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi \end{pmatrix}$$

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where $\Delta^{\dagger} = -\Delta E^{-1}$ and Recursion Operator is

$$\Phi = \begin{pmatrix} \Delta + 2\mu\psi\varphi + 2\psi\Delta^{-1}\varphi & \mu\psi^2 + 2\psi\Delta^{-1}\psi \\ -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi & \Delta^{\dagger} - 2\varphi\Delta^{-1}\psi \end{pmatrix}$$

Bi-Hamiltonian structure for AKNS

The related first three Hamiltonians are

$$H_0 = \mathrm{Tr}(L) = \int_{\mathbb{T}} \psi\varphi\Delta x$$

$$H_1 = \frac{1}{2}\mathrm{Tr}(L^2) = \int_{\mathbb{T}} \left(\varphi\Delta\psi + \frac{1}{2}\mu\psi^2\varphi^2\right)\Delta x$$

$$H_2 = \frac{1}{3}\mathrm{Tr}(L^3) = \int_{\mathbb{T}} \left(\varphi\Delta^2\psi - \psi^2\varphi^2 + \psi\varphi^2 E\psi + \frac{1}{3}\psi\varphi E^{-1}(\psi\varphi)\right.$$

$$\left. + \frac{2}{3}\psi^2\varphi E^{-1}\varphi + \mu\psi^3\varphi^3 - \frac{2}{3}\mu^2\psi^3\varphi^3\right)\Delta x$$

Remark:

Note that when $\mathbb{T} = \mathbb{R}$, or **in the continuous limit of some special time scales**: the Linear Poisson Tensor

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$$\pi_1 = \begin{pmatrix} -2\psi D^{-1}\psi & D + 2\psi D^{-1}\varphi \\ D + 2\varphi \Delta^{-1}\psi & -2\varphi D^{-1}\varphi \end{pmatrix}$$

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and Hamiltonians

$$H_0 = \int_{\mathbb{R}} \psi\varphi dx, \qquad H_1 = \int_{\mathbb{R}} \varphi\psi_x dx$$

Δ-differential Kaup-Broer, $k = 1$

Let

$$L = (1 + \mu v - \mu^2 w)\delta + v + \delta^{-1} w.$$

The first flow

$$\frac{dv}{dt_1} = (\mu v - \mu^2 w)\Delta v + \mu \Delta E^{-1}(w(\mu v - \mu^2 w)),$$

$$\frac{dw}{dt_1} = \Delta E^{-1}\left(\mu v w - \mu^2 w^2\right)$$

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The linear Poisson tensor is given by

$$\pi_0 = \begin{pmatrix} \tilde{u}\Delta\mu - \mu\Delta^\dagger \tilde{u} & \tilde{u}\Delta \\ -\Delta^\dagger \tilde{u} & 0 \end{pmatrix}$$

Δ-differential Kaup-Broer

The Quadratic Poisson Tensor

$$\pi_1 = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}$$

where

$$\begin{aligned}
\pi_{11} &= \tilde{u}\Delta\mu v - \mu v\Delta^\dagger u + \tilde{u}\Delta\tilde{u} \\
&- \tilde{u}\Delta^\dagger\tilde{u} + \tilde{u}\Delta\tilde{u}\Delta\mu - \mu\Delta^\dagger\tilde{u}\Delta^\dagger\tilde{u} + \mu\tilde{u}w\Delta\mu - \mu\Delta^\dagger\mu\tilde{u}w \\
\pi_{12} &= \tilde{u}\Delta v + \tilde{u}\Delta\tilde{u}\Delta + \mu\tilde{u}w\Delta - \mu\Delta^\dagger\tilde{u}w \\
\pi_{21} &= -v\Delta^\dagger\tilde{u} - \Delta^\dagger\tilde{u}\Delta^\dagger\tilde{u} + \tilde{u}w\Delta\mu - \Delta^\dagger\mu\tilde{u}w \\
\pi_{22} &= \tilde{u}w\Delta - \Delta^\dagger\tilde{u}w
\end{aligned}$$

Δ-differential Kaup-Broer

The Hamiltonians are

$$H_0 = \int_{\mathbb{T}} w \, \Delta x$$

$$H_1 = \int_{\mathbb{T}} \left(vw - \frac{1}{2}\mu w^2 \right) \Delta x$$

$$H_2 = \int_{\mathbb{T}} \left(w^2 + v^2 w + \frac{2}{3} w \Delta v + \frac{2}{3}\mu v w \Delta v - \frac{2}{3}\mu^2 w^2 \Delta v - \frac{2}{3}\mu^2 w^3 \right) \Delta x$$

$$\vdots \; .$$

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THANK YOU FOR YOUR ATTENTION!

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The recursion operators of the finite field N^{th} order Lax hierarchies can be constructed solving

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$$L_{t_{n+N}} = L_{t_n} L + [R, L] \tag{7}$$

For $k = 0$, with the remainder in the form

$$R = a_{N-1}\delta^{N-1} + \cdots + a_0 + \sum_s a_{-1,s}\delta^{-1}\varphi_s, \tag{8}$$

Recursion Operators

The recursion operators of the finite field N^{th} order Lax hierarchies can be constructed solving

$$L_{t_{n+N}} = L_{t_n} L + [R, L] \tag{10}$$

For $k = 0$, with the remainder in the form

$$R = a_{N-1}\delta^{N-1} + \cdots + a_0 + \sum_s a_{-1,s}\delta^{-1}\varphi_s, \tag{11}$$

For $k = 1$,

$$R = a_N\delta^N + \cdots + a_0 + \sum_s a_{-1,s}\delta^{-1}\varphi_s. \tag{12}$$

The appropriate reminder is found for AKNS

$$R = \Delta^{-1}\left(\mu^{-1} u_{t_n}\right) - \psi_{t_n}\delta^{-1}\varphi. \tag{13}$$

The appropriate reminder is found for AKNS

$$R = \Delta^{-1}\left(\mu^{-1} u_{t_n}\right) - \psi_{t_n} \delta^{-1} \varphi. \tag{15}$$

Thus the recursion formula

$$\begin{pmatrix} u - \mu^{-1} & \phi E & \psi E^{-1} \\ \psi + \psi\Delta^{-1}\mu^{-1} & \Delta + u + \psi\Delta^{-1}\varphi & \psi\Delta^{-1}\psi \\ -\varphi\Delta^{-1}\mu^{-1} & -\varphi E\Delta^{-1}\varphi & u - \Delta E^{-1} - \varphi E \Delta^{-1}\psi \end{pmatrix} \tag{16}$$

when $\mu \neq 0$.

Otherwise, with the constraint

$$\begin{pmatrix} \Delta + u + \mu\psi\varphi + 2\psi\Delta^{-1}\varphi & \mu\psi^2 + 2\psi\Delta^{-1}\psi \\ -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi & \Delta^{\dagger} - 2\varphi\Delta^{-1}\psi. \end{pmatrix} \tag{17}$$

Otherwise, with the constraint

$$\begin{pmatrix} \Delta + u + \mu\psi\varphi + 2\psi\Delta^{-1}\varphi & \mu\psi^2 + 2\psi\Delta^{-1}\psi \\ -\mu\varphi^2 - 2\varphi\Delta^{-1}\varphi & \Delta^\dagger - 2\varphi\Delta^{-1}\psi. \end{pmatrix} \tag{19}$$

When $\mathbb{T} = \mathbb{R}$, the recursion formula reduces to

$$\begin{pmatrix} D + 2\psi D^{-1}\varphi & 2\psi D^{-1}\psi \\ -2\varphi D^{-1}\varphi & -D_x - 2\varphi D_x^{-1}\psi \end{pmatrix}. \tag{20}$$